Hydrodynamic structure of the augmented Born-Infeld equations

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Abstract

The Born-Infeld system is a nonlinear version of Maxwell's equations. We first show that, by using the energy density and the Poynting vector as additional independent variables, the BI system can be augmented as a 10×10 system of hyperbolic conservation laws. The resulting augmented system has some similarity with Magnetohydrodynamics (MHD) equations and enjoy remarkable properties (existence of a convex entropy, galilean invariance, full linear degeneracy). In addition, the propagation speeds and the characteristic fields can be computed in a very easy way, in contrast with the original BI equations. Then, we investigate several limit regimes of the augmented BI equations, by using a relative entropy method going back to Dafermos, and recover, the Maxwell equations for low fields, some pressureless MHD equations for high fields, and pressureless gas equations for very high fields.

1 Introduction

The Born-Infeld equations were originally designed [BI] as a nonlinear correction to the linear Maxwell equations allowing finite electrostatic fields for point charges. After the emergence of Quantum Electrodynamics, the Born-Infeld model became obsolete, although its remarkable properties had not been forgotten by theoretical physicists. More recently, the BI model came back as an active player in the field of D-brane and string theory [Po]. We refer to G. Boillat [BDLL] and D. Serre [Se] for some mathematical analysis of the BI equations and to G. Gibbons [Gi] for their relevance in contemporary high energy Physics. The BI model was also an attempt [dB] to

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describe matter in a purely electromagnetic way. Particles were thought to correspond to zones of very intense electromagnetic energy. The BI equations are nonlinear hyperbolic evolution PDEs (see [Ma], [AG], [Se], [Da], ..., for this class of PDEs). They were designed on purpose to be 'linearly degenerate' (see [Ma], [Da], [Se] for the concept of linear degeneracy) so that shock waves would not form and, this way, no further microscopical theory would be needed to complete the model. Solutions of linearly degenerate system of hyperbolic conservation laws are in general believed to be smooth or to blow up in sup norm, not in Lipschitz norm (see [Ma]). Thus, solutions of the BI equations are very likely to provide peaky solutions, behaving as particles. Of course, there are different and more classical ways to derive particle equations out of wave equations, mainly through stationary phase arguments or, more recently, by weak convergence techniques (such as H-measures, Wigner transforms etc..., cf. [Ta], [GMMP]...). (See also [Ta2] for a recent view on the relationship between waves and particles.) The goal of this paper is to provide some mathematical confirmation that the BI model establishes a nonlinear transition between wave and particle behaviours according to the intensity of the electromagnetic field.
The main steps of our analysis are as follows
1) First, we lift the original 6×6 BI system to a 10×10 system of conservation laws, by adding the conservation of both the energy and the Poynting vector as additional conservation laws. The resulting augmented BI system (ABI) provides, in a natural way, a set of equations coupling the electromagnetic field and a virtual fluid having the electromagnetic energy as mass density and the Poynting vector as momentum. This can also be seen as a mathematical formalization of the classical idea in Physics (at least at Born's time) that matter could be of pure electromagnetic origin. Of course, the apparent coupling in the ABI equations is somewhat artificial. Indeed, among all solutions of the ABI system, only those with initial conditions valued in a specific 6 dimensional algebraic submanifold of $\mathbf{R^{10}}$ -that we call the BI manifold- genuinely correspond to the original BI equations. For them, the 'fluid part' is entirely subordinate to the 'electromagnetic part'. However, if we complete, through weak convergence (say in L^2), the set of initial conditions valued in the BI manifold, we get all (say L^2) functions valued in the closed *convex* hull of the BI manifold. But, it turns out that this closed convex hull is fully 10 dimensional! So, for most 'generalized' initial conditions obtained after weak completion, the fluid part and the electromagnetic part look independent.

2) The ABI system enjoys remarquable properties, keeps the linear degeneracy of the BI system, but has the Galilean invariance of fluid mechanics, which provides an entropy function (in the sense of the theory of hyperbolic conservation laws [Da]) of almost quadratic nature. This enables us to use a relative entropy method (going back at least to Dafermos, see [Da]) for a rigorous asymptotic analysis of the ABI system in three different regimes, according to the strength of the electromagnetic field. When fields are low, we recover the linear Maxwell equations, on large time intervals. When fields are high, on short time intervals, we recover (depending on the relative strengths of the electric and the magnetic fields) either a pressureless gas system (corresponding to purely inertial non interacting particles) or a pressureless MHD system (describing the motion of non interacting strings). Let us point out that these asymptotic results are essentially incomplete, for two reasons.
First, we assume that the ABI system admits global in time solutions, which is not known -as for essentially all multidimensional systems of hyperbolic conservation laws- except for initial data depending on only one space variable. (However, we believe that the BI equations, among all systems of conservation laws, reach the highest standards, in particular full linear degeneracy, to admit global smooth solutions.) Let us emphasize, however, that our error estimates do not involve *any* a priori bound on the solutions of the ABI system.
Next, our results require the existence of smooth solutions for the limit equations. There is no problem for the linear Maxwell equations, obtained in the low field regime. However, for the high field limits, the limit equations only provide smooth solutions for short times.

2 The augmented Born-Infeld system

2.1 A short review of nonlinear Maxwell theories

For this subsection and the next one, we refer to Boillat [BDLL] and Gibbons [Gi]. Generally speaking, nonlinear Maxwell equations can be obtained by varying a Lagrangian of form

$$\int L(E,B)dxdt,$$

with respect to (E,B) subject to

$$\partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0. \tag{1}$$

Here E and B are time dependent vector fields in $\mathbf{R^3}$ and the 'Lagrangian density' L has to be defined. The resulting system of equations combines (1) and

$$\partial_t D = \nabla \times H, \quad \nabla \cdot D = 0, \quad D = \frac{\partial L}{\partial E}(E,B), \quad H = -\frac{\partial L}{\partial B}(E,B). \qquad (2)$$

which is a nonlinear substitute for the usual linear Maxwell equations. The obtained equations can be explicitely written as evolution equations in the variable (D,B), after introducing the 'energy (or Hamiltonian) density'

$$h(D,B) = \sup_E E \cdot D - L(E,B),$$

and setting

$$E = \frac{\partial h}{\partial D}(D,B), \quad H = \frac{\partial h}{\partial B}(D,B).$$

Of course, Maxwell equations correspond to

$$L(E,B) = \frac{E^2 - B^2}{2}, \quad h(D,B) = \frac{B^2 + D^2}{2}.$$

2.2 The Born-Infeld model

Let us introduce the 'electromagnetic tensor' F

$$\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \qquad (3)$$

and the Minkowski metric tensor g

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (4)$$

The Born-Infeld Lagrangian is given by

$$L(E,B) = -\sqrt{-det(g+F)} = -\sqrt{1 + B^2 - E^2 - (E \cdot B)^2}.$$

Born, Infeld and their followers found out that this Lagrangian density L is characterized by the following properties :
1) L depends only on $B^2 - E^2$ and $E \cdot B$ which are the invariants of the electromagnetic tensor.
2) The equations are 'self-dual', i.e. unaffected by the change of unknown $(D, B) \to (-B, D)$.
2) The corresponding nonlinear Maxwell equations are hyperbolic and linearly degenerate.
Notice that the classical Maxwell theory is recovered in the limit $E << 1$, $B << 1$.
The Born-Infeld energy density h is given (after tedious calculations) by

$$h = \sqrt{1 + B^2 + D^2 + |D \times B|^2}, \tag{5}$$

where $|\cdot|$ stands for the Euclidean norm. We get, for E and H, the following expressions

$$E = \frac{\partial h}{\partial D} = \frac{D + B \times P}{h}, \quad H = \frac{\partial h}{\partial B} = \frac{B - D \times P}{h}, \tag{6}$$

where

$$P = D \times B \tag{7}$$

is the Poynting vector. (Notice that P is also $E \times H$!) Thus, the BI equations can be written

$$\partial_t D + \nabla \times (\frac{-B + D \times P}{h}) = \partial_t B + \nabla \times (\frac{D + B \times P}{h}) = 0, \tag{8}$$

$$\nabla \cdot D = \nabla \cdot B = 0.$$

The energy density h satisfies the additional conservation law

$$\partial_t h + \nabla . P = 0. \tag{9}$$

A classical sufficient condition for a system of conservation laws to be hyperbolic is the existence of an additional conservation laws for a smooth strictly convex function (usually called 'entropy') of the conserved unknowns (see [Da]). Indeed, this property implies that the system is symmetrizable, and, therefore hyperbolic (see [Ma], [AG], [Da]...). Here, h is a strictly convex function of B and D only in a neighborhood of the origin, but not in the large. Therefore, it is not obvious that the BI equations are hyperbolic in the large, from the entropy point of view.

2.3 Lifting of the BI system

Although h, defined by (5), is not a globally convex function of B and D, it is trivially a convex function of B, D and $P = D \times B$. Thus it is natural to look at the evolution equation satisfied by the Poynting vector P. After tedious calculations, we get

Proposition 2.1 *Any smooth solution (D, B) of the BI system satisfies the additional conservation laws,*

$$\partial_t P + \nabla.(\frac{P \otimes P - B \otimes B - D \otimes D}{h}) = \nabla(\frac{1}{h}), \qquad (10)$$

where h and P are defined by (5,7).

The proof is given in Appendix A. So, adding equations (8), (9), (10), we get a 10×10 system of conservation laws in the variables (B, D, h, P) which we call augmented BI (ABI) equations. Of course, all solutions (B, D) of the BI equations solve this augmented system, provided h, P are defined by (5,7). So the BI equations correspond to the ABI system, with special initial conditions, valued in the 6 dimensional submanifold of $\mathbf{R^{10}}$ (5, 7), that we call the BI manifold. At this point, it is not clear that the new system is hyperbolic. However, we prove

Theorem 2.2 *Let us introduce the smooth strictly convex function*

$$(D, B, P, h > 0) \to S(D, B, P, h) = \frac{1 + B^2 + D^2 + P^2}{2h}. \qquad (11)$$

Then, all smooth solutions of the ABI system (8-9-10) satisfy the additional conservation law

$$\partial_t S + \nabla.(\frac{SP}{h}) = \nabla.\{\frac{P - D \times B + (B \cdot P)B + (D \cdot P)D}{h^2}\}, \qquad (12)$$

which makes the ABI system symmetrizable and hyperbolic. In addition, the ABI system is linearly degenerate.

The proof of (12) is given in Appendix B. Let us recall that a system of conservation laws is automatically symmetrizable and hyperbolic whenever it admits a smooth strictly convex entropy function [Da]. The linear degeneracy of the ABI system will be investigated in the next subsection.

Remark 1

The entropy function S is just... $h/2$ (written in a different way) along the BI manifold (5), (7)!

Remark 2

A similar situation occurs in elastodynamics where the original equations lack the existence of a convex entropy. However, they can be augmented, by adding additional conservation laws. The resulting equations do have a strictly convex entropy. (See a discussion in [Da], [DST], and also [GG] for related topics.)

Remark 3

An intermediate extension of the BI system can be obtained by i) adding to (8) the conservation of P (10), ii) relaxing (7). Then, we get a 9×9 system of conservation laws, with unknowns D, B, P. Here h is *not* an independent variable, but rather a smooth strictly convex function of D, B, P defined by

$$h = \sqrt{1 + B^2 + D^2 + P^2}. \tag{13}$$

The corresponding conservation law reads

$$\partial_t h + \nabla \cdot P - \nabla \cdot (\frac{P - D \times B}{h^2}) + \nabla .(\frac{(D \cdot P)D + (B \cdot P)B}{h^2}) = 0 \tag{14}$$

(and not (9)!). Thus h is an entropy and the 9×9 system is automatically symmetrizable and hyperbolic. However the computation of propagation speeds and characteristic fields is cumbersome.

2.4 Propagation speeds and characteristic fields

In contrast with the BI system, the computation of propagation speeds and characteristic fields for the ABI system is straightforward. Since the ABI system is clearly isotropic with respect to the x variable, it is enough to consider solutions which do not depend on x_2, x_3. Notice first that

$$\partial_t B_1 = \partial_t D_1 = 0, \quad \partial_1 B_1 = \partial_1 D_1 = 0,$$

immediately follows from (8), which means that B_1 and D_1 are just constant. Thus, we just have to consider the resulting one-dimensional 8×8 ABI system defined by :

$$\partial_t h + \partial_1 P_1 = 0, \tag{15}$$

$$\partial_t P_1 + \partial_1(\frac{P_1^2 - (1 + D_1^2 + B_1^2)}{h}) = 0, \tag{16}$$

$$\partial_t D_2 + \partial_1(\frac{B_3 + D_2 P_1 - D_1 P_2}{h}) = 0, \tag{17}$$

$$\partial_t D_3 + \partial_1(\frac{-B_2 + D_3 P_1 - D_1 P_3}{h}) = 0, \tag{18}$$

$$\partial_t B_2 + \partial_1(\frac{-D_3 + B_2 P_1 - B_1 P_2}{h}) = 0, \tag{19}$$

$$\partial_t B_3 + \partial_1(\frac{D_2 + B_3 P_1 - B_1 P_3}{h}) = 0, \tag{20}$$

$$\partial_t P_2 + \partial_1(\frac{P_1 P_2 - D_1 D_2 - B_1 B_2}{h}) = 0, \tag{21}$$

$$\partial_t P_3 + \partial_1(\frac{P_1 P_3 - D_1 D_3 - B_1 B_3}{h}) = 0. \tag{22}$$

Since $Z = \sqrt{1 + {B_1}^2 + {D_1}^2}$ is a constant, the one-dimensional ABI system uncouples. Equations (15,16) form a well known system which describes a compressible isentropic gas with pressure $p = -Z^2/h^2$, often called a Chaplygin gas. This 2×2 system is linearly degenerate and the propagation speeds are

$$\lambda_+ = \frac{P_1 + Z}{h}, \quad \lambda_- = \frac{P_1 - Z}{h}.$$

Next, we observe that, once P_1, h are known, equations (17,18, 19,20,21,22) form a 6×6 *linear symmetric* system of conservation laws. The propagation speeds λ are solutions of

$$\xi^2(\xi^2 - Z^2)^2 = 0, \quad \xi = \lambda - \frac{P_1}{h}. \tag{23}$$

Thus, we get again the previous propagations speeds λ_+ and λ_-, which, therefore, have multiplicity 3 each, and, in addition,

$$\lambda_0 = \frac{P_1}{h},$$

with multiplicity 2. The corresponding characteristic fields can be easily computed and the propagation speeds are unchanged along them, which means that the ABI system is fully linearly degenerate. (This also follows from Boillat's theorem [Da], since λ_-, λ_+ and λ_0 are multiple roots -with constant multiplicity- of equation (23).) Notice that these easy calculations *do* apply to the original BI system, just by restriction on the BI manifold (5), (7)! So lifting the original 6×6 system to a 10×10 system has the effect of a drastically simplified algebra.

2.5 Hydrodynamic features of the ABI system

The ABI system has a lot of similarities with Hydrodynamics and, more specifically, Magnetohydrodynamics (MHD), for which we refer, for instance, to [BDLL] or[Da]. Indeed, h and P may be interpreted as the mass density and the impulse of the fluid, which corresponds to the old physical idea that matter may be of pure electromagnetic nature. Introducing the corresponding velocity field v

$$v = \frac{P}{h}, \tag{24}$$

we can write the ABI system as follows

$$\partial_t h + \nabla.(hv) = 0, \tag{25}$$

$$\partial_t(hv) + \nabla.(hv \otimes v - \frac{B \otimes B - D \otimes D}{h}) = \nabla(\frac{1}{h}), \tag{26}$$

$$\partial_t B + \nabla.\{v \otimes B - B \otimes v\} + \nabla \times (\frac{D}{h}), \tag{27}$$

$$\partial_t D + \nabla.\{v \otimes D - D \otimes v\} - \nabla \times (\frac{B}{h}) = 0.$$

Also notice that, in spite of the fact that the original BI system is fully compatible with special relativity, the ABI system is just as Galilean as classical fluid Mechanics! Indeed, a change of frame $(t, x) \to (t, x + ut)$, where u is any fixed 3 dimensional vector leaves the ABI system unchanged provided we set

$$(B, D, h, v)(t, x + ut) = (B, D, h, v)(t, x) + (0, 0, 0, u). \tag{28}$$

Notice that, in sharp contrast, E defined by (6), changes as

$$E(t, x + ut) = E(t, x) + u \times B(t, x).$$

Remark

The Galilean nature of the augmented system seems to single out the BI model among all other electromagnetic models. In particular, augmenting the Maxwell equations by adding the conservation laws for the energy h and the Poynting vector $P = hv$ would lead to the following system

$$\partial_t h + \nabla.(hv) = 0, \quad \partial_t(hv) - \nabla.(B \otimes B + D \otimes D) + \nabla h = 0,$$

$$\partial_t D - \nabla \times B = \partial_t B + \nabla \times D = 0, \quad \nabla \cdot D = \nabla \cdot B = 0,$$

which does not share the Galilean invariance of classical fluid mechanics and does not exhibit a genuinely coupled structure.

2.6 Convex hull of the BI manifold

Due to the linear degeneracy of the ABI system, one may speculate that this system is 'weakly stable'. (This would mean that all weak limits of the ABI system -with suitable uniform bounds- are still solutions of the system.) As a matter of fact, this is true for special solutions depending on a single space variable, as shown in Appendix C, following [Se]. Therefore, it is natural to investigate which initial conditions can be weakly approximated (say, in the L^2 sense) by (say, smooth) initial conditions valued in the BI manifold $(5), (7)$. From elementary convex analysis, we know that such inital conditions are nothing but all (say L^2) functions that are valued in the *closed convex hull* of the BI manifold.

Theorem 2.3 *The closed convex hull of the six dimensional BI manifold (5,7) in* $\mathbf{R}^{10}$ *has dimension* 10. *It contains the convex set*

$$\{h \geq 1 + |D| + |B| + |P|\}$$

and is contained in $\{h \geq \sqrt{1 + D^2 + B^2 + P^2}\}$.

A proof is provided in Appendix D. From this result, we infer that, through weak completion, we may consider, for the ABI system, all kinds of initial conditions with full dimensionality, where the 'fluid variables' (h, P) are clearly distinct from the 'electromagnetic variables' (B, D).

3 Asymptotic Analysis : Results

In this section, several asymptotic regimes of the ABI system are investigated. For low fields, we recover the Maxwell equations. Conversely, for high fields, depending on the chosen scaling, we recover either a pressureless Magnetohydrodynamics system or a pressureless gas dynamics system. For simplicity, we only consider spatially periodic solutions of period 1 and all space integrals will be performed on the unit cube. We say that a collection of Lebesgue integrable functions $(D, B, h \geq 0, P)$ is an entropy weak solution of the ABI system if

$$t \to \int \frac{1 + B^2 + D^2 + P^2}{2h} dx$$

is a bounded non increasing function of $t \geq 0$ and equations (8,9,10) are satisfied in the sense of distributions. Notice that we do not require these solutions to satisfy the corresponding 'entropy inequality'

$$\partial_t S + \nabla.(\frac{SP}{h}) - \nabla.\{\frac{P - D \times B + (B \cdot P)B + (D \cdot P)D}{h^2}\} \leq 0.$$

In particular, we do not require

$$(\frac{1 + B^2 + D^2 + P^2}{h})\frac{P}{h}$$

to be integrable. As pointed out in the introduction, the global existence of such solutions is not known, as for nearly all other multidimensional systems of hyperbolic conservation laws (except for initial data depending only on one space variable, as discussed in Appendix D). Thus our asymptotic results are essentially incomplete. However, our error estimates *do not* require any a priori bound on such solutions. In addition, due to the linear degeneracy of the BI and ABI systems and their very peculiar structure, it is reasonable to conjecture that both systems admit global in time smooth solutions for any smooth initial data.

Let us first state our results, in the next three subsections. Our proofs are postponed to section 4.

3.1 High field limit toward pressureless MHD

Let us first consider solutions of the ABI system for which B, P and h are of high intensity and scale as $\lambda >> 1$, while D stays of order 1. Notice that such a scaling is compatible with the BI manifold (5), (7). (Just take B of order λ and D of order 1.) The rescaled fields

$$\tilde{B} = B/\lambda, \quad \tilde{P} = P/\lambda, \quad \tilde{h} = h/\lambda,$$

are formally approximate solution, up to $O(1/\lambda)$ error, to the homogeneous system derived from the ABI system (8,9,10)

$$\partial_t B + \nabla \times (\frac{B \times P}{h}) = 0, \tag{29}$$

$$\partial_t h + \nabla . P = 0, \tag{30}$$

$$\partial_t P + \nabla . (\frac{P \otimes P - B \otimes B}{h}) = 0. \tag{31}$$

The corresponding 'homogeneous' BI manifold, obtained by rescaling the original BI manifold (5), (7), is given by :

$$B \cdot P = 0, \quad h = \sqrt{B^2 + P^2}, \tag{32}$$

Naturally, this manifold is preserved by the equations (which can be directly checked). System (29,30,31) can be seen as a pressureless version of the classical MHD equations. It is not obvious that it admits local smooth solutions. However, at least for particular initial data, this system turns out to be integrable by writting its (smooth) solutions as superpositions of non-interacting strings, as explained in Appendix E. We have

Theorem 3.1 *Let $(B^*, P^*, h^*) = (h^* b^*, h^* v^*, h^*)$ be a smooth solution of the 'pressureless MHD system' (29,30,31) defined on a finite time interval $[0, T]$. Let $D^* = h^* d^*$ be a smooth solution of the companion linear equation*

$$\partial_t D^* + \nabla . (\frac{P^* \otimes D^* - D^* \otimes P^*}{h^*}) + \nabla \times \frac{B^*}{h^*} = 0$$

(where B^*, P^*, h^* are known). Let $(D, B, P, h) = (hd, hb, hv, h)$ be a weak entropy solution of the ABI system (8,9,10). Then, there is a constant $C \geq 1$ depending only on T and (D^*, B^*, P^*, h^*), such that, for all $\lambda >> 1$,

$$\frac{1}{\int h dx} \int \frac{h}{2} \{ |v - v^*|^2 + |b - b^*|^2 + |d - \frac{d^*}{\lambda}|^2 + (\frac{1}{h} - \frac{1}{\lambda h^*})^2 \} dx \leq C \frac{1}{\lambda^4} \tag{33}$$

holds true for all $t \in [0, T]$, provided it holds true at $t = 0$ with $C = 1$. In addition, we then have, for all $t \in [0, T]$,

$$\int \{ |P - \lambda P^*| + |B - \lambda B^*| + |D - D^*| + |1 - \frac{h}{\lambda h^*}| \} dx \leq C' \tag{34}$$

for some other constant C' depending only on T and (D^*, B^*, P^*, h^*).

3.2 Very high field limit toward pressureless gas dynamics

We now consider solutions of the ABI system for which both B and D are of high intensity and scale as $\lambda >> 1$, while P and h are even higher of size λ^2. (Again, this scaling is compatible with the BI manifold (5), (7).) The rescaled fields

$$\tilde{P} = P/\lambda^2, \quad \tilde{h} = h/\lambda^2,$$

are formally approximate solution, up to $O(1/\lambda^2)$ error, to the new homogeneous system derived from the ABI system (8,9,10)

$$\partial_t h + \nabla . P = 0, \tag{35}$$

$$\partial_t P + \nabla .(\frac{P \otimes P}{h}) = 0. \tag{36}$$

The corresponding 'homogeneous' BI manifold, obtained by rescaling the original BI manifold (5), (7), is just

$$h = |P| \tag{37}$$

and is preserved by the equations. This homogeneous system describes a pressureless gas ('dust'), and, more specifically, a gas of 'photons', once restricted along the homogeneous BI manifold. Geometrically, it describes a continuum of particles moving along straight lines (rays) at constant speed without interactions. (Along the homogeneous manifold, this speed is one.) Smooth solutions exist for short times and usually blow up, because of ray focusing. See more details on Appendix E. We have

Theorem 3.2 *Let $(P^*, h^*) = (h^* v^*, h^*)$ be a smooth solution of the pressureless gas equations (35,36) defined on a finite time interval $[0, T]$. Let $B^* = h^* b^*$, $D^* = h^* d^*$ be smooth solutions of the companion linear equations*

$$\partial_t D^* + \nabla .(\frac{P^* \otimes D^* - D^* \otimes P^*}{h^*}) = 0,$$

$$\partial_t B^* + \nabla .(\frac{P^* \otimes B^* - B^* \otimes P^*}{h^*}) = 0,$$

(where h^*, P^* are known). Let $(D, B, P, h) = (hd, hb, hv, h)$ be a weak entropy solution of the ABI system (8,9,10). Then, there is a constant $C \geq 1$ depending only on T and (D^*, B^*, P^*, h^*), such that for all $\lambda >> 1$,

$$\frac{1}{\int h dx} \int \frac{h}{2} \{ |v - v^*|^2 + |b - \frac{b^*}{\lambda}|^2 + |d - \frac{d^*}{\lambda}|^2 + (\frac{1}{h} - \frac{1}{\lambda^2 h^*})^2 \} dx \leq C \frac{1}{\lambda^8} \tag{38}$$

holds true for all $t \in [0,T]$, provided it holds true at $t = 0$ with $C = 1$. In addition, we have, for all $t \in [0,T]$,

$$\int \{|P - \lambda^2 P^*| + |B - \lambda B^*| + |D - \lambda D^*| + |1 - \frac{h}{\lambda^2 h^*}|\}dx \leq C' \tag{39}$$

for some other constant C' depending only on T and (D^*, B^*, P^*, h^*).

3.3 Low field limit toward Maxwell's equations

Finally, let us consider the opposite regime when the fields are weak. More specifically, we now consider solutions of the ABI system for which both B and D are of law intensity and scale as $\lambda << 1$, while P is even lower of size λ^2. (In addition, we assume here $h-1$ to be of order λ^2, in full compatiblity with the BI manifold (5), (7).) The rescaled fields

$$\tilde{D} = D/\lambda, \quad \tilde{B} = B/\lambda,$$

are formally approximate solution, up to $O(\lambda^2)$ error, to the classical linear Maxwell equations

$$\partial_t B + \nabla \times D = 0, \quad \partial_t D - \nabla \times B = 0. \tag{40}$$

We have

Theorem 3.3 *Let (D^*, B^*) be a global smooth solution of the Maxwell equations (40). Let $(D, B, P, h) = (hd, hb, hv, h)$ be a weak entropy solution of the ABI system (8,9,10). Then, there is a constant $C \geq 1$ depending only on (D^*, B^*), such that, for all $\lambda << 1$,*

$$\frac{1}{\int h dx} \int \frac{h}{2}\{v^2 + |b - \lambda b^*|^2 + |d - \lambda d^*|^2 + (\frac{1}{h} - 1)^2\}dx \leq C\lambda^4 \tag{41}$$

holds true for all $t \in [0, 1/\lambda]$, provided it holds true at $t = 0$ with $C = 1$. In addition, we then have, for all $t \in [0, 1\lambda]$,

$$\int \{|P| + |B - \lambda B^*| + |D - \lambda D^*| + |h - 1|\}dx \leq C'\lambda^2 \tag{42}$$

for some other constant C' depending only on (D^*, B^*).

4 Asymptotic analysis : Proofs

4.1 The abstract relative entropy method

Following [Da], chapter 5.2, let

$$\partial_t U + \nabla . G(U) = 0, \tag{43}$$

be a system of conservation laws admitting a smooth strictly convex entropy $U \to S(U)$. Let U and U' be two solutions of (43), both spatially periodic. We assume U' to be smooth and U to be a weak entropy solutions (or, more precisely, to be a weak solution with $t \to \int S(U)dx$ non increasing). Then, we have

$$\frac{d}{dt}\int \eta(U,U')dx \leq \int \nabla(DS(U')) \cdot Z(U,U')dx, \tag{44}$$

where

$$\eta(U,U') = S(U) - S(U') - DS(U') \cdot (U - U') \tag{45}$$

is the relative entropy, and Z is defined by

$$Z(U,U') = G(U) - G(U') - DG(U') \cdot (U - U'). \tag{46}$$

Notice that we can easily adapt Dafermos' calculation to the case when U' is only an approximate solution

$$\partial_t U' + \nabla . G(U') = r, \tag{47}$$

where r is the error term. Then, we get

$$\frac{d}{dt}\int \eta(U,U')dx \leq \tag{48}$$

$$\int \nabla(DS(U')) \cdot Z(U,U')dx + \int D^2S(U')(r \otimes (U - U'))dx.$$

4.2 The relative entropy method for the augmented BI system

For the ABI system, we use notations

$$U = (D,B,P,h) = (hd,hb,hv,h), \quad U' = (D',B',P',h') = (h'd',h'b',h'v'),$$

where U' is an approximate solution with rest $r = (r_d, r_b, r_v, r_h)$. The relative entropy (45) reads

$$\eta(U, U') = \frac{h}{2}\{|v - v'|^2 + |d - d'|^2 + |b - b'|^2 + (\frac{1}{h} - \frac{1}{h'})^2\}. \tag{49}$$

Notice that we get

$$\int \{|P - P'| + |D - D'| + |B - B'| + |1 - h/h'|\}dx \tag{50}$$

$$\leq (1 + ||U'||_\infty)\sqrt{\int hdx}\sqrt{\int \eta(U, U')dx},$$

where $||\cdot||_\infty$ denotes the sup norm, just by using both Cauchy-Schwartz and triangle inequalities. (Indeed

$$|P - P'| = |hv - h'v'| \leq h|v - v'| + |h'v'||1 - h/h'| = h|v - v'| + |P'||1 - h/h'|$$

etc...) Thus the relative entropy controls the L^1 distance between U and U'. From the abstract method, we get

Theorem 4.1 *Let $U = (hd, hb, hv, h)$ be a weak entropy solution of the ABI system on the periodic cube $\mathbf{R^3}/\mathbf{Z^3}$. Let $U' = (h'd', h'b', h'v', h')$ be a smooth approximate solution of the ABI system with rest $r = (r_d, r_b, r_v, r_h)$. Then the relative entropy $\eta(U, U')$, defined by (49), satisfies*

$$\frac{d}{dt}\int \eta(U, U')dx \leq c_0(R_1 + R_2)\int hdx + c_0(1 + L)\int \eta(U, U')dx, \tag{51}$$

where L, R_1 and R_2 are given by

$$L = ||\nabla(d', b', v', d'^2 + b'^2 + v'^2 + 1/h'^2)||_\infty, \tag{52}$$

$$R_1 = ||\frac{r_v^2 + r_d^2 + r_b^2}{h'^2}||_\infty, \tag{53}$$

$$R_2 = ||r_h^2\frac{v'^2 + d'^2 + b'^2 + 1/h'^2}{h'^2}||_\infty \tag{54}$$

and c_0 is a purely numerical constant.

4.3 Proof of Theorem 4.1

All component of Z, defined by (46), correspond to one of the conservation laws of the ABI system. Those corresponding to equations (8) are given by

$$Z_{B_i,x_j} = h\{(b_i - b_i')(v_j - v_j') - (b_j - b_j')(v_i - v_i') \tag{55}$$

$$+\epsilon_{ijk}(d_k - d_k')(\frac{1}{h} - \frac{1}{h'})\}$$

$$Z_{D_i,x_j} = h\{(d_i - d_i')(v_j - v_j') - (d_j - d_j')(v_i - v_i') \tag{56}$$

$$-\epsilon_{ijk}(b_k - b_k')(\frac{1}{h} - \frac{1}{h'})\},$$

where ϵ_{ijk} is the signature symbol for ijk. The components corresponding to equation (10) are given by

$$Z_{P_i,x_j} = h\{(v_i - v_i')(v_j - v_j') - (d_j - d_j')(d_i - d_i') \tag{57}$$

$$-(b_j - b_j')(b_i - b_i') - (\frac{1}{h} - \frac{1}{h'})^2\delta_{ij}\},$$

where δ_{ij} is the Kronecker symbol. Finally, the component Z_{h,x_j} corresponding to the (linear) equation (9) is null, as expected from definition (46). It follows that

$$\int ||Z(U,U')||dx \leq c_0 \int \eta(U,U')dx, \tag{58}$$

where c_0 denotes, from now on, any purely numerical constant (here depending only on the choice of the matrix norm $||\cdot||$). As a consequence, we get from (48)

$$\frac{d}{dt}\int \eta(U,U')dx \leq c_0||\nabla(DS(U'))||_\infty \int \eta(U,U')dx + R, \tag{59}$$

where

$$R = \int D^2S(U')(r \otimes (U - U'))dx. \tag{60}$$

Let us first estimate R. From formula (11), we get

$$\frac{\partial^2 S}{\partial P_i \partial P_j}(U') = \frac{\partial^2 S}{\partial D_i \partial D_j}(U') = \frac{\partial^2 S}{\partial B_i \partial B_j}(U') = \frac{1}{h'}\delta_{ij},$$

$$\frac{\partial^2 S}{\partial P_i \partial D_j}(U') = \frac{\partial^2 S}{\partial D_i \partial B_j}(U') = \frac{\partial^2 S}{\partial B_i \partial P_j}(U') = 0,$$

$$\frac{\partial^2 S}{\partial P_i \partial h}(U') = \frac{-v_i'}{h'}, \quad \frac{\partial^2 S}{\partial D_i \partial h}(U') = \frac{-d_i'}{h'}, \quad \frac{\partial^2 S}{\partial B_i \partial h}(U') = \frac{-b_i'}{h'},$$

$$\frac{\partial^2 S}{\partial h \partial h}(U') = \frac{v'^2 + b'^2 + d'^2 + \frac{1}{h'^2}}{h'}.$$

Thus, R is given, after few rearrangements, by

$$R = \int \frac{r_v \cdot (v - v') + r_d \cdot (d - d') + r_b \cdot (b - b')}{h'} h dx \tag{61}$$

$$- \int r_h \frac{v' \cdot (v - v') + d' \cdot (d - d') + b' \cdot (b - b') + 1/h'(1/h' - 1/h)}{h'} h dx.$$

So we get, by Cauchy-Schwarz inequality and (49),

$$R \leq c_0 \int \eta(U, U') dx + c_0 (R_1 + R_2) \int h dx$$

where R_1 and R_2 are defined by (53,54). Going back to (59), we observe that (by definition (11))

$$DS(U') = (d', b', v', -\frac{d'^2 + b'^2 + v'^2 + 1/h'^2}{2}), \tag{62}$$

and finally deduce (51). This completes the proof of Theorem 4.1.

4.4 Proof of Theorem 3.1

The proof is an application of Theorem 4.1, Let us define an approximate solution $(D', B', P', h') = (h'd', h'b', h'v', h')$ of the ABI system by setting

$$B' = \lambda B^*, \quad P' = \lambda P^*, \quad h' = \lambda h^*, \quad D' = D^*, \quad b' = b^*, \quad v' = v^*, \quad d' = d^*/\lambda.$$

For this approximate solution, by definition (52), constant L is of order $O(1)$. Moreover, the error terms read

$$r_b = \nabla \times d' = O(1/\lambda), \quad r_d = r_h = 0, \quad r_v = \nabla.(h'd' \otimes d') + \nabla(1/h') = O(1/\lambda).$$

Thus, by definition (53,54),

$$R_2 = 0, \quad R_1 = O(1/\lambda^4).$$

It follows that the (rescaled) relative entropy

$$\frac{1}{\int hdx}\int \frac{h}{2}\{|v-v'|^2+|b-b'|^2+|d-d'|^2+(\frac{1}{h}-\frac{1}{h'})^2\}dx$$

stays of order $O(1/\lambda^4)$ for all $t\in[0,T]$ if it does so at time $t=0$. Since, the relative entropy can also be written

$$\int \frac{h}{2}\{|v-v^*|^2+|b-b^*|^2+|d-\frac{d^*}{\lambda}|^2+(\frac{1}{h}-\frac{1}{\lambda h^*})^2\}dx,$$

we have obtained (33). Finally, since $\int hdx$ is a conserved quantity of size $O(\lambda)$ and $||U'||_\infty = O(\lambda)$, we deduce from inequality (50) that (34) holds true.
the proof is now complete.

4.5 Proof of Theorem 3.2

The proof is again an application of Theorem 4.1. We first define an approximate solution $(D',B',P',h')=(h'd',h'b',h'v',h')$ of the ABI system by setting

$$D'=\lambda D^*,\quad B'=\lambda B^*,\quad P'=\lambda^2 P^*,\quad h'=\lambda^2 h^*,$$

$$d'=d^*/\lambda,\quad b'=b^*/\lambda,\quad v'=v^*.$$

For this approximate solution, by definition (52), constant L is of order $O(1)$. Moreover, the error terms read

$$r_d=-\frac{1}{\lambda}\nabla\times b'=O(1/\lambda^2),\quad r_b=\frac{1}{\lambda}\nabla\times d'=O(1/\lambda^2),\quad r_h=0,$$

$$r_v=\frac{1}{\lambda^2}(\nabla.(h'(d'\otimes d'+b'\otimes b'))+\nabla(1/h'))=O(1/\lambda^2).$$

Thus, by definition (53,54),

$$R_2=0,\quad R_1=O(1/\lambda^8).$$

It follows that the rescaled relative entropy

$$\frac{1}{\int hdx}\int \frac{h}{2}\{|v-v'|^2+|b-b'|^2+|d-d'|^2+(\frac{1}{h}-\frac{1}{h'})^2\}dx$$

stays of order $O(1/\lambda^8)$ for all $t \in [0,T]$ if it does so at time $t=0$. Since, the relative entropy can also be written

$$\int \frac{h}{2}\{|v-v^*|^2+|b-b^*/\lambda|^2+|d-d^*/\lambda|^2+(\frac{1}{h}-\frac{1}{\lambda^2 h^*})^2\}dx,$$

we have obtained (38). Last, $\int h dx$ is a conserved quantity of size $O(\lambda^2)$ and $||U'||_\infty = O(\lambda^2)$, thus (39) follows from (50). This completes the proof.

4.6 Proof of Theorem 3.3

Again Theorem 4.1 is used. Here we define an approximate solution

$$(D',B',P',h')=(h'd',h'b',h'v',h')$$

of the ABI system by setting

$$h'=1, \quad P'=v'=0, \quad D'=d'=\lambda D^*, \quad B'=b'=\lambda B^*,$$

where $\lambda << 1$. For this approximate solution, by definition (52), constant L is of order $O(\lambda)$. Moreover, the error terms read

$$r_d=r_b=r_h=0, \quad r_v=\lambda^2\nabla.(d'\otimes d'+b'\otimes b')=O(\lambda^2).$$

Thus, by definition (53,54),

$$R_2=0, \quad R_1=O(\lambda^4).$$

It follows that the rescaled relative entropy

$$\frac{1}{\int h dx}\int \frac{h}{2}\{|v-v'|^2+|b-b'|^2+|d-d'|^2+(\frac{1}{h}-\frac{1}{h'})^2\}dx$$

stays of order $O(1/\lambda^4)$ for all $t \in [0,T]$ if it does so at time $t=0$. Since, the relative entropy can also be written

$$\int \frac{h}{2}\{v^2+|b-\lambda b^*|^2+|d-\lambda d^*|^2+(\frac{1}{h}-1)^2\}dx,$$

we have obtained (41). Since $\int h dx$ is a conserved quantity of size $O(1)$ and $||U'||_\infty = O(1)$, it follows from (50) that (42) holds true, which completes the proof.

Appendix A : proof of Proposition 2.1

Notice first that, for a general nonlinear Maxwell theory, we have

$$\partial_t(D \times B) = \nabla \cdot (H \otimes B + E \otimes D) + \nabla.(h - D \cdot E - B \cdot H). \qquad (63)$$

For the BI system, H and E are given by

$$E = \frac{D + B \times P}{h}, \quad H = \frac{B - D \times P}{h}.$$

where $P = D \times B$. Thus

$$h - D \cdot E - B \cdot H = \frac{1}{h}(h^2 - D \cdot (D + B \times P) - B \cdot (B - D \times P))$$

$$= \frac{1}{h}(1 + P^2 - 2(D \times B) \cdot P) = \frac{1}{h}(1 - P^2),$$

since $P = D \times B$. Next,

$$H \otimes B + E \otimes D = \frac{1}{h}(B \otimes B + D \otimes D + T),$$

where

$$T_{ij} = -(D \times P)_i B_j + (B \times P)_i D_j$$

$$= (D_i B_k - D_k B_i)(D_j B_k - D_k B_j) = -P_i P_j + P^2 \delta_{ij}$$

(since $P = D \times B$). So we have obtained

$$\partial_t(D \times B) = \nabla(\frac{B \otimes B + D \otimes D - P \otimes P}{h}) + \nabla.(\frac{1}{h}),$$

which exactly is (10). This completes the proof.

Appendix B : proof of Theorem 2.2

We have to show that S defined by (11) satisfies (12) for all smooth solutions of the ABI system. For this proof, we use notations $u_{,i}$ for $\partial u / \partial x_i$, as well as the classical signature symbols ϵ_{ijk} and implicit summation on repeated indices. We also use notations $b = B/h$, $d = D/h$, $v = P/h$. We have

$$\partial_t S + \frac{B^2 + D^2 + P^2 + 1}{2h^2} h_t = b_i B_{i,t} + d_i D_{i,t} + v_i P_{i,t}$$

$$= -\epsilon_{ijk} b_i d_{k,j} - b_i (h(b_i v_j - b_j v_i))_{,j} + \epsilon_{ijk} d_i b_{k,j} - d_i (h(d_i v_j - d_j v_i))_{,j}$$
$$-v_i (h v_i v_j - h b_i b_j - h d_i d_j)_{,j} + v_j (h^{-1})_{,j}$$
$$= (\epsilon_{ijk} d_i b_k)_{,j} - (b^2/2 + d^2/2 + v^2/2)_{,j} h v_j - (b^2 + d^2 + v^2)(h v_j)_j$$
$$+ h (b_i b_j + d_i d_j) v_{i,j}$$

(using that $\nabla \cdot (hb) = \nabla \cdot (hd) = 0$)

$$+ v_i (h b_i b_j + h d_i d_j)_{,j} + h v_j (h^{-2}/2)_{,j}$$
$$= \nabla \cdot (b \times d) + \nabla (h(b(b \cdot v) + d(d \cdot v))) - \nabla \cdot ((b^2/2 + d^2/2 + v^2/2) h v)$$
$$-1/2 (b^2 + d^2 + v^2 + h^{-2})(h v_j)_{,j} + (h v_j h^{-2}/2)_{,j}.$$

So, we have obtained (since $h_{,t} + (hv_j)_{,j} = 0$),

$$\partial_t S = \nabla \cdot (b \times d) + \nabla \cdot (h(b(b \cdot v) + d(d \cdot v))) - \nabla \cdot ((b^2/2 + d^2/2 + v^2/2 - h^{-2}/2) h v),$$
$$= \nabla \cdot (b \times d) + \nabla \cdot (h(b(b \cdot v) + d(d \cdot v))) - \nabla \cdot (S h v - h^{-2} h v)$$
$$= \nabla \cdot (\frac{B \times D}{h^2}) + \nabla \cdot (\frac{B(B \cdot P) + D(D \cdot P)}{h^2}) - \nabla \cdot (SP - \frac{P}{h^2}),$$

which is exactly (12) and completes the proof.

Appendix C : Partial results in one space dimension

Existence of global smooth solutions

Let us consider the 8×8 one-dimensional ABI system (15, 16, 17, 18, 19, 20, 21, 22). For simplicity, we denote by x the one-dimensional space variable and we only consider spatially periodic solutions of period 1. As already observed, this system uncouples. The fluid part (15,16) is just the one-dimensional Chaplygin gas equations, for which the propagation speeds are

$$\lambda_- = \frac{P_1 - 1}{h}, \quad \lambda_+ = \frac{P_1 + 1}{h},$$

and solve

$$(\partial_t + \lambda_- \partial_x) \lambda_+ = 0, \quad (\partial_t + \lambda_+ \partial_x) \lambda_- = 0. \tag{64}$$

This 2×2 system is studied in detail in Serre's book [Se]. In particular, it is shown to be integrable and admit global smooth solutions for smooth initial data, under the following additional condition

$$\Lambda_1 \leq \lambda_-(t = 0, x) \leq \Lambda_2 < \Lambda_3 \leq \lambda_+(t = 0, x) \leq \Lambda_4,$$

for some constants Λ_i, $i = 1, 2, 3, 4$. Once h and P_1 are known, the six remaining equations (17, 18, 19, 20, 21, 22) form a *linear symmetric* system of conservation laws with smooth coefficients, which, therefore, admits global smooth solutions for smooth initial data (see [Ma], [AG]...).

Proposition 4.2 *The one-dimensional ABI system admits global spatially periodic smooth solutions $(D, B, h > 0, P)$ for all smooth initial data such that*

$$\Lambda_1 \leq \frac{P_1(0,x) - 1}{h(0,x)} \leq \Lambda_2 < \Lambda_3 \leq \frac{P_1(0,x) + 1}{h(0,x)} \leq \Lambda_4, \tag{65}$$

for some constants Λ_i, $i = 1, 2, 3, 4$.

Weak stability

Let us now restrict ourself to special solutions to the one-dimensional ABI system of the following form

$$D = (0, D(t,x), 0), \quad B = (0, 0, B(t,x)), \quad h = h(t,x), \quad P = (P(t,x), 0, 0),$$

with obvious notational abuses. The resulting special one-dimensional ABI system is

$$\partial_t D + \partial_x(\frac{B + DP}{h}) = \partial_t B + \partial_x(\frac{D + BP}{h}) = 0, \tag{66}$$

$$\partial_t h + \partial_x P = 0. \tag{67}$$

$$\partial_t P + \partial_x(\frac{P^2 - 1}{h}) = 0, \tag{68}$$

with 'entropy'

$$S(D, B, h, P) = \frac{1 + D^2 + B^2 + P^2}{h}.$$

By using, again, Serre's results we have the following 'weak stability' result

Proposition 4.3 *Let $(D_n, B_n, h_n > 0, P_n)$ be a sequence of global spatially periodic smooth solutions of the special 1-dimensional ABI system (66,67,68), such that*

$$\Lambda_1 \leq \frac{P_n - 1}{h_n} \leq \Lambda_2 < \Lambda_3 \leq \frac{P_n + 1}{h_n} \leq \Lambda_4, \tag{69}$$

$$\int \frac{1 + D_n^2 + B_n^2 + P_n^2}{h_n} dx \leq \Lambda_5, \tag{70}$$

for some constants Λ_i, $i = 1, 2, 3, 4, 5$, *at time* $t = 0$.
Then (D_n, B_n, h_n, P_n) *has at least a subsequence that converges to some limit* (D, B, h, P) *(where the convergence holds true in the space of continuous functions of* t *valued in* L^2 *equipped with the weak topology) and this limit is a (weak) solution of the ABI system.*

Proof

First observe that the initial bounds propagate. From (69), which remains true for all $t \geq 0$, we deduce that

$$0 < \inf_{n,t} h_n(t,x) \leq \sup_{n,t} h_n(t,x) < +\infty.$$

The initial entropy bound (70) also propagates for all $t \geq 0$, which implies

$$\sup_{n,t} \int (D_n^2 + B_n^2 + P_n^2)(t,x)dx < +\infty.$$

Thus the solutions are uniformly bounded in $L^\infty(\mathbf{R}, \mathbf{L^2}(\mathbf{R}/\mathbf{Z}))$ and their time derivatives are bounded in the sense of distributions in x, uniformly in t, which is enough for compactness in the space of all continuous functions of $t \geq 0$, valued in $L^2(\mathbf{R}/\mathbf{Z})$, equipped with the weak topology.
Next, following Serre in his analysis of equations (64), we know that (h, P) is a weak solution of the Chaplygin gas equations (67,68). By applying (three times) the Murat-Tartar div-curl lemma (see [Da], [Se]...) to system (66,67,68), we deduce some weak convergence for nonlinear combinations of (D_n, B_n, h_n, P_n), namely

$$\frac{D_n^2 - B_n^2}{h_n} \to \frac{D^2 - B^2}{h},$$

$$\frac{B_n P_n + D_n}{h_n} \to \frac{BP + D}{h},$$

$$\frac{D_n P_n + B_n}{h_n} \to \frac{DP + B}{h}.$$

This is enough to deduce that the limit (B, D) still satisfies (66) which completes the proof.

Appendix D : Proof of Theorem 2.3

Let us first introduce the indicator function $I(D, B, P, h)$ of the BI manifold, with values 0 inside the BI manifold and $+\infty$ ouside. To get the closed convex hull of the BI manifold, it is enough to compute the bidual function I^{**}. We have

$$I^{**}(D, B, P, h) = \sup_{\delta,\beta,\pi,\eta} D \cdot \delta + B \cdot \beta + P \cdot \pi + h\eta - I^*(\delta, \beta, \pi, \eta),$$

where

$$I^*(\delta, \beta, \pi, \eta) = \sup_{D,B,P,h} D \cdot \delta + B \cdot \beta + P \cdot \pi + h\eta - I(D, B, P, h)$$

$$= \sup_{D,B} D \cdot \delta + B \cdot \beta + D \times B \cdot \pi + \eta\sqrt{1 + D^2 + B^2 + |D \times B|^2},$$

by definition of I as the indicator function of the BI manifold (5,7).
First, we observe that $I^*(\delta, \beta, \pi, \eta) = +\infty$ whenever $\eta > 0$.
(Indeed, let us assume $\eta > 0$. If $\beta \neq 0$, by setting $D = 0$, $B = \beta n$ and letting $n \to +\infty$, we get $I^*(\delta, \beta, \pi, \eta) = +\infty$. By symmetry, we get the same result whenever $\delta \neq 0$. Finally if $\delta = \beta = 0$, we set $B = D = (n, 0, 0)$, let n go to ∞ and again get $I^*(\delta, \beta, \pi, \eta) = +\infty$.)
Next, for $\eta < 0$, we have

$$I^*(\delta, \beta, \pi, \eta) = -\eta K^*(\frac{\delta}{-\eta}, \frac{\beta}{-\eta}, \frac{\pi}{-\eta}),$$

where

$$K^*(\delta, \beta, \pi) = \sup_{D,B} D \cdot \delta + B \cdot \beta + D \times B \cdot \pi - \sqrt{1 + D^2 + B^2 + |D \times B|^2}. \quad (71)$$

It follows that

$$I^{**}(D, B, P, h) = \sup_{\eta<0}(-\eta)(-h + \sup_{\delta,\beta,\pi} D \cdot \delta + B \cdot \beta + P \cdot \pi - K^*(\delta, \beta, \pi))$$

$$= \sup_{\eta<0}(-\eta)(-h + K^{**}(D, B, P))$$

takes value 0 or $+\infty$ whenever $h \geq K^{**}(D, B, P)$ or not. This exactly means that the convex hull of the BI manifold is just $\{h \geq K^{**}(D, B, P)\}$, with

$$K^{**}(D, B, P) = \sup_{\delta,\beta,\pi} D \cdot \delta + B \cdot \beta + P \cdot \pi - K^*(\delta, \beta, \pi), \quad (72)$$

where K^* is defined by (71). Just by setting $D = B = 0$ in (71), we immediately get

$$K^*(\delta,\beta,\pi) \geq -1.$$

Next, we claim that $K^* = +\infty$ unless $|\beta|$, $|\delta|$ and $|\pi|$ are bounded by 1. Indeed, we have (by setting $D = 0$ in definition (71))

$$K^*(\delta,\beta,\pi) \geq \sup_B B\cdot\beta - \sqrt{1+B^2} = -\sqrt{1-\beta^2},$$

if $|\beta| \leq 1$ and $+\infty$ otherwise. Similarly,

$$K^*(\delta,\beta,\pi) \geq -\sqrt{1-\delta^2},$$

if $|\delta| \leq 1$ and $+\infty$ otherwise. Finally, assume $|\pi| > 1$ and let π^* be a unit vector orthogonal to π. Set $B = n\pi^*$, $D = n\pi^* \times \pi$ in (71). Then, we get $D \times B = n^2\pi$ and

$$K^*(\delta,\beta,\pi) \geq \sup_n \{n\pi^* \times \pi \cdot \delta + n\pi^* \cdot \beta + n^2\pi^2$$

$$-\sqrt{1 + n^2\pi^{*2} + (n\pi^* \times \pi)^2 + n^4\pi^2}\} = \sup_n n^2(\pi^2 - |\pi|) = +\infty.$$

So, K^* is infinite, unless $|\beta|$, $|\delta|$ and $|\pi|$ are bounded by 1, and bounded from below by -1, we deduce from definition (72) that

$$K^{**}(D,B,P,h) \leq \sup_{|\delta|\leq 1, |\beta|\leq 1, |\pi|\leq 1} D\cdot\delta + B\cdot\beta + P\cdot\pi + 1$$

$$= 1 + |D| + |B| + |P|.$$

Thus, the closed convex hull of the BI manifold, which is defined by inequality $h \geq K^{**}(D,B,P,h)$ certainly contains the 10 dimensional convex set defined by inequality $h \geq 1 + |D| + |B| + |P|$. Finally, from (71), it follows that

$$K^*(\delta,\beta,\pi) \leq \sup_{D,B,P} D\cdot\delta + B\cdot\beta + P\cdot\pi - \sqrt{1+D^2+B^2+P^2},$$

which implies

$$K^{**}(D,B,P) \geq \sqrt{1+D^2+B^2+P^2},$$

since $\sqrt{1+D^2+B^2+P^2}$ is convex. Thus, the closed convex hull of the BI manifold is contained in $\{h \geq \sqrt{1+D^2+B^2+P^2}\}$ and the proof is now complete.

Appendix E : integrability of the pressureless systems

The integrability of the pressureless MHD system (29,30,31) follows from the following observation. Let $L > 0$ and

$$X : (t, s, a) \in \mathbf{R}_+ \times (\mathbf{R}/\mathbf{LZ}) \times (\mathbf{R}/\mathbf{Z})^{\mathbf{2}} \rightarrow \mathbf{X(t,s,a)} \in (\mathbf{R}/\mathbf{Z})^{\mathbf{3}}$$

be a collection (parameterized by a) of (classical) strings subject to the one-dimensional wave equation

$$\partial_{tt} X = \partial_{ss} X,$$

on the inverval $0 \leq s \leq L$, with periodic boundary conditions at $s = 0$ and $s = L$. Let us introduce the following fields

$$h(t, x) = \int \delta(x - X(t, s, a)) ds da, \tag{73}$$

$$B(t, x) = \int \partial_s X(t, s, a) \delta(x - X(t, s, a)) ds da, \tag{74}$$

$$P(t, x) = \int \partial_t X(t, s, a) \delta(x - X(t, s, a)) ds da, \tag{75}$$

which are well defined as Borel measures. Then, we easily get, in the sense of distributions

$$\nabla \cdot B = 0, \quad \partial_t h + \nabla \cdot P = 0,$$

$$\partial_t B(t, x) = \nabla \cdot \int (\partial_s X \otimes \partial_t X - \partial_t X \otimes \partial_s X)(t, s, a) \delta(x - X(t, s, a)) ds da.$$

$$\partial_t P(t, x) = \nabla \cdot \int (\partial_s X \otimes \partial_s X - \partial_t X \otimes \partial_t X)(t, s, a) \delta(x - X(t, s, a)) ds da. \tag{76}$$

Let us prove, for instance, the last identity. We have

$$\partial_t P(t, x) = \int \partial_{tt} X(t, s, a) \delta(x - X(t, s, a)) ds da$$

$$- \int \partial_t X(t, s, a) (\nabla \delta)(x - X(t, s, a)) \cdot \partial_t X(t, s, a) ds da.$$

Thus,

$$\partial_t P(t, x) + \int \partial_t X(t, s, a) (\nabla \delta)(x - X(t, s, a)) \cdot \partial_t X(t, s, a) ds da$$

$$= \int \partial_{ss} X(t, s, a) \delta(x - X(t, s, a)) ds da$$

(since each $X(\cdot,\cdot,a)$ solves the one dimensional linear wave equation)

$$= \int \partial_s X(t,s,a)(\nabla\delta)(x - X(t,s,a)) \cdot \partial_s X(t,s,a) dsda,$$

(by integration by part in s), which completes the proof of (76).
Let us now assume that, at time $t = 0$,

$$(s,a) \in (\mathbf{R}/\mathbf{LZ}) \times (\mathbf{R}/\mathbf{Z})^{\mathbf{2}} \to \mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a}) \in (\mathbf{R}/\mathbf{Z})^{\mathbf{3}}$$

is a diffeomorphism, which makes $h(t,x)$ a strictly positive smooth function and B and P two smooth vector fields. Since the one-dimensional wave equation keeps solutions smooth, the diffeomorphism property will stay true for at least a short time interval $[0,T]$, more precisely as long as h stays strictly positive. Then, for each $t \in [0,T]$, we have

$$\int (\partial_t X \otimes \partial_s X - \partial_s X \otimes \partial_t X)\delta(x - X(t,s,a))dsda = \frac{P \otimes B - B \otimes P}{h},$$

$$\int (\partial_t X \otimes \partial_t X - \partial_s X \otimes \partial_s X)\delta(x - X(t,s,a))dsda = \frac{P \otimes P - B \otimes B}{h},$$

which makes (h,B,P) a (local) smooth solution of the pressureless MHD system.
Now, let us check that there is, at least, a non trivial set of initial data $(h,B,P)(t=0)$, belonging to the homogeneous BI manifold (32), for which we can satisfy the diffeomorphism property. We set

$$h(t=0,x) = L, \quad B(t=0,x) = (0,0,1), \tag{77}$$

$$P(t=0,x) = \sqrt{L^2-1}(\cos\phi(x), \sin\phi(x), 0),$$

where ϕ is a smooth function on $(\mathbf{R}/\mathbf{Z})^{\mathbf{3}}$, which is compatible with (32), provided that $L > 1$. We observe that

$$X(t=0,s,a) = (a, s/L), \quad \partial_t X(t=0,s,a) = (1/L)P(t=0,a,s/L),$$

is consistent with definitions (74), (75) and (77). Geometrically speaking, this corresponds to a 3-dimensional 'harp' made of straight vertical strings. Then, we explicitely solve the one-dimensional wave equation and get

$$X(t,s,a) = (a + \frac{1}{2L}\int_{s-t}^{s+t} P(0,a,s'/L)ds', \frac{s}{L}),$$

which defines a diffeomorphism at least for a sufficiently short time interval $[0, T]$. Therefore, the corresponding fields (h, B, P) defined by (73), (74) and (75) are solutions to the pressureless MHD system on $[0, T]$ for the prescribed initial data (77).
Let us finally consider the pressureless gas equations (35), (36) along the 'homogeneous' BI manifold, (37). The integrability is very easy to check. We just consider a family of particle trajectories $t \to X(t, a)$, parameterized by $a \in (\mathbf{R}/\mathbf{Z})^{\mathbf{3}}$, with no acceleration

$$\partial_{tt} X(t, a) = 0, \tag{78}$$

and define

$$h(t, x) = \int \delta(x - X(t, a)) da, \tag{79}$$

$$P(t, x) = \int \partial_t X(t, a) \delta(x - X(t, a)) da. \tag{80}$$

As long as $a \to X(t, a)$ is a diffeomorphism of $(\mathbf{R}/\mathbf{Z})^{\mathbf{3}}$, (h, P) is a smooth solution to the pressureless gas system. A typical example of solutions, compatible with the homogeneous BI manifold (37), is provided by

$$X(t, a) = a + tv(a)$$

where v is a smooth function on $(\mathbf{R}/\mathbf{Z})^{\mathbf{3}}$, valued in the unit sphere $\mathbf{S}^{\mathbf{2}}$. The corresponding initial conditions for (h, P) are

$$h(0, x) = 1, \quad P(0, x) = v(x) \in \mathbf{S}^{\mathbf{2}}$$

Such a solution corresponds to a gas of photons with uniform initial density.

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References

[AG] S. Alinhac, P. Gérard, *Opérateurs pseudo-différentiels et théorème de Nash-Moser, InterEditions, (CNRS), Meudon, 1991.*

[BDLL] G. Boillat. C. Dafermos, P. Lax, T.P. Liu, *Recent mathematical methods in nonlinear wave propagation, Lecture Notes in Math., 1640, Springer, Berlin, 1996*

[BI] M. Born, L. Infeld, *Foundations of the new field theory, Proc. Roy. Soc. London, A 144 (1934) 425-451.*

[dB] L. de Broglie, *cours professé à l'IHP, 1956-1957*

[Da] C. Dafermos, *Hyperbolic conservation laws in continuum physics, Grundlehren der Mathematischen Wissenschaften, 325, Springer-Verlag, Berlin, 2000.*

[DST] S. Demoulini, D. Stuart, A. Tzavaras, *A variational approximation scheme for three-dimensional elastodynamics with polyconvex energy, Arch. Ration. Mech. Anal. 157 (2001), no. 4, 325-344.*

[GMMP] P. Gérard, P. Markowich, N. Mauser, F. Poupaud, *Homogenization limits and Wigner transforms, Comm. Pure Appl. Math. 50 (1997), no. 4, 323-379.*

[Gi] G.W. Gibbons, *Aspects of Born-Infeld Theory and String/M-Theory, hep-th/0106059.*

[GG] S. Godunov, V. Gordienko, *Complicated structures of Galilean-invariant conservation laws, J. Appl. Mech. Tech. Phys. 43 (2002), no. 2, 175-189.*

[Ma] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables,* Applied Mathematical Sciences, 53. Springer-Verlag, New York, 1984.

[Po] J. Polchinski, *String theory. Vol. I.,* Cambridge University Press, 1998.

[Se] D. Serre, *Systems of conservation laws, 2, ch. 9.6 and 10.1,* Cambridge University Press, Cambridge, 2000.

[Se2] D. Serre, *work in preparation.*

[Ta] L. Tartar, *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations,* Proc. Roy. Soc. Edinburgh Sect. A 115 (1990), no. 3-4, 193-230.

[Ta2] L. Tartar, *Oscillations and concentration effects in partial differential equations: why waves may behave like particles, XVII CEDYA: Congress on Differential Equations and Applications (Salamanca, 2001), 179-219.*