ASYMPTOTIC STABILITY OF NON-PLANAR RIEMANN SOLUTIONS FOR A SPECIAL CLASS OF MULTI-D SYSTEMS OF CONSERVATION LAWS

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ABSTRACT. We study the asymptotic behavior of entropy solutions of the Cauchy problem for multidimensional systems of conservation laws of the form $\partial_t u + \sum_{\alpha=1}^d \partial_\alpha (g^\alpha(|u|)\, u) = 0$, where the g^α are real smooth functions defined in $[0,+\infty)$, when the initial data are perturbations of two states nonplanar Riemann data. Specifically, if $R_0(x)$ is such Riemann data and $\psi \in L^\infty(\mathbb{R}^d)^n$ satisfies $\psi(Tx) \to 0$ in $L^1_{loc}(\mathbb{R}^d)^n$, as $T \to \infty$, then an entropy solution, $u(x,t)$, of the Cauchy problem with $u(x,0) = R_0(x) + \psi(x)$ satisfies $u(\xi t, t) \to R(\xi)$ in $L^1_{loc}(\mathbb{R}^d)^n$, as $t \to \infty$, where $R(x/t)$ turns out to be the unique self-similar entropy solution of the corresponding Riemann problem.

1. INTRODUCTION

We consider the initial-value problem for multidimensional systems of conservation laws of the form

$$\partial_t u + \sum_{\alpha=1}^{d} \partial_\alpha (g^\alpha(|u|)u) = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^d, \tag{1.1}$$

where $g^\alpha : [0,+\infty) \to \mathbb{R}$ are smooth functions and $u \in \mathbb{R}^n$, with given initial data

$$u(x,t)|_{t=0} = u_0(x), \tag{1.2}$$

for which we assume, on the onset,

$$u_0 \in L^\infty(\mathbb{R}^d)^n. \tag{1.3}$$

The one-dimensional version of (1.1) was considered by Keyfitz and Kranzer [26], who studied the corresponding Riemann problem. Later on many other authors studied several different questions associated with the Keyfitz-Kranzer system including the Cauchy problem, asymptotic behavior, uniqueness, etc. (see, *e.g.*, [28, 19, 31, 5, 6, 20]). It has also been pointed out connections between these systems and magnetohydrodynamics.

Systems of the form (1.1) constitute a very special class in the general universe of multidimensional nonlinear hyperbolic systems of conservation laws, given by

$$\partial_t u + \sum_{\alpha=1}^{d} \partial_\alpha F^\alpha(u) = 0, \tag{1.4}$$

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where $u \in \mathbb{R}^n$, $F^\alpha : V \subseteq \mathbb{R}^n \to \mathbb{R}^n$ are smooth maps and for all $\xi = (\xi_1, \dots, \xi_d)$ the matrix

$$\sum_{\alpha=1}^{d} \xi_\alpha \nabla F^\alpha(u),$$

is diagonalizable everywhere in V, which is the hyperbolicity condition.

We recall that a real smooth function $\eta(u)$ is called an entropy for (1.4), with associated entropy flux $q(u) = (q_1(u), \dots, q_d(u))$, if $\eta(u)$ and $q(u)$ satisfy

$$\nabla q^\alpha(u) = \nabla \eta(u) F^\alpha(u), \qquad \alpha = 1, \dots, d. \tag{1.5}$$

It is well known that the existence of an entropy-entropy flux pair implies the hyperbolicity of (1.4), as first observed by Godunov [24] and Friedrichs and Lax [23]. For a discussion about the relation between entropies and the stability of classical solutions of systems of conservation laws in several space variables we refer to [14, 15]. For an account on the basic concepts and recent developments of the theory of conservation laws we refer to [13, 29].

We recall now the usual definition of entropy solution of (1.4),(1.2).

Definition 1.1. A bounded measurable function $u(x,t)$ is said to be an entropy solution of (1.4),(1.2) in $\Pi_T = \mathbb{R}^d \times (0,T)$ if for all entropy-entropy flux pair (η, q) with η convex and all nonnegative smooth function $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$, vanishing for $t = T$, we have

$$\int_{\Pi_T} \eta(u)\partial_t \varphi + \sum_{\alpha=1}^{d} q^\alpha(u)\partial_\alpha \varphi \, dx\, dt + \int_{\mathbb{R}^d} \eta(u_0)\varphi(x,0)\, dx \ge 0. \tag{1.6}$$

For the special class of multidimensional systems (1.1) it is natural to consider a different concept of entropy solution, namely, that of renormalized entropy solution, as we will call here. This class neither contains nor is contained in the class of entropy solutions as characterized by Definition 1.1. It is defined as follows.

Definition 1.2. Let $u(x,t)$ be bounded measurable function satisfying $r(x,t) := |u(x,t)| \ge \delta$, for some $\delta > 0$. We say that u is a *renormalized entropy solution* of (1.1),(1.2) in $\Pi_T = \mathbb{R}^d \times (0,T)$ if it satisfies:

(1) (1.6), for all entropy-entropy flux pair (η, q) with η verifying

$$\frac{\partial^2 \eta(u)}{\partial r^2} \ge 0, \qquad \text{for all } u \in \{u \in \mathbb{R}^n \,:\, |u| \ge \delta\}$$

and all nonnegative smooth function $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$, vanishing for $t = T$.

(2) Setting $f^\alpha(r) = r g^\alpha(r)$, $\omega(x,t) = u(x,t)/r(x,t)$, for all $H \in C(S^{n-1})$ and all smooth function $\phi \in C_0^\infty(\mathbb{R}^{d+1})$, vanishing for $t = T$, we have

$$\int_{\Pi_T} rH(\omega)\partial_t \phi + \sum_{\alpha=1}^{d} f^\alpha(r)H(\omega)\partial_\alpha \phi \, dx\, dt + \int_{\mathbb{R}^d} r_0(x)H(\omega_0(x))\phi(x,0)\, dx = 0. \tag{1.7}$$

Observe that, since any smooth function of $r = |u|$, $h(r)$, is an entropy for (1.1), with associated entropy fluxes $\int^r h'(s) f^{\alpha\prime}(s)\, ds$, $\alpha = 1, \dots, d$, as can be easily checked, and since taking $H(\omega) = 1$ in (1.7) implies that $r(x,t)$ is a weak solution of the problem

$$\partial_t r + \sum_{\alpha=1}^{d} \partial_\alpha f^\alpha(r) = 0, \tag{1.8}$$

$$r(x,t)|_{t=0} = r_0(x), \tag{1.9}$$

we conclude that, if $u(x,t)$ is a renormalized entropy solution of (1.1),(1.2), $r(x,t) = |u(x,t)|$ is the unique entropy solution of (1.8),(1.9) (cf., [27]). We remark that the strong assumption of the initial data in (1.9) in $L^1_{\rm loc}(\mathbb{R}^d)$ follows from the facts we just mentioned, by using, for instance, the general properties of L^∞ divergence-measure fields (cf., [11]).

For the study of the asymptotic behavior of renormalized entropy solutions to (1.1),(1.2) we consider initial data of the form

$$u_0(x) = R_0(x) + \psi(x), \tag{1.10}$$

where

$$R_0(x) = \begin{cases} u_L, & \text{if } x_\alpha < 0, \text{ for some } \alpha \in \{1,\dots,d\} \\ u_R, & \text{if } x_\alpha > 0, \text{ for all } \alpha \in \{1,\dots,d\} \end{cases}, \tag{1.11}$$

and $\psi \in L^\infty(\mathbb{R}^d)^n$ with

$$\lim_{T\to\infty} \psi(Tx) = 0 \quad \text{in } L^1_{\rm loc}(\mathbb{R}^d)^n. \tag{1.12}$$

The main purpose of this paper is to prove the following result.

Theorem 1.1. *Let $u(x,t)$ be a suitable solution to (1.1),(1.2), with u_0 satisfying (1.3), (1.10), (1.11), (1.12), and $|u_0(x)| \ge \delta > 0$. Assume further that $\frac{dg^\alpha}{dr}(r) > 0$ and $\frac{d^2 f^\alpha(r)}{dr^2} > 0$, for all $r > 0$ and $\alpha = 1,\dots,d$. Then $u(x,t)$ satisfies*

$$u(\xi t, t) \longrightarrow R(\xi), \quad \text{as } t \to \infty, \text{ in } L^1_{loc}(\mathbb{R}^d)^n, \tag{1.13}$$

where $R(x/t)$ is a well determined self-similar suitable entropy solution of the Riemann problem (1.1),(1.11). Moreover, $R(x/t)$ is the only self-similar suitable entropy solution of (1.1),(1.11).

Remark 1.1. As it will become clear from the proof of the above theorem, the hypotheses $\frac{dg^\alpha}{dr}(r) > 0$ and $\frac{d^2 f^\alpha(r)}{dr^2} > 0$, $\alpha = 1,\dots,d$, could be dropped and the result would still hold with a similar proof. We impose them here only because they simplify the structure of the entropy solution of the non-planar two-states Riemann problem for the scalar equation (1.8), and the resulting description of $R(x/t)$.

The proof of Theorem 1.1 is given in Section 2. For a similar result concerning a class of multidimensional systems of viscous conservation laws we refer to [22].

For initial data satisfying $r_0 \in BV_{loc}(\mathbb{R}^d)$ the existence of renormalized entropy solutions of (1.1),(1.2) was recently obtained in [2] as an almost direct consequence of the theory of L. Ambrosio on transport equations with coefficients in BV (see [1]). In fact, in [2] it is shown the existence of a weak solution $u(x,t)$ such that $r(x,t)$ is an entropy solution of (1.8),(1.9) and (1.7) holds for all $H \in C(S^{n-1})^n$, satisfying $|H(\omega)| = 1$, $\omega \in S^{n-1}$. However, the fact that (1.7) holds for all $H \in C(S^{n-1})$ follows clearly from their proof. On the other hand, these properties suffice to describe renormalized entropy solutions due to the following simple lemma characterizing the entropies for system (1.1).

Lemma 1.1. *All smooth entropies of (1.1) defined for $\delta \le r \le M$, for some positive constants δ, M, are of the form $h(r) + rH(\omega)$, with $r = |u|$ and $\omega = u/|u|$, for appropriate smooth functions h and H.*

Proof. It will become clear from the proof that it sufficies to consider the entropies restricted to a half-space, say, $u_n \geq 0$. We observe first that the Jacobian matrices $\nabla(g^\alpha(|u|)u)$, $\alpha = 1, \dots, d$, possess the eigenvalues $\lambda_1(r) = g^\alpha(r)$ and $\lambda_2^\alpha(r) = d(g^\alpha(r)r)/dr$, the first one with multiplicity $n-1$. The set $\{\omega_1, \dots, \omega_{n-1}, r\}$ is a functionally independent family of Riemann invariants for (1.1), that is, for smooth solutions of (1.1) we have

$$\begin{cases} \partial_t \omega_i + \sum_{\alpha=1}^d \lambda_1^\alpha(r)\partial_\alpha \omega_i = 0, \ i = 1, \dots, n-1, \\ \partial_t r + \sum_{\alpha=1}^d \lambda_2^\alpha(r)\partial_\alpha r = 0, \end{cases} \tag{1.14}$$

In this way, η is an entropy for (1.1) if and only if it is a solution of the hyperbolic systems in the variables $\omega_1, \dots, \omega_{n-1}, r$, for $\alpha = 1, \dots, d$,

$$\begin{cases} q_{\omega_i}^\alpha = \lambda_1^\alpha \eta_{\omega_i}, \ i = 1, \dots, n-1, \\ q_r^\alpha = \lambda_2^\alpha \eta_r. \end{cases} \tag{1.15}$$

Clearly, any function of the form $h(r) + rH(\omega)$ solves (1.15), $\alpha = 1, \dots, d$. In general, we solve these systems by first eliminating q^α, obtaining the $n-1$ equations

$$r\eta_{r\omega_i} - \eta_{\omega_i} = 0, \tag{1.16}$$

which do not depend on the g^α's, and then going back to (1.15) to obtain the q^α's. Hence, a solution for these systems is uniquely determined on the region

$$\mathcal{S} = \{u \in \mathbb{R}^n \, : \, \sum_{i=1}^{n-1} \omega_i^2 \leq 1, \ \omega_n \geq 0, \ \delta \leq r \leq M\},$$

from the values prescribed for η on the surface

$$\mathcal{S}_{r=\frac{M+\delta}{2}} = \{u \in \mathcal{S} \, : \, r = \frac{M+\delta}{2}\},$$

and on the line

$$\mathcal{S}_{\omega=(0,\dots,0,1)} = \{u \, : \, \omega = (0, \dots, 0, 1), \ \delta \leq r \leq M\}.$$

Now, suppose we prescribe

$$\eta|\mathcal{S}_{r=\frac{M+\delta}{2}} = \Phi(\omega),$$
$$\eta|\mathcal{S}_{\omega=(0,\dots,0,1)} = \Psi(r),$$

with the compatibility condition

$$\Psi(\frac{M+\delta}{2}) = \Phi(0, \dots, 0, 1).$$

Let us try a solution of the form

$$\eta(u) = h(r) + rH(\omega),$$

for some smooth functions h and H with

$$h(\frac{M+\delta}{2}) = 0.$$

We then easily find

$$H(\omega) = \frac{2}{M+\delta}\Phi(\omega),$$

and

$$h(r) = \Psi(r) - \frac{2r}{M+\delta}\Phi(0, \dots, 0, 1).$$

□

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. The one-dimensional analogue of Theorem 1.1 was obtained in [20]. The method used in [20] was extended and improved in [7]. The extension of the same techniques to multidimensional problems was given in [21, 22]. Here we follow procedures analogous to those in [22]. The proof becomes simpler due to the decoupling provided by (1.8),(1.9).

The main point in the strategy is to prove the relation

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |u(\xi t,t)-R(\xi)|\,dt=0, \qquad \text{in } L^1_{\rm loc}(\mathbb{R}^d). \tag{2.1}$$

An important aspect of (2.1) is its equivalence to the convergence in $L^1_{\rm loc}(\mathbb{R}^{d+1}_+)$ of the scaling sequence $\{u^T\}$, defined by $u^T(x,t)=u(Tx,Tt)$, to $R(x/t)$, when $T\to+\infty$. This fact is frequently useful when trying to prove (2.1). Once (2.1) is proved, a standard procedure established in [7] is then used to strengthen (2.1) into (1.13). This strengthening is similar to the ones encountered in [8, 25].

Another main tool in the analysis is provided by the integral equation (1.7), which says that for all function $H(\omega)$, defined and continuous over the sphere S^{n-1}, we have

$$\partial_t(rH(\omega))+\sum_{\alpha=1}^d\partial_\alpha(f^\alpha(r)H(\omega))=0, \tag{2.2}$$

$$rH(\omega)(x,t)|_{t=0}=r_0H(\omega_0)(x). \tag{2.3}$$

Equation (2.2) is to be taken in the sense of distributions, as usual, and (2.3) is to be regarded in the sense of the normal traces of L^∞ divergence-measure fields (see [11]). We will also make extensive use of the Gauss-Green theorem for L^∞ divergence-measure fields proved in [11].

By Galilean invariance, without loss of generality, we may assume $g^\alpha(r)>0$, for all $\delta\le r\le M$, $\alpha=1,\dots,d$, which we do henceforth. The hypothesis

$$\frac{d^2f^\alpha(r)}{dr^2}>0, \qquad \text{for all } r>0. \tag{2.4}$$

implies a simplification in the structure of the solution to the corresponding Riemann problem for the scalar equation for the dependent variable r, that is,

$$\partial_t r+\sum_{\alpha=1}^d\partial_\alpha f^\alpha(r)=0, \tag{2.5}$$

$$r(x,t)|_{t=0}=\rho_0(x):=\begin{cases} r_L:=|u_L|, & \text{if } x_\alpha<0, \text{ for some } \alpha\in\{1,\dots,d\}\\ r_R:=|u_R|, & \text{if } x_\alpha>0, \text{ for all } \alpha\in\{1,\dots,d\}\end{cases}. \tag{2.6}$$

Let us denote by $\rho(x/t)$ the solution of the Riemann problem (2.5),(2.3). The fact that we have

$$r(\xi t,t)\to\rho(\xi), \qquad \text{in } L^1_{\rm loc}(\mathbb{R}^d), \tag{2.7}$$

follows directly from the global L^1-stability of the solution of (1.8),(1.9) (see [27]), since we have

$$\text{(2.8)} \quad \int_{|\xi|<c} |r(\xi t,t)-\rho(\xi)|\,d\xi = t^{-d}\int_{|x|<ct} |r(x,t)-\rho(x/t)|\,dx$$

$$\le t^{-d}\int_{|x|<(c+K)t} |r_0(x)-\rho_0(x)|\,dx = \int_{|x|<c+K} |r_0(tx)-\rho_0(x)|\,dx \to 0 \quad \text{as } t\to\infty,$$

where we also used the hypotheses (1.10), (1.12), and the self-similarity of $\rho_0(x)$.

Naturally, to obtain (2.1), in view of (2.8), it will be enough to prove

$$\text{(2.9)} \qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T |\omega(\xi t,t)-\varpi(\xi)|\,dt = 0, \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^d),$$

where $|\varpi(\xi)|=1$, a.e. $\xi\in\mathbb{R}^d$. Hence, defining $R(\xi)=\rho(\xi)\varpi(\xi)$, we automatically conclude that $R(x/t)$ is a renormalized entropy solution of the Riemann problem (1.1),(1.11), and is the only such self-similar solution.

To obtain (2.9) we will integrate (2.2) over a family of suitable domains in $\mathbb{R}^{d+1}_+$, using repeatedly the Gauss-Green Theorem for divergence-measure fields, to obtain, roughly speaking, the convergence in L^1_{loc} of $\omega^T(x,t)$ to $\varpi(x/t)$ over each self-similar hypersurface from a family filling out almost all $\mathbb{R}^{d+1}_+$. This is also the main idea in [22].

We next introduce several notations defining sets which will play a role in the forthcoming developments. So, let K, N, T_0 be given fixed positive numbers. We denote:

$$\begin{aligned}
&\Omega^+(\xi)=\{(x,t)\,:\,x_1>\xi_1 t,\dots,x_d>\xi_d t\},\\
&\Omega^-(\xi)=\{(x,t)\,:\,x_i<\xi_\alpha t, \text{ for some } \alpha\in\{1,\dots,d\}\},\\
&|x|_\infty=\max\{|x_\alpha|\,:\,i=1,\dots,d\},\\
&\Omega(T)=\{(x,t)\,:\,|x|_\infty\le NT_0T+K(T_0T-t),\ 0\le t\le T_0T\},\\
&\Omega^\pm(\xi,T)=\Omega^\pm(\xi)\cap\Omega(T),\\
&\nu^\alpha[\xi]=(0,\dots,\underbrace{-1}_{\alpha\text{-th}},0,\dots,0,\xi_\alpha)\in\mathbb{R}^{d+1},\\
&\partial\Omega^+(\xi)=\cup_{\alpha=1}^d S_\alpha(\xi), \quad \text{where } \nu^\alpha[\xi]\perp S_\alpha(\xi)\\
&S_\alpha(\xi,T)=S_\alpha(\xi)\cap\Omega(T),\\
&\partial^0\Omega^+(\xi,T)=\cup_{\alpha=1}^d S_\alpha(\xi,T),\\
&S_\alpha^\tau(\xi,T)=S_\alpha(\xi,T)\cap\{(t,x)\,:\,t=\tau\},\\
&\Omega_\tau(T)=\Omega(T)\cap\{(t,x)\,:\,t=\tau\},\\
&\Omega_\tau^\pm(\xi,T)=\Omega^\pm(\xi,T)\cap\{(t,x)\,:\,t=\tau\}.
\end{aligned}$$

With the help of some of the sets above we will describe the self-similar vector function $R(x/t)$ which will be proven to be both a renormalized entropy solution to the Riemann problem (1.1),(1.11) and the asymptotic limit of renormalized entropy solutions $u(x,t)$ of (1.1),(1.2) with initial data satisfying (1.10), (1.11) and (1.12). As we saw in the proof of Lemma 1.1, the Jacobian matrices $\nabla(g^\alpha(|u|)u)$ possess two eigenvalues $\lambda_1^\alpha(u)=g^\alpha(r)$, with multiplicity $n-1$, and $\lambda_2^\alpha(u)=df^\alpha(r)/dr$. The first $n-1$ eigenfamilies are linearly degenerate since the corresponding eigenvalue λ_1^α does not depend on the Riemann invariants $\omega_1(u),\dots,\omega_n(u)$, whose gradients span an $n-1$-dimensional left-eigenspace, for each u with $|u| \geq \delta$. Observe also that the hypotheses over $g^\alpha(r)$ imply, in particular, that $\lambda_1^\alpha(r) < \lambda_2^\alpha(r)$.

Now, as we said above, under the assumption (2.4), the solution of the Riemann problem (2.5),(2.6) has a very simple structure and is defined as follows. If $r_L > r_R$ then it is given by

$$\rho(\xi) = \begin{cases} r_L, & \text{for } (\xi t, t) \in \Omega^-(\xi^0) \text{ for all } t > 0, \\ r_R, & \text{for } (\xi t, t) \in \Omega^+(\xi^0) \text{ for all } t > 0, \end{cases} \tag{2.10}$$

where ξ^0 is defined by

$$\xi_\alpha^0 = (f^\alpha(r_R) - f^\alpha(r_L))/(r_R - r_L), \qquad \alpha = 1, \dots, d. \tag{2.11}$$

So, in this case we have a shock front along the surface $\partial^0\Omega^+(\xi_0)$.

On the other hand, if $r_R > r_L$ it is given by

$$\rho(\xi) = \begin{cases} r_L, & \text{for } (\xi t, t) \in \Omega^-(\xi^-) \text{ for all } t > 0, \\ r_R, & \text{for } (\xi t, t) \in \Omega^+(\xi^+) \text{ for all } t > 0, \\ \gamma(\sigma), & \text{for } (\xi t, t) \in \partial^0\Omega^+(\xi(\sigma)) \text{ for all } t > 0, \\ & \text{and } {f^1}'(r_L) \leq \sigma \leq {f^1}'(r_R), \end{cases} \tag{2.12}$$

where

$$\xi_\alpha^- = {f^\alpha}'(r_L), \quad \xi_\alpha^+ = {f^\alpha}'(r_R), \qquad \alpha = 1, \dots, d, \tag{2.13}$$

$$\gamma(\sigma) = [{f^1}']^{-1}(\sigma), \tag{2.14}$$

and $\sigma \mapsto \xi(\sigma)$, for ${f^1}'(r_L) \leq \sigma \leq {f^1}'(r_R)$, is defined by

$$\xi_1(\sigma) = \sigma, \quad \xi_\alpha(\sigma) = {f^\alpha}'(r(\gamma(\sigma))), \ \alpha = 2, \dots, d. \tag{2.15}$$

Now, $R(\xi) = \rho(\xi)\varpi(\xi)$, is defined by setting

$$\varpi(\xi) = \begin{cases} \omega_L, & \text{for } (\xi t, t) \in \Omega^-(\xi^{cd}) \text{ for all } t > 0, \\ \omega_R, & \text{for } (\xi t, t) \in \Omega^+(\xi^{cd}) \text{ for all } t > 0, \end{cases} \tag{2.16}$$

where

$$\xi_\alpha^{cd} = g^\alpha(r_L), \quad \alpha = 1, \dots, d. \tag{2.17}$$

Observe that due to the hypothesis that $dg^\alpha(r)/dr > 0$ we always have

$$\xi_\alpha^{cd} < \xi_\alpha^0, \qquad \alpha = 1, \dots, d, \tag{2.18}$$

if $r_L > r_R$ (shock case), that is,

$$g^\alpha(r_L) < \frac{f^\alpha(r_L) - f^\alpha(r_R))}{r_L - r_R}, \qquad \alpha = 1, \dots, d,$$

and if $r_L < r_R$ (rarefaction case)

$$\xi_\alpha^{cd} < \xi_\alpha^-, \qquad \alpha = 1, \dots, d, \tag{2.19}$$

since, obviously,

$$g^\alpha(r_L) < {f^\alpha}'(r_L), \qquad \alpha = 1, \dots, d.$$

We define the map $\sigma \mapsto \xi(\sigma)$, for $-\infty < \sigma < +\infty$, in two different ways depending whether we are in the shock case or in the rarefaction case. In the first case, we define $\xi(\sigma)$ as follows

(2.20)
$$\xi(\sigma) = \begin{cases} \frac{\sigma}{g^1(r_L)}(g^1(r_L), \dots, g^d(r_L)), & -\infty < \sigma \le \xi_1^{cd}, \\ \Big(\sigma, \frac{\sigma-\xi_1^{cd}}{\xi_1^0-\xi_1^{cd}}(\xi_2^0 - \xi_2^{cd}) + \xi_2^{cd}, \dots, \frac{\sigma-\xi_1^{cd}}{\xi_1^0-\xi_1^{cd}}(\xi_d^0 - \xi_d^{cd}) + \xi_d^{cd}\Big), & \xi_1^{cd} < \sigma < \xi_1^0, \\ \frac{\sigma}{\xi_1^0}(\xi_1^0, \dots, \xi_d^0), & \xi_1^0 \le \sigma < \infty. \end{cases}$$

In the rarefaction case, we define $\xi(\sigma)$ by

(2.21)
$$\xi(\sigma) = \begin{cases} \frac{\sigma}{g^1(r_L)}(g^1(r_L), \dots, g^d(r_L)), & -\infty < \sigma \le \xi_1^{cd}, \\ \Big(\sigma, \frac{\sigma-\xi_1^{cd}}{\xi_1^- -\xi_1^{cd}}(\xi_2^- - \xi_2^{cd}) + \xi_2^{cd}, \dots, \frac{\sigma-\xi_1^{cd}}{\xi_1^- -\xi_1^{cd}}(\xi_d^- - \xi_d^{cd}) + \xi_d^{cd}\Big), & \xi_1^{cd} < \sigma < \xi_1^-, \\ \big(\sigma, f^{2'}(\gamma(\sigma)), \dots, f^{d'}(\gamma(\sigma))\big), & \xi_1^- \le \sigma \le \xi_1^+, \\ \frac{\sigma}{\xi_1^+}(\xi_1^+, \dots, \xi_d^+), & \xi_1^+ \le \sigma < \infty, \end{cases}$$

where $\gamma(\sigma)$ is defined in (2.14).

We can now start our analysis. We choose K such that

$$K > \sup_{\substack{\delta \le r \le M \\ \alpha = 1, \dots, d}} g^\alpha(r). \tag{2.22}$$

Now, integrating (2.2) over certain $\Omega^\pm(\xi, T)$, using the Gauss-Green formula and (2.22), we obtain for almost all ξ

$$\int_{\Omega^\pm_{T_0 T}(\xi,T)} rH(\omega)\,dx - \int_{\Omega_0^\pm(\xi,T)} r_0 H(\omega_0)\,dx \pm \sum_{\alpha=1}^d \int_{S_\alpha(\xi,T)} (\xi_\alpha - g^\alpha(r))H(\omega) r\,d\mathcal{H}^d \le 0. \tag{2.23}$$

Hence, if $H(\omega)$ is a Lipschitz continuous nonnegative function such that $H(\omega_L) = 0$, we get from (2.23), after dividing it by T^d, taking the lim sup and using (1.12),

$$\limsup_{T\to\infty} \sum_{\alpha=1}^d \frac{1}{T^d} \int_{S_\alpha(\xi,T)} (-\xi_\alpha + g^\alpha(r))H(\omega) r\,d\mathcal{H}^d \le 0. \tag{2.24}$$

Analogously, if $H(\omega)$ is a Lipschitz continuous nonnegative function such that $H(\omega_R) = 0$, we get from (2.23), after dividing it by T^d, taking the lim sup and using (1.12),

$$\limsup_{T\to\infty} \sum_{\alpha=1}^d \frac{1}{T^d} \int_{S_\alpha(\xi,T)} (\xi_\alpha - g^\alpha(r))H(\omega) r\,d\mathcal{H}^d \le 0. \tag{2.25}$$

Motivated by (2.24), (2.25) we introduce the mesures $\mu^\xi_{\alpha,T}$ defined on $\mathcal{S} = \{u \in \mathbb{R}^n : \delta \le |u| \le M\}$ such that for all $h \in C([\delta, M])$ and $H \in C(S^{n-1})$ we have

$$\langle \mu^\xi_{\alpha,T}, h(r)H(\omega)\rangle = c_\alpha^{-1}(\xi)\frac{1}{T^d}\int_{S_\alpha(\xi,T)} h(r)H(\omega)\,d\mathcal{H}^d, \qquad \alpha = 1, \dots, d, \tag{2.26}$$

where $c_\alpha(\xi)$ is a normalizing positive constant such that $\langle \mu^\xi_{\alpha,T}, 1\rangle = 1$.

Now, in (2.24) we consider $H(\omega) = |\omega - \omega_L|$, $\xi = \xi(\sigma)$, with $-\infty < \sigma < \xi_1^{cd}$ and write it as

$$\limsup_{T\to\infty} \sum_{\alpha=1}^{d} c_\alpha(\xi(\sigma))\big\langle \mu_{\alpha,T}^{\xi(\sigma)}, (-\xi_\alpha(\sigma) + g^\alpha(r))r|\omega - \omega_L|\big\rangle \le 0. \tag{2.27}$$

Now, given any sequence $T_k \to \infty$ we may obtain a subsequence, still denoted by T_k, and probability measures $\mu_\alpha^{\xi(\sigma)}$, $\alpha = 1, \dots, d$, such that $\mu_{\alpha,T_k}^{\xi(\sigma)} \rightharpoonup \mu_\alpha^{\xi(\sigma)}$. So, we have

$$\sum_{\alpha=1}^{d} c_\alpha(\xi(\sigma))\big\langle \mu_{\alpha}^{\xi(\sigma)}, (-\xi_\alpha(\sigma) + g^\alpha(r))r|\omega - \omega_L|\big\rangle \le 0. \tag{2.28}$$

Since, due to the strong convergence of $r^T(x,t)$ to $\rho(x/t)$, we necessarily have

$$\mu_\alpha^{\xi(\sigma)} = \delta_{\rho(\xi(\sigma))}(r) \otimes \nu_\alpha^{\xi(\sigma)}(\omega), \tag{2.29}$$

for some probability measures $\nu_\alpha^{\xi(\sigma)}$ defined on S_+^{n-1}, (2.28) is only possible if

$$\nu_\alpha^{\xi(\sigma)}(\omega) = \delta_{\omega_L}, \qquad \text{for all } \alpha = 1, \dots, d,$$

where, as usual, we denote by δ_v the Dirac measure concentrated at v. This then implies that, for almost all $-\infty < \sigma < \xi_1^{cd}$, we have

$$\lim_{T\to\infty} \sum_{\alpha=1}^{d} \frac{1}{T^d} \int_{S_\alpha(\xi(\sigma),T)} |u(x,t) - R(\xi(\sigma))|\, d\mathcal{H}^d = 0. \tag{2.30}$$

Similarly, taking in (2.25) $H(\omega) = |\omega - \omega_R|$ and $\xi = \xi(\sigma)$, for $\sigma > \xi_1^{cd}$, writing it as

$$\limsup_{T\to\infty} \sum_{\alpha=1}^{d} c_\alpha(\xi(\sigma))\big\langle \mu_{\alpha,T}^{\xi(\sigma)}, (\xi_\alpha(\sigma) - g^\alpha(r))r|\omega - \omega_R|\big\rangle \le 0, \tag{2.31}$$

considering a sequence $T_k \to \infty$, passing to a subsequence, we obtain probability measures $\mu_\alpha^{\xi(\sigma)}$, $\alpha = 1, \dots, d$, satisfying (2.29), for some probability measures $\nu_\alpha^{\xi(\sigma)}$ defined on S^{n-1}, and such that

$$\sum_{\alpha=1}^{d} c_\alpha(\xi(\sigma))\big\langle \mu_{\alpha}^{\xi(\sigma)}, (\xi_\alpha(\sigma) - g^\alpha(r))r|\omega - \omega_R|\big\rangle \le 0, \tag{2.32}$$

which is only possible if

$$\nu_\alpha^{\xi(\sigma)}(\omega) = \delta_{\omega_R}, \qquad \text{for all } \alpha = 1, \dots, d.$$

This again implies (2.30) for almost all $\sigma > \xi_1^{cd}$.

Now, taking a countable families of N's and T_0's increasing indefinitely, from (2.30) we conclude, using Fubini's theorem, that $u^T(x,t) \to R(x/t)$ in $L^1_{\rm loc}(\mathbb{R}_+^{d+1})$, which, as we said above, is equivalent to (2.1).

As it was mentioned above, (2.1) can be improved to (1.13). The latter is achieved using a method established in [7], motivated by [25]. The strategy is similar to the one for obtaining the decay of periodic solutions in L^1_{loc}, from the decay in time-average along rays (see [8]), and is based on the existence of a strictly convex entropy for (1.1), say, $\eta(u) = |u|^2$. Namely, we use the entropy inequalities, the convergence in time-average, and a result connecting both resembling a continuous version of Hardy's convergence theorem (cf. [33], p.156), to get the usual convergence in time (1.13). The details of the one dimensional case are given in [7], while the the multidimensional case can be seen in [10]. The basic steps are as follows. Given a smooth strictly convex entropy $\eta(u)$ for (1.1), we consider its quadratic part around v

$$\alpha(u,v)=\eta(u)-\eta(v)-\nabla\eta(v)(u-v).$$

In the space of the variables ξ, we take any ball B over which $R(\xi)$ is Lipschitz. Using the equation (1.1) for $u(\xi t,t)$ and $R(\xi)$ we obtain, after integration over B,

$$\frac{d}{dt}\int_B \alpha(u(\xi t,t),R(\boldsymbol{\xi}))\,d\xi \le \frac{C}{t},$$

for some constant $C>0$, independent of t. Now, (2.1) implies

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\int_B \alpha(u(\xi t,t),R(\xi))\,d\xi\right)\,dt=0.$$

Therefore, the nonnegative function $Y(t)=\int_B \alpha(u(\xi t,t),R(\xi))\,d\xi$ satisfies

$$\frac{d}{dt}Y(t)\le\frac{C}{t}, \tag{2.33}$$

and

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T Y(t)\,dt=0, \tag{2.34}$$

and an easy argument shows that (2.33) and (2.34) imply

$$\lim_{t\to\infty}Y(t)=0. \tag{2.35}$$

Now, by Besicovitch's covering theorem, any compact set $\mathcal{K}\subseteq\mathbb{R}^d$ can be covered by a finite number of disjoint countable families of balls B where $R(\xi)$ is Lipschitz, and so (2.35) implies (1.13).

The fact that $R(\xi)$ is the only self-similar weak entropy solution of satisfying (2.5),(2.6) and (2.2),(2.3) follows immediately from the above analysis. Theorem 1.1 is proved.

Final Remarks

In the case where $r_0\in BV_{loc}(\mathbb{R}^d)$, Remark 2.8 in [2], using the notation adopted therein, implies that the flows Φ^T, associated with the fields $(r^T,f^1(r^T),\dots,f^d(r^T))$, converge in $L^{1,\infty}_{loc}(\mathbb{R}^{d+1}_+)$ to the flow Φ, associated with $(r,f^1(r),\dots,f^d(r))$. Nevertheless, this fact by itself does not allow us to conclude that $u^T\to R$ in $L^1_{loc}(\mathbb{R}^{d+1}_+)$. We also recall that this type of stability cannot hold in the general L^∞ case, in view of the counter-example presented by Bressan [3].

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