STABILITY CONDITIONS
FOR STRONG RAREFACTION WAVES

MARTA LEWICKA

Contents

Abstract. In this paper we study a number of algebraic conditions connected with the stability of strictly hyperbolic $n \times n$ systems of conservation laws in one space dimension

$$u_t + f(u)_x = 0.$$

Such conditions yield existence and continuity of the flow of solutions in the vicinity of the reference solution. Our main concern is a single rarefaction wave having arbitrarily large strength.

1. Introduction

In this paper we study a number of algebraic conditions connected with the stability of strictly hyperbolic $n \times n$ systems of conservation laws in one space dimension:

$$u_t + f(u)_x = 0. \tag{1.1}$$

The well-posedness of (1.1) has been the subject of vast research in recent years; for an overview see [B, D, HR]. While most of the analysis ([BLY] and more recently [BiB]) has been carried out in the setting of initial data

$$u(0, x) = \bar{u}(x) \tag{1.2}$$

having small total variation, at the same time results of [BC, Scho, Le1, Le3] point out that for the stability of patterns containing large waves, extra assumptions are required. The stability conditions, in essence, aim at providing an estimate on the distance between a reference solution u_0 and another solution to (1.1) which is

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viewed as a perturbation of u_0. They refer to the existence of weights with respect to which the flow generated by the linearized problem

$$u_t + \mathrm{D}f(u_0)u_x = 0 \tag{1.3}$$

becomes a contraction; the linearization in (1.3) is taken at states attained by u_0. Under these assumptions the existence of global solutions and their continuous dependence on initial data has been proven in the vicinity of patterns containing only noninteracting shocks [Le1] or being a single rarefaction wave [Le3].

The objective of this paper is a more detailed study of the stability conditions arising when u_0 contains rarefactions. With respect to the case with only shocks present [Le2], the main difficulty here stems from the change of weights along rarefaction curves. This accounts for the change of location of perturbation waves of different characteristic families as they pass through each rarefaction fan. Hence we mainly focus on the case when u_0 is a single rarefaction wave of arbitrarily large strength. The stability conditions are presented in section 2.

We now introduce the main hypothesis and set the notation.

(H1) $\left[\right.$ The system (1.1) is strictly hyperbolic in a domain $\Omega \subset \mathbf{R}^n$ to be specified later. More precisely, for each $u \in \Omega$ the Jacobian matrix $\mathrm{D}f(u)$ of the smooth flux $f : \Omega \longrightarrow \mathbf{R}^n$ has n distinct and real eigenvalues: $\lambda_1(u) < \ldots < \lambda_n(u)$.

Let $\{r_i(u)\}_{i=1}^n$ be the basis of right eigenvectors of $\mathrm{D}f$ having unit length:

$$\mathrm{D}f(u)r_i(u) = \lambda_i(u)r_i(u), \quad ||r_i(u)|| = 1.$$

Call $\{l_i(u)\}_{i=1}^n$ the dual basis of left eigenvectors so that $\langle r_i(u), l_j(u)\rangle = \delta_{ij}$ for all $i, j : 1 \ldots n$ and all $u \in \Omega$.

Fix $k : 1 \ldots n$ and consider an integral curve $\mathcal{R}_k$ of the vector field r_k:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{R}_k(\theta) = r_k(\mathcal{R}_k(\theta)), \\ &u_l = \mathcal{R}_k(0), \quad u_r = \mathcal{R}_k(\Theta), \qquad \Theta > 0. \end{aligned} \tag{1.4}$$

$\mathcal{R}_k$ is called the rarefaction curve joining the left and right states $u_l, u_r \in \Omega$. For a small $\epsilon > 0$ we define the domain:

$$\Omega = \Omega_\epsilon = \{u \in \mathbf{R}^n : \quad ||u - \mathcal{R}_k(\theta)|| < \epsilon \quad \text{for some } \theta \in [0, \Theta]\}. \tag{1.5}$$

We further assume that:

(H2) $\left[\right.$ In Ω, each characteristic field $i : 1 \ldots n$ is either linearly degenerate: $\langle \mathrm{D}\lambda_i, r_i\rangle \equiv 0$, or it is genuinely nonlinear which means that $\langle \mathrm{D}\lambda_i, r_i\rangle > 0$. The k-th characteristic field is assumed to be genuinely nonlinear.

The piecewise smooth, self-similar function, called the centered rarefaction wave is given by:

$$u_0(t,x) = \left\{ \begin{array}{cl} u_l & \text{if } x < t\lambda_k(u_l) \\ \mathcal{R}_k(\theta) & \text{if } x = t\lambda_k(\mathcal{R}_k(\theta)), \quad \theta \in [0,\Theta] \\ u_r & \text{if } x > t\lambda_k(u_r) \end{array} \right. \tag{1.6}$$

and provides an entropy admissible solution of (1.1) [Sm, D].

We now recall the main results of [Le3].

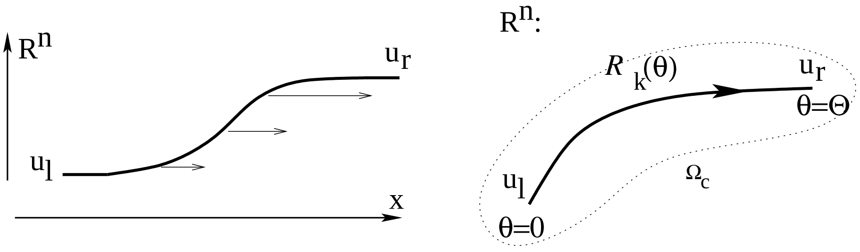

FIGURE 1.1

Theorem 1.1. *Assume that (H1), (H2) and the BV stability condition (BV) hold. For $c, \delta > 0$ let $\mathcal{E}_{c,\delta}$ denote the set of all continuous functions $\bar{u}$ satisfying:*

(i) $\bar{u}(x) \in \Omega_c$ *for all* $x \in \mathbf{R}$,

(ii) $\lim_{x\to-\infty} \bar{u}(x) = u_l$ *and* $\lim_{x\to\infty} \bar{u}(x) = u_r$,

(iii) $|TV(\bar{u}) - |\mathcal{R}_k|| < \delta$, *where* $|\mathcal{R}_k|$ *is the arc-length of the rarefaction curve* $\mathcal{R}_k(\theta)$, $\theta \in [0, \Theta]$.

There exists $c, \delta > 0$ such that for every $\bar{u} \in$ cl $\mathcal{E}_{c,\delta}$, where cl *denotes the closure in L^1_{loc}, the Cauchy problem (1.1) (1.2) has a global entropy admissible solution $u(t, x)$.*

Theorem 1.2. *Assume that (H1), (H2) and the L^1 stability condition (L1) are satisfied. Then there exists a closed domain $\mathcal{D} \subset L^1_{loc}(\mathbf{R}, \Omega)$, containing all continuous functions $\bar{u}$ satisfyling (i), (ii), (iii) in Theorem 1.1, for some $c, \delta > 0$, and there exists a semigroup $S : \mathcal{D} \times [0, \infty) \longrightarrow \mathcal{D}$ such that:*

(i) $||S(\bar{u}, t) - S(\bar{v}, s)||_{L^1} \leq L \cdot (|t - s| + ||\bar{u} - \bar{v}||_{L^1})$ *for all* $\bar{u}, \bar{v} \in \mathcal{D}$, *all* $t, s \geq 0$ *and a uniform constant L, depending only on the system (1.1),*

(ii) *for all $\bar{u} \in \mathcal{D}$, the trajectory $t \mapsto S(\bar{u}, t)$ is the solution to (1.1) (1.2) given in Theorem 1.1.*

The paper is constructed as follows. In section 2 we present conditions (BV) and (L1), we also introduce a weaker condition which is sufficient for the solvability of Riemann problems in Ω. In section 3 we prove that our conditions are one stronger than the other, while sections 4, 5 and 6 gather their various properties. In particular, in section 5 we display an interesting connection between the weighted stability conditions and the Riccati equation in case $n = 3$. Section 7 contains examples complementing our work. In section 8 we restate some results of sections 2 and 3, in the context of a general pattern u_0 containing several strong shocks and rarefaction waves.

2. STABILITY CONDITIONS FOR STRONG RAREFACTIONS

Define the square $(n-1)$-dimensional production matrix function:

$$\mathbf{P}(\theta) = [p_{ij}(\theta)]_{\substack{i,j:1\dots n,\\ i,j\neq k}} \quad \text{for} \quad \theta \in [0, \Theta],$$

$$p_{ij}(\theta) = \begin{cases} |\langle l_j, [r_i, r_k]\rangle(\mathcal{R}_k(\theta))| & \text{if} \quad i \neq j, \\ \mathrm{sgn}(k - i) \cdot \langle l_i, [r_i, r_k]\rangle(\mathcal{R}_k(\theta)) & \text{if} \quad i = j, \end{cases} \tag{2.1}$$

where $[r_i, r_k] = \mathrm{D}r_i \cdot r_k - \mathrm{D}r_k \cdot r_i$ stands for the Lie bracket of the vector fields r_i and r_k. We have the following:

(BV) $\left[\begin{array}{l} BV \text{ Stability Condition: There exist positive smooth functions} \\ w_1 \dots w_{k-1}, w_{k+1} \dots w_n : [0,\Theta] \to \mathbf{R}_+ \text{ such that} \\ \mathbf{P}(\theta) \cdot \begin{bmatrix} w_1(\theta) \\ \vdots \\ w_{k-1}(\theta) \\ w_{k+1}(\theta) \\ \vdots \\ w_n(\theta) \end{bmatrix} < \begin{bmatrix} w_1'(\theta) \\ \vdots \\ w_{k-1}'(\theta) \\ -w_{k+1}'(\theta) \\ \vdots \\ -w_n'(\theta) \end{bmatrix} \quad \text{for every } \theta \in (0,\Theta), \\ \text{and the above vector inequality is understood componentwise.} \end{array}\right.$

Define the mass production matrix function:

$$
\begin{aligned}
\mathbf{M}(\theta) &= [m_{ij}(\theta)]_{\substack{i,j:1\dots n,\\ i,j\neq k}}, \quad \text{for } \theta \in [0,\Theta], \\
m_{ij}(\theta) &= \begin{cases} p_{ij}(\theta) \cdot \dfrac{|\lambda_j - \lambda_k|}{|\lambda_i - \lambda_k|}(\mathcal{R}_k(\theta)) & \text{if } i \neq j, \\ p_{ij}(\theta) + \dfrac{\mathrm{D}\lambda_i \cdot r_k}{|\lambda_i - \lambda_k|}(\mathcal{R}_k(\theta)) & \text{if } i = j. \end{cases}
\end{aligned}
\tag{2.2}
$$

Then, we have:

(L1) $\left[\begin{array}{l} L^1 \text{ Stability Condition: There exist positive smooth functions} \\ w_1 \dots w_{k-1}, w_{k+1} \dots w_n : [0,\Theta] \to \mathbf{R}_+ \text{ such that the inequality in (BV)} \\ \text{is satisfied with } \mathbf{M}(\theta) \text{ replacing the matrix } \mathbf{P}(\theta). \end{array}\right.$

A version of (L1), where all weights w_i are linear, was introduced in [BM]. Condition (L1) is more general, as can be seen from Example 7.3, compare also Remark 7.4. On the other hand, (L1) holds if and only if it is satisfied with constant and equal weights, for some rescaling of the coordinate system $\{r_i\}_{i=1}^n$ (see Corollary 4.2).

In section 3 we will prove that (L1) is stronger than the condition (BV). Below we introduce a third stability condition, guaranteing the existence result of the type of Theorem 1.1, in the context of the Riemann initial data.

Define the $n \times n$ transport matrix function $\mathbf{T}(\theta)$ to be the solution of the following ODE system:

$$
\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{T}(\theta) = \mathrm{D}r_k(\mathcal{R}_k(\theta)) \cdot \mathbf{T}(\theta), \quad \theta \in [0,\Theta], \\ \mathbf{T}(0) = \mathrm{Id}_n. \end{cases}
\tag{2.3}
$$

Also, for any $\theta_1, \theta_2 \in [0,\Theta]$ with $\theta_1 \leq \theta_2$, let $F(\theta_1,\theta_2)$ be the $n \times n$ matrix whose columns $c_i(\theta_1,\theta_2) \in \mathbf{R}^n$, $i : 1 \dots n$ are given by:

$$
\begin{aligned}
c_i(\theta_1,\theta_2) = \mathbf{T}(\theta_2) \cdot \mathbf{T}(\theta_1)^{-1} \cdot r_i(\mathcal{R}_k(\theta_1)) \quad &\text{for } i : 1 \dots k-1, \\
c_i(\theta_1,\theta_2) = r_i(\mathcal{R}_k(\theta_2)) \quad &\text{for } i : k \dots n.
\end{aligned}
\tag{2.4}
$$

We may now set:

(F) $\left[\begin{array}{l} \text{Finiteness Condition: For every } \theta_1, \theta_2 \in [0,\Theta] \text{ with } \theta_1 \leq \theta_2, \text{ the} \\ \text{matrix } F(\theta_1,\theta_2) \text{ is invertible.} \end{array}\right.$

Theorem 2.1. *Assume (H1), (H2) and let the Finiteness Condition (F) hold. There exist $\epsilon, \delta > 0$ such that for every $u^-, u^+ \in \Omega_\epsilon$ with $\lambda_k(u^+) - \lambda_k(u^-) > -\delta$, the Riemann problem (1.1) (1.2) with:*

$$\bar{u} = u(0,x) = \begin{cases} u^- & x < 0, \\ u^+ & x > 0, \end{cases} \tag{2.5}$$

has the unique self-similar solution, attaining states insinde Ω_ϵ. The solution is composed of $n-1$ weak waves of families $1 \dots k-1, k+1 \dots n$, and a k-th rarefaction wave.

Proof. By a standard argument the assumptions (H1) and (H2) imply the assertion for $u^-, u^+ \in \Omega_\epsilon$ such that $|\lambda_k(u^+) - \lambda_k(u^-)| < \delta$, if only δ and ϵ are small [L, B]. We will prove that the invertibility of $F(0, \Theta)$ is sufficient for the solvability of (1.1) (2.5) whenever $||u^- - u_l|| < \delta$ and $||u^+ - u_r|| < \delta$ with a small $\delta > 0$. By a compactness argument, the proof will be then complete.

For each $i : 1 \dots n$ and $u \in \Omega$, call $\sigma \mapsto \mathcal{S}_i(u, \sigma)$ and $\sigma \mapsto \mathcal{R}_i(u, \sigma)$ the i-th shock and the i-th rarefaction curves through the point u [L, Sm]. In particular, by (1.4), we have $\mathcal{R}_k(u_l, \theta) = \mathcal{R}_k(\theta)$. Both curves are defined at least locally, that is for $\sigma \in (-\epsilon, \epsilon)$ and have second order contact at $\sigma = 0$. The i-th wave curve $\sigma \mapsto \mathcal{W}_i(u, \sigma)$ is obtained by taking the positive part of $\mathcal{R}_i$ ($\sigma \geq 0$) and the negative part of $\mathcal{S}_i$ ($\sigma < 0$).

Define an auxiliary $\mathcal{C}^2$ function $G(u^-, u^+, \sigma_1 \dots \sigma_n) \in \mathbf{R}^n$, whose arguments stay close to $u_l, u_r, \sigma_i = 0$ for $i \neq k$ and $\sigma_k = \Theta$, respectively:

$$\begin{aligned} G(u^-, u^+, \sigma_1 \dots \sigma_n) = \mathcal{W}_n(\sigma_n) \dots \circ \mathcal{W}_{k+1}(\sigma_{k+1}) \circ \mathcal{R}_k(\sigma_k) \\ \circ \mathcal{W}_{k-1}(\sigma_{k-1}) \dots \circ \mathcal{W}_1(u^-, \sigma_1) - u^+. \end{aligned}$$

Notice that by (1.4) the function $\mathcal{R}_k(u, \sigma)$ is defined on $\Omega_\epsilon \times (-\epsilon, \Theta + \epsilon)$ for a small $\epsilon > 0$. We clearly have:

$$\frac{\partial G}{\partial(\sigma_1 \dots \sigma_n)}(u_l, u_r, \sigma_i = 0 \text{ for } i \neq k \text{ and } \sigma_k = \Theta) = F(0, \Theta),$$

as $\mathrm{d}/\mathrm{d}\sigma \mathcal{W}_i(u, 0) = r_i(u)$ and $\mathrm{d}/\mathrm{d}\sigma \mathcal{R}_k(u, 0) = r_k(u)$ for every $u \in \Omega$. Since $F(0, \Theta)$ is invertible, by implicit function theorem we conclude the result. ■

Remark 2.2. We have used the following property of the matrix $\mathbf{T}(\theta)$:

$$\mathbf{T}(\theta) \cdot r_i(u_l) = \lim_{\epsilon \to 0} \frac{\mathcal{R}_k(u_l + \epsilon r_i(u_l), \theta) - \mathcal{R}_k(\theta)}{\epsilon} \tag{2.6}$$

For $i < k$, the left hand side of (2.6) is equal to $c_i(0, \theta)$. Thus the first $k-1$ columns of the finiteness matrix $F(\theta_1, \theta_2)$ are equal to the eigenvectors at $\mathcal{R}_k(\theta_1)$ corresponding to characteristic families $i < k$ (slow modes), transported by the flow of the ODE (1.4) to the point $\mathcal{R}_k(\theta_2)$. The condition (F) simply says that this set of vectors can be completed by the remaining right eigenvectors at $\mathcal{R}_k(\theta_2)$ (that is, the eigenvectors corresponding to the fast modes $i \geq k$) to form a basis of $\mathbf{R}^n$. Obviously, the k-th column c_k in (2.4) can be computed by any of the two formulae because the flow of (1.4) preserves the k-th eigenvector: $\mathbf{T}(\theta_2) \cdot \mathbf{T}(\theta_1)^{-1} \cdot r_k(\mathcal{R}_k(\theta_1)) = r_k(\mathcal{R}_k(\theta_2))$.

We have shown that the invertibility of $F(0, \Theta)$ implies the solvability of any Riemann problem (1.1) (2.5) close to the initial data ($u^- = u_l, u^+ = u_r$). This condition is strictly weaker than (F), as shown by the Example 7.1. Also, it follows from Example 7.1 that (F) is a nontrivial condition.

3. A proof of (L1) $\Rightarrow$ (BV) $\Rightarrow$ (F)

In this section we prove the basic relation among the three stability conditions from section 2. We first establish an abstract lemma on matrix analysis.

Lemma 3.1. *Let* $\tilde{\mathbf{P}}(\theta) = [\tilde{p}_{ij}(\theta)]_{i,j:1\dots n}$ *be a continuous* $n \times n$ *matrix function, defined on an interval* $[0, \Theta]$. *Fix* $k : 1 \dots n$ *and define an associated matrix function* $\hat{\mathbf{P}}(\theta) = [\hat{p}_{ij}(\theta)]_{i,j:1\dots n}$ *by:*

$$\hat{p}_{ij}(\theta) = \begin{cases} |\tilde{p}_{ij}(\theta)| & \text{if } \; i \neq j, \\ (\mathrm{sgn}\ (i-k)) \cdot \tilde{p}_{ii}(\theta) & \text{if } \; i = j. \end{cases}$$

Assume that there exist positive smooth functions $w_1 \dots w_n : [0, \Theta] \to \mathbf{R}_+$ *such that the following vector inequality is satisfied componentwise:*

$$\hat{\mathbf{P}}(\theta) \cdot \begin{bmatrix} w_1(\theta) \\ \vdots \\ w_n(\theta) \end{bmatrix} < \begin{bmatrix} w_1'(\theta) \\ \vdots \\ w_{k-1}'(\theta) \\ -w_k'(\theta) \\ \vdots \\ -w_n'(\theta) \end{bmatrix} \quad \text{for every } \theta \in (0, \Theta). \tag{3.1}$$

Then we have:

(i) *Let* $b : [0, \Theta] \longrightarrow \mathbf{R}^n$, $b(\theta) = (b_1(\theta) \dots b_n(\theta))$ *satisfy:*

$$\frac{\mathrm{d}}{\mathrm{d}\theta} b(\theta) = b(\theta)^t \cdot \tilde{\mathbf{P}}(\theta) \quad \text{for } \theta \in [0, \Theta], \tag{3.2}$$

$$\sum_{i=1}^{n} |b_i(0)| > 0. \tag{3.3}$$

The above implies that:

$$\sum_{i<k} \Big(|b_i(\Theta)| w_i(\Theta) - |b_i(0)| w_i(0) \Big) > \sum_{i \geq k} \Big(|b_i(\Theta)| w_i(\Theta) - |b_i(0)| w_i(0) \Big). \tag{3.4}$$

(ii) *Calling* B *the solution of the matrix differential equation:*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\theta} B(\theta) = \tilde{\mathbf{P}}(\theta) \cdot B(\theta), \quad \theta \in [0, \Theta], \\ B(0) = \mathrm{Id}_n, \end{cases} \tag{3.5}$$

the $(k-1) \times (k-1)$ *principal minor of* $B(\Theta)$ *is invertible.*

Proof. (i). Using (3.2), (3.3) and (3.1) we obtain:

$$\begin{aligned}
&\sum_{i<k}(\text{sgn } b_i)\cdot(b_i\cdot w_i)' - \sum_{i\geq k}(\text{sgn } b_i)\cdot(b_i\cdot w_i)'\\
&\qquad > \sum_{i=1}^n\left((\text{sgn } b_i)\cdot(\text{sgn } (k-i))\cdot w_i\cdot\sum_{j=1}^n b_j\tilde p_{ji}\right) + \sum_{i=1}^n\left(|b_i|\cdot\sum_{j=1}^n w_j\hat p_{ij}\right)\\
&\qquad = \left[\sum_{i\neq j}|b_i|w_j\hat p_{ij} + (\text{sgn } b_j)(\text{sgn } (k-j))\cdot b_iw_j\tilde p_{ij}\right] \qquad (3.6)\\
&\qquad\qquad + \left[\sum_{i=1}^n |b_i|w_i\hat p_{ii} + (\text{sgn } (k-i))\cdot|b_i|w_i\tilde p_{ii}\right]\\
&\qquad \geq \left[\sum_{i\neq j}|b_iw_j\hat p_{ij}| - |b_iw_j\tilde p_{ij}|\right] + \left[\sum_{i=1}^n |b_i|w_i(\hat p_{ii} + (\text{sgn } (k-i))\cdot\tilde p_{ii})\right].
\end{aligned}$$

Since $\hat p_{ii} = -(\text{sgn } (k-i))\tilde p_{ii}$ for every $i : 1\ldots n$, and $|\tilde p_{ij}| = |\hat p_{ij}|$ for $i\neq j$, we conclude that the right hand side of (3.6) is nonnegative, and thus:

$$\forall\ \theta\in[0,\Theta]\quad \sum_{i<k}(\text{sgn } b_i)(\theta)\cdot(b_i\cdot w_i)'(\theta) > \sum_{i\geq k}(\text{sgn } b_i)(\theta)\cdot(b_i\cdot w_i)'(\theta). \tag{3.7}$$

Applying $\int_0^\Theta \mathrm{d}\theta$ to both sides of (3.7) we now arrive at (3.4).

(ii). We fix $k>1$ and argue by contradiction. If the $(k-1)\times(k-1)$ principal minor of $B(\Theta)$ was singular, then there would exist $b : [0,\Theta] \longrightarrow \mathbf{R}^n$ satisfying (3.2), (3.3) together with:

$$\forall\ i\geq k\quad b_i(0)=0 \quad\text{ and }\quad \forall\ i<k\quad b_i(\Theta)=0. \tag{3.8}$$

In view of (3.4), the condition (3.8) now implies

$$-\sum_{i<k}|b_i(0)|w_i(0) > \sum_{i\geq k}|b_i(\Theta)|w_i(\Theta),$$

which is clearly a contradiction, as the weights $\{w_i\}$ are all positive functions. ■

Theorem 3.2. *(BV) ⇒ (F).*

Proof. It suffices to show that the existence of positive weights in (BV) implies the invertibility of the matrix $F(0,\Theta)$.

For $\theta\in[0,\Theta]$, let $R(\theta)$ denote the $n\times n$ matrix whose columns are the right eigenvectors of the matrix $\mathrm{D}f(\mathcal{R}_k(\theta))$. Obviously $R(\theta)$ is non-singular and the rows of its inverse $R(\theta)^{-1}$ provide the basis of left eigenvectors $\{l_i(\mathcal{R}_k(\theta))\}$. It is easily seen that the invertibility of $F(0,\Theta)$ is equivalent to the invertibility of the product $R(\Theta)^{-1}\cdot F(0,\Theta)$, which is in turn equivalent to the following condition:

(3.9) The $(k-1)\times(k-1)$ principal minor of $R(\Theta)^{-1}\cdot\mathbf{T}(\Theta)\cdot R(0)$ is invertible.

Recall that the transport matrix function $\mathbf{T}$ is defined in (2.3).

Let $\tilde{\mathbf{P}}(\theta) = [\tilde p_{ij}(\theta)]_{i,j:1\ldots n}$ be the $n\times n$ matrix function, with its coefficients given by:

$$\tilde p_{ij}(\theta) = \langle l_j,[r_k,r_i]\rangle\,(\mathcal{R}_k(\theta)),\quad \theta\in[0,\Theta].$$

Let

$$B(\theta) = R(\theta)^{-1} \cdot \mathbf{T}(\theta) \cdot R(0). \tag{3.10}$$

We will show that B satisfies (3.5) on $[0, \Theta]$. Indeed, one has:

$$B(0) = R(0)^{-1} \cdot \mathbf{T}(0) \cdot R(0) = R(0)^{-1} \cdot R(0) = \mathrm{Id}_n.$$

Using (3.10) and (2.3) we calculate:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta} B(\theta) &= \left\{ \frac{\mathrm{d}}{\mathrm{d}\theta} \left[R(\theta)^{-1}\right] \cdot \mathbf{T}(\theta) + R(\theta)^{-1} \cdot \frac{\mathrm{d}}{\mathrm{d}\theta} \mathbf{T}(\theta) \right\} \cdot R(0) \\ &= \left\{ -R(\theta)^{-1} \cdot \frac{\mathrm{d}}{\mathrm{d}\theta} [R(\theta)] \cdot R(\theta)^{-1} \cdot \mathbf{T}(\theta) + R(\theta)^{-1} \cdot \mathrm{D}r_k(\theta) \cdot \mathbf{T}(\theta) \right\} \cdot R(0) \\ &= \left\{ -R(\theta)^{-1} \cdot \frac{\mathrm{d}}{\mathrm{d}\theta} [R(\theta)] + R(\theta)^{-1} \cdot \mathrm{D}r_k(\theta) \cdot R(\theta) \right\} \cdot R(\theta)^{-1} \cdot \mathbf{T}(\theta) \cdot R(0). \end{aligned} \tag{3.11}$$

Since clearly :

$$\tilde{\mathbf{P}}(\theta) = R(\theta)^{-1} \cdot \left[\mathrm{D}r_k(\theta) \cdot R(\theta) - \frac{\mathrm{d}}{\mathrm{d}\theta} R(\theta) \right],$$

we conclude in view of (3.11) and (3.10) that B satisfies the differential equation in (3.5).

On account of (3.9), it remains thus to prove that the condition (BV) implies:

(3.12) The $(k-1) \times (k-1)$ principal minor of $B(\Theta)$ is invertible.

Let $\hat{\mathbf{P}}(\theta) = [\hat{p}_{ij}(\theta)]_{i,j:1\ldots n}$ be given by the formula in (2.1), for every $\theta \in [0, \Theta]$. Note that the k-th row of $\hat{\mathbf{P}}(\theta)$ contains only zero elements. It is then easy to see that the condition (BV) is equivalent to the existence of positive smooth weights $w_1 \ldots w_n : [0, \Theta] \longrightarrow \mathbf{R}_+$ such that (3.1) holds. Indeed, one implication is trivial, and the converse one is obtained by taking

$$w_k(\theta) = \epsilon \cdot (\Theta + 1 - \theta),$$

with $\epsilon > 0$ small enough. Now (3.1) implies (3.12) by Lemma 3.1 and our proof is complete. ∎

Remark 3.3. The implication (F) $\Rightarrow$ (BV) is not true, as shown by Example 7.5.

We end this section by an easy observation.

Theorem 3.4. *(L1) $\Rightarrow$ (BV).*

Proof. Assume that (L1) holds. For $i \neq k$ define

$$\tilde{w}_i(\theta) = |\lambda_i(\theta) - \lambda_k(\theta)| \cdot w_i(\theta), \quad \theta \in [0, \Theta]. \tag{3.13}$$

We claim that (BV) is satisfied with weights $\{\tilde{w}_i\}_{i\neq k}$ as in (3.13). Indeed, for every $i \neq k$ we have:

$$
\begin{aligned}
&\left(\sum_{j\neq k} p_{ij}\tilde{w}_j\right) - (\mathrm{sgn}\ (k-i))\cdot \tilde{w}_i' \\
&\quad = \left(\sum_{j\neq i,k} p_{ij}\cdot|\lambda_j-\lambda_k|\cdot\tilde{w}_j\right) + p_{ii}\cdot|\lambda_i-\lambda_k|\cdot\tilde{w}_i \\
&\qquad\qquad - \Big(\langle \mathrm{D}\lambda_k, r_k\rangle w_i - \langle \mathrm{D}\lambda_i, r_k\rangle w_i + (\lambda_k-\lambda_i)w_i'\Big) \\
&\quad = |\lambda_i-\lambda_k|\cdot\left\{\left(\sum_{j\neq i,k} p_{ij}\cdot\frac{|\lambda_j-\lambda_k|}{|\lambda_i-\lambda_k|}\cdot\tilde{w}_j\right) + p_{ii}w_i + \frac{\langle \mathrm{D}\lambda_i, r_k\rangle}{|\lambda_i-\lambda_k|}\cdot w_i\right\} \\
&\qquad - \langle \mathrm{D}\lambda_k, r_k\rangle w_i \\
&\quad = |\lambda_i-\lambda_k|\cdot\left\{\left(\sum_{j\neq i,k} m_{ij}w_j\right) + m_{ii}w_i - (\mathrm{sgn}\ (k-i))\cdot w_i'\right\} \\
&\qquad - \langle \mathrm{D}\lambda_k, r_k\rangle w_i,
\end{aligned}
\tag{3.14}
$$

the last equality being a consequence of (2.2). The right hand side of (3.14) is clearly negative, in view of (L1) and the genuine nonlinearity of the k-th characteristic field. This proves the theorem. ∎

4. Miscellaneous properties of (BV) and (L1)

In this section we gather several useful properties of the BV and L^1 stability conditions. We mainly focus on (BV) because (L1) has the same structure, and consequently results on (BV) can be easily translated for (L1) (see Theorem 4.6).

The next theorem states that the condition (BV) is independent of the scaling of eigenvectors $\{r_i\}_{i=1}^n$ in Ω.

Theorem 4.1. *For every $i : 1 \dots n$ and $u \in \Omega$, define*

$$\tilde{r}_i(u) = \alpha_i(u)\cdot r_i(u),$$

where each rescaling function $\alpha_i : \Omega \longrightarrow \mathbf{R}_+$ is positive and smooth. Call $\{\tilde{l}_i\}_{i=1}^n$ the dual basis to $\{\tilde{r}_i\}_{i=1}^n$ and let $\tilde{\mathcal{R}}_k$ be the corresponding reparametrisation of $\mathcal{R}_k$:

$$
\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}s}\tilde{\mathcal{R}}_k(s) = \tilde{r}_k(\tilde{\mathcal{R}}_k(s)), \\
u_l = \tilde{\mathcal{R}}_k(0), \quad u_r = \tilde{\mathcal{R}}_k(S), \qquad S > 0.
\end{gathered}
$$

Then (BV) holds if and only there exists smooth positive weights $\{\tilde{w}_i(s)\}_{i\neq k}$, defined along the reparametrised rarefaction; $s \in [0, S]$, such that the appropriate vector inequality as in (BV) holds.

Proof. Fix $s \in [0, S]$ and let $\theta \in [0, \Theta]$ be such that $\mathcal{R}_k(\theta) = \tilde{\mathcal{R}}_k(s)$. For every $i, j \neq k$ we have:

$$\begin{aligned}\langle \tilde{l}_j, [\tilde{r}_i, \tilde{r}_i] \rangle (\tilde{\mathcal{R}}_k(s)) &= \left\langle \frac{1}{\alpha_j} l_j,\ \alpha_i\alpha_k \cdot \mathrm{D}r_i \cdot r_k + \alpha_k \cdot \langle \mathrm{D}\alpha_i, r_k \rangle \cdot r_i \right. \\ &\qquad \left. - \alpha_i\alpha_k \cdot \mathrm{D}r_k \cdot r_i - \alpha_i \cdot \langle \mathrm{D}\alpha_k, r_i \rangle \cdot r_k \right\rangle (\mathcal{R}_k(\theta)) \\ &= \left\{ \frac{\alpha_i}{\alpha_j}\alpha_k \cdot \langle l_j, [r_i, r_i] \rangle + \delta_{ij} \frac{\alpha_k}{\alpha_j} \cdot \langle \mathrm{D}\alpha_i, r_k \rangle \right\} (\mathcal{R}_k(\theta)).\end{aligned} \tag{4.1}$$

Define

$$\tilde{w}_i(s) = \alpha_i(\mathcal{R}_k(\theta)) \cdot w_i(\theta). \tag{4.2}$$

Since $\mathrm{d}\theta/\mathrm{d}s = \alpha_k(\tilde{\mathcal{R}}_k(s))$, by (4.1), (4.2) and (2.1) it follows for every $i \neq k$:

$$\begin{aligned}&\left(\sum_{j \neq i,k} \tilde{w}_j(s) \cdot |\langle \tilde{l}_j, [\tilde{r}_i, \tilde{r}_k] \rangle (\tilde{\mathcal{R}}_k(s))| \right) \\ &\qquad + \tilde{w}_i(s) \cdot (\mathrm{sgn}\ (k-i)) \cdot \langle \tilde{l}_i, [\tilde{r}_i, \tilde{r}_k] \rangle (\tilde{\mathcal{R}}_k(s)) - (\mathrm{sgn}\ (k-i)) \cdot \tilde{w}_i'(s) \\ &= \left(\sum_{j \neq i,k} w_j(\theta) \cdot |\alpha_i\alpha_k \cdot \langle l_j, [r_i, r_k] \rangle| (\mathcal{R}_k(\theta)) \right) \\ &\qquad + w_i(\theta) \cdot (\mathrm{sgn}\ (k-i)) \cdot (\alpha_i\alpha_k \langle l_j, [r_i, r_k] \rangle) (\mathcal{R}_k(\theta)) \\ &\qquad + w_i(\theta) \cdot (\mathrm{sgn}\ (k-i)) \cdot (\alpha_k \langle \mathrm{D}\alpha_i, r_k \rangle) (\mathcal{R}_k(\theta)) \\ &\quad - (\mathrm{sgn}\ (k-i)) \cdot \Big\{ w_i'(\theta) \cdot (\alpha_i\alpha_k)(\mathcal{R}_k(\theta)) + w_i(\theta) \cdot (\alpha_k \langle \mathrm{D}\alpha_i, r_k \rangle)(\mathcal{R}_k(\theta)) \Big\} \\ &= (\alpha_i\alpha_k)(\mathcal{R}_k(\theta)) \cdot \left\{ \left(\sum_{j \neq k} p_{ij}(\theta) w_j(\theta) \right) - (\mathrm{sgn}\ (k-i)) \cdot w_i'(\theta) \right\}.\end{aligned} \tag{4.3}$$

Recalling that all the rescalings α_i are positive, we obtain that the negativity of the left hand side in (4.3) is equivalent to the inequality in (BV). This finishes the proof. ∎

Corollary 4.2. *The condition (BV) is equivalent to the following one. There exist smooth rescaling of eigenvectors $\{r_i\}_{i \neq k}$ along $\mathcal{R}_k$, given by functions $\gamma_i : [0, \Theta] \longrightarrow \mathbf{R}_+$ such that calling*

$$\tilde{r}_i(\mathcal{R}_k(\theta)) = \gamma_i(\theta) \cdot r_i(\mathcal{R}_k(\theta)) \ \textit{for}\ i \neq k \quad \textit{and} \quad \tilde{r}_k = r_k,$$

one has for every $i \neq k$ and every $\theta \in [0, \Theta]$:

$$\left(\sum_{j \neq k,i} |\langle \tilde{l}_j, [\tilde{r}_i, \tilde{r}_k] \rangle (\mathcal{R}_k(\theta))| \right) + (\mathrm{sgn}\ (k-i)) \cdot \langle \tilde{l}_i, [\tilde{r}_i, \tilde{r}_k] \rangle (\mathcal{R}_k(\theta)) < 0. \tag{4.4}$$

Above, the vectors $\{\tilde{l}_i(\mathcal{R}_k(\theta))\}_{i=1}^n$ are the dual basis to $\{\tilde{r}_i(\mathcal{R}_k(\theta))\}_{i=1}^n$.

Proof. If (BV) holds, then one may take

$$\gamma_i(\theta) = \frac{1}{w_i(\theta)} \qquad \text{for } i \neq k,\ \theta \in [0, \Theta].$$

On the other hand, if the functions γ_i are given, take $\alpha_i : \Omega \longrightarrow \mathbf{R}_+$ to be any smooth positive reparametrisation such that

$$\alpha_i(\mathcal{R}_k(\theta)) = \gamma_i(\theta), \qquad \theta \in [0, \Theta].$$

Since the eigenvectors r_k are not to be rescaled, both implications follow now from Theorem 4.1. ∎

Theorem 4.3. *The stability condition (BV) is satisfied in either of the following cases.*

(i) $k = 1$ *or* n, *that is when the wave in (1.6) is of the extreme characteristic field.*
(ii) Θ *is sufficiently small, that is when the wave in (1.6) is weak.*

Proof. (i). To fix the ideas, assume that $k = n$. Let Z is any constant $(n-1)\times(n-1)$ matrix whose components are strictly bigger than those of the matrix $\mathbf{P}(\theta)$, for all $\theta \in [0, \Theta]$. Take $w = (w_1 \dots w_{k-1}, w_{k+1} \dots w_n)$ to be the solution of:

$$w' = Z \cdot w, \qquad w_i(0) = 1 \text{ for } i \neq k, \tag{4.5}$$

Since the fundamental solution of (4.5) has all its components positive, each w_i must be a positive function and consequently the inequality in (BV) holds.

(ii). Define $Z(\theta) = \mathbf{P}(\theta) + \mathrm{Id}_{n-1}$, for $\theta \in [0, \Theta]$. The initial-value problem:

$$Z(\theta) \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_{k-1} \\ w_{k+1} \\ \vdots \\ w_n \end{bmatrix} (\theta) = \begin{bmatrix} w_1' \\ \vdots \\ w_{k-1}' \\ -w_{k+1}' \\ \vdots \\ -w_n' \end{bmatrix} (\theta), \qquad w_i(0) = 1 \text{ for all } i \neq k,$$

has a local solution, remaining positive on some interval $[0, \epsilon]$, and therefore satisfying (BV). ∎

Recall that the system (1.1) is said to have a coordinate system of Riemann invariants [D, Sm, S] if there exist smooth functions $v_1 \dots v_n : \Omega \longrightarrow \mathbf{R}$ such that:

$$\langle \mathrm{D}v_i, r_j \rangle(u) \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases} \qquad \text{for every } u \in \Omega. \tag{4.6}$$

Using the Frobenius theorem, one can prove (see [D]) that (4.6) implies

$$[r_i, r_j](u) \in \text{span}\,\{r_i, r_j\} \qquad \text{for all } i, j : 1 \dots n,\ u \in \Omega.$$

Hence the matrix $\mathbf{P}(\theta)$ is diagonal for every $\theta \in [0, \Theta]$ and the inequality in (BV) becomes decoupled. Notice now that for any continuous function $a : [0, \Theta] \longrightarrow \mathbf{R}$, the differential inequality $w'(\theta) \lesseqgtr a(\theta) w(\theta)$ admits a positive solution $w(\theta) = \exp\left[\int_0^\theta a(s)\mathrm{d}s \mp \theta\right]$.

We have thus proved:

Theorem 4.4. *If (1.1) admits a system of Riemann invariants then (BV) is satisfied, for every* $k : 1 \dots n$.

Remark 4.5. It is well known that every 2×2 hyperbolic system of conservation laws has a coordinate system of Riemann invariants. Therefore any rarefaction wave in such systems satisfies (BV), which is obviously also a consequence of Theorem 4.3 (i).

We now restate the results of this section in the context of condition (L1), the detailed verification is left to the reader.

Theorem 4.6. *The following assertions are true.*

(i) *The L^1 stability condition is independent of the scaling of the eigenvectors $\{r_i\}_{i=1}^n$ in Ω. In particular, it is equivalent to the condition formulated as in Corollary 4.2 with the inequality (4.4) replaced by:*

$$\left(\sum_{j\neq k,i} \left|(\lambda_j - \lambda_k)\cdot\langle \tilde{l}_j, [\tilde{r}_i, \tilde{r}_k]\rangle\right|(\mathcal{R}_k(\theta))\right) + \Big((\lambda_k - \lambda_i)\cdot\langle \tilde{l}_i, [\tilde{r}_i, \tilde{r}_k]\rangle\Big)(\mathcal{R}_k(\theta)) + \langle D\lambda_i, r_k\rangle(\mathcal{R}_k(\theta)) < 0.$$

(ii) *Any extreme field ($k = 1$ or n) rarefation, or a weak (Θ small) rarefaction satisfies (L1).*

(iii) *If (1.1) has a coordinate system of Riemann invariants then (L1) holds for every $k : 1 \dots n$.*

In [Le3], the proof of Theorem 1.2 used the form of the mass production coefficients as in (2.2). They may be simplified as follows:

Lemma 4.7. *For all $\theta \in [0, \Theta]$ and all $i \neq j$ distinct from k there holds:*

$$m_{ij}(\theta) = |\langle l_j, \mathrm{D}r_i \cdot r_k\rangle(\mathcal{R}_k(\theta))|, \tag{4.7}$$

$$m_{ii}(\theta) = \mathrm{sgn}\ (k-i)\cdot\langle l_i, \mathrm{D}r_i \cdot r_k\rangle(\mathcal{R}_k(\theta)). \tag{4.8}$$

Proof. Recall the following useful identity ([D], pg 126):

$$\forall j,k \quad \langle \mathrm{D}\lambda_j, r_k\rangle \cdot r_j - \langle \mathrm{D}\lambda_k, r_j\rangle \cdot r_k = \mathrm{D}f\cdot[r_j, r_k] - \lambda_j \mathrm{D}r_j \cdot r_k + \lambda_k \mathrm{D}r_k \cdot r_j. \tag{4.9}$$

Multiplying (4.9) by a left eigenvector l_i we obtain:

$$\forall i \notin \{j,k\} \quad (\lambda_i - \lambda_j)\cdot\langle l_i, \mathrm{D}r_j \cdot r_k\rangle = (\lambda_i - \lambda_k)\cdot\langle l_i, \mathrm{D}r_k \cdot r_j\rangle, \tag{4.10}$$

$$\forall j \neq k \quad \langle \mathrm{D}\lambda_j, r_k\rangle = (\lambda_k - \lambda_j)\cdot\langle l_j, \mathrm{D}r_k \cdot r_j\rangle, \tag{4.11}$$

Now (4.7) follows directly from (4.10) and (4.8) is a consequence of (4.11). ■

5. Discussion of the case $n = 3,\ k = 2$

In view of Theorem 4.3 (i) every rarefaction wave (1.4) in a solution to a 2×2 system (1.1) as well as both the slowest and the fastest waves in any $n \times n$ system, is BV (and L^1) stable. In this section we focus on intermediate field rarefactions in 3×3 systems. In particular, we show the natural correspondence between the conditions in section 2 and the solvability of certain associated Riccati equations. Using this approach we derive several sufficient conditions for (BV) (or (L1)).

Our study relies on a number of abstract matrix analysis results.

Lemma 5.1. *Let $a, b, c, d : [0, \Theta] \longrightarrow \mathbf{R}$ be continuous functions, b and c nonnegative. Then the vector inequality:*

$$\begin{bmatrix} a(\theta) & b(\theta) \\ c(\theta) & d(\theta) \end{bmatrix} \cdot \begin{bmatrix} w_1(\theta) \\ w_2(\theta) \end{bmatrix} < \begin{bmatrix} w_1'(\theta) \\ -w_2'(\theta) \end{bmatrix}, \qquad \theta \in (0, \Theta) \tag{5.1}$$

has a positive solution $w_1, w_2 : [0, \Theta] \longrightarrow \mathbf{R}_+$ iff the Riccati equation:

$$v'(\theta) = b(\theta) + [a(\theta) + d(\theta)] \cdot v(\theta) + c(\theta) \cdot v(\theta)^2, \qquad \theta \in (0, \Theta) \tag{5.2}$$

has a positive solution $v : [0, \Theta] \longrightarrow \mathbf{R}_+$.

Proof. **1.** If (5.1) holds, then the positive function v can be defined as w_1/w_2. Hence:

$$v' = \frac{w_1'}{w_2} - v \cdot \frac{w_2'}{w_2} > \frac{a \cdot w_1 + b \cdot w_2}{w_2} + v \cdot \frac{c \cdot w_1 + d \cdot w_2}{w_2} = b + [a + d] \cdot v + c \cdot v^2.$$

2. On the other hand, if (5.2) is satisfied for some positive function v, then the inequality

$$w'(\theta) > \epsilon + b(\theta) + [a(\theta) + d(\theta)] \cdot w(\theta) + c(\theta) \cdot w(\theta)^2$$

also has a positive solution $w : [0, \Theta] \longrightarrow \mathbf{R}_+$ if $\epsilon > 0$ is small enough. Define:

$$\begin{aligned} w_2(\theta) &= \exp\left(-\int_0^\theta \frac{\epsilon}{w(s)} + d(s) + c(s)w(s)\mathrm{d}s\right), \\ w_1(\theta) &= w(\theta) \cdot w_2(\theta). \end{aligned}$$

It follows that:

$$\begin{aligned} w_1' - aw_1 - bw_2 &= w'w_2 + ww_2' - aww_2 - bw_2 \\ &= w_2 \cdot (w' + w \cdot (\ln w_2)' - aw - b) \\ &= w_2 \cdot (w' - w \cdot (\epsilon/w + d + cw) - aw - b) \\ &= w_2 \cdot \left(w' - \epsilon - b - (a + d) \cdot w - cw^2\right) > 0 \end{aligned}$$

and

$$w_2' + cw_1 + dw_2 = w_2 \cdot ((\ln w_2)' + cw + d) = -w_2 \cdot \epsilon/w < 0.$$

Therefore, (5.1) holds. ∎

Remark 5.2. In the setting of Lemma 5.1, one can see that $v : [0, \Theta] \longrightarrow \mathbf{R}$ satisfies (5.2) iff the function $w : [0, \Theta] \longrightarrow \mathbf{R}$ defined by:

$$w(\theta) = v(\theta) \cdot \exp\left(-\int_0^\theta (a + d)(s)\mathrm{d}s\right)$$

is a solution of the Riccati equation:

$$\begin{aligned} w'(\theta) =& b(\theta) \cdot \exp\left(-\int_0^\theta (a + d)(s)\mathrm{d}s\right) \\ &+ c(\theta) \cdot \exp\left(\int_0^\theta (a + d)(s)\mathrm{d}s\right) \cdot w(\theta)^2. \end{aligned} \tag{5.3}$$

Thus conditions in Lemma 5.1 are both equivalent to the following one: The initial value problem (5.3) with $w(0) = 0$ has the solution defined on $[0, \Theta]$.

Lemma 5.3. *Let $b, c : [0, \Theta] \longrightarrow \mathbf{R}_+$ be continuous nonnegative functions. Assume that*

$$\int_0^\Theta c(\theta) \int_0^\theta b(s)\mathrm{d}s\mathrm{d}\theta < 1. \tag{5.4}$$

Then the initial value problem:

$$\begin{cases} w'(\theta) = b(\theta) + c(\theta) \cdot w(\theta)^2, \\ w(0) = 0 \end{cases} \tag{5.5}$$

has the solution w defined on the entire interval $[0, \Theta]$.

Proof. As in the proof of Lemma 5.1, it is easy to see that the solvability of (5.5) is equivalent to the existence of positive solutions $w_1, w_2 : [0, \Theta] \longrightarrow \mathbf{R}_+$ of the following system of two ODEs:

$$\begin{cases} w_1' = bw_2, \\ w_2' = -cw_1. \end{cases} \tag{5.6}$$

Indeed, take z to be a positive solution of the equation in (5.5) and define $w_2(\theta) = \int_0^\theta c(s)z(s)\mathrm{d}s$, $w_1(\theta) = z(\theta)w_2(\theta)$. On the other hand, given w_1 and w_2, the function $z = w_1/w_2$ clearly satisfies the ODE in (5.5).

We will prove that assuming (5.4), the solution to (5.6) with initial data:

$$w_1(0) = 1, \qquad w_2(0) = C, \tag{5.7}$$

satisfies $w_2(\theta) > 0$ for all $\theta \in [0, \Theta]$ if only $C > 0$ is large enough. Since consequently $w_1 > 0$, the proof will be complete. We have:

$$\begin{aligned} w_2(\theta) &= C - \int_0^\theta c(s)w_1(s)\mathrm{d}s = C - \int_0^\theta c(s)\left[1 + \int_0^s b(\tau)w_2(\tau)\mathrm{d}\tau\right]\mathrm{d}s \\ &= C - \int_0^\theta c(s)\mathrm{d}s - \int_0^\theta c(s)\int_0^s b(\tau)w_2(\tau)\mathrm{d}\tau\mathrm{d}s. \end{aligned} \tag{5.8}$$

Take $\epsilon \in (0, 1)$ and $C > 0$ such that

$$\int_0^\Theta c(\theta)\int_0^\theta b(s)\ \mathrm{d}s\ \mathrm{d}\theta \le \epsilon \quad \text{and} \quad C - \int_0^\Theta c(\theta)\mathrm{d}\theta > \epsilon C.$$

To obtain a contradiction, suppose that

$$\min_{[0,\Theta]} w_2 \le 0. \tag{5.9}$$

Then, by (5.8):

$$\begin{aligned} \max_{[0,\Theta]} w_2 = w_2(\theta_{max}) &\le C - \int_0^{\theta_{max}} c(s)\mathrm{d}s \\ &\qquad - \left(\min_{[0,\Theta]} w_2\right) \cdot \int_0^{\theta_{max}} c(s)\int_0^s b(\tau)\mathrm{d}\tau\mathrm{d}s \\ &\le C - \epsilon \cdot \min_{[0,\Theta]} w_2, \end{aligned} \tag{5.10}$$

$$
\begin{aligned}
\min_{[0,\Theta]} w_2 = w_2(\theta_{min}) \geq C - \int_0^{\theta_{min}} c(s)\mathrm{d}s \\
- \left(\max_{[0,\Theta]} w_2\right) \cdot \int_0^{\theta_{min}} c(s) \int_0^s b(\tau)\mathrm{d}\tau\mathrm{d}s \\
> \epsilon C - \epsilon \cdot \max_{[0,\Theta]} w_2 .
\end{aligned} \tag{5.11}
$$

Combining (5.10) and (5.11) we arrive at:

$$\max_{[0,\Theta]} w_2 < C - \epsilon \cdot \left(\epsilon C - \epsilon \cdot \max_{[0,\Theta]} w_2\right),$$

which is equivalent to:

$$\max_{[0,\Theta]} w_2 < C.$$

This contradicts (5.7) and thus we see that (5.9) cannot hold. The proof is done. ■

By Lemma 5.1, Remark 5.2 and Lemma 5.3, we obtain:

Theorem 5.4. *When $n = 3$ and $k = 2$, then:*

(i) *The stability condition (BV) is equivalent to the existence of a positive solution $v : [0, \Theta] \longrightarrow \mathbf{R}_+$ of the Riccati equation:*

$$v'(\theta) = p_{13}(\theta) + (p_{11}(\theta) + p_{33}(\theta)) \cdot v(\theta) + p_{31}(\theta) \cdot v(\theta)^2. \tag{5.12}$$

(ii) *In particular, (BV) is satisfied, if:*

$$\int_0^\Theta \int_0^\theta e^{\int_s^\theta p_{11}+p_{33}} \cdot p_{13}(s) \cdot p_{31}(\theta)\mathrm{d}s\mathrm{d}\theta < 1. \tag{5.13}$$

Remark 5.5. Condition (5.13) is certainly satisfied if p_{13} or p_{31} are equal to 0. We also see that in this case (5.12) becomes the Bernoulli or the linear equation, respectively. On the other hand, in general (5.13) is strictly weaker than the condition postulated in Theorem 5.4 (i). Indeed, when $p_{11} = p_{33} = 0$ and $p_{13}(\theta) = b > 0$, $p_{31}(\theta) = c > 0$ are constant functions, then the solution to (5.12) takes the form:

$$v(\theta) = \sqrt{b/c} \cdot \mathrm{tg}\left(\sqrt{bc}\theta + \mathrm{arctg}\left(v(0)/\sqrt{b/c}\right)\right).$$

Therefore the condition in (i) is here equivalent to: $\Theta\sqrt{bc} < \pi/2$, while (5.13) reduces to: $\Theta^2 \cdot bc/2 < 1$. The former inequality is obviously less restrictive than the latter one.

In view of the above analysis, determining the BV stability of intermediate rarefactions in 3×3 systems of conservation laws reduces to evaluating the position of the blow-up time of the solution to (5.5). In particular the inequality (5.4) provides a sufficient condition for the blow-up to occur after the time Θ. Another proof of this result has been communicated to me by professor Ray Redheffer [R2].

Using the analysis in [R1] one can find other interesting sufficient and necessary conditions in this line. For example [R2], if $c'(0) = 0$ then

$$bc + \frac{1}{2}\left(\frac{c'}{c}\right)' - \frac{1}{4}\left(\frac{c'}{c}\right)^2 < \frac{\pi^2}{4} \qquad \text{on } [0,1] \tag{5.14}$$

implies that the corresponding solution exists on $[0,1]$. On the other hand, if (5.14) holds with a converse inequality then the blow-up occurs at some point $\theta \leq \Theta = 1$. It can be checked that the conditions (5.14) and (5.13) are independent.

As remerked in section 4, the respective results concerning the L^1 stability condition can be easily recovered. In particular, we have:

Theorem 5.6. *When $n=3$ and $k=2$, both assertions of Theorem 5.4 remain valid also for the condition (L1), if we replace the coefficients p_{ij} in (5.12) and (5.13) by the mass production matrix coefficients m_{ij} given in (2.2).*

6. A remark for the case $n > 3$

When $n=3$, the numbers p_{11}, p_{33}, p_{13} and $p_{31}(\theta)$ playing role in various conditions derived in the previous section, can be seen (in view of (2.1) and standard Taylor estimates [Sm]) as transmission and reflection coefficients, in the interactions of small perturbation of families 1 and 3 with parts of the rarefaction wave $\mathcal{R}_k$ (located at θ). In this section we present a generalisation of Theorem 5.4 (ii) to a particular case of $n \times n$ systems (1.1) in which both transmission matrices are zero.

Lemma 6.1. *Let k, n be natural numbers and $1 < k < n$. Let $B(\theta)$ and $C(\theta)$ be two continuous matrix functions defined on $[0,\Theta]$, with all its entries nonnegative, and of dimensions $(n-k)\times(k-1)$ and $(k-1)\times(n-k)$, respectively. Assume that*

$$\left\| \int_0^\Theta \int_0^\theta B^t(s) \cdot C^t(\theta) \mathrm{d}s\mathrm{d}\theta \right\|_1 < 1, \tag{6.1}$$

where the norm of a $m \times m$ matrix $X = [x_{ij}]_{i,j:1\ldots m}$ is defined by

$$\| X \|_1 = \max_{j:1\ldots m} \sum_{i=1}^m |x_{ij}|.$$

Then there exist positive functions $w_1 \ldots w_{k-1}, w_{k+1} \ldots w_n : [0,\Theta] \longrightarrow \mathbf{R}_+$ such that

$$B(\theta) \cdot \begin{bmatrix} w_{k+1} \\ \vdots \\ w_n \end{bmatrix}(\theta) \quad < \quad \begin{bmatrix} w_1' \\ \vdots \\ w_{k-1}' \end{bmatrix}(\theta) \tag{6.2}$$

$$C(\theta) \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_{k-1} \end{bmatrix}(\theta) \quad < \quad -\begin{bmatrix} w_{k+1}' \\ \vdots \\ w_n' \end{bmatrix}(\theta), \tag{6.3}$$

componentwise, for all $\theta \in (0,\Theta)$.

Proof. We will prove that under the condition (6.1), the system of ODEs obtained by replacing the inequalities signs in (6.2) (6.3) by equalities has a positive solution $w_1 \ldots w_{k-1}, w_{k+1} \ldots w_n$ on $[0,\Theta]$. This will clearly complete the proof, since the inequality in (6.1) is strict.

Let $w_i(0) = 1$ for all $i < k$, and $w_i(0) = C$ for all $i > k$ and some constant $C > 0$. Notice that the positivity of $w_1 \ldots w_{k-1}$ is now implied by the positivity of

$w_{k+1}\dots w_n$. We have, for every $\theta\in[0,\Theta]$:

$$\begin{aligned}\begin{bmatrix} w_{k+1}\\ \vdots \\ w_n\end{bmatrix}(\theta) &= \begin{bmatrix} w_{k+1}\\ \vdots \\ w_n\end{bmatrix}(0) - \int_0^\theta C(s)\cdot\begin{bmatrix} w_1\\ \vdots \\ w_{k-1}\end{bmatrix}(s)\mathrm{d}s\\ &= \begin{bmatrix} w_{k+1}\\ \vdots \\ w_n\end{bmatrix}(0) - \int_0^\theta C(s)\mathrm{d}s\cdot\begin{bmatrix} w_1\\ \vdots \\ w_{k-1}\end{bmatrix}(0)\\ &\qquad - \int_0^\theta C(s)\int_0^s B(\tau)\cdot\begin{bmatrix} w_{k+1}\\ \vdots \\ w_n\end{bmatrix}(\tau)\mathrm{d}\tau\mathrm{d}s.\end{aligned}\tag{6.4}$$

To prove that $w_{k+1}\dots w_n$ remain positive we argue by contradiction. Assume there exists $\theta_0\in[0,\Theta]$ such that:

$$\forall\theta\in[0,\theta_0)\quad\forall i>k\quad w_i(\theta)>0\quad\text{and}\quad\exists s>k\quad w_s(\theta_0)=0.\tag{6.5}$$

Then, for every $\theta\in[0,\theta_0)$ and every $i<k$ there holds $w_i(\theta)>0$. Hence:

$$\forall\theta\in[0,\theta_0]\quad\forall i>k\quad w_i(\theta)\le w_i(0)=C.$$

Consequently by (6.4):

$$\begin{aligned}0=w_s(\theta_0) &\ge C-\int_0^\Theta\sum_{j=1}^{k-1}C_{ij}(s)\mathrm{d}s - C\cdot\int_0^{\theta_0}\int_0^s\sum_{j=1}^{k-1}\left(C(s)\cdot B(\tau)\right)_{ij}\mathrm{d}\tau\mathrm{d}s\\ &\ge C-\left\|\int_0^\Theta C^t(s)\mathrm{d}s\right\|_1 - C\cdot\left\|\int_0^\Theta\int_0^\theta B^t(s)\cdot C^t(\theta)\mathrm{d}s\mathrm{d}\theta\right\|_1.\end{aligned}\tag{6.6}$$

The right hand side of (6.6) is strictly positive for a large constant C, by (6.1). This contradiction proves that θ_0 in (6.5) does not exist and the lemma follows. ■

Recall now the definition (2.1) and take

$$A=[p_{ij}]_{i,j:1\dots k-1},\qquad B=[p_{ij}]_{\substack{i:1\dots k-1,\\ j:k+1\dots n}}$$
$$C=[p_{ij}]_{\substack{i:k+1\dots n,\\ j:1\dots k-1}}\qquad D=[p_{ij}]_{i,j:k+1\dots n}.$$

We see that if A and D are zero matrices then the condition (6.1) clearly implies (BV). Both this condition and (5.13) were postulated in [Scho] to be sufficient for the existence result as in Theorem 1.1. Using Lemma 6.1 to appropriate blocks of the mass production matrix $\mathbf{M}$, it is also not difficult to find the respective condition implying the L^1 stability,

In the general case, when A and D are not necessarily zero, one expects the following condition to be sufficient for (BV) to hold:

(6.7)
$$\left\|\int_0^\Theta\int_0^\theta\left[X^D(\theta)\cdot C(\theta)\cdot\left(X^{-A}(\theta)\right)^{-1}\cdot X^{-A}(s)\cdot B(s)\cdot\left(X^D(s)\right)^{-1}\right]^t\mathrm{d}s\mathrm{d}\theta\right\|_1<1,$$

where X^{-A} and X^D are the fundamental solutions of the ODEs:

$$\begin{cases} \left(X^{-A}\right)' = -X^{-A}\cdot A, \\ X^{-A}(0) = \mathrm{Id}_{k-1} \end{cases} \qquad \begin{cases} \left(X^{D}\right)' = X^{D}\cdot D, \\ X^{D}(0) = \mathrm{Id}_{n-k}. \end{cases}$$

By a change of variables, (6.7) becomes (6.1) (now with different matrices C and B) and Lemma 6.1 can be used to recover (BV) under additional assumptions. Namely, the integrand matrix in (6.7) should have nonnegative components and the fundamental matrix $\left(X^D(\theta)\right)^{-1}$ should have positive diagonal and non-negative off-diagonal components, for each θ. This is the case when, for example, the transmission matrices A and D are diagonal.

7. Examples

In this section we present a number of examples complementing the analysis in sections 2–6. We will usually define a strictly hyperbolic matrix $\mathcal{A}(u)$, for u in a neighbourhood of $\mathcal{R}_k$ given by the equation (1.4). We set $\Theta = 1$. The right and left eigenvetors $\{r_i\}_{i=1}^n$, $\{l_i\}_{i=1}^n$ of $\mathcal{A}(u)$ will be used to compute the coefficients in $\mathbf{P}(\theta)$ or $\mathbf{T}(\theta)$. We will not necessarily have $\mathcal{A}(u) = \mathrm{D}f(u)$ for some smooth flux f.

Example 7.1. *$F(0,\Theta)$ is invertible but $F(\theta_1,\theta_2)$ is not, for some $0<\theta_1<\theta_2<\Theta$. Thus, in particular, the condition (F) is not satisfied.*

Let $n=3, k=2$. Set $\mathcal{A}$ to be any strictly hyperbolic 3×3 matrix with the eigenvectors given by:

$$r_1(x,y,z) = [\cos 2y,\ 0,\ \sin 2y]^t, \qquad r_2(x,y,z) = [0,\ -1,\ 0]^t,$$
$$r_3(x,y,z) = [-\sin y,\ 0,\ \cos y]^t.$$

Take $\mathcal{R}_2(\theta) = (0, 1-\theta, 0)$. Obviously $\mathbf{T} = \mathrm{Id}_3$. Therefore the matrix $F(0,1) = [r_1(0,1,0), r_2, r_3(0,0,0)]$ is invertible, but $F(1-\pi/4, 1)$ is not as $r_1(0,\pi/4,0) = r_3(0,0,0) = [0,0,1]^t$. ■

Remark 7.2. In Example 7.1 take $r_2(x,y,z) = [0,1,0]^t$. Consider the rarefaction $\mathcal{R}_2(\theta) = (0,\theta,0)$ defined on $[0,1]$ and joining the same states as before, but in the reverse order. Using the analysis in section 5 one can prove that the condition (BV) is now equivalent to the existence of the non-negative solution to the problem:

$$\begin{cases} v'(y) = \dfrac{2}{\cos y} - 3(\tan y)v(y) + \dfrac{1}{\cos y}v(y)^2, \qquad y\in[0,1], \\ v(0) = 0. \end{cases}$$

The author used Maple to check that the solution exists on the whole interval $[0,1]$. Thus, in particular, (F) is satisfied along the "inverse rarefaction curve" (with respect to Example 7.1) $\mathcal{R}_2(\theta)$.

Example 7.3. *The condition (BV) is satisfied but the weights $\{w_i\}_{i=1}^n$ cannot be taken to be linear.*

Indeed, if we requested the weights $\{w_i\}_{i\neq k}$ in (BV) to be linear, then the condition would no longer be invariant under rescalings of the eigenvector basis (compare Theorem 4.1). Let $n=2, k=2$. Take $\mathcal{A}(u)$ to be any smooth strictly hyperbolic 2×2 matrix whose right eigenvectors r_1, r_2 satisfy:

$$r_1(\theta,0) = [\sqrt{1-\exp(2\theta-4)},\ \exp(\theta-2)]^t, \qquad r_2(\theta,0) = [1\ ,0]^t.$$

By Theorem 4.3 (i), the condition (BV) must be satisfied for any rarefaction in this system. Take $\mathcal{R}_2(\theta) = (\theta, 0)$ and calculate:

$$\begin{aligned} p_{11}(\theta) &= \langle \mathrm{D}r_1(\theta,0) \cdot r_2(\theta,0), l_1(\theta,0)\rangle \\ &= \left[\mathrm{d}\sqrt{1-\exp(2\theta-4)}/\mathrm{d}\theta,\ \exp(\theta-2)\right] \cdot \left[\begin{array}{c} 0 \\ \exp(2-\theta) \end{array} \right] = 1. \end{aligned}$$

If $w_1 > 0$ in (BV) could be taken linear, we would then have:

$$p_{11} \cdot (w_1(0) + w_1' \cdot \theta) < w_1'.$$

This inequality, however, fails to be true on the interval $[1 - w_1(0)/w_1', 1)$. ■

Remark 7.4. Note that all elements of the production matrix in Example 7.3 are nonnegative. This shows that the condition (BV) is indeed stronger that the BV stability version of the L^1 stability condition (3.44) from [BM], where all the second order coefficients p_{ij} (including the diagonal elements p_{ii}) are taken in the absolute value, and the existence of a linear positive solution $\{w_i\}_{i=1}^n$ to the corresponding vector inequality is asked. On the other hand, the existence of linear weights satisfying the inequality in (BV) with a matrix $\mathbf{P}$ with bigger components clearly implies our BV stabilty condition, which thus can be seen as a generalization of the argument in [BM].

Example 7.5. *The condition (F) is satisfied but (BV) is not.*

Let $n = 3, k = 2$. Take $\mathcal{A}(u = (x,y,z))$ to be a smooth 3×3 strictly hyperbolic matrix whose eigenvectors are given by:

$$r_1(x,y,z) = [1,0,0]^t, \quad r_2(x,y,z) = [az, 1, ax]^t, \quad r_3(x,y,z) = [0,0,1]^t,$$

with some $a > \pi/2$. Consider the rarefaction curve $\mathcal{R}_2(\theta) = (0,\theta,0)$. It is easy to calculate that the producion matrix $\mathbf{P}$ has the form:

$$\mathbf{P}(\theta) = \left[\begin{array}{cc} 0 & a \\ a & 0 \end{array} \right].$$

By Remark 5.5, the condition (BV) is thus equivalent to $|a| < \pi/2$ and so it is not satisfied.

We will show that (F) is however satisfied. Since

$$\mathrm{D}r_2(\mathcal{R}_2(\theta)) = \left[\begin{array}{ccc} 0 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{array} \right],$$

we have:

$$\mathbf{T}(\theta) = \exp(\theta \cdot \mathrm{D}r_2) = \left[\begin{array}{ccc} \cosh(a\theta) & 0 & \sinh(a\theta) \\ 0 & 1 & 0 \\ \sinh(a\theta) & 0 & \cosh(a\theta) \end{array} \right].$$

Fix $0 < \theta_1 < \theta_2 < 1$. Using a version of (3.9), we see that the matrix $F(\theta_1, \theta_2)$ is invertible iff the first row - first column element of $\mathbf{T}(\theta_1)^{-1} \cdot \mathbf{T}(\theta_2)$ is nonzero. Noting that $\det \mathbf{T}(\theta) = 1$, this element can be easily computed as:

$$\cosh(a\theta_1)\cosh(a\theta_2) - \sinh(a\theta_1)\sinh(a\theta_2) = \cosh(a\theta_1 - a\theta_2) > 0.$$

■

Example 7.6. The study of plane waves in a half space occupied by a hyperelastic solid leads to the following 6×6 system of hyperbolic conservation laws [TT]:

$$\begin{cases} S_x - \rho_0 V_t = 0, \\ V_x - G \cdot S_t = 0. \end{cases} \tag{7.1}$$

Here $S = (s_1, s_2, s_3)$ and $V = (v_1, v_2, v_3)$ are unknown quantities whose evolution is governed by a symmetric 3×3 matrix G containing appropriate derivatives of a sufficiently regular constitutive function $W(\sigma = s_1, \ \tau^2 = s_2^2 + s_3^2)$. The constant ρ_0 is positive. The derivation of the system, its physical relevance and the related details can be found in [TT]. We are merely interested in verifying the BV stability condition for the rarefaction waves generated from the four intermediate characteristic fields of (7.1). Taking

$$W(\sigma, \tau^2) = \frac{\alpha}{2}\sigma^2 + \frac{\beta}{6}\sigma^3 + \frac{\delta}{4}(\tau^2)^2 \tag{7.2}$$

after a number of calculations [Mu] one arrives at explicit forms of the production matrices $\mathbf{P}$, corresponding to different rarefaction curves (which may be bounded or unbounded, depending on the initial data and the parameters of the system). Although the matrices $\mathbf{P}$ are 5×5 and in general with nonconstant coefficients, by their specific structure the inequality in (BV) can be reduced to studying different Riccati equations of the form:

$$v'(\theta) = \frac{A}{B \pm \theta} \cdot (a + bv(\theta) + cv^2(\theta)). \tag{7.3}$$

By a change of variable (7.3) is equivalent to

$$v'(s) = (a + bv(s) + cv^2(s)). \tag{7.4}$$

Since in each case $a, c > 0, \ b < 0$ and $b^2 - 4ac \geq 0$, the right hand side of (7.4) has a positive root. Thus (7.4) has a (trivial) positive solution existing for all s. Based on this observation one obtains the BV stability of all rarefaction waves in the model (7.1) with the constitutive function (7.2). Incorporating the term $\sigma\tau^2$ in W may lead to a more complicated analysis [Mu]. ■

8. Stability conditions for general patterns of non-interacting large waves

In section 2 we have shown that for a single k-rarefaction the invertibility of the matrix $F(0, \Theta)$ implies the assertion of Theorem 2.1 with (u^-, u^+) close to the extreme states of the reference pattern u_0 in (1.6). For a single k-shock the corresponding property follows from the Majda stability condition [M]. It turns out that in case of multiple waves an additional finiteness condition, accounting for the mutual influence of the strong waves in u_0 ir required. The analysis related to the case with strong shocks was the contents of [Le1, Le2].

Below we study the similar problem for a general pattern u_0 of M shock and rarefaction waves of different characteristic families. We also state the respective BV stability condition and prove a useful generalization of Theorem 3.2.

Let $M + 1$ (with $2 \leq M \leq n$) distinct states $\{u_0^q\}_{q=0}^M$ in $\mathbf{R}^n$ be given. Assume that the Riemann problem (u_0^0, u_0^M) for (1.1) has a self-similar solution composed of M (large) waves $\{u_0^{q-1}, u_0^q\}_{q=1}^M$. For each $q : 1 \ldots M$, the q-th wave joining states (u_0^{q-1}, u_0^q) is said to belong to i_q-th characteristic family and all families $i_1 < i_2 < \ldots < i_M$ are genuinely nonlinear. The waves can be of two types:

(i) Stable rarefaction waves, that is:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{R}_{i_q}(\theta) = r_{i_q}\left(\mathcal{R}_{i_q}(\theta)\right), \\ &u_0^{q-1} = \mathcal{R}_{i_q}(0), \quad u_0^q = \mathcal{R}_{i_q}(\Theta_q), \quad \Theta_q > 0, \end{aligned} \tag{8.1}$$

and the matrix $F_q(0, \Theta_q)$, defined as in (2.4) (2.3) with the field number i_q replacing k, is invertible.

(ii) Lax compressive, Majda stable shocks [L, M]. That is, calling Λ^q the speed of the shock we have:

$$\Lambda^q \cdot (u_0^q - u_0^{q-1}) = f(u_0^q) - f(u_0^{q-1}), \tag{8.2}$$

$$\lambda_{i_q-1}(u_0^{q-1}) < \Lambda^q < \lambda_{i_q}(u_0^{q-1}) \quad \text{and} \quad \lambda_{i_q}(u_0^q) < \Lambda^q < \lambda_{i_q+1}(u_0^q), \tag{8.3}$$

$$\det\Big[r_1(u_0^{q-1}) \ldots r_{i_q-1}(u_0^{q-1}),\ u_0^q - u_0^{q-1},\ r_{i_q+1}(u_0^q) \ldots r_n(u_0^q)\Big] \neq 0 \tag{8.4}$$

We moreover assume that in a sufficiently small neighbourhood of the set of states in $\mathbf{R}^n$ attained by u_0, the system (1.1) is strictly hyperbolic, with each characteristic family genuinely nonlinear or linearly degenerate.

For each $q : 0 \ldots M$, let Ω^q be an open neighbourhood of the state u_0^q. According to [Le2], for each shock (u_0^{q-1}, u_0^q) conditions (8.2) (8.3) (8.4) imply (and by the shock compressibility are essentially equivalent to) the existence of a constitutive function $\Psi^q : \Omega^{q-1} \times \Omega^q \longrightarrow \mathbf{R}^{n-1}$ whose zero locus is composed of pairs of states that can be joined by a stable i_q shock. Moreover the following $n-1$ vectors are linearly independent:

$$\left\{\frac{\partial \Psi^q}{\partial u^{q-1}}(u_0^{q-1}, u_0^q) \cdot r_i(u_0^{q-1})\right\}_{i=1}^{i_q-1} \cup \left\{\frac{\partial \Psi^q}{\partial u^q}(u_0^{q-1}, u_0^q) \cdot r_i(u_0^q)\right\}_{i=i_q+1}^{n}. \tag{8.5}$$

In case (u_0^{q-1}, u_0^q) is a stable rarefaction wave as in (i), the corresponding function Ψ^q can be defined:

$$\Psi^q(u^{q-1}, u^q) = (\sigma_1 \ldots \sigma_{k-1}, \sigma_{k+1} \ldots \sigma_n), \tag{8.6}$$

where $\{\sigma_i\}_{i=1}^n$ stand for the strengths of the waves in the solution of the Riemann problem (u^{q-1}, u^q); compare Theorem 2.1 and its proof.

For each $q : 1 \ldots M$ define a $(n-1) \times (n-1)$ matrix C_q whose negative first $i_q - 1$ columns, and last $n - i_q$ columns are the vectors in (8.5). Notice that for rarefactions $C_q = \mathrm{Id}_{n-1}$ and thus C_q is invertible for each q. Call

$$\begin{aligned} F_q^{left} &= -C_q^{-1} \cdot \frac{\partial \Psi^q}{\partial u^{q-1}}(u_0^{q-1}, u_0^q) \cdot \Big[r_{i_q}(u_0^{q-1}) \ldots r_n(u_0^{q-1})\Big], \\ F_q^{right} &= C_q^{-1} \cdot \frac{\partial \Psi^q}{\partial u^q}(u_0^{q-1}, u_0^q) \cdot \left[r_1(u_0^q) \ldots r_{i_q}(u_0^q)\right]. \end{aligned} \tag{8.7}$$

By an argument as in the proof of Theorem 2.1 we see that the $(n-1) \times i_q$ matrix F_q^{right} expresses strengths of the weak outgoing waves in terms of strengths of waves perturbing the right state of the Riemann problem (u_0^{q-1}, u_0^q). Analogously, the $(n-1) \times (n - i_q + 1)$ matrix F_q^{left} corresponds to perturbations of u_0^{q-1} in the same Riemann problem.

Define now the square $M\cdot(n-1)$ dimensional finiteness matrix $\mathbf{F}$:

$$\mathbf{F} = \begin{bmatrix} [\Theta] & F_1^{right} & & & \\ F_2^{left} & [\Theta] & F_2^{right} & & \\ & F_3^{left} & [\Theta] & F_3^{right} & \\ & & \ddots & \ddots & \\ & & & F_M^{left} & [\Theta] \end{bmatrix}, \tag{8.8}$$

where $[\Theta]$ stands for the $(n-1)\times(n-1)$ zero matrix. The following is a generalisation of Theorem 2.1.

(8.9) FINITENESS CONDITION: 1 is not an eigenvalue of the matrix $\mathbf{F}$.

Theorem 8.1. *In the above setting, let the condition (8.9) hold. Then any Riemann problem $(u^-,u^+)\in\Omega^0\times\Omega^M$ for (1.1) has a unique self-similar solution attaining $n+1$ states, consequtively connected by $(n-M)$ weak waves and M strong waves (shocks or rarefactions) joining states in different sets Ω^q.*

Proof. Define an auxiliary function

$$G:\left(\Omega^0\times\Omega^1\times\ldots\times\Omega^M\right)\times \\ I^{i_1-1}\times I^{i_2-i_1-1}\times I^{i_3-i_2-1}\times\ldots\times I^{i_M-i_{M-1}-1}\times I^{n-i_M}\longrightarrow\mathbf{R}^{M\cdot(n-1)},$$

$$\begin{aligned} G\Big((u^-,u^1,u^2\ldots u^{M-1},u^+),\\ (\sigma_1,\sigma_2\ldots\sigma_{i_1-1}),(\sigma_{i_1+1}\ldots\sigma_{i_2-1})\ldots(\sigma_{i_M+1}\ldots\sigma_n)\Big)\\ =\Psi^1\Big(\mathcal{W}_{i_1-1}(\sigma_{i_1-1})\ldots\circ\mathcal{W}_1(u^-,\sigma_1),\ u^1\Big),\\ \Psi^2\Big(\mathcal{W}_{i_2-1}(\sigma_{i_2-1})\ldots\circ\mathcal{W}_{i_1+1}(u^1,\sigma_{i_1+1}),\ u^2\Big),\\ \ldots\\ \Psi^M\Big(\mathcal{W}_{i_M-1}(\sigma_{i_M-1})\ldots\circ\mathcal{W}_{i_{M-1}+1}(u^{M-1},\sigma_{i_{M-1}+1}),\ u^M\Big), \end{aligned}$$

where

$$u^+=\mathcal{W}_n(\sigma_n)\ldots\circ\mathcal{W}_{i_M+1}(u^M,\sigma_{i_M+1}),$$

and I denotes a small interval in $\mathbf{R}$, containing 0. Call A the $M\cdot(n-1)$ dimensional square matrix that is the derivative of G with respect to the variables $(u^1\ldots u^{M-1}),(\sigma_1\ldots\sigma_n)$ at the point $\big((u_0^0\ldots u_0^M),(0\ldots0)\big)$. We will show that A is invertible iff the condition (8.9) holds, which by implicit function theorem will complete the proof.

Note first, that the invertibility of A is equivalent to the invertibility of the following matrix (which without loss of generality we also call A), of the same dimension:

$$
A = \begin{bmatrix} A_1 & B_1^r & & & & \\ & B_1^l & A_2 & B_2^r & & \\ & & & B_2^l & & \\ & & & \ddots & \ddots & \\ & & & & A_M & \widetilde{A}_M \end{bmatrix}. \tag{8.10}
$$

Here

$$
A_q = \begin{cases} \dfrac{\partial \Psi^1}{\partial u^0}\,(u_0^0, u_0^1) \cdot \left[r_1(u_0^0) \dots r_{i_1 - 1}(u_0^0)\right] & \text{for } q = 1 \\ \\ \dfrac{\partial \Psi^q}{\partial u^{q-1}}\,(u_0^{q-1}, u_0^q) \cdot \left[r_{i_{q-1}+1}(u_0^{q-1}) \dots r_{i_q - 1}(u_0^{q-1})\right] & \text{for } q : 2 \dots M \end{cases}
$$

and

$$
\begin{aligned}
\widetilde{A}_M &= \frac{\partial \Psi^M}{\partial u^M}\,(u_0^{M-1}, u_0^M) \cdot \left[r_{i_M+1}(u_0^M) \dots r_n(u_0^M)\right], \\
B_q^l &= \frac{\partial \Psi^q}{\partial u^{q-1}}\,(u_0^{q-1}, u_0^q) \cdot \left[r_1(u_0^{q-1}) \dots r_n(u_0^{q-1})\right], \\
B_q^r &= \frac{\partial \Psi^q}{\partial u^q}\,(u_0^{q-1}, u_0^q) \cdot \left[r_1(u_0^{q-1}) \dots r_n(u_0^{q-1})\right].
\end{aligned}
$$

Introducing (8.7) in (8.10) and permuting the columns of A we observe that A is invertible iff the following matrix (which we again denote by A) is invertible:

$$
A = \begin{bmatrix} -C_1 & C_1 \cdot F_1^{right} & & & \\ C_2 \cdot F_2^{left} & -C_2 & C_2 \cdot F_2^{right} & & \\ & & \ddots & \ddots & \\ & & & C_M \cdot F_M^{left} & -C_M \end{bmatrix}. \tag{8.11}
$$

Multiplying A by the square block matrix:

$$
\begin{bmatrix} C_1^{-1} & & & \\ & C_2^{-1} & & \\ & & \ddots & \\ & & & C_M^{-1} \end{bmatrix},
$$

we conclude that the invertibility of A in (8.11) is equivalent to the invertibility of $\mathbf{F} - \mathrm{Id}_{M \cdot (n-1)}$ and hence equivalent to (8.9). ■

Remark 8.2. Let (u_0^{q-1}, u_0^q) be a stable i_q- rarefaction wave. After neglecting the i_q-th rows of the two matrices:

$$
\begin{aligned}
&F(0, \Theta_q)^{-1} \cdot \mathbf{T}_q(\Theta_q) \cdot \Big[r_{i_q}(u_0^{q-1}), r_{i_q+1}(u_0^{q-1}) \dots r_n(u_0^{q-1})\Big], \\
&F(0, \Theta_q)^{-1} \cdot \Big[r_1(u_0^q) \dots r_{i_q-1}(u_0^q), r_{i_q}(u_0^q)\Big],
\end{aligned} \tag{8.12}
$$

they become respectively F_q^{left} and F_q^{right}.

We now formulate the following:

(8.13) BV Stability Condition for the wave pattern u_0

There exist positive continuous weights $\{w_i(u)\}_{i=1}^n$ defined on the set of states u attained by the reference solution u_0 (that is, at the isolated endpoints of shocks and along the rarefaction curves), such that for every $q : 1 \dots M$ the following holds.

(i) If (u_0^{q-1}, u_0^q) is a shock then

$$\left|F_q^{left}\right|^t \cdot \begin{bmatrix} w_1(u_0^{q-1}) \\ \vdots \\ w_{i_q-1}(u_0^{q-1}) \\ w_{i_q+1}(u_0^q) \\ \vdots \\ w_n(u_0^q) \end{bmatrix} < \begin{bmatrix} w_{i_q}(u_0^{q-1}) \\ \vdots \\ w_n(u_0^{q-1}) \end{bmatrix}$$

$$\text{and} \qquad \left|F_q^{right}\right|^t \cdot \begin{bmatrix} w_1(u_0^{q-1}) \\ \vdots \\ w_{i_q-1}(u_0^{q-1}) \\ w_{i_q+1}(u_0^q) \\ \vdots \\ w_n(u_0^q) \end{bmatrix} < \begin{bmatrix} w_1(u_0^q) \\ \vdots \\ w_{i_q}(u_0^q) \end{bmatrix},$$

where the components of a matrix $|A|$ are ment to be absolute values of the components of A, and the above vector inequality is understood componentwise.

(ii) If (u_0^{q-1}, u_0^q) is a rarefaction then the corresponding BV stability condition (BV) is satisfied, with the production matrix $\mathbf{P}_q$ defined by (2.1) along the rarefaction curve $\mathcal{R}_q$.

Based on the results of [BM, Le1, Le3], we conjecture that the condition (8.13) implies the BV stability of the pattern u_0, in the sense of Theorem 1.1. Also, a similar weighted L^1 stability condition can be easily formulated and will imply the existence of a continuous flow of solutions, as in Theorem 1.2. Our final result is:

Theorem 8.3. *In the above setting, the condition (8.13) implies the solvability of any Riemann problem in the vicinity of* $(u_0(1, x_1), u_0(1, x_2))$, *for any* $x_1 < x_2$.

Proof. In view of Theorem 8.1, it is enough to show that (8.13) implies (8.9). By Lemma 3.3 from [Le2] and Remark 8.2, this will be achieved provided we prove the inequalities in (8.13) (i) for each rarefaction (u_0^{q-1}, u_0^q). But this indeed follows from Lemma 3.1 (i), applied to the matrix $\tilde{\mathbf{P}}$ as in the proof of Theorem 3.2. ∎

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Marta Lewicka, University of Chicago, Department of Mathematics, Eckhart Hall, 1118 E. 58th Street, Chicago, IL 60637, USA

E-mail address: `lewicka@math.uchicago.edu`