A Sharp Decay Estimate for Positive Nonlinear Waves

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Abstract. We consider a strictly hyperbolic, genuinely nonlinear system of conservation laws in one space dimension. A sharp decay estimate is proved for the positive waves in an entropy weak solution. The result is stated in terms of a partial ordering among positive measures, using symmetric rearrangements and a comparison with a solution of Burgers' equation with impulsive sources.

1 - Introduction

Consider a strictly hyperbolic system of n conservation laws

$$u_t + f(u)_x = 0 \tag{1.1}$$

and assume that all characteristic fields are genuinely nonlinear. Call $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of the Jacobian matrix $A(u) \doteq Df(u)$. We shall use bases of left and right eigenvectors $l_i(u)$, $r_i(u)$ normalized so that

$$\nabla\lambda_i(u)\, r_i(u) \equiv 1\,, \qquad\qquad l_i(u)\, r_j(u) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{1.2}$$

Given a function $u : I\!R \mapsto I\!R^n$ with small total variation, following [BC], [B] one can define the measures μ^i of i-waves in u as follows. Since $u \in BV$, its distributional derivative $D_x u$ is a Radon measure. We define μ^i as the measure such that

$$\mu^i \doteq l_i(u) \cdot D_x u \tag{1.3}$$

restricted to the set where u is continuous, while, at each point x where u has a jump, we define

$$\mu^i\big(\{x\}\big) \doteq \sigma_i\,, \tag{1.4}$$

where σ_i is the strength of the i-wave in the solution of the Riemann problem with data $u^- = u(x-)$, $u^+ = u(x+)$. In accordance with (1.2), if the solution of the Riemann problem contains the intermediate states $u^- = \omega_0, \omega_1, \ldots, \omega_n = u^+$, the strength of the i-wave is defined as

$$\sigma_i \doteq \lambda_i(\omega_i) - \lambda_i(\omega_{i-1}). \tag{1.5}$$

Observing that

$$\sigma_i = l_i(u^+) \cdot (u^+ - u^-) + O(1) \cdot |u^+ - u^-|^2,$$

we can find a vector $l_i(x)$ such that

$$\big|l_i(x) - l_i\big(u(x+)\big)\big| = \mathcal{O}(1) \cdot \big|u(x+) - u(x-)\big|, \tag{1.6}$$

$$\sigma_i = l_i(x) \cdot \big(u(x+) - u(x-)\big). \tag{1.7}$$

We can thus define the measure μ^i equivalently as

$$\mu^i \doteq l_i \cdot D_x u\,, \tag{1.8}$$

where $l_i(x) = l_i\big(u(x)\big)$ at points where u is continuous, while $l_i(x)$ is some vector which satisfies (1.6)-(1.7) at points of jump. For all $x \in I\!R$ there holds

$$\Big|l_i(x) - l_i\big(u(x)\big)\Big| = \mathcal{O}(1) \cdot \big|u(x+) - u(x-)\big|\,. \tag{1.9}$$

We call μ^{i+}, μ^{i-} respectively the positive and negative parts of μ^i, so that

$$\mu^i = \mu^{i+} - \mu^{i-}, \qquad\qquad |\mu^i| = \mu^{i+} + \mu^{i-}. \tag{1.10}$$

It is our purpose to prove a sharp estimate on the decay of the density of the measures μ^{i+}. This will be achieved by introducing a partial ordering within the family of positive Radon measures. In the following, $meas(A)$ denotes the Lebesgue measure of a set A.

Definition 1. *Let μ, μ' be two positive Radon measures. We say that $\mu \preceq \mu'$ if and only if*

$$\sup_{meas(A)\le s} \mu(A) \;\le\; \sup_{meas(B)\le s} \mu'(B) \qquad\qquad \textit{for every } s > 0\,. \tag{1.11}$$

In some sense, the above relation means that μ' is more singular than μ. Namely, it has a greater total mass, concentrated on regions with higher density. Notice that the usual order relation

$$\mu \le \mu' \qquad \text{if and only if} \qquad \mu(A) \le \mu'(A) \quad \text{for every} \quad A \subset I\!R \tag{1.12}$$

is much stronger. Of course $\mu \le \mu'$ implies $\mu \preceq \mu'$, but the converse does not hold.

Following [BC], [B], together with the measures μ^i we define the Glimm functionals

$$V(u) \doteq \sum_i |\mu^i|(I\!R)\,, \tag{1.13}$$

$$Q(u) \doteq \sum_{i<j} \big(|\mu^j| \otimes |\mu^i|\big)\big\{(x,y)\,;\; x<y\big\} + \sum_i \big(\mu^{i-} \otimes |\mu^i|\big)\big\{(x,y)\,;\; x \neq y\big\}\,. \tag{1.14}$$

Let now $u = u(t,x)$ be an entropy weak solution of (1.1). If the total variation of u is small and the constant C_0 is large enough, it is well known that the quantities

$$Q(t) \doteq Q\big(u(t)\big)\,, \qquad\qquad \Upsilon(t) \doteq V\big(u(t)\big) + C_0\, Q\big(u(t)\big) \tag{1.15}$$

are non-increasing in time. The decrease in Q controls the amount of interaction, while the decrease in Υ controls both the interaction and the cancellation in the solution.

An accurate estimate on the measure μ_t^{i+} of positive i-waves in $u(t,\cdot)$ will be obtained by a comparison with a solution of Burgers' equation with source terms.

Theorem 1. *For some constant κ and for every small BV solution $u = u(t,x)$ of the system (1.1) the following holds. Let $w = w(t,x)$ be the solution of the scalar Cauchy problem with impulsive source term*

$$w_t + (w^2/2)_x = -\kappa\, \mathrm{sgn}(x) \cdot \frac{dQ}{dt}\,, \tag{1.16}$$

$$w(0,x) = \mathrm{sgn}(x) \cdot \sup_{meas(A)<2|x|} \frac{\mu^{i+}(A)}{2}\,. \tag{1.17}$$

Then, for every $t \geq 0$,

$$\mu_t^{i+} \preceq D_x w(t)\,. \tag{1.18}$$

As shown in the next section, the initial data in (1.17) represents the *odd rearrangement* of the function $v_i(x) \doteq \mu_0^{i+}\big(]-\infty, x]\big)$. The above theorem improves the earlier estimate derived in [BC]. For a scalar conservation law with strictly convex flux, a classical decay estimate was proved by Oleinik [O]. In the case of genuinely nonlinear systems, results related to the decay of nonlinear waves were also obtained in [GL], [L1], [L2], [BG]. An application of the present analysis will appear in [BY], where Theorem 1 is used to estimate the rate of convergence of vanishing viscosity approximations.

2 - Lower semicontinuity

Let μ be a positive Radon measure on $I\!\!R$, so that $\mu \doteq D_x v$ is the distributional derivative of some bounded, non-decreasing function $v : I\!\!R \mapsto I\!\!R$. We can decompose

$$\mu = \mu^{\text{sing}} + \mu^{ac}$$

as the sum of a singular and an absolutely continuous part, w.r.t. Lebesgue measure. The absolutely continuous part corresponds to the usual derivative $z \doteq v_x$, which is a non-negative $\mathbf{L}^1$ function defined at a.e. point. We shall denote by $\hat z$ the *symmetric rearrangement* of z, i.e. the unique even function such that

$$\hat z(x) = \hat z(-x)\,, \qquad\qquad \hat z(x) \geq \hat z(x') \quad \text{if} \quad 0 < x < x'\,, \tag{2.1}$$

$$\text{meas}\Big(\big\{x\,;\ \hat z(x) > c\big\}\Big) = \text{meas}\Big(\big\{x\,;\ z(x) > c\big\}\Big) \qquad\qquad \text{for every } c > 0\,. \tag{2.2}$$

Moreover, we define the *odd rearrangement* of v as the unique function $\hat v$ such that (fig. 1)

$$\hat v(-x) = -\hat v(x)\,, \qquad\qquad \hat v(0+) = \frac{1}{2}\mu^{\text{sing}}(I\!\!R)\,, \tag{2.3}$$

$$\hat v(x) = \hat v(0+) + \int_0^x z(y)\,dy \qquad \text{for } x > 0\,. \tag{2.4}$$

By construction, the function $\hat v$ is convex for $x < 0$ and concave for $x > 0$.

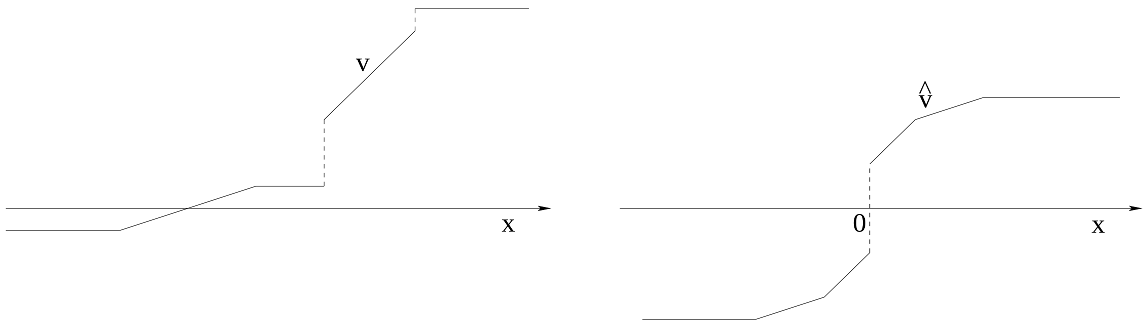

figure 1

The relation between the odd rearrangement $\hat v$ and the partial ordering (1.10) is clarified by the following result, which is an easy consequence of the definitions.

Proposition 1. *Let $\mu = D_x v$ be a positive Radon measure. Define the odd function $w : I\!\!R \mapsto I\!\!R$ by setting*

$$w(s) \doteq \sup_{meas(A) \leq 2s} \frac{\mu(A)}{2} \qquad\qquad \textit{for all} \ \ s > 0\,, \tag{2.5}$$

while $w(s) \doteq -w(-s)$ *for* $s<0$. *Then* $w=\hat{v}$ *is precisely the odd rearrangement of* v *and*

$$\mu \preceq D_s w\,. \tag{2.6}$$

Two more results will be used in the sequel. By the restriction of a measure μ to a set J, we mean the measure

$$(\mu\lfloor J)(A) \doteq \mu(A\cap J)\,.$$

Proposition 2. *Let* μ, μ' *be positive measures. Consider any finite partition* $I\!R = J_1 \cup \cdots \cup J_N$. *If the restrictions of* μ, μ' *to each set* J_ℓ *satisfy* $\mu\lfloor J_\ell \preceq \mu'\lfloor J_\ell$, *then* $\mu \preceq \mu'$.

Proposition 3. *Assume that* $\mu \preceq D_s w$ *for some nondecreasing odd function* w. *If* $|\mu^\sharp - \mu|(I\!R) \le \varepsilon$, *then*

$$\mu^\sharp \preceq D_s \left[w + \mathrm{sgn}(s)\cdot \frac{\varepsilon}{2}\right]\,.$$

The next result is concerned with the lower semicontinuity of the partial ordering $\preceq$ w.r.t. weak convergence of measures.

Proposition 4. *Consider a sequence of measures* μ_ν *converging weakly to a measure* μ. *Assume that the positive parts satisfy* $\mu_\nu^+ \preceq Dw_\nu$ *for some odd, nondecreasing functions* $s\mapsto w_\nu(s)$, *concave for* $s>0$. *Let* w *be the odd funtion such that*

$$w(s) \doteq \liminf_{\nu\to\infty} w_\nu(s) \qquad \text{for} \quad s>0\,.$$

Then the positive part of μ *satisfies*

$$\mu^+ \preceq D_s w\,. \tag{2.7}$$

Proof. By possibly taking a subsequence, we can assume that $w_\nu(s) \to w(s)$ for all $s\neq 0$. Moreover, we can assume the weak convergence

$$\mu_\nu^+ \rightharpoonup \tilde{\mu}^+\,, \qquad\qquad \mu_\nu^- \rightharpoonup \tilde{\mu}^-\,,$$

for some positive measures $\tilde{\mu}^+$, $\tilde{\mu}^-$. We thus have

$$\mu = \tilde{\mu}^+ - \tilde{\mu}^-\,, \qquad\qquad \mu^+ \le \tilde{\mu}^+\,, \qquad \mu^- \le \tilde{\mu}^-\,. \tag{2.8}$$

By (2.8) it suffices to prove that $\tilde{\mu}^+ \preceq D_s w$, i.e.

$$\mathrm{meas}\,(A) \le 2s \qquad \Longrightarrow \qquad \tilde{\mu}^+(A) \le 2w(s)\,, \tag{2.9}$$

for every $s > 0$ and every Borel measurable set $A \subset I\!R$. If (2.9) fails, there exists $s > 0$ and a set A such that

$$\text{meas}\,(A) = 2s\,, \qquad \tilde\mu^+(A) > 2w(s) = 2\lim_{\nu\to\infty} w_\nu(s).$$

Since w is continuous for $s > 0$, we can choose an open set $A' \supseteq A$ such that, setting $s' \doteq \text{meas}\,(A')/2$, one has $2w(s') < \tilde\mu^+(A)$. By the weak convergence $\mu_\nu^+ \rightharpoonup \tilde\mu^+$ one obtains

$$\tilde\mu^+(A') \leq \liminf_{\nu\to\infty} \mu_\nu^+(A') \leq 2w(s') < \tilde\mu^+(A)\,,$$

reaching a contradiction. Hence (2.9) must hold. □

Toward the proof of Theorem 1 we shall need a lower semicontinuity property for wave measures, similar to what proved in [BaB]. In the following, C_0 is the same constant as in (1.15).

Lemma 1. *Consider a sequence of functions u_ν with uniformly small total variation and call μ_ν^{i+} the corresponding measures of positive i-waves. Let $s \mapsto w_\nu(s)$, $\nu \geq 1$, be a sequence of odd, nondecreasing functions, concave for $s > 0$, such that*

$$\mu_\nu^{i+} \preceq D_s\Big[w_\nu + C_0\,\text{sgn}(s)\big(Q_0 - Q(u_\nu)\big)\Big] \tag{2.10}$$

for some Q_0. Assume that $u_\nu \to u$ and $w_\nu \to w$ in $\mathbf{L}^1_{\text{loc}}$. Then the measure of positive i-waves in u satisfies

$$\mu^{i+} \preceq D_s\Big[w + C_0\,\text{sgn}(s)\big(Q_0 - Q(u)\big)\Big]\,. \tag{2.11}$$

Proof. The main steps follow the proof of Theorem 10.1 in [B].

1. By possibly taking a subsequence we can assume that $u_\nu(x) \to u(x)$ for every x and that the measures of total variation converge weakly, say

$$|\mu_\nu| \doteq \big|D_x u_\nu\big| \rightharpoonup \mu^\sharp \tag{2.12}$$

for some positive Radon measure $\mu^\sharp$. In this case one has $\mu^\sharp \geq |\mu|$, in the sense of (1.12).

2. Let any $\varepsilon > 0$ be given. Since the total mass of $\mu^\sharp$ is finite, one can select finitely many points $y_1, \ldots, y_N$ such that

$$\mu^\sharp\big(\{x\}\big) < \varepsilon, \qquad\qquad \text{for all} \quad x \notin \{y_1, \ldots, y_N\}. \tag{2.13}$$

We now choose disjoint open intervals $I_k \doteq]y_k - \rho,\ y_k + \rho[$ such that

$$\mu^\sharp\big(I_k \setminus \{y_k\}\big) < \frac{\varepsilon}{N} \qquad\qquad k = 1, \ldots, N. \tag{2.14}$$

Moreover, we choose $R > 0$ such that

$$\bigcup_{k=1}^{N} I_k \subset [-R, R], \qquad \mu^\sharp\big(]-\infty, -R] \cup [R, \infty[\big) < \varepsilon. \tag{2.15}$$

Because of (2.13), we can now choose points $p_0 < -R < p_1 < \cdots < R < p_r$ which are continuity points for u and for every u_ν, such that

$$\mu^\sharp\big(\{p_h\}\big) = 0, \qquad u_\nu(p_h) \to u(p_h) \qquad \text{for all} \quad h = 0, \ldots, r, \tag{2.16}$$

and such that either

$$p_h - p_{h-1} < \frac{\varepsilon}{N}, \qquad p_{h-1} < y_k < p_h, \qquad [p_{h-1}, p_h] \subset I_k, \tag{2.17}$$

for some $k \in \{1, \ldots, N\}$, or else

$$|\mu|\big([p_{h-1}, p_h]\big) \leq \mu^\sharp\big([p_{h-1}, p_h]\big) < \varepsilon. \tag{2.18}$$

Call $J_h \doteq [p_{h-1}, p_h]$. If (2.18) holds, by weak convergence for some ν_0 sufficiently large one has

$$|\mu_\nu|(J_h) < \varepsilon \qquad \text{for all} \quad \nu \geq \nu_0. \tag{2.19}$$

On the other hand, if (2.17) holds, from (2.14) it follows

$$|\mu|\big(J_h \setminus \{y_k\}\big) \leq \mu^\sharp\big(J_h \setminus \{y_k\}\big) < \frac{\varepsilon}{N}. \tag{2.20}$$

In the remainder of the proof, the main strategy is as follows.

- On the intervals $J_{h(k)}$ containing a point y_k of large oscillation, we first replace each u_ν by a piecewise constant function $\bar{u}_\nu$ having a single jump at y_k. The relations between the corresponding measures μ^i_ν and $\bar{\mu}^i_\nu$ are given by Lemma 10.2 in [B]. Then we take the limit as $\nu \to \infty$.
- On the remaining intervals J_h with small oscillation, we replace the left eigenvectors $l_i(u_\nu)$ by a constant vector $l_i(u^*_h)$. Then we use Proposition 4 to estimate the limit as $\nu \to \infty$.

3. We first take care of the intervals J_h containing a point y_k of large oscillation, so that (2.17) holds. For each $k = 1, \ldots, N$, let $h = h(k) \in \{1, \ldots, r\}$ be the index such that $y_k \in J_h \doteq [p_{h-1}, \, p_h]$. For every $\nu \geq 1$ consider the function

$$\bar{u}_\nu(x) \doteq \begin{cases} u_\nu(x) & \text{if} \ \ x \notin \cup_k J_{h(k)}, \\ u_\nu(p_{h(k)-1}) & \text{if} \ \ x \in]p_{h(k)-1}, \ y_k[\,, \\ u_\nu(p_h) & \text{if} \ \ x \in \ [y_k, \ p_{h(k)}]. \end{cases}$$

Observe that all functions $u, \bar u_\nu$ are continuous at every point $p_0, \ldots, p_r$ and have jumps at $y_1, \ldots, y_N$. Call $\bar\mu^i_\nu$, $i = 1, \ldots, n$, the corresponding measures, defined as in (1.8) with u replaced by $\bar u_\nu$. Clearly $\bar\mu^i_\nu = \mu^i_\nu$ outside the intervals $J_{h(k)}$ of large oscillation. By Lemma 10.2 at p.203 in [B], there holds

$$Q(\bar u_\nu) \leq Q(u_\nu), \qquad\qquad V(\bar u_\nu) + C_0\, Q(\bar u_\nu) \leq V(u_\nu) + C_0 \cdot Q(u_\nu),$$

$$\bar\mu^{i+}_\nu(I\!R) - \mu^{i+}_\nu(I\!R) \ \leq\ C_0 \big[Q(u_\nu) - Q(\bar u_\nu)\big].$$

As a consequence, from (2.10) we deduce

$$\bar\mu^{i+}_\nu \preceq D_s\Big[T^\varepsilon w_\nu + C_0 \,\mathrm{sgn}(s)\big(Q_0 - Q(\bar u_\nu)\big)\Big]\,, \tag{2.21}$$

where

$$T^\varepsilon w(s) \doteq \begin{cases} w(s+\varepsilon/2) & \text{if} \quad s > 0,\\ w(s-\varepsilon/2) & \text{if} \quad s < 0.\end{cases}$$

Indeed, all the mass which in μ^{i+}_ν lies on the set

$$\Omega \doteq \bigcup_{k=1}^{N} J_{h(k)}\,, \qquad\qquad J_h \doteq [p_{h-1},\ p_h]$$

is replaced in $\bar\mu^{i+}_\nu$ by point masses at $y_1, \ldots, y_N$. We obtain (2.21) by observing that, by (2.17), $meas(\Omega) < \varepsilon$. Moreover, the increase in the total mass is $\leq C_0\big[Q(u_\nu) - Q(\bar u_\nu)\big]$.

Since $u_\nu(p_h) \to u(p_h)$ for every h, there holds

$$\begin{aligned}\Big|\mu^i\big(\{y_k\}\big) - \bar\mu^i_\nu\big(\{y_k\}\big)\Big| &= \mathcal{O}(1) \cdot \Big\{\big|u(y_k-) - u(p_{h(k)-1})\big| + \big|u(y_k+) - u(p_{h(k)})\big| \\ &\qquad + \big|u(p_{h(k)-1}) - u_\nu(p_{h(k)-1})\big| + \big|u(p_{h(k)}) - u_\nu(p_{h(k)})\big|\Big\} \\ &= \mathcal{O}(1) \cdot \frac{\varepsilon}{N}\end{aligned} \tag{2.22}$$

for each $k = 1, \ldots, N$ and all ν sufficiently large. By construction we also have

$$|\bar\mu^i_\nu|\big(J_{h(k)} \setminus \{y_k\}\big) = 0, \qquad\qquad |\mu^i|\big(J_{h(k)} \setminus \{y_k\}\big) = \mathcal{O}(1) \cdot \frac{\varepsilon}{N}\,. \tag{2.23}$$

4. Next, call $\mathcal{S} \doteq \big\{h\,;\ \mu^\sharp(J_h) < \varepsilon\big\}$ the family of intervals where the oscillation of every u_ν is small, so that (2.18) holds. If $h \in \mathcal{S}$, for every $x, y \in J_h$ and ν sufficiently large, one has

$$\big|u_\nu(x) - u_\nu(y)\big| \leq |\mu_\nu|(J_h) < \varepsilon,$$

$$\big|u(x) - u(y)\big| \leq |\mu|(J_h) \leq \mu^\sharp(J_h) < \varepsilon.$$

Set $u_h^* \doteq u(p_h)$. By the pointwise convergence $u_\nu(p_h) \to u(p_h)$ and the two above estimates it follows

$$\big|u_\nu(x) - u_h^*\big| < \varepsilon, \qquad \big|u(x) - u_h^*\big| < \varepsilon, \qquad \text{for all } \ x \in J_h\,. \tag{2.24}$$

5. We now introduce the measures $\hat\mu_\nu^i$ such that

$$\hat\mu_\nu^i \doteq l_i(u_h^*) \cdot D_x u_\nu$$

restricted to each interval J_h, $h \in \mathcal{S}$ where the oscillation is small, while

$$\hat\mu_\nu^i = \bar\mu_\nu^i$$

on each interval $J_h = J_{h(k)}$ where the oscillation is large. Observe that the restriction of $\hat\mu_\nu^i$ to $J_{h(k)}$ consists of a single mass at the point y_k. Namely, $\hat\mu_\nu^i\big(\{y_k\}\big)$ is precisely the size of the i-th wave in the solution of the Riemann problem with data $u^- = u_\nu(p_{h(k)-1})$, $u^+ = u_\nu(p_{h(k)})$.

We define $\hat w_\nu$ as the non-decreasing odd function such that

$$\hat w_\nu(s) \ \doteq \sup_{meas(A)\le 2s} \frac{\hat\mu_\nu^{i+}(A)}{2}\,, \qquad s > 0. \tag{2.25}$$

By possibly taking a further subsequence we can assume the convergence

$$Q(\bar u_\nu) \to \overline{Q}\,, \qquad \hat\mu_\nu^i \rightharpoonup \hat\mu^i\,, \qquad \hat w_\nu(s) \to \hat w(s)\,.$$

Using (2.16), we can apply Proposition 4 on each interval J_h and obtain

$$\hat\mu^{i+} \preceq D_s \hat w\,. \tag{2.26}$$

6. Observe that, by (2.24) and (2.19),

$$|\hat\mu_\nu^i - \mu_\nu^i|(J_h) = \mathcal{O}(1) \cdot \varepsilon\, \mu^\sharp(J_h) \qquad h \in \mathcal{S}\,, \tag{2.27}$$

From (2.21) and the definition of $\hat w_\nu$ at (2.25) it thus follows

$$\hat w_\nu(s) \le T^\varepsilon w_\nu(s) + C_0\big[Q_0 - Q(\bar u_\nu)\big] + \mathcal{O}(1) \cdot \varepsilon \qquad s > 0\,. \tag{2.28}$$

Letting $\nu \to \infty$ we obtain

$$\hat w(s) \le T^\varepsilon w(s) + C_0[Q_0 - \overline{Q}] + \mathcal{O}(1) \cdot \varepsilon \qquad s > 0\,, \tag{2.29}$$

$$\overline{Q} = \lim_{\nu\to\infty} Q(\bar u_\nu) \ge \lim_{\nu\to\infty} Q(u_\nu) - \mathcal{O}(1)\cdot\varepsilon \ge Q(u) - \mathcal{O}(1)\cdot\varepsilon\,, \tag{2.30}$$

because of the lower semicontinuity of the functional $u \mapsto Q(u)$. From (2.26), (2.29) and (2.30) we deduce

$$\hat\mu^{i+} \preceq D_s\Big[T^\varepsilon w + \mathrm{sgn}(s)\,\big(C_0[Q_0 - Q(u)] + \mathcal{O}(1)\cdot\varepsilon\big)\Big]\,.$$

By (2.22)–(2.24), our construction of the measure $\hat\mu^i$ achieves the property

$$\big|\mu^{i+} - \hat\mu^{i+}\big|(I\!R) = \mathcal{O}(1) \cdot \varepsilon\,.$$

Hence, by Proposition 3,

$$\mu^{i+} \preceq D_s\Big[T^\varepsilon w + \mathrm{sgn}(s)\,\big(C_0[Q_0 - Q(u)] + \mathcal{O}(1)\cdot\varepsilon\big)\Big]\,.$$

Since $\varepsilon > 0$ was arbitrary, this proves (2.11). $\square$

3 - A decay estimate

The second basic ingredient in the proof is the following lemma, which refines the estimate in [BC].

Lemma 2. *For some constant $\kappa > 0$ the following holds. Let $u = u(t,x)$ be any entropy weak solution of (1.1), with initial data $u(0,x) = \bar u(x)$ having small total variation. Then the measure μ_t^{i+} of positive i-waves in $u(t,\cdot)$ can be estimated as follows.*

Let $w : [0,\tau[\, \times I\!R \mapsto I\!R$ be the solution of Burgers' equation

$$w_t + (w^2/2)_x = 0 \tag{3.1}$$

with initial data

$$w(0,x) = \mathrm{sgn}(x) \cdot \sup_{meas(A)\le 2|x|} \frac{\mu_0^{i+}(A)}{2}\,. \tag{3.2}$$

Set

$$w(\tau,x) = w(\tau-,\ x) + \kappa\,\mathrm{sgn}(x)\cdot \Big[Q(\bar u) - Q\big(u(\tau)\big)\Big]\,. \tag{3.3}$$

Then

$$\mu_\tau^{i+} \preceq D_x w(\tau)\,. \tag{3.4}$$

Proof. The main steps follow the proof of Theorem 10.3 in [B]. We first prove the estimate (3.3) under the additional hypothesis:

(H) There exist points $y_1 < \cdots < y_m$ such that the initial data $\bar u$ is smooth outside such points, constant for $x < y_1$ and $x > y_m$, and the derivative component $l_i(u)\,u_x$ is constant on each interval $]y_\ell,\,y_{\ell+1}[$. Moreover, the Glimm functional $t\mapsto Q\big(u(t)\big)$ is continuous at $t=\tau$.

1. The solution $u = u(t,x)$ can be obtained as limit of front tracking approximations. In particular, we can consider a particular converging sequence $(u_\nu)_{\nu\ge1}$ of ε_ν-approximate solutions with the following additional properties:

(i) Each i-rarefaction front x_α travels with the characteristic speed of the state on the right:

$$\dot x_\alpha = \lambda_i\big(u(x_\alpha+)\big).$$

(ii) Each i-shock front x_α travels with a speed strictly contained between the right and the left characteristic speeds:

$$\lambda_i\big(u(x_\alpha+)\big) < \dot x_\alpha < \lambda_i\big(u(x_\alpha-)\big). \tag{3.5}$$

(iii) As $\nu \to \infty$, the interaction potentials satisfy

$$Q\big(u_\nu(0,\cdot)\big) \to Q(\bar u). \tag{3.6}$$

2. Let u_ν be an approximate solution constructed by the front tracking algorithm. By a *(generalized) i-characteristic* we mean an absolutely continuous curve $x = x(t)$ such that

$$\dot x(t) \in \big[\lambda_i(u_\nu(t,x-)),\ \lambda_i(u_\nu(t,x+))\big]$$

for a.e. t. If u_ν satisfies the above properties (i)-(ii), then the i-characteristics are precisely the polygonal lines $x : [0,\tau] \mapsto I\!R$ for which the following holds. For a suitable partition $0 = t_0 < t_1 < \cdots < t_m = \tau$, on each subinterval $[t_{j-1}, t_j]$ either $\dot x(t) = \lambda_i\big(u_\nu(t,x)\big)$, or else x coincides with a wave-front of the i-th family. For a given terminal point $\bar x$ we shall consider the *minimal backward i-characteristic* through $\bar x$, defined as

$$y(t) = \min\big\{x(t)\,;\quad x \ \text{ is an } i\text{-characteristic},\quad x(\tau) = \bar x\big\}.$$

Observe that $y(\cdot)$ is itself an i-characteristic. By (3.5), it cannot coincide with an i-shock front of u on any nontrivial time interval.

In connection with the exact solution u, we define an i-characteristic as a curve

$$t \mapsto x(t) = \lim_{\nu\to\infty} x_\nu(t)$$

which is the limit of i-characteristics in a sequence of front tracking solutions $u_\nu \to u$.

3. Let $\varepsilon > 0$ be given. If the assumption (H) holds, the measure μ_τ^{i+} of i-waves in $u(\tau)$ is supported on a bounded interval and is absolutely continuous w.r.t. Lebesgue measure. We can thus find a piecewise constant function ψ^τ with jumps at points $x_1(\tau) < \bar x_2(\tau) < \ldots < \bar x_N(\tau)$ such that

$$\int \left|\frac{d\mu_\tau^{i+}}{dx} - \psi^\tau\right| dx < \varepsilon\,, \qquad \int_{x_j(\tau)}^{x_{j+1}(\tau)} \left(\frac{d\mu_\tau^{i+}}{dx} - \psi^\tau\right) dx = 0 \qquad j = 1,\ldots,N-1\,. \tag{3.7}$$

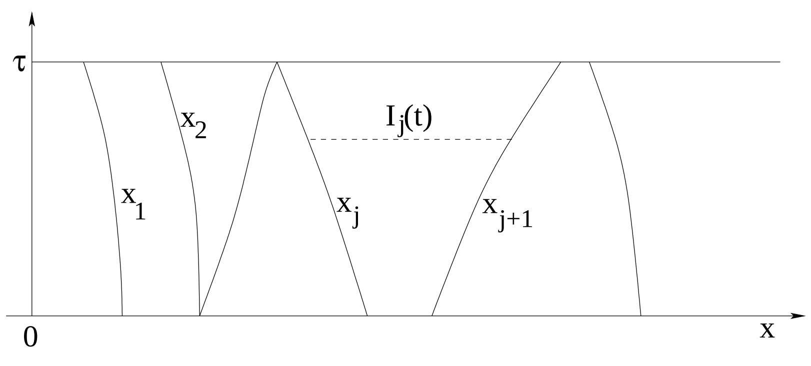

figure 2

To prove the lemma in this special case, relying on Proposition 2, it thus suffices to find i-characteristics $t \mapsto x_j(t)$ such that the following holds (fig. 2)

(i) For each $j = 1, \ldots, N$, the function ψ^τ is constant on the interval $\big]x_j(\tau),\, x_{j+1}(\tau)\big[$ and (3.7) holds. Moreover, either $x_j(0) = x_{j+1}(0)$, or else the derivative component $\psi^0 \doteq l^i(u)u_x(0,\cdot)$ is constant on the interval $\big]x_j(0),\, x_{j+1}(0)\big[$.

(ii) An estimate corresponding to (3.3)-(3.4) holds restricted to each subinterval $\big[x_j(\tau),\, x_{j+1}(\tau)\big[$.

We need to explain in more detail this last statement. Define

$$I_j(t) \doteq \big[x_j(t),\, x_{j+1}(t)\big[\,, \qquad\qquad \Delta_j \doteq \big\{(t,x)\,;\;\; t \in [0,\tau]\,,\;\; x \in I_j(t)\big\}\,.$$

For each j, we denote by Γ_j the total amount of wave interaction within the domain Δ_j. This is defined as in [B], first for a sequence of front tracking approximations u_ν, then taking a limit as $\nu \to \infty$. Furthermore, we define the constant values

$$\psi_j^\tau \doteq \psi^\tau(x) \qquad\qquad x \in I_j(\tau)\,,$$

$$\psi_j^0 \doteq \psi^0(x) \qquad\qquad x \in I_j(0)\,,$$

Call

$$\sigma_j^0 \doteq \lim_{t \to 0+} \mu^{i+}\big(I_j(t)\big)$$

the initial amount of positive i-waves inside the interval I_j.

For each interval I_j, we consider on one hand the function w_j^τ corresponding to (3.2)-(3.3), namely

$$w_j^\tau(s) \doteq \min\left\{\sigma_j^0\,,\; \frac{s}{\tau + (\psi_j^0)^{-1}}\right\} + \kappa\Gamma_j \cdot \mathrm{sgn}(s)\,.$$

Here $(\psi_j^0)^{-1} \doteq 0$ in the case where $x_j(0) = x_{j+1}(0)$. This may happen when the initial data has a jump at $x_j(0)$, and the corresponding measure μ^{i+} has a Dirac mass (with infinite density) at that point.

On the other hand, we look at the nondecreasing, odd function η_j such that

$$\eta_j(s) \doteq \min\Big\{\psi_j^\tau\, s,\;\; \psi_j^\tau\big[x_{j+1}(\tau) - x_j(\tau)\big]\Big\} \qquad\qquad s > 0\,.$$

Our basic goal is to prove that (fig. 3)

$$\eta_j(s) \le w_j^\tau(s) \qquad\qquad \text{for all } s > 0\,. \tag{3.8}$$

Indeed, by (3.7), for $s > 0$ one has

$$\sup_{meas(A) \le 2s} \frac{\mu_\tau^{i+}\big(A \cap I_j(\tau)\big)}{2} \le \eta_j(s) + \varepsilon_j$$

with

$$\sum_j \varepsilon_j < \varepsilon\,.$$

Proving (3.8) for each j will thus imply

$$\mu_\tau^{i+} \preceq w(\tau,x) = w(\tau-,\ x) + \kappa\,\mathrm{sgn}(x)\cdot\Big[Q(\bar u) - Q\big(u(\tau)\big) + \mathcal{O}(1)\cdot\varepsilon\Big]\,.$$

Since $\varepsilon > 0$ was arbitrary, this establishes the lemma under the additional assumptions (H).

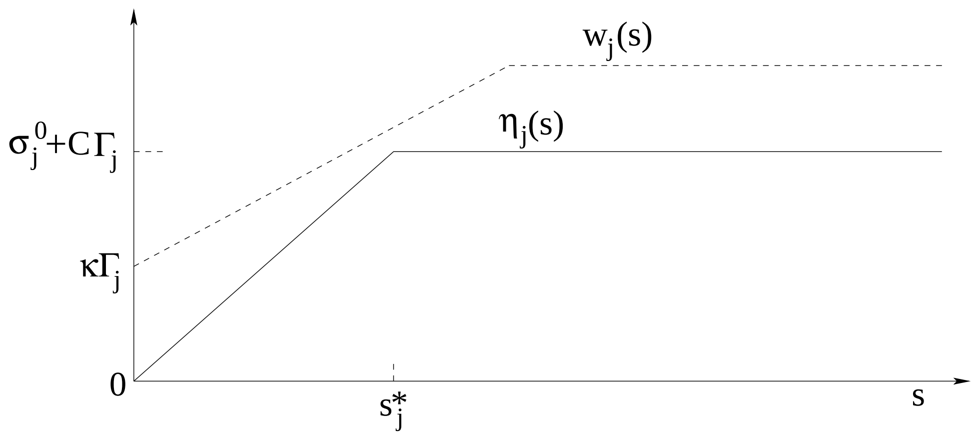

figure 3

4. We now work toward a proof of (3.8), in three cases.

Case 1: $\sigma_j^0 = 0$.

Case 2: $x_j(0) = x_{j+1}(0)$ and $\sigma_j^0 > 0$.

Case 3: $x_j(0) < x_{j+1}(0)$ and $\sigma_j^0 = \big(x_{j+1}(0) - x_j(0)\big)\,\psi_j^0 > 0$.

In Case 1 the proof is easy. Indeed, the total amount of positive i-waves in $I_j(\tau)$ is here bounded by a constant times the total amount of interaction taking place inside the domain Δ_j, i.e.

$$\mu_\tau^{i+}\big(I_j(\tau)\big) \le C_0\cdot\Gamma_j$$

for some constant C_0. On the other hand

$$w_j^\tau(s) = \kappa\Gamma_j\cdot\mathrm{sgn}(s)\,.$$

Choosing $\kappa > C_0$ we achieve (3.8).

5. Since Case 2 can be obtained from Case 3 in the limit as $x_{j+1} - x_j \to 0$, we shall only give a proof for Case 3.

We can again distinguish two cases. If the amount of interaction Γ_j is large compared with the initial amount of i-waves, say

$$\Gamma_j \geq \frac{1}{6C_0}\sigma_j^0 ,$$

then the bound (3.8) is readily achieved choosing $\kappa > 8C_0$. Indeed, for $s > 0$ we have

$$\eta_j(s) \leq \frac{1}{2}\,\mu_\tau^{i+}\big(I_j(\tau)\big) \leq C_0\Gamma_j + \sigma_j^0 \leq 7C_0\,\Gamma_j\,.$$

The more difficult case to analyse is when Γ_j is small, say

$$\Gamma_j < \sigma_j^0/6C_0\,. \tag{3.9}$$

Looking at figure 3, it clearly suffices to prove (3.8) for the single value

$$s = s_j^* \doteq \frac{x_{j+1}(\tau) - x_j(\tau)}{2}\,.$$

Equivalently, calling

$$z_j(t) \doteq x_{j+1}(t) - x_j(t)$$

the length of the interval $I_j(t)$ and

$$\sigma_j^\tau \doteq \mu_\tau^{i+}\big(I_j(\tau)\big) = z_j(\tau)\,\psi_j^\tau$$

the total amount of positive i-waves inside $I_j(\tau)$, we need to show that

$$\sigma_j^\tau \leq 2\kappa\Gamma_j + \min\left\{\sigma_j^0\,,\quad \frac{2s_j^*}{\tau + (\psi_j^0)^{-1}}\right\}\,. \tag{3.10}$$

By the approximate conservation of i-waves over the region Δ_j, we can write

$$\sigma_j^\tau \leq \sigma_j^0 + C_0\Gamma_j\,. \tag{3.11}$$

Using (3.11) in (3.10), our task is reduced to showing that

$$\sigma_j^\tau \leq 2\kappa\Gamma_j + \frac{2s_j^*}{\tau + (\psi_j^0)^{-1}} \tag{3.12}$$

for a suitably large constant κ. Because of (3.11), it suffices to show that

$$\begin{aligned} z_j(\tau) &\geq (\sigma_j^0 - C'\Gamma_j)\big(\tau + (\psi_j^0)^{-1}\big) \\ &= \big[z_j(0) + \tau\sigma_j^0\big] - C'\big(\tau + (\psi_j^0)^{-1}\big)\,\Gamma_j \end{aligned} \tag{3.13}$$

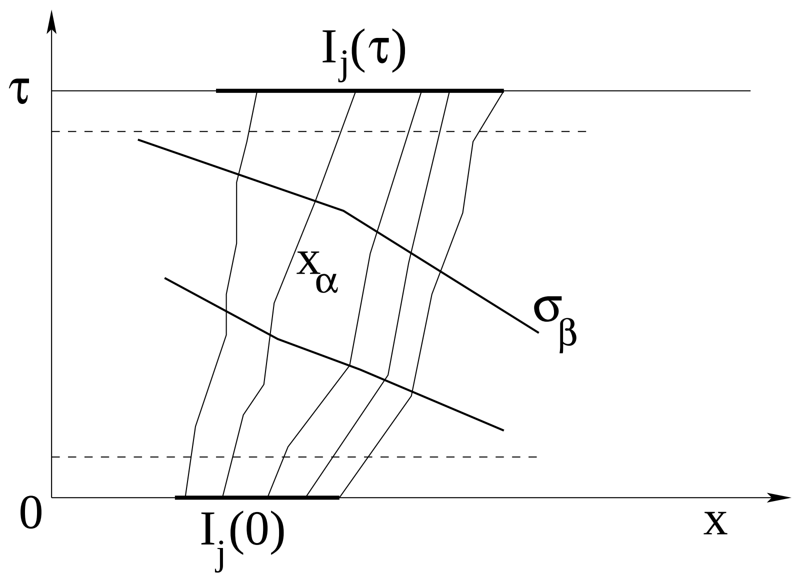

figure 4

for a suitable constant C'.

6. We now prove (3.13). Notice that, by genuine nonlinearity and the normalization (1.2), if no other waves were present in the region Δ_j we would have $\Gamma_j = 0$ and

$$\frac{d}{dt} z_j(t) \equiv \sigma_j^0 .$$

In this case, the equality would hold in (3.13).

To handle the general case, we represent the solution u as a limit of front tracking approximations u_ν, where for each $\nu \geq 1$ the function $u_\nu(0, \cdot)$ contains exactly ν rarefaction fronts equally spaced along the interval $I_j(0)$. Each of these fronts has initial strength $\sigma_\alpha(0) = \sigma_j^0/\nu$. For $\alpha = 1, \ldots, \nu$, let $y_\alpha(t) \in I_j(t)$ be the location of one of these fronts at time $t \in [0, \tau]$, and let $\sigma_\alpha(t) > 0$ be its strength. Moreover, call

$$J_\alpha(t) \doteq \big[y_\alpha(t)\,,\ y_{\alpha+1}(t)\big[\,, \qquad\qquad \Delta_\alpha \doteq \big\{(t,x)\,;\quad t \in [0,\tau]\,,\quad x \in J_\alpha(t)\big\}\,,$$

and let Γ_α be the total amount of interaction in u_ν taking place inside the domain Δ_α.

We define a subset of indices $\mathcal{I} \subseteq \{1, \ldots, \nu\}$ by setting $\alpha \in \mathcal{I}$ if

$$5C_0 \Gamma_\alpha > \sigma_\alpha(0) = \sigma_j^0/\nu\,. \tag{3.14}$$

Observe that, if $\alpha \notin \mathcal{I}$, then

$$\left| \frac{\sigma_\alpha(t)}{\sigma_\alpha(0)} - 1 \right| < \frac{1}{2} \qquad \text{for all } t \in [0, \tau]\,.$$

In particular, if $\alpha, \alpha+1 \notin \mathcal{I}$, then the interval $J_\alpha(t)$ is well defined for all $t \in [0, \tau]$. Its length

$$z_\alpha(t) \doteq y_{\alpha+1}(t) - y_\alpha(t)$$

satisfies the differential inequality

$$\frac{d}{dt} z_\alpha(t) \geq W_\alpha(t) - C_1 \cdot \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta| \tag{3.15}$$

for some constant C_1. Here

$$\begin{aligned} W_\alpha(t) &\doteq \big[\text{amount of } i\text{-waves inside the interval } J_\alpha(t)\big] \\ &\geq \sigma_\alpha(0) - C_0 \Gamma_\alpha \,, \end{aligned} \tag{3.16}$$

while $\mathcal{C}_\alpha(t)$ refers to the set of all wave fronts of different families which are crossing the interval J_α at time t. Calling W'_α the total amount of waves of families $\neq i$ which lie inside $J_\alpha(0)$, we can now write

$$\int_0^\tau \left(\sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta| \right) dt \leq \left(\max_{t \in [0,\tau]} z_\alpha(t) \right) \cdot \frac{2\nu}{\sigma_j^0} \cdot \Gamma_\alpha + \mathcal{O}(1) \cdot \tau \Gamma_\alpha + \mathcal{O}(1) \cdot \left(\frac{z_j(0)+1}{\nu} \right) W'_\alpha \,. \tag{3.17}$$

Indeed, by strict hyperbolicity, every front σ_β of a different family can spend at most a time $\mathcal{O}(1) \cdot z_\alpha$ inside J_α. Either it is located inside J_α already at time $t = 0$, or else, when it enters, it crosses y_α or $y_{\alpha+1}$. In this case, since α, $\alpha+1 \notin \mathcal{I}$, by (3.14) it will produce an interaction of magnitude $|\sigma_\beta \, \sigma_\alpha| \geq |\sigma_\beta \cdot \sigma_j^0|/2\nu$. The second term on the right hand side of (3.17) takes care of the new wave fronts which are generated through interactions inside J_α. The last term takes into account wave front of different families that initially lie already inside J_α at time $t = 0$. Integrating (3.15) over the time interval $[0, \tau]$ and using (3.16)-(3.17) one obtains

$$z_\alpha(\tau) \;\geq\; z_\alpha(0) + \tau \frac{\sigma_j^0}{\nu} - \mathcal{O}(1) \cdot \tau \Gamma_\alpha - \mathcal{O}(1) \cdot \left(\max_{t \in [0,\tau]} z_\alpha(t) \right) \cdot \frac{2\nu}{\sigma_j^0} \cdot \Gamma_\alpha - \mathcal{O}(1) \cdot \left(\frac{z_j(0)+1}{\nu} \right) W'_\alpha \,. \tag{3.18}$$

7. To proceed in our analysis, we now show that

$$\max_{t \in [0,\tau]} z_\alpha(t) \leq 2 \, z_\alpha(\tau) \,. \tag{3.19}$$

Indeed, let $\tau' \in [0, \tau]$ be the time where the maximum is attained. If our claim (3.19) does not hold, there would exists a first time $\tau'' \in [\tau', \tau]$ such that $z_\alpha(\tau'') = z_\alpha(\tau')/2$. From (3.15) and the assumption $W_\alpha(t) \geq 0$ it follows

$$\int_{\tau'}^{\tau''} C_1 \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta| \, dt \geq \frac{z_\alpha(\tau')}{2} \,. \tag{3.20}$$

Using the smallness of the total variation, a contradiction is now obtained as follows. Call

$$\Phi(t) \doteq C_0 Q(t) + \sum_{k_\beta \neq i} \phi_{k_\beta}\big(t, x_\beta(t)\big)\, |\sigma_\beta|\,,$$

where the sum ranges over all fronts of strength σ_β located at x_β, of a family $k_\beta \neq i$. The weight functions ϕ_j are defined as

$$\phi_j(t,x) \doteq \begin{cases} 0 & \text{if } \; x > y_{\alpha+1}(t)\,, \\ \dfrac{y_{\alpha+1}(t) - x}{y_{\alpha+1}(t) - y_\alpha(t)} & \text{if } \; x \in \big[y_\alpha(t),\ y_{\alpha+1}(t)\big]\,, \\ 1 & \text{if } \; x < y_\alpha(t)\,, \end{cases}$$

in the case $j > i$, while

$$\phi_j(t,x) \doteq \begin{cases} 1 & \text{if } \; x > y_{\alpha+1}(t)\,, \\ \dfrac{x - y_\alpha(t)}{y_{\alpha+1}(t) - y_\alpha(t)} & \text{if } \; x \in \big[y_\alpha(t),\ y_{\alpha+1}(t)\big]\,, \\ 0 & \text{if } \; x < y_\alpha(t)\,, \end{cases}$$

in the case $j < i$. Because of the term $C_0 Q(t)$, the functional Φ is non-increasing at times of interactions. Moreover, outside interaction times a computation entirely similar to the one at p.213 of [B] now yields

$$-\frac{d}{dt}\Phi(t) \geq \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta| \cdot \frac{c_0}{z(t)}\,, \tag{3.21}$$

for some small constant $c_0 > 0$ related to the gap between different characteristic speeds. From (3.20) and (3.21) respectively we now deduce

$$\int_{\tau'}^{\tau''} \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta|\, dt \geq \frac{z_\alpha(\tau')}{2C_1}\,,$$

$$\int_{\tau'}^{\tau''} \sum_{\beta \in \mathcal{C}_\alpha(t)} |\sigma_\beta|\, dt \leq \int_{\tau'}^{\tau''} \left|\frac{d\Phi(t)}{dt}\right| \cdot \frac{z_\alpha(\tau')}{c_0}\, dt \leq \frac{\Phi(\tau')}{c_0}\, z_\alpha(\tau')\,.$$

Since $\Phi(t) = \mathcal{O}(1) \cdot \text{Tot.Var.}\big\{u(t)\big\}$, by the smallness of the total variation we can assume $\Phi(\tau') < 2C_1/c_0$. In this case, the two above inequalities yield a contradiction.

8. Using (3.19), from (3.18) we obtain

$$\begin{aligned} z_j(\tau) &= \sum_{1 \leq \alpha \leq \nu} z_\alpha(\tau) \geq \sum_{\alpha \notin \mathcal{I}} z_\alpha(\tau) \\ &\geq \sum_{\alpha \notin \mathcal{I}} \left\{ \frac{z_\alpha(0) + \tau\sigma_j^0/\nu}{1 + C_2(\nu/\sigma_j^0)\Gamma_\alpha} - \mathcal{O}(1) \cdot \tau\Gamma_j - \mathcal{O}(1) \cdot \left(\frac{z_j(0)+1}{\nu}\right) W'_\alpha \right\} \\ &\geq \sum_{\alpha \notin \mathcal{I}} \left(z_\alpha(0) + \tau\frac{\sigma_j^0}{\nu}\right)\left(1 - C_2\frac{\nu}{\sigma_j^0}\Gamma_\alpha\right) - \mathcal{O}(1) \cdot \tau\Gamma_j - \mathcal{O}(1) \cdot \frac{z_j(0)+1}{\nu} \\ &\geq \sum_{\alpha \notin \mathcal{I}} \left(z_\alpha(0) + \tau\frac{\sigma_j^0}{\nu}\right) - C_2\frac{z_j(0)}{\sigma_j^0}\Gamma_j - \mathcal{O}(1) \cdot \tau\Gamma_j - \mathcal{O}(1) \cdot \frac{z_j(0)+1}{\nu}\,. \end{aligned} \tag{3.22}$$

By (3.14) the cardinality of the set $\mathcal{I}$ satisfies

$$\#\mathcal{I}\cdot\frac{\sigma_j^0}{5C_0\nu}\leq\sum_{\alpha\in\mathcal{I}}\Gamma_\alpha\leq\Gamma_j\,,$$

hence

$$\frac{\#\mathcal{I}}{\nu}\leq\frac{5C_0}{\sigma_j^0}\Gamma_j\,.$$

In turn, this implies

$$\sum_{\alpha\notin\mathcal{I}}\left(z_\alpha(0)+\tau\frac{\sigma_j^0}{\nu}\right)\geq\big(z_j(0)+\tau\sigma_j^0\big)\left(1-\frac{\#\mathcal{I}}{\nu}\right)\geq\big(z_j(0)+\tau\sigma_j^0\big)-5C_0\Gamma_j\frac{z_j(0)}{\sigma_j^0}\Gamma_j-5C_0\tau\Gamma_j\,. \quad (3.23)$$

Using (3.23) in (3.22), observing that

$$\frac{z_j(0)}{\sigma_j^0}=\frac{x_{j+1}(0)-x_j(0)}{\sigma_j^0}=(\psi_j^0)^{-1}.$$

and letting $\nu\to\infty$ we conclude

$$z_j(\tau)\geq\big(z_j(0)+\tau\sigma_j^0\big)-\mathcal{O}(1)\cdot(\psi_j^0)^{-1}\Gamma_j-\mathcal{O}(1)\cdot\tau\Gamma_j\,.$$

This establishes (3.13),for a suitable constant C'.

9. In the general case, without the assumptions (H), the lemma is proved by an approximation argument. We construct a convergent sequence of initial data $\bar u_\nu\to\bar u$ which satisfy (H) and such that

$$\bar u_\nu\to\bar u\,,\qquad Q(\bar u_\nu)\to Q(\bar u)\,,\qquad \big|\mu_{\nu,0}^{i+}-\mu_0^{i+}\big|\to 0\,.$$

Calling w_ν the solution of (3.1) with initial data

$$w_\nu(0,x)=\mathrm{sgn}(x)\cdot\sup_{meas(A)\leq 2|x|}\frac{\mu_{\nu,0}^{i+}(A)}{2}\,,$$

by the previous analysis we have

$$\mu_{\nu,\tau_\nu}^{i+}\preceq D_x\Big[w_\nu(\tau_\nu-)+\mathrm{sgn}(x)\cdot\big[Q(\bar u_\nu)-Q(u_\nu(\tau_\nu))\big]\Big]\,.$$

Observe that $w_\nu(\tau_\nu-)\to w(\tau-)$ in $\mathbf{L}^1_{\mathrm{loc}}$. Choosing $\kappa\geq C_0$, by the lower semicontinuity result stated in Lemma 1 we now conclude

$$\mu_\tau^{i+}\preceq D_x\Big[w(\tau-)+\kappa\,\mathrm{sgn}(x)\cdot\big[Q(\bar u)-Q(u(\tau))\big]\Big]\,.$$

$\square$

4 - Proof of the main theorem

Using the previous lemmas, we now give a proof of Theorem 1. For a given interval $[0, \tau]$, the solution of the impulsive Cauchy problem (1.17)-(1.18) can be obtaines as follows. Consider a partition $0 = t_0 < t_1 < \cdots < t_N = \tau$. Construct an approximate solution by requiring that $w(0, x) = \hat{v}_i(x)$,

$$w_t + (w^2/2)_x = 0 \tag{4.1}$$

on each subinterval $[t_{k-1}, t_k[$, while

$$w(t_k, x) = w(t_k-,\ x) + \kappa \operatorname{sgn}(x) \cdot \big[Q(t_{k-1}) - Q(t_k)\big] . \tag{4.2}$$

We then consider a sequence of partitions $0 = t_0^\nu < t_1^\nu < \cdots < t_{N_\nu}^\nu = \tau$, and the corresponding solutions w_ν. If the mesh of the partitions approaches zero, i.e.

$$\lim_{\nu\to\infty} \sup_k |t_k^\nu - t_{k-1}^\nu| = 0 ,$$

then the approximate solutions w_ν converge to a unique limit, which yields the solution of (1.17)-(1.18).

Call $\mathcal{F}$ the set of nondecreasing odd functions, concave for $x > 0$. This set is positively invariant for the flow of Burgers' equation (4.1). Moreover, this flow is order preserving. Namely, if $w, w' \in \mathcal{F}$ are solutions of (4.1) with initial data such that $w(0, x) \le w'(0, x)$ for all $x > 0$, then also

$$w(t, x) \le w'(t, x) \qquad\qquad \text{for all } \ t, x > 0 .$$

Equivalently,

$$D_x w(0) \preceq D_x w'(0) \qquad \Longrightarrow D_x w(t) \preceq D_x w'(t)$$

for every $t > 0$. For each fixed ν, we can apply Lemma 2 on each subinterval $[t_{k-1}^\nu, t_k^\nu]$ and obtain

$$\mu_{t_k^\nu}^{i+} \preceq D_x w_\nu(t_k^\nu) \qquad \Longrightarrow \qquad \mu_{t_{k+1}^\nu}^{i+} \preceq D_x w_\nu(t_{k+1}^\nu) .$$

By induction on k, this yields

$$\mu_\tau^{i+} \preceq D_x w_\nu(\tau) , \tag{4.3}$$

where w_ν is the approximate solution constructed according to (4.1)-(4.2). Letting $\nu \to \infty$ and using Lemma 1, we achieve a proof of Theorem 1. □

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