STABILITY OF SOLUTIONS OF QUASILINEAR PARABOLIC EQUATIONS

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ABSTRACT. We bound the difference between solutions u and v of $u_t = a\Delta u + \operatorname{div}_x f + h$ and $v_t = b\Delta v + \operatorname{div}_x g + k$ with initial data φ and ψ, respectively, by $\|u(t,\cdot) - v(t,\cdot)\|_{L^p(E)} \le A_E(t)\|\varphi - \psi\|^{2\rho_p}_{L^\infty(\mathbb{R}^n)} + B(t)(\|a-b\|_\infty + \|\nabla_x \cdot f - \nabla_x \cdot g\|_\infty + \|f_u - g_u\|_\infty + \|h-k\|_\infty)^{\rho_p} |E|^{\eta_p}$. Here all functions a, f, and h are smooth and bounded, and may depend on u, $x \in \mathbb{R}^n$, and t. The functions a and h may in addition depend on ∇u. Identical assumptions hold for the functions that determine the solutions v. Furthermore, $E \subset \mathbb{R}^n$ is assumed to be a bounded set, and ρ_p and η_p are fractions that depend on n and p. The diffusion coefficients a and b are assumed to be strictly positive and the initial data are smooth.

1. Introduction

We show that one can bound the difference between solutions u and v of

$$\begin{aligned} &u_t = a(t,x,u,\nabla u)\Delta u + \operatorname{div}_x \big(f(t,x,u)\big) + h(t,x,u,\nabla u), \quad x \in \mathbb{R}^n,\ 0<t<T,\\ &u(0,x) = \varphi(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &v_t = b(t,x,v,\nabla v)\Delta v + \operatorname{div}_x \big(g(t,x,v)\big) + k(t,x,v,\nabla v), \quad x \in \mathbb{R}^n,\ 0<t<T,\\ &v(0,x) = \psi(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1.2}$$

respectively. The assumptions are that the diffusion coefficients a and b are bounded from below by a strictly positive constant. All functions a, f, h, etc, as well as the initial data φ, etc, are assumed to be smooth and bounded. We are interested in estimating the *local* L^p-norm of $u(t,\cdot) - v(t,\cdot)$ over any *bounded* subset $E \subset \mathbb{R}^n$ in terms of norm differences of the initial data as well as a and b, etc.

In the hyperbolic case, that is, $a = b = 0$, the classical result of Kuznetsov [12] and Lucier [14] (see also [9, Ch. 2]) reads

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R})} \le \|\varphi - \psi\|_{L^1(\mathbb{R})} + t \min\{\text{T.V.}(\varphi), \text{T.V.}(\psi)\}\, \|f-g\|_{\text{Lip}}$$

in the one-dimensional case ($n=1$) where $f = f(u)$, $g = g(u)$ and $h = k = 0$. Here T.V.(ϕ) denotes the total variation of the function ϕ and $\|f\|_{\text{Lip}}$ denotes the

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Lipschitz semi-norm. Recently, Bianchini and Colombo [2] showed flux stability in the case of hyperbolic systems on the line. Indeed, they established the estimate

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^1(\mathbb{R})} \le C\,t\,\|Df-Dg\|_{C^0(\Omega)}$$

for solutions u and v of $u_t+f(u)_x=0$, $v_t+g(u)_x=0$, respectively with $u|_{t=0}=v|_{t=0}$. The usual assumptions on the flux functions and the initial conditions apply, see [2].

The dependence in a of the solution u of the equation

$$u_t-\Delta a(u)=0$$

is treated in [1], assuming only that a is nondecreasing, and thereby allowing degenerate diffusion. However, no explicit stability estimate is provided. Otto [15] studied the equation

$$B(u)_t-\operatorname{div}_x(a(\nabla u,B(u)))+h(B(u))=0$$

with a continuous and monotone nondecreasing B. Under certain assumptions he proved that

$$\|B(u_1(t))-B(u_2(t))\|_1 \le \exp(Lt)\,\|B(u_1(0))-B(u_2(0))\|_1\,.$$

By extending Kružkov's famous doubling of variables method, Bouchut and Perthame [3] showed that

$$\|u_1(t,\cdot)-u_2(t,\cdot)\|_{L^1(\mathbb{R}^n)} \le \left\|u_1^0-u_2^0\right\|_{L^1(\mathbb{R}^n)} + C\,\mathrm{T.V.}(u_1^0)\sqrt{t\,\mathrm{Lip}(a)}$$

when u_j satisfies $u_t+\operatorname{div}_x(f)=\Delta a(u_j)$ with initial data u_j^0, $j=1,2$. Here a is assumed to be Lipschitz and nondecreasing.

Closer to the approach of this paper, Cockburn and Gripenberg [6] established the estimate

$$\|u_1(t,\cdot)-u_2(t,\cdot)\|_{L^1(\mathbb{R}^n)} \le \mathrm{T.V.}(\varphi)\big(t\,\|f_1'-f_2'\|_\infty + 4\sqrt{tn}\left\|\sqrt{a_1'}-\sqrt{a_2'}\right\|_\infty\big)$$

for solutions u_j, $j=1,2$ of

$$u_{j,t}=\operatorname{div}_x(f_j)+\Delta(a_j(u_j)),\quad u_j|_{t=0}=\varphi.$$

Allowing for explicit spatial dependence in the flux function, Evje, Karlsen, and Risebro [8, 11] showed stability for solutions of

$$u_{j,t}+\operatorname{div}_x(k_j(x)f_j(u))=\Delta A_j(u),\quad u_j|_{t=0}=u_j^0,$$

in the sense that

$$\begin{aligned}\|u_1(t,\cdot)-u_2(t,\cdot)\|_{L^1(\mathbb{R}^n)} \le \left\|u_1^0-u_2^0\right\|_{L^1(\mathbb{R}^n)} &+ t\,C\big(\|k_1-k_2\|_{\infty,bv}+\|f_1-f_2\|_{\infty,\mathrm{Lip}}\big)\\ &+\sqrt{t}\,C\left\|\sqrt{A_1'}-\sqrt{A_2'}\right\|_\infty\end{aligned}$$

where $\|\cdot\|_{\infty,bv}$ and $\|\cdot\|_{\infty,\mathrm{Lip}}$ is the sum of the sup-norm and the BV-norm and the sum of sup-norm and the Lipschitz norm, respectively. Here A_j is allowed to be degenerate. Karlsen and Ohlberger [10] established L^1 contractivity of solutions of

$$u_t+\operatorname{div}_x(V(t,x)f(u))=\nabla\cdot(K(t,x)\nabla A(u)))+q(t,x,u).$$

Recently, Chen and Karlsen [5] established the estimate

$$\begin{aligned}\|u_1(t,\cdot)-u_2(t,\cdot)\|_{L^1(\mathbb{R}^n)} \le \left\|u_1^0-u_2^0\right\|_{L^1(\mathbb{R}^n)} &+ t\,C\,\|f_1'-f_2'\|_\infty\\ &+\Big(t\left\|(\sqrt{A_1}-\sqrt{A_2})(\sqrt{A_1}-\sqrt{A_2})^\top\right\|_\infty\Big)^{1/2}\end{aligned}$$

for solutions of $u_{j,t} + \operatorname{div}_x f_j(u_j) = \nabla\cdot(A_j(u_j)\nabla u_j)$ with initial data $u_j|_{t=0} = u_j^0$.

We consider here the strictly parabolic case where the diffusion constant is not allowed to decrease to zero. However, we allow full explicit spatial and temporal dependence in all parameters. In addition, we let the diffusion and source depend explicitly on the gradient of the unknown u. All parameters, including the initial data are assumed to be smooth. Existence of regular bounded solutions is secured by classical results, see [13]. The question is to obtain explicit stability estimates. Our main result reads as follows. Let u and v denote solutions of (1.1) and (1.2), respectively. Then

$$\begin{aligned}\|u(t,\cdot)-v(t,\cdot)\|_{L^p(E)} \le A_E(t)\|\varphi-\psi\|^{2\rho_p}_{L^\infty(\mathbb{R}^n)}\\ &+ B(t)\Big(\|a-b\|_{L^\infty(\mathcal{R}_0)} + \|\nabla_x\cdot f - \nabla_x\cdot g\|_{L^\infty(\mathcal{R})}\\ &+ \|f_u - g_u\|_{L^\infty(\mathcal{R})} + \|h-k\|_{L^\infty(\mathcal{R}_0)}\Big)^{\rho_p}|E|^{\eta_p},\end{aligned}$$

where

$$\rho_p := \begin{cases}\frac{1}{2}, & \text{if } 1\le p\le 2,\\ \frac{1}{p}, & \text{if } 2<p<\infty,\end{cases} \qquad \eta_p := \begin{cases}\frac{2-p}{2p}+\frac{1}{2n}, & \text{if } 1\le p\le 2,\\ \frac{1}{np}, & \text{if } 2<p<\infty,\end{cases}$$

$$A_E(t) := C\begin{cases}\big(|E|^{(2-p)/2p+1/2n} + |E|^{1/p}\big), & \text{if } 1\le p\le 2,\\ (1+t^{(p-2)/p})\big(|E|^{1/np}+|E|^{1/p}\big), & \text{if } 2<p<\infty,\end{cases}$$

$$B(t) := C\begin{cases} t, & \text{if } 1\le p\le 2,\\ (t+t^{2/p}), & \text{if } 2<p<\infty,\end{cases}$$

for any bounded connected set $E\subset\mathbb{R}^n$ with Lipschitz boundary. Here $\mathcal{R}_0 = [0,T]\times E\times[-K_1,K_1]\times[-K_2,K_2]$ and $\mathcal{R} = [0,T]\times E\times[-K_1,K_1]$.

As a particular example we note that for solutions u and v of

$$u_t = a(t,x,u,\nabla u)\Delta u, \quad v_t = b(t,x,v,\nabla v)\Delta v$$

with initial conditions $u|_{t=0}=\varphi$ and $v|_{t=0}=\psi$, we find

$$\begin{aligned}\|u(t,\cdot)-v(t,\cdot)\|_{L^2(E)} &\le C(|E|^{1/2n}+|E|^{1/2})\,\|\varphi-\psi\|_{L^\infty(\mathbb{R}^n)}\\ &\quad + C\,t\,|E|^{1/2n}\,\|a-b\|^{1/2}_{L^\infty(\mathcal{R}_0)}.\end{aligned}$$

Our proof is based on a homotopy argument, inspired by [4]. Introducing

$$\begin{aligned} u_{\theta,t} &= \big(\theta a+(1-\theta)b\big)\Delta u_\theta + \operatorname{div}_x\big(\theta f+(1-\theta)g\big) + \theta h + (1-\theta)k,\\ u_\theta|_{t=0} &= \theta\varphi + (1-\theta)\psi,\end{aligned}$$

we see that $u_0 = u$ and $u_1 = v$. Thus u_θ interpolates between u (for $\theta = 0$) and v (for $\theta=1$). The key estimate establishes that

$$\begin{aligned}\|u(t,\cdot)-v(t,\cdot)\|_{L^p(E)} &\equiv \operatorname{dist}_{L^p(E)}\big(u(t,\cdot),v(t,\cdot)\big)\\ &\le \operatorname{length}_{L^p(E)}\big(u_\theta(t,\cdot)\big)\\ &= \int_0^1 \Big\|\frac{\partial u_\theta}{\partial\theta}(t,\cdot)\Big\|_{L^p(E)} d\theta,\end{aligned}$$

and we establish θ-independent estimates for $\|\partial u_\theta/\partial\theta\|$.

2. Fundamental assumptions

Fix $T > 0$. Let $u = u(t,x)$ and $v = v(t,x)$ be the bounded solution of the quasilinear initial value problem (see [13])

$$\begin{aligned} &u_t = a(t,x,u,\nabla u)\Delta u + \operatorname{div}_x\big(f(t,x,u)\big) + h(t,x,u,\nabla u), \quad x \in \mathbb{R}^n,\ 0 < t < T,\\ &u(0,x) = \varphi(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &v_t = b(t,x,v,\nabla v)\Delta v + \operatorname{div}_x\big(g(t,x,v)\big) + k(t,x,v,\nabla v), \quad x \in \mathbb{R}^n,\ 0 < t < T,\\ &v(0,x) = \psi(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{2.2}$$

respectively. Here

$$f = (f_1, \dots, f_n), g = (g_1, \dots, g_n) \colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n,$$

and

$$\begin{aligned} \operatorname{div}_x\big(f(t,x,u)\big) &= \sum_{j=1}^n \big(f_j(t,x,u,\nabla u)\big)_{x_j}\\ &= \sum_{j=1}^n \Big(f_{j,x_j} + \frac{\partial f_j}{\partial u} u_{x_j}\Big)\\ &= \nabla_x \cdot f + f_u \cdot \nabla u. \end{aligned}$$

Observe that $\nabla_x \cdot f$ is a scalar. The divergence operator div_x always acts on the spatial variables only. By $\nabla_q a$ (similarly for b, h, and k) we denote the gradient of a with respect to the final n variables (where ∇u usually sits). Our fundamental assumptions are

($\mathcal{H}_1$) the viscous coefficients a and b are of class $C^3([0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ such that

$$\begin{aligned} &0 < a_* \le a(\cdot,\cdot,\cdot,\cdot) \le a^* < \infty, \qquad \|a\|_{C^3([0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n)} \le k_1,\\ &0 < b_* \le b(\cdot,\cdot,\cdot,\cdot) \le b^* < \infty, \qquad \|b\|_{C^3([0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n)} \le k_1, \end{aligned} \tag{2.3}$$

for some positive constants $a_*,\ a^*,\ b_*,\ b^*,\ k_1$;

($\mathcal{H}_2$) the convective terms f and g are of class $C^3([0,T] \times \mathbb{R}^n \times \mathbb{R})$ and the source terms h and k are of class $C^3([0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ such that for all $i,\ j,\ l \in \{1, \dots, n\}$ and any $\Phi \in \{f_1, \dots, f_n, g_1, \dots, g_n, h, k\}$ the following quantities

$$\begin{aligned} &\Big\|\frac{\partial \Phi}{\partial x_i}\Big\|_{L^\infty}, \Big\|\frac{\partial^2 \Phi}{\partial x_i \partial x_j}\Big\|_{L^\infty}, \Big\|\frac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_l}\Big\|_{L^\infty},\\ &\Big\|\frac{\partial \Phi}{\partial u}\Big\|_{L^\infty}, \Big\|\frac{\partial^2 \Phi}{\partial u^2}\Big\|_{L^\infty}, \Big\|\frac{\partial^2 \Phi}{\partial x_i \partial u}\Big\|_{L^\infty}, \|\nabla_q h\|_{L^\infty}, \|\nabla_q k\|_{L^\infty}, \end{aligned} \tag{2.4}$$

are all bounded by a positive constant k_2;

($\mathcal{H}_3$) the initial data φ and ψ are of class $C^2(\mathbb{R}^n)$ such that

$$\|\varphi\|_{C^2(\mathbb{R}^n)}, \|\psi\|_{C^2(\mathbb{R}^n)} \le k_3, \tag{2.5}$$

for a positive constant k_3.

Lemma 2.1 ($\mathbf{L^\infty}$-bounds on u and v). *Fix $T > 0$. By* [13] *there exist positive constants K_1, K_2, K_3 such that*

$$
\begin{aligned}
\|u\|_{L^\infty([0,T]\times\mathbb{R}^n)}, \|v\|_{L^\infty([0,T]\times\mathbb{R}^n)} &\le K_1,\\
\Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^\infty([0,T]\times\mathbb{R}^n)}, \Big\|\frac{\partial v}{\partial x_i}\Big\|_{L^\infty([0,T]\times\mathbb{R}^n)} &\le K_2,\\
\Big\|\frac{\partial^2 u}{\partial x_i \partial x_j}\Big\|_{L^\infty([0,T]\times\mathbb{R}^n)}, \Big\|\frac{\partial^2 v}{\partial x_i \partial x_j}\Big\|_{L^\infty([0,T]\times\mathbb{R}^n)} &\le K_3,
\end{aligned}
\tag{2.6}
$$

for all i, $j \in \{1, ..., n\}$ where K_1, K_2, K_3 depend only on T, n, a_*, a^*, b_*, b^*, k_1, k_2, and k_3.

3. The homotopy argument

Our approach is based on the following homotopy argument. Let $0 \le \theta \le 1$. The function u_θ interpolates between the functions u and v. More precisely, denote by u_θ the solution of the quasilinear initial value problem

$$
\begin{aligned}
u_{\theta,t} = &\big(\theta a(t,x,u_\theta,\nabla u_\theta) + (1-\theta) b(t,x,u_\theta,\nabla u_\theta)\big)\Delta u_\theta\\
&+ \mathrm{div}_x \big(\theta f(t,x,u_\theta) + (1-\theta) g(t,x,u_\theta)\big)\\
&+ \theta h(t,x,u_\theta,\nabla u_\theta) + (1-\theta) k(t,x,u_\theta,\nabla u_\theta), \quad x \in \mathbb{R}^n,\ 0 < t < T,\\
u_\theta(0,x) = &\ \theta\varphi(x) + (1-\theta)\psi(x), \quad x \in \mathbb{R}^n.
\end{aligned}
\tag{3.1}
$$

Clearly

$$u_0 = v, \qquad u_1 = u.$$

Indeed

$$\theta \longmapsto u_\theta(t,\cdot)$$

is a curve joining $v(t,\cdot)$ and $u(t,\cdot)$, and

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^p(E)} \equiv \mathrm{dist}_{L^p(E)}\big(u(t,\cdot), v(t,\cdot)\big) \le \mathrm{length}_{L^p(E)}\big(u_\theta(t,\cdot)\big), \tag{3.2}$$

for each $0 \le t \le T$, $E \subset \mathbb{R}^n$ measurable set and $1 \le p \le \infty$.

Lemma 3.1 ($\mathbf{L^\infty}$-bounds on $\mathbf{u_\theta}$). *By* [13, Theorem V 8.1]*, there exist positive constants K_1, K_2, K_3 depending only on T, n, a_*, a^*, b_*, b^*, k_1, k_2 and k_3 such that*

$$
\begin{aligned}
\|u_\theta\|_{L^\infty([0,T]\times\mathbb{R}^n)} &\le K_4,\\
\Big\|\frac{\partial u_\theta}{\partial x_i}\Big\|_{L^\infty([0,T]\times\mathbb{R}^n)} &\le K_5,\\
\Big\|\frac{\partial^2 u_\theta}{\partial x_i \partial x_j}\Big\|_{L^\infty([0,T]\times\mathbb{R}^n)} &\le K_6,
\end{aligned}
\tag{3.3}
$$

for each $0 \le \theta \le 1$ and i, $j \in \{1, ..., n\}$.

Lemma 3.2 (Smoothness of $\theta \mapsto u_\theta$). *Assume $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and $(\mathcal{H}_3)$. The curve*

$$\theta \in [0,1] \longmapsto u_\theta(t,\cdot) \in C^2(\mathbb{R}^n)$$

is of class C^1. In particular, we infer

$$\mathrm{length}_{L^p(E)}\big(u_\theta(t,\cdot)\big) = \int_0^1 \Big\|\frac{\partial u_\theta}{\partial \theta}(t,\cdot)\Big\|_{L^p(E)} d\theta, \tag{3.4}$$

for each $0 \le t \le T$ and $E \subset \mathbb{R}^n$ measurable set.

Proof. Consider the map

$$
\begin{aligned}
F\colon \mathcal{D} &\longrightarrow C^\infty\big(]0,T[\times\mathbb{R}^n\big)\cap C^2\big([0,T]\times\mathbb{R}^n\big),\\
F(\theta,\omega)(t,x) &:= \frac{\partial\omega}{\partial t}(t,x) - \Big(\theta a\big(t,x,\omega(t,x),\nabla\omega(t,x)\big)\\
&\qquad\qquad + (1-\theta)b\big(t,x,\omega(t,x),\nabla\omega(t,x)\big)\Big)\Delta\omega(t,x)\\
&\quad - \operatorname{div}_x\big(\theta f(t,x,\omega(t,x)) + (1-\theta)g(t,x,\omega(t,x))\big)\\
&\quad - \big(\theta h(t,x,\omega(t,x),\nabla\omega(t,x)) + (1-\theta)k(t,x,\omega(t,x),\nabla\omega(t,x))\big),
\end{aligned}
$$

where

$$
\mathcal{D} := \Big\{(\theta,\omega)\in[0,1]\times C^\infty\big(]0,T[\times\mathbb{R}^n\big)\cap C^2\big([0,T]\times\mathbb{R}^n\big) \mid \omega(0,\cdot)=\theta\varphi+(1-\theta)\psi\Big\}.
$$

From the definition of u_θ,

$$
F(\theta,u_\theta)\equiv 0, \qquad 0\le\theta\le 1. \tag{3.5}
$$

Observe that F is of class C^1 and

$$
\begin{aligned}
\frac{\partial F}{\partial\theta}(\theta,\omega) &= \big(b(t,x,\omega,\nabla\omega) - a(t,x,\omega,\nabla\omega)\big)\Delta\omega\\
&\quad + \operatorname{div}_x\big(g(t,x,\omega) - f(t,x,\omega)\big) + k(t,x,\omega,\nabla\omega) - h(t,x,\omega,\nabla\omega).
\end{aligned}
$$

To compute

$$
\frac{\partial F}{\partial\omega}(\theta,\omega)\big[(\theta',z)\big] = \frac{\partial F}{\partial\varepsilon}(\theta,\omega+\varepsilon z)\Big|_{\varepsilon=0}
$$

we find

$$
\begin{aligned}
F(\theta,\omega+\varepsilon z) &= \frac{\partial\omega}{\partial t} + \varepsilon\frac{\partial z}{\partial t} - \big(\theta a\big(t,x,\omega+\varepsilon z,\nabla\omega+\varepsilon\nabla z\big)\\
&\qquad + (1-\theta)b(t,x,\omega+\varepsilon z,\nabla\omega+\varepsilon\nabla z)\big)\,(\Delta\omega+\varepsilon\Delta z)\\
&\quad - \operatorname{div}_x\big(\theta f(t,x,\omega+\varepsilon z) + (1-\theta)g(t,x,\omega+\varepsilon z)\big)\\
&\quad - \big(\theta h(t,x,\omega+\varepsilon z,\nabla\omega+\varepsilon\nabla z) + (1-\theta)k(t,x,\omega+\varepsilon z,\nabla\omega+\varepsilon\nabla z)\big),\\
\frac{\partial F}{\partial\varepsilon}(\theta,\omega+\varepsilon z) &= \frac{\partial z}{\partial t} - \big(\theta a(t,x,\omega+\varepsilon z,\nabla\omega+\varepsilon\nabla z)\\
&\qquad + (1-\theta)b(t,x,\omega+\varepsilon z,\nabla\omega+\varepsilon\nabla z)\big)\Delta z\\
&\quad - \Big(\theta\frac{\partial a}{\partial\omega}(t,x,\omega+\varepsilon z,\nabla\omega+\varepsilon\nabla z)\\
&\qquad + (1-\theta)\frac{\partial b}{\partial\omega}(t,x,\omega+\varepsilon z,\nabla\omega+\varepsilon\nabla z)\Big)z\,(\Delta\omega+\varepsilon\Delta z)\\
&\quad - \Big(\theta\nabla_q a(t,x,\omega+\varepsilon z,\nabla\omega+\varepsilon\nabla z)\\
&\qquad + (1-\theta)\nabla_q b(t,x,\omega+\varepsilon z,\nabla\omega+\varepsilon\nabla z)\Big)\cdot\nabla z\;(\Delta\omega+\varepsilon\Delta z)\\
&\quad - \operatorname{div}_x\Big(\Big(\theta\frac{\partial f}{\partial\omega}(t,x,\omega+\varepsilon z) + (1-\theta)\frac{\partial g}{\partial\omega}(t,x,\omega+\varepsilon z)\Big)z\Big)\\
&\quad - \Big(\theta\frac{\partial h}{\partial\omega}(t,x,\omega+\varepsilon z,\nabla\omega+\varepsilon\nabla z)\\
&\qquad + (1-\theta)\frac{\partial k}{\partial\omega}(t,x,\omega+\varepsilon z,\nabla\omega+\varepsilon\nabla z)\Big)z
\end{aligned}
$$

$$- \Big(\theta \nabla_q h(t,x,\omega+\varepsilon z, \nabla\omega+\varepsilon\nabla z) \\ + (1-\theta)\nabla_q k(t,x,\omega+\varepsilon z,\nabla\omega+\varepsilon\nabla z)\Big)\cdot\nabla z.$$

Thus

$$\begin{aligned}
\frac{\partial F}{\partial\omega}(\theta,\omega)\big[(\theta',z)\big] = \frac{\partial z}{\partial t} &- \big(\theta a(t,x,\omega,\nabla\omega) + (1-\theta) b(t,x,\omega,\nabla\omega)\big)\Delta z \\
&- \left(\theta\frac{\partial a}{\partial\omega}(t,x,\omega,\nabla\omega) + (1-\theta)\frac{\partial b}{\partial\omega}(t,x,\omega,\nabla\omega)\right) z\Delta\omega \\
&- (\theta\nabla_q a(t,x,\omega,\nabla\omega) + (1-\theta)\nabla_q b(t,x,\omega,\nabla\omega))\cdot\nabla z\,\Delta\omega \\
&- \operatorname{div}_x\left(\Big(\theta\frac{\partial f}{\partial\omega}(t,x,\omega) + (1-\theta)\frac{\partial g}{\partial\omega}(t,x,\omega)\Big)z\right) \\
&- \Big(\theta\frac{\partial h}{\partial\omega}(t,x,\omega,\nabla\omega) + (1-\theta)\frac{\partial k}{\partial\omega}(t,x,\omega,\nabla\omega)\Big)z \\
&- \Big(\theta\nabla_q h(t,x,\omega,\nabla\omega) + (1-\theta)\nabla_q k(t,x,\omega,\nabla\omega)\Big)\cdot\nabla z, \\
&\qquad (\theta,\omega),(\theta',z)\in\mathcal{D}.
\end{aligned}$$

Observe that $(\theta',z)\in\mathcal{D}$ satisfies the equation

$$\frac{\partial F}{\partial\omega}(\theta,\omega)\big[(\theta',z)\big] = \zeta$$

if and only if z is solution of the linear initial value problem

$$\begin{aligned}
z_t = {} & \big(\theta a(t,x,\omega,\nabla\omega) + (1-\theta)b(t,x,\omega,\nabla\omega)\big)\Delta z \\
& + \big(\theta a_\omega(t,x,\omega,\nabla\omega) + (1-\theta)b_\omega(t,x,\omega,\nabla\omega)\big)\Delta\omega z \\
& + \big(\theta\nabla_q a(t,x,\omega,\nabla\omega) + (1-\theta)\nabla_q b(t,x,\omega,\nabla\omega)\big)\cdot\nabla z\,\Delta\omega \\
& + \operatorname{div}_x\Big(\big(\theta f_\omega(t,x,\omega) + (1-\theta)g_\omega(t,x,\omega)\big)z\Big) \\
& + \big(\theta h_\omega(t,x,\omega,\nabla\omega) + (1-\theta)k_\omega(t,x,\omega,\nabla\omega)\big)z \\
& + \big(\theta\nabla_q h(t,x,\omega,\nabla\omega) + (1-\theta)\nabla_q k(t,x,\omega,\nabla\omega)\big)\cdot\nabla z + \zeta(t,x), \\
& \qquad x\in\mathbb{R}^n,\ 0<t<T, \\
z(0,x) = {} & \theta'\varphi(x) + (1-\theta')\psi(x), \quad x\in\mathbb{R}^n.
\end{aligned}$$

Since this problem is well-posed (see [13, Theorem IV 5.1]), $\frac{\partial F}{\partial\omega}(\theta,\omega)$ is invertible. By the implicit function theorem, the curve $\theta\longmapsto u_\theta$ is of class C^1 and clearly (3.4) holds. This concludes the proof. □

Differentiating equation (3.1) with respect to θ, we have

$$\begin{aligned}
\frac{\partial^2 u_\theta}{\partial t\partial\theta} = {} & \big(\theta a(t,x,u_\theta,\nabla u_\theta) + (1-\theta)b(t,x,u_\theta,\nabla u_\theta)\big)\Delta\left(\frac{\partial u_\theta}{\partial\theta}\right) \\
& + \left(\theta\frac{\partial a}{\partial u}(t,x,u_\theta,\nabla u_\theta) + (1-\theta)\frac{\partial b}{\partial u}(t,x,u_\theta,\nabla u_\theta)\right)\Delta u_\theta\,\frac{\partial u_\theta}{\partial\theta} \\
& + (\theta\nabla_q a(t,x,u_\theta,\nabla u_\theta) + (1-\theta)\nabla_q b(t,x,u_\theta,\nabla u_\theta))\cdot\nabla\left(\frac{\partial u_\theta}{\partial\theta}\right)\,\Delta u_\theta \\
& + \big(a(t,x,u_\theta,\nabla u_\theta) - b(t,x,u_\theta,\nabla u_\theta)\big)\Delta u_\theta
\end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&+\left(\frac{\partial f}{\partial u}(t,x,u_\theta)-\frac{\partial g}{\partial u}(t,x,u_\theta)\right)\nabla u_\theta+\nabla_x\cdot f(t,x,u_\theta)-\nabla_x\cdot g(t,x,u_\theta)\\
&+\left(\theta\frac{\partial f}{\partial u}(t,x,u_\theta)+(1-\theta)\frac{\partial g}{\partial u}(t,x,u_\theta)\right)\cdot\nabla\left(\frac{\partial u_\theta}{\partial\theta}\right)\\
&+\left(\theta\nabla_x\cdot\frac{\partial f}{\partial u}(t,x,u_\theta)+(1-\theta)\nabla_x\cdot\frac{\partial g}{\partial u}(t,x,u_\theta)\right)\frac{\partial u_\theta}{\partial\theta}\\
&+\left(\theta\frac{\partial^2 f}{\partial u^2}(t,x,u_\theta)+(1-\theta)\frac{\partial^2 g}{\partial u^2}(t,x,u_\theta)\right)\cdot\nabla u_\theta\,\frac{\partial u_\theta}{\partial\theta}\\
&+h(t,x,u_\theta,\nabla u_\theta)-k(t,x,u_\theta,\nabla u_\theta)\\
&+\left(\theta\frac{\partial h}{\partial u}(t,x,u_\theta,\nabla u_\theta)+(1-\theta)\frac{\partial k}{\partial u}(t,x,u_\theta,\nabla u_\theta)\right)\frac{\partial u_\theta}{\partial\theta}\\
&+\left(\theta\nabla_q h(t,x,u_\theta,\nabla u_\theta)+(1-\theta)\nabla_q k(t,x,u_\theta,\nabla u_\theta)\right)\cdot\nabla\left(\frac{\partial u_\theta}{\partial\theta}\right).
\end{aligned}
$$

Denoting

$$
\begin{aligned}
z_\theta(t,x)&:=\frac{\partial u_\theta}{\partial\theta},\\
\alpha(\theta,t,x)&:=\theta a+(1-\theta)b,\\
\beta(\theta,t,x)&:=\big(\theta\nabla_q a+(1-\theta)\nabla_q b\big)\Delta u_\theta+\theta\frac{\partial f}{\partial u}+(1-\theta)\frac{\partial g}{\partial u}\\
&\quad+\big(\theta\nabla_q h+(1-\theta)\nabla_q k\big),\\
\gamma(\theta,t,x)&:=\left(\theta\frac{\partial a}{\partial u}+(1-\theta)\frac{\partial b}{\partial u}\right)\Delta u_\theta\\
&\quad+\theta\nabla_x\cdot\frac{\partial f}{\partial u}+(1-\theta)\nabla_x\cdot\frac{\partial g}{\partial u}+\theta\frac{\partial h}{\partial u}+(1-\theta)\frac{\partial k}{\partial u}\\
&\quad+\left(\theta\frac{\partial^2 f}{\partial u^2}+(1-\theta)\frac{\partial^2 g}{\partial u^2}\right)\cdot\nabla u_\theta,\\
\sigma(\theta,t,x)&:=\big(a-b\big)\Delta u_\theta+\left(\frac{\partial f}{\partial u}-\frac{\partial g}{\partial u}\right)\cdot\nabla u_\theta\\
&\quad+\nabla_x\cdot f-\nabla_x\cdot g+h-k,
\end{aligned}
$$

for each $0\le\theta\le1,\ t\ge0,\ x\in\mathbb{R}^n$, there results

$$
\begin{aligned}
\frac{\partial z_\theta}{\partial t}&=\alpha(\theta,t,x)\Delta z_\theta+\beta(\theta,t,x)\cdot\nabla z_\theta+\gamma(\theta,t,x)z_\theta+\sigma(\theta,t,x),\\
&\qquad\qquad 0\le\theta\le1,\ 0<t<T,\ x\in\mathbb{R}^n. \qquad (3.7)
\end{aligned}
$$

Moreover, observe that

$$
z_\theta(0,x)=\varphi(x)-\psi(x),\qquad 0\le\theta\le1,\ x\in\mathbb{R}^n. \tag{3.8}
$$

Lemma 3.3 ($\mathbf{L^\infty}$**-bounds on** α**,** β**,** γ**).** *From the definition of* α*,* (2.3) *and* (3.3)*, we have*

$$
0<\alpha_*\le\alpha(\cdot,\cdot,\cdot)\le\alpha^*,\qquad \|\nabla\alpha\|_{L^\infty}\le k_1(1+K_5+nK_6) \tag{3.9}
$$

where

$$
\alpha_*:=\min\{a_*,b_*\},\qquad \alpha^*:=\max\{a^*,b^*\}.
$$

Moreover, from the definition of β *and* (2.4)*, we infer*

$$\|\beta\|_{L^\infty} = \sup_{j=1,\dots,n} \|\beta_j\|_{L^\infty} \leq K_7, \tag{3.10}$$

where

$$K_7 := nk_1K_6 + 2k_2.$$

Finally, from the definition of γ*,* (2.3)*,* (2.4) *and* (3.3)*, we find*

$$\|\gamma\|_{L^\infty} \leq K_8, \tag{3.11}$$

where

$$K_8 := nk_1K_6 + (n+1+nK_5)k_2.$$

Lemma 3.4 ($\mathbf{L^\infty}$-bounds on $\mathbf{z_\theta}$). *Assume* $(\mathcal{H}_1)$*,* $(\mathcal{H}_2)$*, and* $(\mathcal{H}_3)$*. There exists a positive constant* C_1 *depending only on* T*,* n*,* a_**,* a^**,* b_**,* b^**,* k_1*,* k_2*, and* k_3 *such that*

$$\|z_\theta(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C_1 t + \|\varphi - \psi\|_{L^\infty(\mathbb{R}^n)}, \tag{3.12}$$

for each $0 \leq t \leq T$ *and* $0 \leq \theta \leq 1$*.*

Proof. To simplify the notation we let w denote the solution of (3.7), that is,

$$w_t = \alpha\Delta w + \beta\cdot\nabla w + \gamma w + \sigma, \quad w|_{t=0} = w_0. \tag{3.13}$$

Linearity implies that

$$w = w_1 + w_2$$

where w_1 and w_2 solve

$$\begin{aligned} w_{1,t} &= \alpha\Delta w_1 + \beta\cdot\nabla w_1 + \gamma w_1, \quad w_1|_{t=0} = w_0,\\ w_{2,t} &= \alpha\Delta w_2 + \beta\cdot\nabla w_2 + \gamma w_2 + \sigma, \quad w_2|_{t=0} = 0, \end{aligned}$$

respectively. We infer from [13, p. 389] that

$$\begin{aligned} w_1(t,x) &= \int_{\mathbb{R}^n} G(t,0,x,\xi)w_0(\xi)\,d\xi,\\ w_2(t,x) &= \int_0^t\int_{\mathbb{R}^n} G(t,\tau,x,\xi)\sigma(\tau,\xi)\,d\xi d\tau, \end{aligned}$$

where G is the Green's function. For $t\in[0,T]$ for some fixed T positive we find

$$|w_2(t,x)| \leq C\,t\,\|\sigma\|_\infty.$$

Introduce $z = w_1 - w_0$ which satisfies the equation for w_2 with $\sigma = \alpha\Delta w_0 + \beta\operatorname{div}_x w_0 + \gamma w_0$. Thus

$$|w_1(t,x)| \leq |z(t,x)| + |w_0(x)| \leq \|w_0\|_\infty + C\,t\,\|\alpha\Delta w_0 + \beta\cdot\nabla w_0 + \gamma w_0\|_\infty.$$

□

Observe that in the previous lemma, the smoothness of the initial condition enters in a crucial way. With less regularity we get the familiar $O(t^{1/2})$ behavior near $t = 0$ (see, e.g., [9, Sec. 4.4]).

4. Stability of quasilinear parabolic equations

We begin with the following lemma.

Lemma 4.1 (Poincaré-type inequality). *There exists a positive constant Λ_0, depending only on n, such that*

$$\int_B |f|^2\,dx \le \Lambda_0\,|B|^{2/n}\int_B |\nabla f|^2\,dx + \Lambda_0\,|B|^{1/n}\int_{\partial B}|f|^2\,dx, \tag{4.1}$$

for each $f \in C^2(\mathbb{R}^n)$ and $B \subset \mathbb{R}^n$ bounded connected set with Lipschitz boundary. In the case $n = 1$ we mean

$$\int_{\partial B}|f|^2\,dx = |f(x_0)|^2,$$

for some $x_0 \in B$.

The proof of this lemma is more or less classical (see [16, Theorem A.9] and [7, Lemma A.2]) and the dependence of the coefficients on the measure of the domain is consequence of a standard rescaling argument.

Now we prove the key estimate in the L^2-norm for the map z_θ.

Lemma 4.2 (Case $\mathbf{p = 2}$: Energy estimate). *Assume $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and $(\mathcal{H}_3)$. Then there exists a positive constant C_2 depending only on $T, n, a_*, a^*, b_*, b^*, k_1, k_2$, and k_3 such that*

$$\begin{aligned}\|z_\theta(t,\cdot)\|_{L^2(E)} \le{}& C_2(|E|^{1/2n} + |E|^{1/2})\|\varphi-\psi\|_{L^\infty(\mathbb{R}^n)}\\ &+ C_2\,t\,|E|^{1/2n}\Big(\|a-b\|_{L^\infty(\mathcal{R}_0)} + \|\nabla_x\cdot f - \nabla_x\cdot g\|_{L^\infty(\mathcal{R})}\\ &+ \|f_u - g_u\|_{L^\infty(\mathcal{R})} + \|h-k\|_{L^\infty(\mathcal{R}_0)}\Big)^{1/2},\end{aligned} \tag{4.2}$$

for each $0 \le t \le T$, $0 \le \theta \le 1$ and $E \subset \mathbb{R}^n$ bounded connected set with Lipschitz boundary. Here $\mathcal{R}_0 = [0,T]\times E\times[-K_1,K_1]\times[-K_2,K_2]$ and $\mathcal{R} = [0,T]\times E\times[-K_1,K_1]$.

Proof. Let $B \subset \mathbb{R}^n$ be a ball and $0 < t < T$. Then by (3.7) we find

$$\begin{aligned}\frac{d}{dt}\int_B \frac{1}{2}z_\theta^2(t,x)\,dx &= \int_B z_\theta z_{\theta,t}\,dx\\ &= \int_B \alpha z_\theta\Delta z_\theta\,dx + \int_B z_\theta\beta\cdot\nabla z_\theta\,dx + \int_B \gamma z_\theta^2\,dx + \int_B \sigma z_\theta\,dx.\end{aligned} \tag{4.3}$$

Observe that, by (3.11),

$$\int_B \gamma z_\theta^2\,dx \le K_8\int_B z_\theta^2\,dx, \tag{4.4}$$

and, by (3.10),

$$\begin{aligned}\int_B z_\theta\beta\cdot\nabla z_\theta\,dx &\le \frac{1}{\alpha_*}\int_B |\beta|^2\,z_\theta^2\,dx + \frac{\alpha_*}{4}\int_B|\nabla z_\theta|^2\,dx\\ &\le \frac{K_7^2}{\alpha_*}\int_B z_\theta^2\,dx + \frac{\alpha_*}{4}\int_B |\nabla z_\theta|^2\,dx.\end{aligned} \tag{4.5}$$

By Lemma 3.4 and (2.5),

$$\int_B \sigma z_\theta\,dx \le \int_B |\sigma|\,|z_\theta|\,dx = \int_B \sqrt{|\sigma|}\big(\sqrt{|\sigma|}\,|z_\theta|\big)\,dx$$

$$
\begin{aligned}
&\le \frac{1}{2}\int_B |\sigma|\,dx + \frac{1}{2}\int_B |\sigma| z_\theta^2\,dx \qquad (4.6)\\
&\le \frac{1}{2}|B|\ \|\sigma\|_{L^\infty(\mathcal{R})} + C_1^2\,|B|\,t^2\|\sigma\|_{L^\infty(\mathcal{R})}\\
&\qquad + \|\varphi-\psi\|^2_{L^\infty(\mathbb{R}^n)}\,|B|\ \|\sigma\|_{L^\infty(\mathcal{R})}\\
&\le \frac{K_9}{2}|B|\ \|\sigma\|_{L^\infty(\mathcal{R})} + C_1^2\,|B|\,t^2\|\sigma\|_{L^\infty(\mathcal{R})},
\end{aligned}
$$

where

$$K_9 := 1 + 8k_3^2.$$

Moreover, by the divergence theorem we have

$$
\begin{aligned}
\int_B \alpha z_\theta \Delta z_\theta\,dx &= \int_{\partial B}\alpha z_\theta(\nabla z_\theta\cdot\nu)\,dx - \int_B(\nabla\alpha\cdot\nabla z_\theta)z_\theta\,dx - \int_B \alpha\,|\nabla z_\theta|^2\,dx\\
&\le \int_{\partial B}\alpha z_\theta(\nabla z_\theta\cdot\nu)\,dx + \frac{1}{2\alpha_*}\int_B|\nabla\alpha|^2\,z_\theta^2\,dx \qquad (4.7)\\
&\quad + \frac{\alpha_*}{2}\int_B|\nabla z_\theta|^2\,dx - \alpha_*\int_B|\nabla z_\theta|^2\,dx\\
&= \int_{\partial B}\alpha z_\theta(\nabla z_\theta\cdot\nu)\,dx + \frac{1}{2\alpha_*}\int_B|\nabla\alpha|^2\,z_\theta^2\,dx - \frac{\alpha_*}{2}\int_B|\nabla z_\theta|^2\,dx\\
&\le \int_{\partial B}\alpha z_\theta(\nabla z_\theta\cdot\nu)\,dx + \frac{\|\nabla\alpha\|^2_{L^\infty}}{2\alpha_*}\int_B z_\theta^2\,dx - \frac{\alpha_*}{2}\int_B|\nabla z_\theta|^2\,dx,
\end{aligned}
$$

where ν is the external normal to ∂B and in the case $n=1$, $\partial B = \{x_1, x_2\}$, $x_1 < x_2$, we mean

$$\int_{\partial B}\alpha z_\theta(\nabla z_\theta\cdot\nu)\,dx = \alpha(\theta,t,x_2)z_\theta(t,x_2)z_{\theta,x}(t,x_2) - \alpha(\theta,t,x_1)z_\theta(t,x_1)z_{\theta,x}(t,x_1).$$

Substituting (4.4), (4.5), (4.6), (4.7) in (4.3) we obtain

$$
\begin{aligned}
\frac{d}{dt}\int_B \frac{1}{2}z_\theta^2(t,x)\,dx &\le -\frac{\alpha_*}{4}\int_B|\nabla z_\theta|^2\,dx + \int_{\partial B}\alpha z_\theta(\nabla z_\theta\cdot\nu)\,dx\\
&\quad + \left(K_8 + \frac{K_7^2}{\alpha_*} + \frac{\|\nabla\alpha\|^2_{L^\infty}}{2\alpha_*}\right)\int_B z_\theta^2\,dx \qquad (4.8)\\
&\quad + \frac{K_9}{2}|B|\ \|\sigma\|_{L^\infty(\mathcal{R})} + C_1^2\,|B|\,t^2\|\sigma\|_{L^\infty(\mathcal{R})}.
\end{aligned}
$$

By Lemma 4.1 and the assumptions on B,

$$-\int_B|\nabla z_\theta|^2\,dx \le -\frac{1}{\Lambda_0\,|B|^{2/n}}\int_B z_\theta^2\,dx + \frac{1}{|B|^{1/n}}\int_{\partial B}z_\theta^2\,dx, \qquad (4.9)$$

so by Lemma 3.4, (4.8) and (4.9),

$$
\begin{aligned}
\frac{d}{dt}\int_B\frac{1}{2}z_\theta^2(t,x)\,dx &\le \left(K_8 + \frac{K_7^2}{\alpha_*} + \frac{\|\nabla\alpha\|^2_{L^\infty}}{2\alpha_*} - \frac{\alpha_*}{4\Lambda_0\,|B|^{2/n}}\right)\int_B z_\theta^2\,dx\\
&\quad + \frac{K_9\,|B|}{2}\|\sigma\|_{L^\infty(\mathcal{R})} + C_1^2\,|B|\,t^2\|\sigma\|_{L^\infty(\mathcal{R})}\\
&\quad + \int_{\partial B}\alpha z_\theta(\nabla z_\theta\cdot\nu)\,dx + \frac{\alpha_*}{4\,|B|^{1/n}}\int_{\partial B}z_\theta^2\,dx \qquad (4.10)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(K_8 + \frac{K_7^2}{\alpha_*} + \frac{\|\nabla\alpha\|_{L^\infty}^2}{2\alpha_*} - \frac{\alpha_*}{4\Lambda_0 \left|B\right|^{2/n}}\right)\int_B z_\theta^2\,dx\\
&\quad + \frac{K_9\left|B\right|}{2}\|\sigma\|_{L^\infty(\mathcal{R})} + C_1^2\left|B\right| t^2\|\sigma\|_{L^\infty(\mathcal{R})}\\
&\quad + \int_{\partial B}\alpha z_\theta(\nabla z_\theta\cdot\nu)\,dx + \frac{\alpha_* C_1^2\,t^2}{2\left|B\right|^{1/n}} + \frac{\alpha'\|\varphi-\psi\|_{L^\infty(\mathbb{R}^n)}^2}{2\left|B\right|^{1/n}},
\end{aligned}
$$

for some constant $\alpha' > 0$ assuming that, say, e.g., $|\partial B| \leq 1$. We will eventually choose $|B| < \delta < 1$ sufficiently small (maybe dependent on $\|\sigma\|_{L^\infty(\mathcal{R})}$) and Λ sufficiently large (independent of $\|\sigma\|_{L^\infty(\mathcal{R})}$) so that

$$
C_1^2\ |B|\ t^2\|\sigma\|_{L^\infty(\mathcal{R})} + \frac{\alpha_* C_1^2 t^2}{2\left|B\right|^{1/n}} \leq \frac{\Lambda t^2}{2\left|B\right|^{1/n}}\|\sigma\|_{L^\infty(\mathcal{R})}. \tag{4.11}
$$

Furthermore,

$$
\frac{K_9\left|B\right|}{2}\left\|\sigma\right\|_{L^\infty(\mathcal{R})} + \int_{\partial B}\alpha z_\theta(\nabla z_\theta\cdot\nu)\,dx \leq \frac{\Lambda}{2}\left|B\right|\left\|\sigma\right\|_{L^\infty(\mathcal{R})}.
$$

There exists $\omega > 0$ (independent of $\|\sigma\|_{L^\infty(\mathcal{R})}$) such that

$$
\frac{\alpha_*}{4\Lambda_0\left|B\right|^{2/n}} - K_8 - \frac{K_7^2}{\alpha_*} - \frac{\|\nabla\alpha\|_{L^\infty}^2}{2\alpha_*} \geq \frac{\omega}{2\left|B\right|^{2/n}}. \tag{4.12}
$$

Substituting (4.11) and (4.12) in (4.10), we have

$$
\begin{aligned}
\frac{d}{dt}\int_B z_\theta^2(t,x)\,dx &\leq -\frac{\omega}{\left|B\right|^{2/n}}\int_B z_\theta^2(t,x)\,dx + \Lambda\left|B\right|\ \left\|\sigma\right\|_{L^\infty(\mathcal{R})}\\
&\quad + \frac{\Lambda t^2}{\left|B\right|^{1/n}}\left\|\sigma\right\|_{L^\infty(\mathcal{R})} + \frac{\alpha'}{\left|B\right|^{1/n}}\left\|\varphi-\psi\right\|_{L^\infty(\mathbb{R}^n)}^2.
\end{aligned} \tag{4.13}
$$

By the Gronwall inequality and (3.8), we have

$$
\begin{aligned}
\int_B z_\theta^2(t,x)\,dx &\leq \exp\big(-\frac{\omega t}{\left|B\right|^{2/n}}\big)\int_B z_\theta^2(0,x)\,dx\\
&\quad + \Lambda\exp\big(-\frac{\omega t}{\left|B\right|^{2/n}}\big)\int_0^t \exp\big(\frac{\omega\tau}{\left|B\right|^{2/n}}\big)\left|B\right|\ \|\sigma\|_{L^\infty(\mathcal{R})}d\tau\\
&\quad + \exp\big(-\frac{\omega t}{\left|B\right|^{2/n}}\big)\int_0^t \exp\big(\frac{\omega\tau}{\left|B\right|^{2/n}}\big)\frac{\Lambda\tau^2}{\left|B\right|^{1/n}}\|\sigma\|_{L^\infty(\mathcal{R})}d\tau\\
&\quad + \exp\big(-\frac{\omega t}{\left|B\right|^{2/n}}\big)\int_0^t \exp\big(\frac{\omega\tau}{\left|B\right|^{2/n}}\big)\frac{\alpha'}{\left|B\right|^{1/n}}\left\|\varphi-\psi\right\|_{L^\infty(\mathbb{R}^n)}^2 d\tau\\
&\leq \exp\big(-\frac{\omega t}{\left|B\right|^{2/n}}\big)\int_B \big(\varphi(x)-\psi(x)\big)^2\,dx \qquad (4.14)\\
&\quad + \Lambda\frac{\left|B\right|^{1+2/n}}{\omega}\|\sigma\|_{L^\infty(\mathcal{R})}\left(1-\exp\big(-\frac{\omega t}{\left|B\right|^{2/n}}\big)\right)\\
&\quad + \frac{\Lambda t^2}{\omega}\|\sigma\|_{L^\infty(\mathcal{R})}\left|B\right|^{1/n}\left(1-\exp\big(-\frac{\omega t}{\left|B\right|^{2/n}}\big)\right)
\end{aligned}
$$

$$+\frac{\alpha'}{\omega}\,\|\varphi-\psi\|^2_{L^\infty(\mathbb{R}^n)}\,|B|^{1/n}\left(1-\exp\big(-\frac{\omega t}{|B|^{2/n}}\big)\right).$$

Observe that,

$$\frac{|B|^{2/n}}{\omega}\left(1-\exp\big(-\frac{\omega t}{|B|^{2/n}}\big)\right)\le t,\quad 1-\exp\big(-\frac{\omega t}{|B|^{2/n}}\big)\le 1,$$
$$\exp\big(-\frac{\omega t}{|B|^{2/n}}\big)\le 1,\qquad t\ge 0,$$

and, by (2.3), (2.4) and Remark 3.1,

$$\begin{aligned}\|\sigma\|_{L^\infty(\mathcal{R})}\le K_0\big(&\|a-b\|_{L^\infty(\mathcal{R})}+\|\nabla_x\cdot f-\nabla_x\cdot g\|_{L^\infty(\mathcal{R})}\\ &+\|f_u-g_u\|_{L^\infty(\mathcal{R})}+\|h-k\|_{L^\infty(\mathcal{R})}\big),\qquad 0\le t\le T,\end{aligned}$$

for some positive constant K_0, then, from (4.14) and since $|B|<1$,

$$\begin{aligned}\int_B z_\theta^2(t,x)\,dx\le&\int_B\big(\varphi(x)-\psi(x)\big)^2\,dx\\ &+K_0\Lambda\left(1+\frac{1}{\omega}\right)|B|^{1/n}\;t^2\big(\|a-b\|_{L^\infty(\mathcal{R})}+\|\nabla_x\cdot f-\nabla_x\cdot g\|_{L^\infty(\mathcal{R})}\\ &+\|f_u-g_u\|_{L^\infty(\mathcal{R})}+\|h-k\|_{L^\infty(\mathcal{R})}\big)+\frac{\alpha'\,|B|^{1/n}}{\omega}\,\|\varphi-\psi\|^2_{L^\infty(\mathbb{R}^n)}\\ \le&\left(|B|+\frac{\alpha'}{\omega}\,|B|^{1/n}\right)\|\varphi-\psi\|^2_{L^\infty(\mathbb{R}^n)}\\ &+K_0\Lambda\left(1+\frac{1}{\omega}\right)|B|^{1/n}\;t^2\big(\|a-b\|_{L^\infty(\mathcal{R})}+\|\nabla_x\cdot f-\nabla_x\cdot g\|_{L^\infty(\mathcal{R})}\\ &\quad+\|f_u-g_u\|_{L^\infty(\mathcal{R})}+\|h-k\|_{L^\infty(\mathcal{R})}\big).\end{aligned}\tag{4.15}$$

Let now $\tilde{E}\supset E$ be a connected set such that interior of $\tilde{E}$ contains the closure of E, $\mathrm{dist}(\partial\tilde{E},\partial E)>0$, and $|\tilde{E}|=2\,|E|$. Since the closure of E is compact, we can cover it with finitely many balls $B_1,\dots,B_m\subset\mathbb{R}^n$, that is, $E\subset\cup_jB_j$. We may choose the balls such that $\cup_jB_j$ is contained in the interior of $\tilde{E}$, and thus

$$|\cup_jB_j|\le\sum_{j=1}^k|B_j|\le|\tilde{E}|=2\,|E|\,.$$

We assume that both $|\partial B_j|\le 1$ and $|B_j|\le\delta<1$. Thus the result (4.15) holds and we may sum the inequality over all balls $B_1,\dots,B_m\subset\mathbb{R}^n$, which yields

$$\begin{aligned}\int_E z_\theta^2(t,x)\,dx\le C&\left(|E|+|E|^{1/n}\right)\|\varphi-\psi\|^2_{L^\infty(\mathbb{R}^n)}\\ &+C\,|E|^{1/n}\;t^2\big(\|a-b\|_{L^\infty(\mathcal{R})}+\|\nabla_x\cdot f-\nabla_x\cdot g\|_{L^\infty(\mathcal{R})}\\ &\quad+\|f_u-g_u\|_{L^\infty(\mathcal{R})}+\|h-k\|_{L^\infty(\mathcal{R})}\big)\end{aligned}\tag{4.16}$$

which proves (4.2). □

This proves the following result.

Theorem 4.3. *Fix $T>0$. Let $u=u(t,x)$ and $v=v(t,x)$ be the classical solution of (2.1) and (2.2), respectively, with $a=a(t,x,y,q)$ and $b=b(t,x,y,q)$ satisfying*

$(\mathcal{H}_1)$, $f = f(t,x,y)$, $g = g(t,x,y)$, $h = h(t,x,y,q)$, and $k = k(t,x,y,q)$ satisfying $(\mathcal{H}_2)$, and φ and ψ satisfying $(\mathcal{H}_3)$. Then there exists a positive constant C depending only on T, n, a_*, a^*, b_*, b^*, k_1, k_2, and k_3 such that

$$\begin{aligned}\|u(t,\cdot)-v(t,\cdot)\|_{L^2(E)} \le C\big(|E|^{1/2n}+|E|^{1/2}\big)\|\varphi-\psi\|_{L^\infty(\mathbb{R}^n)}\\ + C\,t\Big(\|a-b\|_{L^\infty(\mathcal{R}_0)}+\|\nabla_x\cdot f-\nabla_x\cdot g\|_{L^\infty(\mathcal{R})}\\ +\|f_u-g_u\|_{L^\infty(\mathcal{R})}+\|h-k\|_{L^\infty(\mathcal{R}_0)}\Big)^{1/2}|E|^{1/2n},\end{aligned} \tag{4.17}$$

for all $0\le t\le T$ with $\mathcal{R} := [0,T]\times E\times[-K_1,K_1]$, $\mathcal{R}_0 := [0,T]\times E\times[-K_1,K_1]\times[-K_2,K_2]$ where $E\subset\mathbb{R}^n$ is bounded connected set with Lipschitz boundary.

Proof. Direct consequence of (3.2), (3.4) and Lemmas 4.2, 5.1, and 5.2. □

5. Estimates in $L^p(E)$

We want to extend the estimate of Theorem 4.3 to general p.

Lemma 5.1 (Case $\mathbf{1\le p<2}$). *Assume $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and $(\mathcal{H}_3)$. There exists a positive constant C_3 depending only on T, n, a_*, a^*, b_*, b^*, k_1, k_2 and k_3 such that*

$$\begin{aligned}\|z_\theta(t,\cdot)\|_{L^p(E)} \le C_3\left(|E|^{(2-p)/(2p)+1/2n}+|E|^{1/p}\right)\|\varphi-\psi\|_{L^\infty(\mathbb{R}^n)}\\ + C_3\,t\Big(\|a-b\|_{L^\infty(\mathcal{R}_0)}+\|\nabla_x\cdot f-\nabla_x\cdot g\|_{L^\infty(\mathcal{R})}\\ +\|f_u-g_u\|_{L^\infty(\mathcal{R})}+\|h-k\|_{L^\infty(\mathcal{R}_0)}\Big)^{1/2}|E|^{(2-p)/(2p)+1/2n},\end{aligned} \tag{5.1}$$

for each $0\le t\le T$, $E\subset\mathbb{R}^n$ bounded connected set with Lipschitz boundary, $0\le\theta\le1$ and $1\le p<2$.

Proof. By the Hölder inequality,

$$\begin{aligned}\|z_\theta(t,\cdot)\|^p_{L^p(E)} &= \int_E z_\theta^p(t,x)\,dx\\ &\le |E|^{1/q'}\,\|z_\theta^p(t,\cdot)\|_{L^q(E)}\\ &= |E|^{1/q'}\left(\int_E z_\theta^{pq}(t,x)\,dx\right)^{1/q},\end{aligned} \tag{5.2}$$

with

$$q := \frac{2}{p}, \qquad q' := \frac{2}{2-p}.$$

So, by (5.2),

$$\begin{aligned}\|z_\theta(t,\cdot)\|^p_{L^p(E)} &\le |E|^{(2-p)/2}\left(\int_E z_\theta^2(t,x)\,dx\right)^{p/2}\\ &\le |E|^{(2-p)/2}\,\|z_\theta(t,\cdot)\|^p_{L^2(E)},\end{aligned}$$

then, by Lemma 4.2,

$$\begin{aligned}\|z_\theta(t,\cdot)\|_{L^p(E)} \le C_2\left(|E|^{(2-p)/(2p)+1/2n}+|E|^{1/p}\right)\|\varphi-\psi\|_{L^\infty(\mathbb{R}^n)}\\ + C_3t\Big(\|a-b\|_{L^\infty(\mathcal{R})}+\|\nabla f-\nabla g\|_{L^\infty(\mathcal{R})}\end{aligned}$$

$$+ \|f_u - g_u\|_{L^\infty(\mathcal{R})} + \|h-k\|_{L^\infty(\mathcal{R})}\Big)^{1/2} |E|^{(2-p)/(2p)+1/2n},$$

This concludes the proof. □

Lemma 5.2 (Case p > 2). *Assume $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and $(\mathcal{H}_3)$. There exists a positive constant C_4 depending only on T, n, a_*, a^*, b_*, b^*, k_1, k_2 and k_3 such that*

$$\begin{aligned}
\|z_\theta(t,\cdot)\|_{L^p(E)} &\le C_4(1+t^{(p-2)/p})\big(|E|^{1/np} + |E|^{1/p}\big)\|\varphi-\psi\|^{2/p}_{L^\infty(\mathbb{R}^n)} \\
&\quad + C_4(t+t^{2/p})\Big(\|a-b\|_{L^\infty(\mathcal{R})} + \|\nabla_x\cdot f - \nabla_x \cdot g\|_{L^\infty(\mathcal{R})} \\
&\quad + \|f_u - g_u\|_{L^\infty(\mathcal{R})} + \|h-k\|_{L^\infty(\mathcal{R})}\Big)^{1/p} |E|^{1/(np)},
\end{aligned} \tag{5.3}$$

for each $0 \le t \le T$, $E \subset \mathbb{R}^n$ bounded connected set with Lipschitz boundary, $0 \le \theta \le 1$ and $2 < p < \infty$.

Proof. Observe that

$$\begin{aligned}
\|z_\theta(t,\cdot)\|^p_{L^p(E)} &= \int_E z_\theta^p(t,x)\,dx \\
&\le \|z_\theta(t,\cdot)\|^{p-2}_{L^\infty(\mathbb{R}^n)} \int_E z_\theta^2(t,x)\,dx \\
&= \|z_\theta(t,\cdot)\|^{p-2}_{L^\infty(\mathbb{R}^n)} \|z_\theta(t,\cdot)\|^2_{L^2(E)}.
\end{aligned}$$

Since $2/p$, $(p-2)/p < 1$, by Lemmas 3.4 and 4.2, we have

$$\begin{aligned}
\|z_\theta(t,\cdot)\|_{L^p(E)} &\le \|z_\theta(t,\cdot)\|^{(p-2)/p}_{L^\infty(\mathbb{R}^n)} \|z_\theta(t,\cdot)\|^{2/p}_{L^2(E)} \\
&\le \Big(C_1^{(p-2)/p}\, t^{(p-2)/p} + \|\varphi-\psi\|^{(p-2)/p}_{L^\infty(\mathbb{R}^n)}\Big) \\
&\quad \times \Big[C_2^{2/p}\big(|E|^{1/np} + |E|^{1/p}\big)\|\varphi-\psi\|^{2/p}_{L^\infty(\mathbb{R}^n)} \\
&\quad + C_2^{2/p}\, t^{2/p}\, |E|^{1/np} \Big(\|a-b\|_{L^\infty(\mathcal{R})} + \|\nabla_x\cdot f - \nabla_x\cdot g\|_{L^\infty(\mathcal{R})} \\
&\quad + \|f_u - g_u\|_{L^\infty(\mathcal{R})} + \|h-k\|_{L^\infty(\mathcal{R})}\Big)^{1/p}\Big] \\
&\le \Big(C_1^{(p-2)/p}\, t^{(p-2)/p} + k_4\Big) \\
&\quad \times \Big[C_2^{2/p}\big(|E|^{1/np} + |E|^{1/p}\big)\|\varphi-\psi\|^{2/p}_{L^\infty(\mathbb{R}^n)} \\
&\quad + C_2^{2/p}\, t^{2/p}\, |E|^{1/np} \Big(\|a-b\|_{L^\infty(\mathcal{R})} + \|\nabla_x\cdot f - \nabla_x\cdot g\|_{L^\infty(\mathcal{R})} \\
&\quad + \|f_u - g_u\|_{L^\infty(\mathcal{R})} + \|h-k\|_{L^\infty(\mathcal{R})}\Big)^{1/p}\Big] \\
&= \Big(C_1^{(p-2)/p}\, t^{(p-2)/p} + k_4\Big) C_2^{2/p}\big(|E|^{1/np} + |E|^{1/p}\big) \\
&\quad \times \|\varphi-\psi\|^{2/p}_{L^\infty(\mathbb{R}^n)} + C_2^{2/p}\Big(C_1^{(p-2)/p}\, t + k_4 t^{2/p}\Big)|E|^{1/np} \\
&\quad \times \Big(\|a-b\|_{L^\infty(\mathcal{R})} + \|\nabla_x\cdot f - \nabla_x\cdot g\|_{L^\infty(\mathcal{R})} \\
&\quad + \|f_u - g_u\|_{L^\infty(\mathcal{R})} + \|h-k\|_{L^\infty(\mathcal{R})}\Big)^{1/p},
\end{aligned}$$

where k_4 is a positive constant such that

$$(2k_3)^{(p-2)/p} \le k_4, \qquad 2 < p < \infty.$$

Since the maps

$$2 < p < \infty \longmapsto C_1^{(p-2)/p},\, C_2^{2/p}$$

are bounded the proof is done. □

The following theorem summarizes the result in Theorem 4.3 with the extension to general p.

Theorem 5.3. *Fix $T > 0$. Let $u = u(t,x)$ and $v = v(t,x)$ be the classical solution of (2.1) and (2.2), respectively, with $a = a(t,x,y,q)$ and $b = b(t,x,y,q)$ satisfying $(\mathcal{H}_1)$, $f = f(t,x,y)$, $g = g(t,x,y)$, $h = h(t,x,y,q)$, and $k = k(t,x,y,q)$ satisfying $(\mathcal{H}_2)$, and φ and ψ satisfying $(\mathcal{H}_3)$. Then there exists a positive constant C depending only on T, n, a_*, a^*, b_*, b^*, k_1, k_2, and k_3 such that*

$$\begin{aligned} \|u(t,\cdot) - v(t,\cdot)\|_{L^p(E)} \le A_E(t)\|\varphi - \psi\|^{2\rho_p}_{L^\infty(\mathbb{R}^n)} & \\ + B(t)\Big(\|a-b\|_{L^\infty(\mathcal{R})} + \|\nabla_x \cdot f - \nabla_x \cdot g\|_{L^\infty(\mathcal{R})} & \\ + \|f_u - g_u\|_{L^\infty(\mathcal{R})} + \|h-k\|_{L^\infty(\mathcal{R})}\Big)^{\rho_p} |E|^{\eta_p}, & \end{aligned} \tag{5.4}$$

with $\mathcal{R} := [0,T] \times E \times [-K_1, K_1]$, $\mathcal{R}_0 := [0,T] \times E \times [-K_1, K_1] \times [-K_2, K_2]$. Here

$$\rho_p := \begin{cases} \frac{1}{2}, & \text{if } 1 \le p \le 2, \\ \frac{1}{p}, & \text{if } 2 < p < \infty, \end{cases} \qquad \eta_p := \begin{cases} \frac{2-p}{2p} + \frac{1}{2n}, & \text{if } 1 \le p \le 2, \\ \frac{1}{np}, & \text{if } 2 < p < \infty, \end{cases}$$

$$A_E(t) := C \begin{cases} (|E|^{(2-p)/2p+1/2n} + |E|^{1/p}), & \text{if } 1 \le p \le 2, \\ (1 + t^{(p-2)/p})(|E|^{1/np} + |E|^{1/p}), & \text{if } 2 < p < \infty, \end{cases}$$

$$B(t) := C \begin{cases} t, & \text{if } 1 \le p \le 2, \\ (t + t^{2/p}), & \text{if } 2 < p < \infty, \end{cases}$$

for all $0 \le t \le T$, where $E \subset \mathbb{R}^n$ is bounded connected set with Lipschitz boundary and $1 \le p < \infty$.

Proof. Direct consequence of (3.2), (3.4) and Lemmas 4.2, 5.1, 5.2. □

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