A LEVEL SET METHOD FOR THE COMPUTATION OF MULTIVALUED SOLUTIONS TO QUASI-LINEAR HYPERBOLIC PDES AND HAMILTON-JACOBI EQUATIONS

SHI JIN* AND STANLEY J. OSHER†

Abstract. We develop a level set method for the computation of multivalued solutions to quasi-linear hyperbolic partial differential equations and Hamilton-Jacobi equations in any number of space dimensions. The idea is to define the solution of the quasi-linear hyperbolic PDEs or the gradient of the solution to the Hamilton-Jacobi equations as zero level sets of level set functions. We then derive the evolution equations for the level set functions which, surprisingly, satisfy *linear Liouville equations*. By using the local level set method the cost of each time update for this method is $O(N^d \log N)$ for a d dimensional problem, where N is the number of grid points in each dimension.

Key words. Hyperbolic PDEs, Hamilton-Jacobi equations, level set method, multivalued solution, Liouville equation

AMS subject classifications.

1. Introduction.

Many physical problems require the computations of multivalued solutions. Such problems arise in the computations of dispersive waves [37, 24, 13], geometric optics [9, 27] , semiclassical linear and nonlinear Schrödinger equations [14, 22, 21, 33], multiple arrival in tomography and seismic migration [12, 34, 36], electron beam modulation in vacuum electronic devices (such as Klystron) [18, 25], etc. Although these problems are often, in the continuum limit, described by nonlinear hyperbolic partial differential equations or Hamilton-Jacobi equations, the classical entropy or viscosity solutions are not adequate in describing the post singularity behavior. The underlying physics is often defined by Hamiltonian systems which are nondissipative, often dispersive, and the solution can be superimposed in the phase space. Direct numerical discretizations of these nonlinear PDEs often yield viscosity solutions which are irreversible, thus violating underlying physical principles, such as superposition.

There are two classes of methods used to compute multivalued solutions. A classical technique is ray tracing, which is a Lagrangian method that solves a set of ordinary differential equations in order to trace the wavefronts. This method is easy to implement, but may encounter difficulties in spatial resolution when points close initially may diverge at later times. This results in the loss of numerical acuracy unless a regridding and/or interpolation procedure is implemented from time to time. Another class involves Eulerian methods, which solve partial differential equations on a fixed grid, see e.g. [2],[3]. The new difficulty lies in the fact that solution becomes multivalued in the physical space beyond singularity (caustic) formation.

In recent years, there has developed a growing interest in the computation of multivalued solutions of these nonlinear PDEs using Eulerian methods. Methods based in physical space often use moment closure of the classical Liouville equations. This approach started with the work of Brenier and colleagues [4, 5] and has been studied numerically by Engquist and Runborg [9] and Gosse [16] in the context of geometric optics. For linear Schrödinger equations multivalued solutions were studied

* Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA. jin@math.wisc.edu. Research was supported in part by NSF grant DMS-0196106.

† Level Set Systems Inc., 1058 Embury St., Pacific Palisades, CA 90272. sjo@levelset.com. Research supported by AFOSR Grant No. F49620-01-1-0189.

in Sparber, Markowich and Mauser [33] and Jin and Li [22], and numerical studies of the resulting moment systems were studied in Jin and Li [22], Gosse, Jin and Li [17]. Computations based on moment systems are efficient when the number of phases is low in one space dimension. They become complex when the number of phases becomes large, and in multidimensions. For the computation of wavefronts, a Liouville equation based phase space techniques, using the segmentation projection method, was introduced by Engquist, Runborg and Tornberg [10], or using the level set method, was introduced by Osher et. al [27], [31], [8]. For computations of multiple arrivals using static Hamilton-Jacobi equation, a Liouville equation based method was also introduced by Fomel and Sethian [12]. In all these newly developed Liouville based methods the goal was to compute the *wave front* instead of the solution in the entire physical domain.

In this paper we present a generic level set method for multidimensional quasilinear scalar hyperbolic PDEs and Hamilton-Jacobi equations that allows the computations of the solution in the *entire* physical domain. The basic idea for hyperbolic PDEs can be traced back to Jacobi [7], and was recycled by Osher [26], then Evans, [11], Giga, [15] and Tsai, Giga and Osher [35] for the computation of viscosity solutions. The novelty of this paper is to use this formulation as a general framework for the computation of multivalued solutions, which has never been done before. The basic idea is to build the entire solution– a surface rather than only a curve in two dimension, for example–into the zero level set of level set functions. Moreover, we show how this idea can be extended for general Hamilton-Jacobi equations, which is new (to our knowledge). The key idea here is to build the *gradient* into the zero level set, namely, using the conservation law (the gradient) form of the Hamilton-Jacobi equation, rather than the original Hamilton-Jacobi equation. This allows us to obtain a linear Liouville equation (with variable coefficients) in what resembles phase space. Due to the linearity of the level set Liouvile equation, the singularities of nonlinear PDEs are unfolded and linear superposition is preserved in this "phase" space. The intersection of the zero level sets of d such level set functions in d-space dimensions yield the desired multivalued solutions in the physical space. This follows from the vector level set method introduced in [6] and used in [27], [31], [8]. Both time dependent and static Hamilton-Jacobi equations can be treated in this fashion.

Although the level set equation is defined in the "phase" space, a local level set method can be used [1], [30] since the area of interest is near the zero level set. When the computation is restricted to the region near the zero level set, we have a fast level set method whose computational cost is $O(N^d \log N)$ in d-space dimension where N is the number of grid points in each space dimension.

In Osher, et al [27],[31],[8] the authors were concerned with following a curve in $\Re^2$ or a surface in $\Re^3$ as each point evolves according to the bicharacteristics of the associated Hamiltonian. This is implemented by viewing the set $\Gamma(t) \in \Re^d$ as the common zero level set of a functions of 2d - 1 variables. Each update of the Liouville equation involves therefore $O(N^{d-1} \log N)$ for the eikonal equation and $O(N^d \log N)$ operations for general Hamiltonians.

In Fomel and Sethian [12] the authors compute *all* arrival times for a set of codimension one in $\Re^d$. One level set function is used in a static setting. Thus each update involves $O(N^{2d-1} \log N)$ operations for the eikonal equation.

In our present work we take an entire region of zero codimension in $\Re^2$ or $\Re^3$ (or $\Re^d$) and update it according to the bicharacteristic flow, i.e. we solve an initial value problem for the multivalued solution with data given on a subset of $\Re^d$. We use d

level set functions of 2d variables. Thus every update has complexity $O(N^d \log N)$ in general.

2. A Level Set Equation for Multivalued solutions.

In this section we derive our level set equation for multivalued solutions in multidimensional hyperbolic PDEs and Hamilton-Jacobi equations.

Consider $\mathbf{x} = (x_1, x_2, \cdots x_d) \in \Re^d$.

2.1. Multidimensional hyperbolic PDEs. Let $u(t, \mathbf{x}) \in \Re$ be a scalar satisfying an initial value problem of an d-dimensional first order hyperbolic PDE with source term:

$$\partial_t u + \mathbf{F}(u) \cdot \nabla_{\mathbf{x}} u + q(\mathbf{x}) = 0\,, \tag{2.1}$$

$$u(0, \mathbf{x}) = u_0(\mathbf{x})\,. \tag{2.2}$$

Here $\mathbf{F}(u) : \Re^d \to \Re^d$ is a vector, and $q : \Re^d \to \Re$ is the source term. This equation includes any such hyperbolic PDE in conservative or nonconservative form. We introduce a level set function $\phi(t, \mathbf{x}, p)$ in dimension $d + 1$, whose zero level set is the solution u:

$$\phi(t, \mathbf{x}, p) = 0 \qquad \text{at} \quad p = u(t, \mathbf{x})\,. \tag{2.3}$$

Therefore we evolve the entire solution u as the zero level set of ϕ.

Taking the time derivative of (2.3), one obtains

$$\partial_t \phi + \partial_p \phi \, \partial_t p = 0\,, \tag{2.4}$$

or, by using the equation (2.1) and $p = u$,

$$\partial_t \phi - \partial_p \phi \, \mathbf{F} \cdot \nabla_{\mathbf{x}} u - q(\mathbf{x}) \partial_p \phi = 0\,. \tag{2.5}$$

Taking the gradient in $\mathbf{x}$ on (2.3) one has

$$\nabla_{\mathbf{x}} \phi + \partial_p \phi \, \nabla_{\mathbf{x}} u = 0\,. \tag{2.6}$$

Applying (2.6) in (2.5), one obtains

$$\partial_t \phi + \mathbf{F}(p) \cdot \nabla_{\mathbf{x}} \phi - q(\mathbf{x}) \, \partial_p \phi = 0\,. \tag{2.7}$$

This is our level set equation. It resembles a Liouville equation, which is linear hyperbolic with variable coefficients, in $d + 1$ space variables. Of course $x \in \Re^d$ and $p \in \Re^1$, so this is not a true Liouville equation, for $d > 1$. For the Hamilton-Jacobi equation analyzed below, we will arrive at a true Liouville equation. However, even there, p does not denote phase in the classical sense. Instead it is just the auxiliary set of variables needed to generate the level set motion.

For smooth initial data $u_0(x)$, the initial condition for ϕ can be chosen simply as

$$\phi(0, \mathbf{x}, p) = p - u_0(\mathbf{x})\,. \tag{2.8}$$

However, if the initial data is discontinuous, such as in the Riemann problem, such a choice of the initial level set will miss the line that connects the two constant states, as cam be seen from the exact solution of the Liouville equation which can be solved analytically in 1-d using the method of characteristics. In this case, a good choice for initial level set function is the *signed distance function* to the interface $p = u_0(\mathbf{x})$, which is continuous and will remain continuous for all time due to the linearity of the Liouville equation. Thus our level set method always select the continuous multivalued solution.

2.2. Time dependent Hamilton-Jacobi equations. Consider the time dependent, d-dimensional Hamilton-Jacobi equation

$$\partial_t S + H(\mathbf{x}, \nabla_{\mathbf{x}} S) = 0\,, \tag{2.9}$$
$$S(0, \mathbf{x}) = S_0(\mathbf{x})\,. \tag{2.10}$$

Introduce $\mathbf{u} = (u_1, \cdots, u_d) = \nabla_{\mathbf{x}} S$. Taking the gradient on (2.9), one gets an equivalent (at least for smooth solutions) form of the Hamilton-Jacobi equation

$$\partial_t \mathbf{u} + \nabla_{\mathbf{x}} H(\mathbf{x}, \mathbf{u}) = 0\,, \tag{2.11}$$
$$\mathbf{u}(0, \mathbf{x}) \equiv \mathbf{u}_0(\mathbf{x}) = \nabla_{\mathbf{x}} S_0(\mathbf{x})\,. \tag{2.12}$$

This conservation law formulation and its equivalence with the original Hamilton-Jacobi equation, in the sense of viscosity solution, was discussed in [23] and used to construct numerical schemes for the original Hamilton-Jacobi equation. It can also be used to construct the discontinuous Galerkin method for Hamilton-Jacobi equations [19]. A key condition to guarantee the condition is that $\mathbf{u}$ *remains a gradient*:

$$\mathbf{u} = \nabla_{\mathbf{x}} S\,, \tag{2.13}$$

which ensures that

$$\nabla_{\mathbf{x}} \mathbf{u} = \nabla_{\mathbf{x}}^2 S \tag{2.14}$$

is the Hessian matrix of S thus is *symmetric*, namely,

$$(\nabla_{\mathbf{x}} \mathbf{u})^T = \nabla_{\mathbf{x}} \mathbf{u}\,. \tag{2.15}$$

As will be seen later, this is a critical condition for our derivation of the level set equation.

We use d level set functions $\phi_i = \phi_i(t, \mathbf{x}, \mathbf{p}),\ i = 1, \cdots, d$, where $\mathbf{p} = (p_1, \cdots, p_d) \in \Re^d$, such that the intersection of their zero level sets yields $\mathbf{u}$, namely,

$$\phi_i(t, \mathbf{x}, \mathbf{p}) = 0 \qquad \text{at} \quad \mathbf{p} = \mathbf{u}(t, \mathbf{x})\,, \qquad i = 1, \cdots, d \tag{2.16}$$

We now derive the equation for ϕ_i $(i = 1, \cdots, d)$ as in the preceding subsection.

Taking the time derivative of (2.16), one gets

$$\partial_t \phi_i + \nabla_{\mathbf{p}} \phi_i \cdot \partial_t \mathbf{p} = 0 \tag{2.17}$$

or, by using equation (2.11) and $\mathbf{p} = \mathbf{u}$,

$$\partial_t \phi_i - \nabla_{\mathbf{p}} \phi_i \cdot \nabla_{\mathbf{x}} H(\mathbf{x}, \mathbf{p}) = 0\,. \tag{2.18}$$

Using $\nabla_{\mathbf{x}} H(\mathbf{x}, \mathbf{p}) = \nabla_{\mathbf{x}} \mathbf{u} \nabla_{\mathbf{p}} H + \nabla_{\mathbf{x}} H$ in (2.18) one gets

$$\partial_t \phi_i - \nabla_{\mathbf{p}} \phi_i \cdot \nabla_{\mathbf{x}} \mathbf{u} \nabla_{\mathbf{p}} H - \nabla_{\mathbf{p}} \phi_i \cdot \nabla_{\mathbf{x}} H = 0\,. \tag{2.19}$$

Taking the gradient in $\mathbf{x}$ on (2.16) one has

$$\nabla_{\mathbf{x}} \phi_i + \nabla_{\mathbf{x}} \mathbf{u} \nabla_{\mathbf{p}} \phi_i = 0\,. \tag{2.20}$$

Note

$$\begin{aligned} \nabla_{\mathbf{p}} \phi_i \cdot \nabla_{\mathbf{x}} \mathbf{u} \nabla_{\mathbf{p}} H \quad &= (\nabla_{\mathbf{p}} H)^T (\nabla_{\mathbf{x}} \mathbf{u})^T \nabla_{\mathbf{p}} \phi_i = (\nabla_{\mathbf{p}} H)^T \nabla_{\mathbf{x}} \mathbf{u} \nabla_{\mathbf{p}} \phi_i \\ &= -(\nabla_{\mathbf{p}} H)^T \nabla_{\mathbf{x}} \phi_i \end{aligned}$$

where in the second equality we used the symmetry (2.14) while in the third equality we used (2.20). Thus ϕ_i solves the following equation:

$$\partial_t \phi + \nabla_{\mathbf{p}} H \cdot \nabla_{\mathbf{x}} \phi - \nabla_{\mathbf{x}} H \cdot \nabla_{\mathbf{p}} \phi = 0\,. \tag{2.21}$$

It is the Liouville equation, which is linear hyperbolic with variable coefficients since in (2.21) $H = H(\mathbf{x}, \mathbf{p})$. A convenient initial condition for each $\phi_i,\ i = 1, \ldots, d$ can be taken as:

$$\phi_i(0, \mathbf{x}, \mathbf{p}) = p_i - u_i(\mathbf{x})\,, \tag{2.22}$$

if $\mathbf{u}_0(\mathbf{x})$ is continuous. Otherwise we use the signed distance function, namely, choose $\phi_i(0, \mathbf{x}, \mathbf{p})$ to be the signed distance to the interface $p_i = u_i(\mathbf{x})$.

2.3. Static Hamilton-Jacobi equations. We now derive the level set equations for the d-dimensional static Hamilton-Jacobi equation

$$H(\mathbf{x}, \nabla_{\mathbf{x}} S) = 0\,. \tag{2.23}$$

Introduce $\mathbf{u} = \nabla_{\mathbf{x}} S$. Taking the gradient on (2.23), one gets an equivalent (at least for smooth solutions) form of the Hamilton-Jacobi equation

$$\nabla_{\mathbf{x}} H(\mathbf{x}, \mathbf{u}) = 0\,, \tag{2.24}$$

Again we need that condition

$$\mathbf{u} = \nabla_{\mathbf{x}} S\,, \tag{2.25}$$

which ensures that symmetry of $\nabla_{\mathbf{x}} \mathbf{u} = \nabla_{\mathbf{x}}^2 S$.

We use d level set functions $\phi_i = \phi_i(t, \mathbf{x}, \mathbf{p})$ $(i = 1, \cdots, d)$ such that the intersection of their zero level sets yields $\mathbf{u}$, namely,

$$\phi_i(\mathbf{x}, \mathbf{p}) = 0 \qquad \text{at} \quad \mathbf{p} = \mathbf{u}(t, \mathbf{x})\,, \qquad i = 1, \cdots, d\,. \tag{2.26}$$

We now derive the equation for ϕ_i $(i = 1, \cdots d)$.

Taking the gradient in $\mathbf{x}$ of (2.26) one has

$$\nabla_{\mathbf{x}} \phi_i + \nabla_{\mathbf{x}} \mathbf{u}\, \nabla_{\mathbf{p}} \phi_i = 0\,. \tag{2.27}$$

Taking gradient with respect to $\mathbf{x}$ on equation $H(\mathbf{x}, \mathbf{u}) = 0$ one gets

$$\nabla_{\mathbf{x}} H + \nabla_{\mathbf{x}} \mathbf{u}\, \nabla_{\mathbf{p}} H = 0\,. \tag{2.28}$$

Note

$$\begin{aligned} \nabla_{\mathbf{p}} \phi_i \cdot \nabla_{\mathbf{x}} \mathbf{u}\, \nabla_{\mathbf{p}} H \quad &= (\nabla_{\mathbf{p}} H)^T (\nabla_{\mathbf{x}} \mathbf{u})^T \nabla_{\mathbf{p}} \phi_i = (\nabla_{\mathbf{p}} H)^T \nabla_{\mathbf{x}} \mathbf{u} \nabla_{\mathbf{p}} \phi_i \\ &= -(\nabla_{\mathbf{p}} H)^T \nabla_{\mathbf{x}} \phi_i \end{aligned} \tag{2.29}$$

where in the second equality we used the symmetry of $\nabla_{\mathbf{x}} \mathbf{u}$ while in the third equality we used (2.27). Multiplying (2.28) by $\nabla_{\mathbf{p}} \phi_i$ and using (2.29) give the following equation static Liouville equation

$$\nabla_{\mathbf{p}} H \cdot \nabla_{\mathbf{x}} \phi_i - \nabla_{\mathbf{x}} H \cdot \nabla_{\mathbf{p}} \phi_i = 0\,. \tag{2.30}$$

3. Numerical Discretization.

We need to solve the system (2.21) of d linear equations in 2d unknowns, plus time. This is a decoupled system of linear equations with variable coefficients. Thus we need only consider linear equations of the form

$$\partial_t \psi + a \cdot \nabla_x \psi + b \cdot \nabla_p \psi = 0, x = (x_1, \ldots, x_d), p = (p_1, \ldots, p_d)$$

We shall use the ENO and WENO schemes for Hamilton-Jacobi equations developed in [29],[20], respectively as well as the TVD Runge-Kutta time discretizations of Shu-Osher [32]. We are solving a linear Hamilton-Jacobi equation so the numerical Hamiltonian can be taken to be simple upwind differencing. We use a uniform grid in x with grid spacing Δx, and in p with spacing Δp. For details see [29],[20] or a complete overview in the recent book [28].

We also use a local level set method as in [27],[31],[8], originating in [30]. This lowers the complexity of each update down to $O(N^d)\log N$ where N is the number of points in each direction and N^d represents the number of points on the zero level set of the vector functions $\psi(x,t)$. A similar problem was treated numerically in [27] with a somewhat detailed exposition. We omit the details here.

4. Numerical Examples.

In this section, we present several numerical examples.

4.1. One-dimensional hyperbolic equations.

We use the Burgers' equation with forcing as an example. This equation arises as the semiclassical limit of the linear Schrödinger equation.

$$\partial_t u + u \partial_x u + \partial_x V = 0\,, \tag{4.1}$$

$$u(0,x) = u_0(x)\,, \tag{4.2}$$

where $V(x)$ is the potential.

Example 4.1: A free particle model for a Gaussian pulse. We first consider the following problem which models the evolution in time of a pulse-type signal for (4.1) with zero potential $V(x) \equiv 0$. The initial condition for $x \in [-1,1]$ is

$$u_0(x) = -\sin(\pi x)|\sin(\pi x)|^{k-1}. \tag{4.3}$$

Intuitively, the decreasing interval of the velocity profile will eventually lead to caustics. The examples considered here have been given in [16, 17].

The mesh size and time steps are $\Delta x = \Delta p = 0.01, \Delta t = 0.005$ respectively for all the runs and numerical results are shown at time $t = 0$ and $t = 1$. Periodic boundary conditions are used in the computational domain $[-1,1] \times [-1.5, 1.5]$. We use a larger domain in p to avoid dealing with boundary conditions in the p direction. We use second order ENO method and a second order TVD Runge-Kutta method for time discretization. The numerical results are show in Fig. 4.1.

One cusp caustic: k=1. In this case, the exact solution of the rays contains self-interference, develops a cusp singularity, and hence admits at most three phases.

Two cusps merging into one: k=2. In this case, the initial data u_0 has an inflection point at $x = 0$ and two cusp singularities develop inside the geometric solution which exhibits at most five phases.

The 'square signal': k=100. In this case, the initial data develops two cusp singularities involving five phases for all later time.

Example 4.2: Discontinuous initial data. We consider two Riemann problem. The first one is given by

$$u_0(x,y) = \begin{cases} 1, & x < 0; \\ 0, & x > 0. \end{cases} \tag{4.4}$$

For viscosity solution this initial data yields a shock moving to the right with speed 1/2. Periodic boundary condition is used in a computational domain $[-2,2]\times[-0.1,1.1]$. The mesh size and time steps are $\Delta x = \Delta p = 0.01, \Delta t = 0.005$ respectively. We use a larger domain in p to avoid dealing with boundary conditions in the p direction. The numerical results at $t = 0.5$ is shown in on the top of Fig. 4.2.

The second initial condition is choosen to be

$$u_0(x,y) = \begin{cases} -1, & x < 0; \\ 1, & x > 0. \end{cases} \tag{4.5}$$

This initial data yields a rarefaction wave traveling to the right with the characteristic speed. The exact solution is a line of slope x/t connecting -1 from the left to 1 from the right. Periodic boundary condition is used in a computational domain $[-2,2] \times [-1.1,1.1]$. The mesh size and time steps are $\Delta x = \Delta p = 0.01, \Delta t = 0.005$ respectively. We use a larger domain in p to avoid dealing with boundary conditions in the p direction. The numerical results $t = 0.5$ is shown in the bottom of Fig. 4.2.

In both cases the signed distance functions are used for the initial condition of the level set functions.

Example 4.3: The harmonic oscillator. Consider a confining potential

$$V(x) = \frac{1}{2}x^2,$$

together with the initial conditions:

$$u_0(x) = -\tanh(5x). \tag{4.6}$$

The exact solution can be found by the method of characteristics:

$$x(t) = \cos(t)x_0 + \sin(t)u_0(x_0), \qquad u(x(t),t) = -\sin(t)x_0 + \cos(t)u_0(x_0).$$

This corresponds to a rotation in the phase space at unitary angular speed.

The mesh size and time steps are $\Delta x = \Delta p = 0.01, \Delta t = 0.005$ respectively, and results are shown at time $t = 0.5$. Periodic boundary conditions are used in the computational domain $[-2,2] \times [-2,2]$. We use a larger domain in both x and p to avoid trouble with the boundary conditions. The numerical result is given in Fig.4.3.

In all numerical examples the numerical solutions agree with those given in [17] using ray tracing.

4.2. A two-dimensional hyperbolic equation. As an example we use the two dimensional Burger's equation

$$\partial_t u + u\partial_x u + u\partial_y u = 0\,. \tag{4.7}$$

Example 4.4: A Riemann problem. Consider the initial data:

$$u_0(x,y) = \begin{cases} -1.0, & x > 0, y > 0; \\ -0.2, & x < 0, y > 0; \\ 0.5, & x < 0, y < 0; \\ 0, & x > 0, y < 0. \end{cases} \tag{4.8}$$

We compute the problem over domain $[-1,1]\times[-1,1]\times[-1,1]$ with reflecting boundary conditions. 40 cells are used in each direction and $\Delta t = 0.025$. The numerical solution at $t = 0.8$ is show in Fig. 4.4. One sees a multivalued surface.

Example 4.5: A continuous initial data Consider the following initial condition:

$$u_0(x,y) = 0.45\cos\pi x[\sin(\pi y) - 1]\,. \tag{4.9}$$

We solve the problem over domain $[-1,1]\times[-1,1]\times[-1,1]$ with periodic boundary conditions. 20 cells are used in each direction and $\Delta t = 0.025$. The numerical solution at $t = 1$ is show in Fig. 4.5.

4.3. A two-dimensional Hamilton-Jacobi equations. Consider the eikonal equation:

$$\partial_t S + \frac{1}{2}|\nabla S|^2 = 0\,, \qquad \mathbf{x} \in \Re^2\,. \tag{4.10}$$

This eikonal equation arises as the semiclassical limit of the linear Schrödinger equation, where S is the phase. It is valid before the formation of caustics. Beyond caustics, the velocity $\mathbf{u}(u,v) = \nabla S \in \Re^2$ becomes multivalued [22, 33]. It is the multivalued solution that we are interested in computing here.

Taking the gradient of (4.10), one obtains the momentum equations of pressureless gas equations for $\mathbf{u}$:

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = 0 \tag{4.11}$$

We solve two level set equations (2.21) for two level set functions with initial data given by (2.22), with the Hamiltonian $H(\mathbf{x},\mathbf{u}) = \frac{1}{2}|\mathbf{u}|^2$. The intersection of zero level hypersurfaces of the two level set functions yields the solution u and v.

Consider the following initial condition:

$$u_0(x,y) = 0.45\cos\pi x[\sin(\pi y) - 1]\,, \tag{4.12}$$

$$v_0(x,y) = 0.45\cos\pi y[\sin(\pi x) - 1]\,, \tag{4.13}$$

We solve the problem over a domain $[-1,1]\times[-1,1]\times[-1,1]$ with periodic boundary conditions. 20 cells are used in each direction and $\Delta t = 0.025$. The numerical solution at $t = 1$ is shown in Fig. 4.6, which not only has z-shaped multivalued solution but also a pipe-like surface crossing.

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REFERENCES

[1] D. Adalsteinsson and J.A. Sethian, *A fast level set method for propagating interfaces,* J. Comput. Phys. **118** (1995), 269-277.

[2] J-D. Benamou, *Big ray tracing: Multivalued travel time field computation using viscosity solution of the eikonal equation*, J. Comp. Phys. **128** (1996), 463-474.

[3] J-D. Benamou, *Direct computation of multivalued phase space solutions for Hamilton-Jacobi equations*, Comm. Pure Appl. Math. **52** (1999), 1443-1475.

[4] Y. Brenier, *Averaged multivalued solutions for scalar conservation laws*, SIAM J. Numer. Anal. **21** (1984), 1013 – 1037.

[5] Y. Brenier and L. Corrias, *A kinetic formulation for multibranch entropy solutions of scalar conservation laws*, Ann. Inst. Henri Poincaré **15** (1998), 169-190.

[6] P. Burchard, L.-T. Cheng, B. Merriman and S. Osher, *Motion of curves in three spatial dimensions using a level set approach*, J. Comput. Phys., **170** (2001), 720-741.

[7] C. Caratheodory, Calculus of Varieties of Partial Differential Equations of the First Order, Chelsea, 1982.

[8] L.-T. Cheng, M. Kang, S. Osher, H. Shim and Y.-H. Tsai, *Reflection in a level set framework for geometric optics*, Computer Methods in Engineering and Physics, to appear.

[9] B. Engquist and O. Runborg, *Multi-Phase computations in geometrical optics*, J. Comp. Appl. Math. **74** (1996), 175-192.

[10] B. Engquist, O. Runborg and A.-K. Tornberg, *High frequency wave propagation by the segment projection method*, J. Comp. Phys. **178** (2002) 373–390.

[11] L.-C. Evans, *A geometric interpretation of the heat equation with multivalued initial data*, SIAM J. Math. Anal., **27** (1996), 932-958.

[12] S. Fomel and J.A. Sethian,*Fast-phase space computation of multiple arrivals*, Proc. Nat. Acad. Sci. **99** (2002), 7329-7334.

[13] H. Flaschka, M.G. Forest and D.W. Mclaughlin, *Multi-phase averaging and the inverse spectral solution of the Korteweg-de Vries equation*. Comm. Pure Appl. Math. **33** (1980), 739-784.

[14] M.G. Forest and J.E. Lee, *Geometry and modulation theory for the periodic nonlinear Schrödinger equation*. Oscillation theory, computation, and methods of compensated compactness (Minneapolis, Minn., 1985), 35–69, IMA Vol. Math. Appl., 2, Springer, New York , 1986.

[15] Y. Giga, *Viscosity solutions with shocks*, Hokkaido Univ. Preprint Series in Math (2001) #599.

[16] L. Gosse, *Using K-branch entropy solutions for multivalued geometric optics computations*, J. Comp. Phys., Vol. **180** (2002), 155–182.

[17] L. Gosse, Shi Jin and X.T. Li, *On two moment systems for computing multiphase semiclassical limits of the Schrödinger equation*, Math. Model Meth. Appl. Sci., submitted.

[18] V.L. Granatstein, R.K. Parker and C.M. Armstrong, *Vacuum electronics at the dawn of the twenty-first century,* Proc. IEEE **87** (1999), 702-716.

[19] C. Hu and C.-W. Shu, A discontinuous Galerkin finite element method for Hamilton-Jacobi equations, SIAM J. on Sci. Computing, **21** (1999) 666-690.

[20] G.-S. Jiang and D.-P. Peng, *Weighted ENO schemes for Hamilton-Jacobi equations*, SIAM J. Sci. Comput., **21** (2000), 2126-2143.

[21] Shan Jin, C.D. Levermore and D.W. Mclaughlin, *The semiclassical limit of the defocusing NLS Hierarchy,* Comm. Pure Appl. Math. **52** (1999), 613-654.

[22] Shi Jin, X.T. Li, *Multi-phase computations of the semiclassical limit of the Schrödinger equation and related problems: Whitham vs. Wigner*, Physica D, submitted.

[23] Shi Jin and Z.P. Xin, *Numerical passage from systems of conservation laws to Hamilton-Jacobi equation, and a relaxation scheme*, SIAM J. Num. Anal. **35** (1998), 2385-2404.

[24] P.D. Lax and C.D. Levermore, *The small dispersion limit of the Korteweg-de Vries equation*, I,II,III. Cmm. Pure Appl. Math. **36** (1983), no.3,253-290; no. 5,571-593; no.6,809-830.

[25] X.T. Li, J.G. Wöhlbier, Shi Jin and J.H. Booske, *An Eulerian method for computing multivalued solutions of the Euler-Poisson equations and applications to wave breaking in klystrons*, Phys. Rev. E., submitted.

[26] S. Osher, *A level set formulation for the solution of the Dirichlet problem for Hamilton-Jacobi equations*, SIAM J. Math. Anal **24** (1993), 1145-1152.

[27] S. Osher, L.T. Cheng, M. Kang, H. Shim and Y.H. Tsai, *Geometric optics in a phase space based level set and Eulerian framework*, J. Comp. Phys., **79** (2002), 622-648.

[28] S. Osher and R. Fedkiw, Level set methods and dynamic implicit surfaces, Springer-Verlag, NY, 2002.

[29] S. Osher and C.-W. Shu, *High order essentially nonoscillatory schemes for Hamilton-Jacobi equations*, SIAM J. Numer. Analys., **28** (1991), 907-921.

[30] D. Peng, B. Merriman, S. Osher, H. Zhao and M. Kang, *A PDE based fast local level set method,* J. Comput. Phys., **155** (1999), 410-438.

[31] J. Qian, L.-T. Cheng and S. Osher, *A level set based Eulerian approach for anisotropic wave propagation,* Wave Motion, to appear.

[32] C.-W. Shu and S. Osher, *Efficient implementation of essentially nonoscillatory shock capturing schemes, II*, J. Comp. Phys., **83** (1989), 32-78.

[33] C. Sparber, P. Markowich, N. Mauser, *Multivalued Geometrical optics: Wigner vs. WKB*, Asymptotic Analysis, to appear.

[34] W.W. Symes and J. Qian, *A slowness matching Eulerian method for multivalued solutions of Eikonal equations*, J. Sci. Comp., submitted.
[35] Y.-H. R. Tsai, Y. Giga and S. Osher, *A level set approach for computing discontinuous solutions of Hamilton-Jacobi equations*, Math. Comp. **72** (2003), 159-181.
[36] J. van Trier and W. W. Symes, *Upwind finite-difference calculation of traveltimes*, Geophysics **56** (1991),812-821.
[37] G.B. Whitham, Linear and Nonlinear Waves, Wiley, New York, 1974.

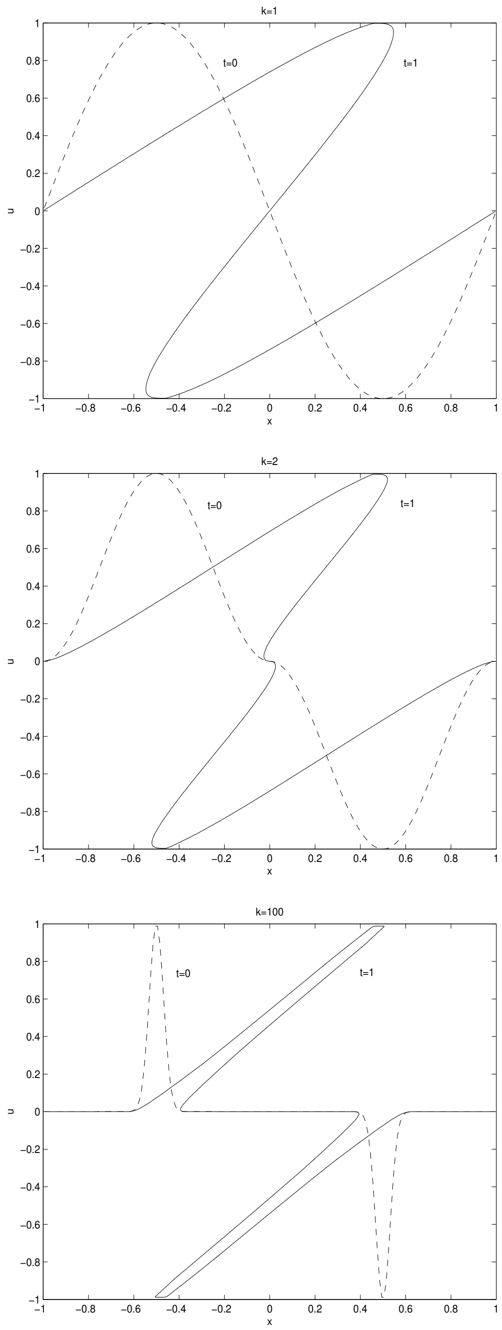


FIG. 4.1. *The contour plots of u for different k's with initial condition (4.3).*

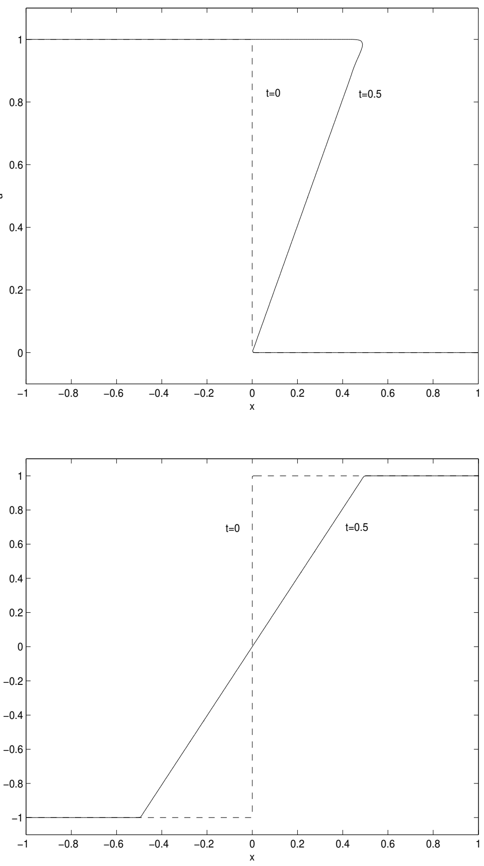


FIG. 4.2. *The contour plots of u for Riemann problem. Top: shock initial data (4.4); bottom: rarefaction initial data (4.5).*

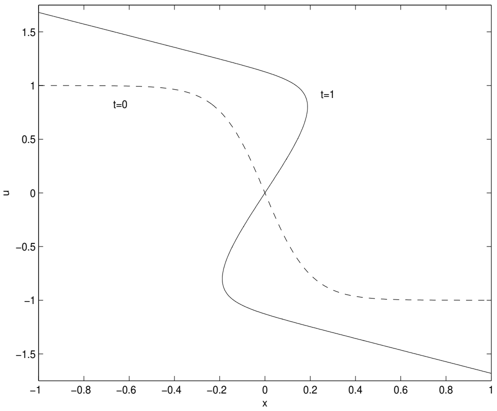


FIG. 4.3. *The numerical solutions of the harmonic oscillator case.*

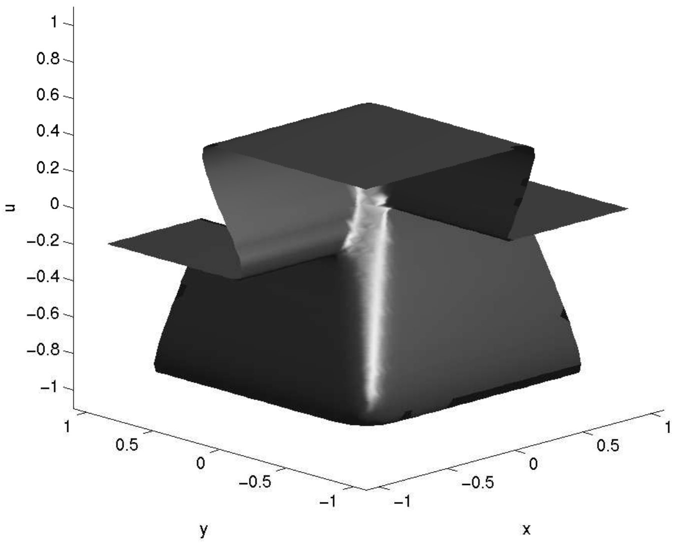


FIG. 4.4. *The numerical solutions of the two-d Burgers' equation (4.7) with initial data (4.8) at* $t = 0.8$.

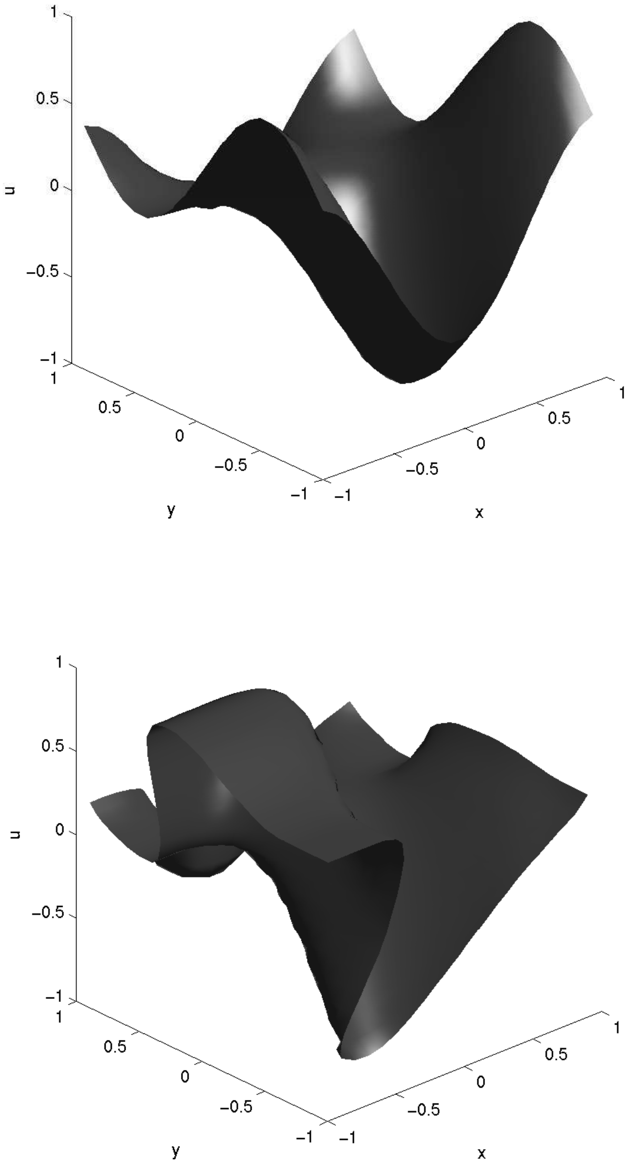
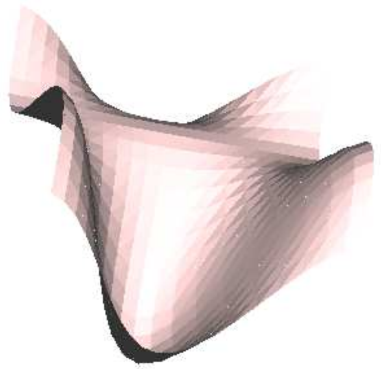
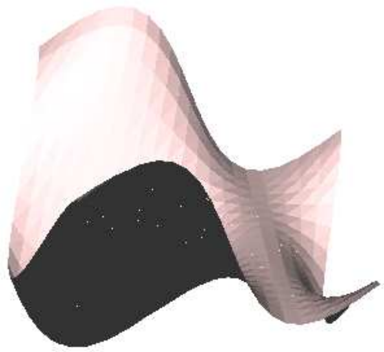


FIG. 4.5. *u for initial data (4.9). Top: initial data; bottom: solution at $t = 1$.*



(a)



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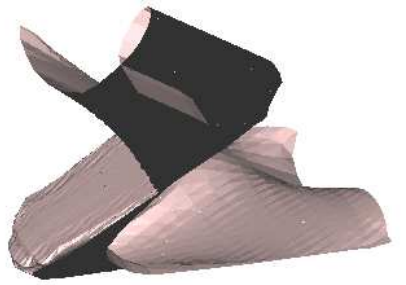
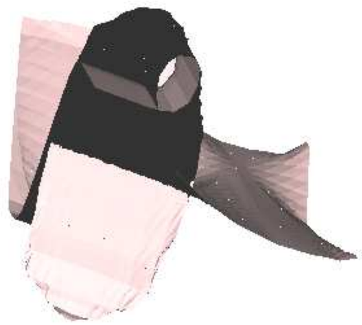


(c)



(d)

FIG. 4.6. *Numerical solutions for the 2d pressureless gas equation (4.11). From top to bottom: (a) u at $t=0$; (b) v at $t=0$; (c) u at $t=1$; (d) v at $t=1$.*