Third order finite volume evolution Galerkin (FVEG) methods for two-dimensional wave equation system

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Abstract

The subject of the paper is the derivation and analysis of third order finite volume evolution Galerkin schemes for the two-dimensional wave equation system. To achieve this the first order approximate evolution operator is considered. A recovery stage is carried out at each level to generate a piecewise polynomial approximation $\tilde{\mathbf{U}}^n = R_h \mathbf{U}^n \in S_h^3$ from the piecewise constant $\mathbf{U}^n \in S_h^0$, to feed into the calculation of the fluxes. We estimate the truncation error and give numerical examples to demonstrate the higher order behaviour of the scheme for smooth solutions.

Key words: hyperbolic systems, wave equation, evolution Galerkin schemes, recovery stage, finite volume.

1 Introduction

Evolution Galerkin methods (EG-methods) were proposed to approximate evolutionary problems of first order hyperbolic systems. In [10] Ostkamp derived such schemes for the approximation of the solution of the wave equation system as well as the Euler equations of gas dynamics in two space dimensions. In [3], [6] Lukáčová, Morton and Warnecke constructed further EG-schemes, namely EG1, EG2 and EG3. In [14] Zahaykah derived the approximate evolution operator of the solution for the wave equation system in three space dimensions. Using these results new 2D EG schemes, namely FREG, SREG and EG4 schemes were derived, cf. [14]. These methods were applied to the Maxwell as well as to the linearized Euler equations, see [9]. The second order finite volume EG-methods have been introduced and studied in [5], [7], [8].

The main objective of this paper is the analysis of third order EG schemes for the wave equation system in two space dimensions. The bicharacteristics theory of linear hyperbolic systems is used to obtain the exact integral equations which are equivalent to the differential form of the system under consideration. Applying certain types of quadratures we derive approximate evolution operators. Projecting these operators on a finite element space of piecewise constants we end with first order evolution Galerkin schemes. To increase the accuracy we carry out a recovery stage before the evolutionary step in the finite volume construction.

The outline of this paper is as follows: in the next section we present the general theory that is used to derive the exact integral equations. In Section 3 we introduce the evolution Galerkin schemes. The exact integral equations as well as the approximate evolution operators for the wave equation system in 2D are given in Section 4. The analysis of third order schemes

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is given in Section 5. Finally in Section 6 numerical tests, which demonstrate third order behaviour of our scheme, are presented.

2 General Theory

In this section we recall the exact integral equations for a general linear hyperbolic system using the concept of bicharacteristics. The general form of the linear hyperbolic system is given as

$$\mathbf{U}_t + \sum_{k=1}^{d} \mathbf{A}_k \mathbf{U}_{x_k} = 0, \;\; \mathbf{x} = (x_1, \ldots, x_d)^T \in \mathbb{R}^d \tag{2.1}$$

where the coefficient matrices $\mathbf{A}_k, k = 1, ..., d$ are elements of $\mathbb{R}^{p\times p}$ and the dependent variables are $\mathbf{U} = (u_1, ..., u_p)^T = \mathbf{U}(\mathbf{x}, t) \in \mathbb{R}^p$. Let $\mathbf{A}(\mathbf{n}) = \sum_{k=1}^{d} n_k \mathbf{A}_k$ be the so-called **matrix pencil** where $\mathbf{n} = (n_1, ..., n_d)^T$ is a directional vector in $\mathbb{R}^d$. Since system (2.1) is hyperbolic then the matrix $\mathbf{A}(\mathbf{n})$ has p real eigenvalues $\lambda_k,\ k = 1, ..., p$ and p corresponding linearly independent right eigenvectors $\mathbf{r}_k = \mathbf{r}_k(\mathbf{n}),\ k = 1, ..., p$. Let $\mathbf{R} = [\mathbf{r}_1|\mathbf{r}_2|...|\mathbf{r}_p]$ be the matrix of right eigenvectors. Then we can define the characteristic variable $\mathbf{W} = \mathbf{W}(\mathbf{n})$ as $\partial\mathbf{W}(\mathbf{n}) = \mathbf{R}^{-1}\partial\mathbf{U}$, where $\partial\mathbf{W}$, $\partial\mathbf{U}$ denote the Jacobian matrices of $\mathbf{W}(\mathbf{x}, t)$, $\mathbf{U}(\mathbf{x}, t)$, respectively; i.e. $\partial\mathbf{W}(\mathbf{n})/\partial t = \mathbf{R}^{-1}\partial\mathbf{U}/\partial t$ and $\partial\mathbf{W}(\mathbf{n})/\partial x_k = \mathbf{R}^{-1}\partial\mathbf{U}/\partial x_k,\ k = 1, 2, \ldots, d$.
Since system (2.1) has constant coefficient matrices $\mathbf{A}_k$ we have $\mathbf{W} = \mathbf{R}^{-1}\mathbf{U}$ or $\mathbf{U} = \mathbf{R}\,\mathbf{W}$. Multiplying equation (2.1) by $\mathbf{R}^{-1}$ from the left we get

$$\mathbf{R}^{-1}\mathbf{U}_t + \sum_{k=1}^{d} \mathbf{R}^{-1}\mathbf{A}_k\mathbf{R}\,\mathbf{R}^{-1}\mathbf{U}_{x_k} = 0. \tag{2.2}$$

Let $\mathbf{B}_k = \mathbf{R}^{-1}\mathbf{A}_k\mathbf{R} = (b_{ij}^k)_{i,j=1}^p$, where $k = 1, 2, ..., d$ then equation (2.2) can be rewritten in the following form

$$\mathbf{W}_t + \sum_{k=1}^{d} \mathbf{B}_k \mathbf{W}_{x_k} = 0.$$

If we introduce the decomposition $\mathbf{B}_k$=$\mathbf{D}_k$+$\mathbf{B}'_k$, where $\mathbf{D}_k$ contains the diagonal part of the matrix $\mathbf{B}_k$, then we get

$$\mathbf{W}_t + \sum_{k=1}^{d} \mathbf{D}_k \mathbf{W}_{x_k} = -\sum_{k=1}^{d} \mathbf{B}'_k \mathbf{W}_{x_k} =: \mathbf{S}. \tag{2.3}$$

The i-th bicharacteristic corresponding to the i-th equation of (2.3) is defined by

$$\frac{d\mathbf{x}_i}{d\tilde{t}} = \mathbf{b}_{ii}(\mathbf{n}) = (b_{ii}^1, b_{ii}^2, ..., b_{ii}^d)^T,$$

where $i = 1, ..., p$. Here b_{ii}^k are the diagonal entries of the matrix $\mathbf{B}_k$, $k = 1, ..., d$, $i = 1, ..., p$. We consider the bicharacteristics backwards in time from $t + \Delta t$ to t, where Δt is a suitable time step. Let the initial conditions be $\mathbf{x}_i(t + \Delta t, \mathbf{n}) = \mathbf{x}$ for all $\mathbf{n} \in \mathbb{R}^d$ and $i = 1, ..., p$. Then the bicharacteristics are given as $\mathbf{x}_i(\tilde{t}, \mathbf{n}) = \mathbf{x} - \mathbf{b}_{ii}(\mathbf{n})(t + \Delta t - \tilde{t})$, $\tilde{t} \in [t, t + \Delta t]$.

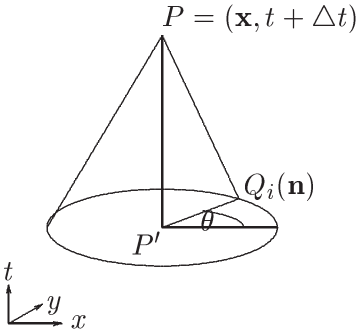

Figure 1: Bicharacteristics along the Mach cone through P and $Q_i(\mathbf{n})$, $d = 2$.

We will integrate the i-th equation of the system (2.3) from the point P down to the point $Q_i(\mathbf{n})$, where the bicharacteristic hits the basic plane, see Figure 1. The point $P \equiv (\mathbf{x}, t+\Delta t) \in \mathbb{R}^p \times \mathbb{R}_+$ is taken to be a fixed point, while $Q_i(\mathbf{n}) = (\mathbf{x}_i(\mathbf{n}, t), t) = (\mathbf{x} - \Delta t \mathbf{b}_{ii}, t)$. Note that bicharacteristics are straight lines because the system is linear with constant coefficients. Now the i-th equation reads

$$\frac{\partial w_i}{\partial t} + \sum_{k=1}^{d} b_{ii}^k \frac{\partial w_i}{\partial x_k} = -\left(\sum_{j=1, i\neq j}^{d} \left(b_{ij}^1 \frac{\partial w_j}{\partial x_1} + b_{ij}^2 \frac{\partial w_j}{\partial x_2} + \ldots + b_{ij}^d \frac{\partial w_j}{\partial x_d} \right) \right) = S_i, \tag{2.4}$$

Taking a vector $\sigma_i = (b_{ii}^1, b_{ii}^2, \ldots, b_{ii}^d, 1)$, we can define the directional derivative

$$\frac{dw_i}{d\sigma_i} = \left(\frac{\partial w_i}{\partial x_1}, \frac{\partial w_i}{\partial x_2}, \ldots, \frac{\partial w_i}{\partial x_d}, \frac{\partial w_i}{\partial t} \right) \cdot \sigma_i = \frac{\partial w_i}{\partial t} + b_{ii}^1 \frac{\partial w_i}{\partial x_1} + b_{ii}^2 \frac{\partial w_i}{\partial x_2} + \ldots + b_{ii}^d \frac{\partial w_i}{\partial x_d}.$$

Hence the i-th equation (2.4) can be rewritten as follows

$$\frac{dw_i}{d\sigma_i} = S_i = - \sum_{j=1, i\neq j}^{d} \left(b_{ij}^1 \frac{\partial w_j}{\partial x_1} + b_{ij}^2 \frac{\partial w_j}{\partial x_2} + \ldots + b_{ij}^d \frac{\partial w_j}{\partial x_d} \right).$$

Now the integration from P to $Q_i(\mathbf{n})$ gives

$$w_i(P) - w_i(Q_i(\mathbf{n})) = S_i', \tag{2.5}$$

where

$$S_i' = \int_t^{t+\Delta t} S_i(\mathbf{x}_i(\tilde{t}, \mathbf{n}), \tilde{t}, \mathbf{n}) d\tilde{t} = \int_0^{\Delta t} S_i(\mathbf{x}_i(\tau, \mathbf{n}), t + \Delta t - \tau, \mathbf{n}) \mathrm{d}\tau.$$

Multiplication of equation (2.5) by $\mathbf{R}$ from the left and $(d-1)-$dimensional integration of the variable $\mathbf{n}$ over the unit sphere O in $\mathbb{R}^d$ leads to the integral representation

$$\mathbf{U}(P) = \mathbf{U}(\mathbf{x}, t + \Delta t) = \frac{1}{|O|} \int_O \mathbf{R}(\mathbf{n}) \begin{pmatrix} w_1(Q_1(\mathbf{n}), \mathbf{n}) \\ w_2(Q_2(\mathbf{n}), \mathbf{n}) \\ w_3(Q_3(\mathbf{n}), \mathbf{n}) \\ \vdots \\ w_p(Q_p(\mathbf{n}), \mathbf{n}) \end{pmatrix} \mathrm{d}O + \tilde{\mathbf{S}}, \tag{2.6}$$

where

$$\tilde{\mathbf{S}} = (\tilde{S}_1, \tilde{S}_2, ..., \tilde{S}_p)^T = \frac{1}{|O|}\int_O \mathbf{R}(\mathbf{n})\mathbf{S}'\mathrm{d}O = \frac{1}{|O|}\int_O \int_0^{\Delta t} \mathbf{R}(\mathbf{n})\mathbf{S}(t+\Delta t - \tau, \mathbf{n})\mathrm{d}\tau\mathrm{d}O$$

and $|O|$ corresponds to the measure of the domain of integration.

3 Evolution Galerkin Schemes

For simplicity let us consider $d = 2$. Let $h > 0$ be the mesh size parameter. We construct a mesh for $\mathbb{R}^2$, which consists of the square mesh cells

$$\Omega_{kl} = \left[(k-\frac{1}{2})h, (k+\frac{1}{2})h\right] \times \left[(l-\frac{1}{2})h, (l+\frac{1}{2})h\right] = \left[x_k - \frac{h}{2}, x_k + \frac{h}{2}\right] \times \left[y_l - \frac{h}{2}, y_l + \frac{h}{2}\right],$$

where $k, l \in \mathbb{Z}$. Let us denote by $H^\kappa(\mathbb{R}^2)$ the Sobolev space of distributions with derivatives up to order κ in L^2 space, where $\kappa \in \mathbb{N}$. Consider the general hyperbolic system given by the equation (2.1). Let us denote by $E(s) : (H^\kappa(\mathbb{R}^2))^p \to (H^\kappa(\mathbb{R}^2))^p$ the exact evolution operator for the system (2.1), i.e.

$$\mathbf{U}(., t+s) = E(s)\mathbf{U}(., t). \tag{3.1}$$

We suppose that S_h^m is an arbitrary finite element space consisting of piecewise polynomials of order $m \geq 0$ with respect to the square mesh; $S_h^m \subset (L^1_{loc}(\mathbb{R}^2))^p$. Assume a constant time step, i.e. $t_n = n\Delta t$. Let $\mathbf{U}^n$ be an approximation in the space S_h^m to the exact solution $\mathbf{U}(., t_n)$ at time $t_n \geq 0$. We consider $E_\tau : (L^1_{loc}(\mathbb{R}^2))^p \to (H^\kappa(\mathbb{R}^2))^p$ to be a suitable approximate evolution operator to the exact evolution operator $E(\tau)$, cf. (3.1). In practice we will use restrictions of E_τ to the subspace S_h^m for $m \geq 0$. We denote by $P_h : (H^\kappa(\mathbb{R}^2))^p \to S_h^m$ the L^2-projection onto S_h^m. Now we can define the general class of evolution Galerkin methods.

Definition 3.2 *Starting from some initial data $\mathbf{U}^0 \in S_h^m$ at time $t = 0$, an evolution Galerkin method (EG-method) is recursively defined by means of*

$$\mathbf{U}^{n+1} = P_h E_\tau \mathbf{U}^n. \tag{3.3}$$

For first order methods we can limit our consideration to the case where S_h^0 is composed of piecewise constant (nonconforming) finite elements and define P_h by the integral averages in the following way

$$P_h\mathbf{U}^n|_{\Omega_{kl}} = \frac{1}{|\Omega_{kl}|}\int_{\Omega_{kl}} \mathbf{U}(x, y, t_n)\mathrm{d}x\mathrm{d}y.$$

Higher order accuracy can be obtained either by taking $m > 0$ with a suitable projection onto S_h^m, or by inserting a recovery stage R_h before the evolution step in equation (3.3) to give

$$\mathbf{U}^{n+1} = P_h E_\tau R_h \mathbf{U}^n. \tag{3.4}$$

Here we have denoted by $R_h : S_h^m \to S_h^r$ a recovery operator, $r > m \geq 0$, and considered our approximate evolution operator E_τ on S_h^r. In what follows we will limit our further considerations to the case where $m = 0$ and $r = 2$. The finite difference approach (3.3) or (3.4) involves the computation of multiple integrals and becomes quite complex for higher order polynomials. To avoid this we will consider higher order evolution Galerkin schemes based on the finite volume formulation instead.

Definition 3.5 *Starting from some initial data* $\mathbf{U}^0 \in S_h^m$*, the finite volume evolution Galerkin method (FVEG) is recursively defined by means of*

$$\mathbf{U}^{n+1} = \mathbf{U}^n - \frac{1}{h}\int_0^{\Delta t} \sum_{j=1}^{2} \delta_{x_j} \mathbf{f}_j(\tilde{\mathbf{U}}^{n+\frac{\tau}{\Delta t}})\, d\tau, \tag{3.6}$$

where $\delta_{x_j}\mathbf{f}_j(\tilde{\mathbf{U}}^{n+\frac{\tau}{\Delta t}})$ *represents an approximation to the edge flux difference and* δ_x *is defined by* $\delta_x v = v(x+\frac{h}{2}) - v(x-\frac{h}{2})$*. The cell boundary value* $\tilde{\mathbf{U}}^{n+\frac{\tau}{\Delta t}}$ *is evolved using the approximate evolution operator* E_τ *to* $t_n+\tau$ *and averaged along the cell boundary, i.e.*

$$\tilde{\mathbf{U}}^{n+\frac{\tau}{\Delta t}} = \sum_{k,l\in\mathbb{Z}} \left(\frac{1}{|\partial\Omega_{kl}|} \int_{\partial\Omega_{kl}} E_\tau R_h \mathbf{U}^n dS \right) \chi_{kl}, \tag{3.7}$$

where χ_{kl} *is the characteristic function of* $\partial\Omega_{kl}$*.*

In this formulation a first order approximation E_τ to the exact operator $E(\tau)$ yields an overall higher order update from $\mathbf{U}^n$ to $\mathbf{U}^{n+1}$. To obtain this approximation in the discrete scheme it is only necessary to carry out a recovery stage at each level to generate a piecewise polynomial approximation $\tilde{\mathbf{U}}^n = R_h\mathbf{U}^n \in S_h^r$ from the piecewise constant $\mathbf{U}^n \in S_h^0$, to feed into the calculation of the fluxes. Later in Section 5 we will show which recovery can be used to achieve a third order approximation. To close this section it is important to note that in the updating step (3.6) some numerical quadratures are used instead of the exact time integration. Similarly, to evaluate the intermediate value $\tilde{\mathbf{U}}^{n+\frac{\tau}{\Delta t}}$ in (3.7) either the two dimensional integrals along the cell-interface and around the Mach cone are evaluated exactly or by means of suitable numerical quadratures.

4 Exact Integral Equations And Approximate Evolution Operators for the Wave Equation System

We will consider the two dimensional wave equation system given as

$$\begin{aligned} \phi_t + c(u_x+v_y) &= 0 \\ u_t + c\phi_x &= 0 \\ v_t + c\phi_y &= 0, \end{aligned} \tag{4.1}$$

where c is a given constant. We denote by $\mathbf{U} = (\phi, u, v)^T$ the vector of conservative unknowns and by $\mathbf{f}_1(\mathbf{U}) = (c\,u, c\,\phi, 0)^T$, $\mathbf{f}_2(\mathbf{U}) = (c\,v, 0, c\,\phi)^T$ flux functions. Then the two dimensional wave equation system (4.1) can be rewritten in the conservation law form

$$\mathbf{U}_t + \mathbf{f}_1(\mathbf{U})_x + \mathbf{f}_2(\mathbf{U})_y = \mathbf{0}. \tag{4.2}$$

Now we recall here the exact integral equations derived in [6]. These will be used in order to construct the so-called EG4 finite difference scheme. The detailed description of the approximate evolution operators for the other EG schemes (EG1, EG2, EG3) are given in [6].
Let $P = (x, y, t+\Delta t)$, $P' = (x, y, t)$, $Q = (x + c\Delta t\cos\theta, y + c\Delta t\sin\theta, t) = (\mathbf{x} + c\Delta t\mathbf{n}(\theta), t)$ and the so-called source term be given as

$$S = c\left[u_x \sin^2\theta - (u_y+v_x)\sin\theta\cos\theta + v_y\cos^2\theta\right], \tag{4.3}$$

then we have

Exact Integral Equations:

$$\phi_P = \frac{1}{2\pi}\int_0^{2\pi} (\phi_Q - u_Q \cos\theta - v_Q \sin\theta)\, d\theta + \tilde{S}_1, \tag{4.4}$$

$$u_P = \frac{1}{2}u_{P'} + \frac{1}{2\pi}\int_0^{2\pi} (-\phi_Q \cos\theta + u_Q \cos^2\theta + v_Q \sin\theta \cos\theta)\, d\theta + \tilde{S}_2, \tag{4.5}$$

$$v_P = \frac{1}{2}v_{P'} + \frac{1}{2\pi}\int_0^{2\pi} (-\phi_Q \sin\theta + u_Q \cos\theta \sin\theta + v_Q \sin^2\theta\, d\theta + \tilde{S}_3, \tag{4.6}$$

where

$$\tilde{S}_1 = \frac{-1}{2\pi}\int_0^{2\pi}\int_0^{\triangle t} S(\mathbf{x} + c\tau\mathbf{n}(\theta), t + \Delta t - \tau, \theta)\, \mathrm{d}\tau\, \mathrm{d}\theta,$$

$$\tilde{S}_2 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\triangle t} \cos\theta S(\mathbf{x} + c\tau\mathbf{n}(\theta), t + \Delta t - \tau, \theta)\, \mathrm{d}\tau\, \mathrm{d}\theta$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\triangle t} \left[c\phi_x(\mathbf{x}, t + \Delta t - \tau)\sin^2\theta - c\phi_y(\mathbf{x}, t + \Delta t - \tau)\sin\theta\cos\theta\right]\, \mathrm{d}\tau\, \mathrm{d}\theta,$$

$$\tilde{S}_3 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\triangle t} \sin\theta S(\mathbf{x} + c\tau\mathbf{n}(\theta), t + \Delta t - \tau, \theta)\, \mathrm{d}\tau\, \mathrm{d}\theta$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\triangle t} \left[c\phi_y(\mathbf{x}, t + \Delta t - \tau)\cos^2\theta - c\phi_x(\mathbf{x}, t + \Delta t - \tau)\sin\theta\cos\theta\right]\, \mathrm{d}\tau\, \mathrm{d}\theta.$$

The above integral equations give us an implicit formulation of the solution $\mathbf{U}^{n+1}$ at the point $P = (x, y, t^{n+1})$. In order to obtain an explicit numerical scheme it is necessary to use some numerical quadratures in order to approximate the time integral from 0 to Δt. Using the backward rectangle rule leads us to an $\mathcal{O}(\Delta t^2)$ approximation of the time integrals appearing in $\tilde{S}_1$, $\tilde{S}_2$ and $\tilde{S}_3$. Further we use the following result [6, Lemma 2.1]

$$\Delta t \int_0^{2\pi} S(t, \theta)\mathrm{d}\theta = \int_0^{2\pi} (u\cos\theta + v\sin\theta)\mathrm{d}\theta.$$

This already yields the approximate evolution operator for the first variable ϕ. Similar expressions can be used for $S\cos\theta$ and $S\sin\theta$ to obtain approximations for u and v, cf. [6]. Note that in these formulae we replaced the spatial derivatives of u, v in S just by u, v themselves.

Approximate evolution operator for EG4:

$$\phi_P = \frac{1}{2\pi}\int_0^{2\pi}(\phi_Q - 2u_Q\cos\theta - 2v_Q\sin\theta)\mathrm{d}\theta + O(\Delta t^2), \tag{4.7}$$

$$u_P = \frac{1}{2\pi}\int_0^{2\pi}(-2\phi_Q\cos\theta + 2u_Q\cos^2\theta + 2v_Q\sin\theta\cos\theta)\mathrm{d}\theta + O(\Delta t^2), \tag{4.8}$$

$$v_P = \frac{1}{2\pi}\int_0^{2\pi}(-2\phi_Q\sin\theta + 2u_Q\sin\theta\cos\theta + 2v_Q\sin^2\theta)\mathrm{d}\theta + O(\Delta t^2). \tag{4.9}$$

5 Third Order Finite Volume EG-Schemes

As we mentioned in Section 3, a mechanism of obtaining higher order evolution Galerkin schemes is to use the approximate evolution operator on a finite dimensional space of piecewise polynomials of higher degree. This can be accomplished by carrying out a recovery stage before the evolutionary step. See, e.g. Sonar [11], [12], for the general theory of higher order schemes for hyperbolic systems using recovery. Second order FVEG methods were studied in [5], [7], [8]. In this section we aim to derive a third order evolution Galerkin scheme for the two dimensional wave equation system. On each mesh cell Ω_{kl} the recovery stage that we use is given by the following biquadratic polynomial

$$R_h\mathbf{U}^n\Big|_{\Omega_{kl}} = \sum_{i=0}^{2}\sum_{j=0}^{2}\frac{D^{ij}\mathbf{U}}{i!j!}(x-x_k)^i(y-y_l)^j, \qquad k,l\in\mathbb{Z}, \tag{5.1}$$

with coefficients $D^{ij}\mathbf{U} = (D^{ij}\phi, D^{ij}u, D^{ij}v) \in \mathbb{R}^3$. Here $\mathbf{U}^n$ denotes a piecewise constant approximation on the square mesh. The constant term of the recovered polynomial is adjusted in such a way that the recovery is conservative, i.e.

$$\frac{1}{h^2}\int_{\Omega_{kl}} R_h\mathbf{U}^n\mathrm{d}x\mathrm{d}y = \frac{1}{h^2}\int_{\Omega_{kl}}\mathbf{U}^n\mathrm{d}x\mathrm{d}y. \tag{5.2}$$

The recovered polynomial $R_h\mathbf{U}^n$ is used only on the cell Ω_{kl}, but uses values from the eight neighbouring cells $\Omega_{k\pm1,l\pm1}$. We also take it to satisfy the following property

$$\frac{1}{h^2}\int_{\Omega_{k\pm1,l\pm1}} R_h\mathbf{U}^n dxdy = \frac{1}{h^2}\int_{\Omega_{k\pm1,l\pm1}}\mathbf{U}^n dxdy. \tag{5.3}$$

This means that the nine coefficients $D^{ij}\mathbf{U}$ of the polynomial $R_h\mathbf{U}^n$ are determined in such a way that they maintain the nine cell averages of $\mathbf{U}^n$ on Ω_{kl} and its eight neighbors. For such a recovery it was proved by Sonar [11, Theorem 4] that on Ω_{kl}

$$\mathbf{U}(x,y,t^n) - R_h\mathbf{U}^n(x,y) = \mathcal{O}(h^3) \tag{5.4}$$

holds for any function $\mathbf{U}(.,t^n) \in C^3(\Omega_{kl})$.

In what follows we will first demonstrate that a recovery stage of the form (5.1) in (3.4) produces a third order of accuracy in the space.

Theorem 5.6 *Let* $\mathbf{U}$ *be a* C^3 *solution of the two dimensional system (4.1). Let* E_τ *be the EG4 approximate evolution operator defined in (4.7)-(4.9) and* R_h *be the recovery operator onto a finite element space of piecewise biquadratic polynomials satisfying (5.2) and (5.3). Then the resulting evolution Galerkin scheme (3.4) is of order* $\mathcal{O}(h^3)$ *in space and of order* $\mathcal{O}(\Delta t^2)$ *in time.*

Proof: Without loss of generality we just consider in detail the first component ϕ, i.e. equation (4.7). The other components may be treated analogously. Then

$$\phi_P = \frac{1}{2\pi}\int_0^{2\pi} (\phi_Q - 2\cos\theta u_Q - 2\sin\theta v_Q)\,\mathrm{d}\theta + \mathcal{O}(\Delta t^2), \tag{5.7}$$

where $P = (x, y, t+\Delta t)$ and $Q = (x + c\Delta t\cos\theta, y + c\Delta t\sin\theta, t)$. We remind that

$$\Omega_{kl} = [x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}] \times [y_{l-\frac{1}{2}}, y_{l+\frac{1}{2}}] \text{ with } x_{k\pm\frac{1}{2}} = x_k \pm \frac{h}{2},\ y_{l\pm\frac{1}{2}} = y_l \pm \frac{h}{2}.$$

Averaging over the square cell Ω_{kl} and using the recovery R_h we get

$$\phi_{kl}^{n+1} = \frac{1}{2\pi h^2}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\int_{y_{l-\frac{1}{2}}}^{y_{l+\frac{1}{2}}}\int_0^{2\pi} (R_h\,\phi_Q^n - 2\cos\theta R_h\,u_Q^n - 2\sin\theta R_h\,v_Q^n)\,\mathrm{d}\theta\,\mathrm{d}y\mathrm{d}x. \tag{5.8}$$

Here $\mathbf{U}^n$ denotes the L^2-projection of the exact solution $\mathbf{U}$ onto a piecewise constant function with respect to our square mesh, obtained by calculating the integral average on each cell. Now it is well known that given a function f of two variables x and y which is sufficiently smooth, it is possible to expand it via the following Taylor-type formula

$$f(x,y) = \sum_{i=0}^{r}\sum_{j=0}^{s} \frac{(x-x_k)^i (y-y_l)^j}{i!j!} f^{ij}(x_k, y_l) + R, \tag{5.9}$$

where $f^{ij} = \frac{\partial^{i+j} f(x_k,y_l)}{\partial x^i \partial y^j}$. The remainder R is given as

$$R = \frac{(x-x_k)^{r+1}}{(r+1)!} f^{r+1,0}(\xi, y) + \frac{(y-y_l)^{s+1}}{(s+1)!} f^{0,s+1}(x,\eta) - \frac{(x-x_k)^{r+1}(y-y_l)^{s+1}}{(r+1)!(s+1)!} f^{r+1,s+1}(\xi,\eta),$$

with ξ and η being the same in all terms in which they occur, see e.g. Stancu [13]. Let $C^{r+1,s+1}$ be a space of functions having $r+1$ and $s+1$ continuous derivatives in the $x-$, $y-$ direction, respectively. Assume that ϕ, u, $v \in C^{r+1,s+1}$ then the following integrals

$$\frac{1}{2\pi\,c\,\Delta t}\int_0^{2\pi} (\phi^{r+1,0}(\xi,y) - 2\cos\theta u^{r+1,0}(\xi,y) - 2\sin\theta v^{r+1,0}(\xi,y))_Q c\Delta t \mathrm{d}\theta,$$
$$\frac{1}{2\pi\,c\,\Delta t}\int_0^{2\pi} (\phi^{0,s+1}(x,\eta) - 2\cos\theta u^{0,s+1}(x,\eta) - 2\sin\theta v^{0,s+1}(x,\eta))_Q c\Delta t \mathrm{d}\theta,$$
$$\frac{1}{2\pi\,c\,\Delta t}\int_0^{2\pi} (\phi^{r+1,s+1}(\xi,\eta) - 2\cos\theta u^{r+1,s+1}(\xi,\eta) - 2\sin\theta v^{r+1,s+1}(\xi,\eta))_Q c\Delta t \mathrm{d}\theta,$$

are bounded. Thus, averaging ϕ_P in equation (5.7) on the square cell Ω_{kl} and approximating the functions ϕ, u and v by using the Taylor-type formula (5.9) with $r = s = 2$ we get

$$
\begin{aligned}
\phi_P \;=\; & \frac{1}{2\pi h^2}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\int_{y_{l-\frac{1}{2}}}^{y_{l+\frac{1}{2}}}\int_0^{2\pi}\left(\left(\sum_{i=0}^{2}\sum_{j=0}^{2}\frac{(x-x_k)^i(y-y_l)^j}{i!j!}\phi^{i,j}(x_k,y_l)\right)_Q\right.\\
&-2\cos\theta\left(\sum_{i=0}^{2}\sum_{j=0}^{2}\frac{(x-x_k)^i(y-y_l)^j}{i!j!}u^{i,j}(x_k,y_l)\right)_Q\\
&\left.-2\sin\theta\left(\sum_{i=0}^{2}\sum_{j=0}^{2}\frac{(x-x_k)^i(y-y_l)^j}{i!j!}v^{i,j}(x_k,y_l)\right)_Q\right)\mathrm{d}\theta\mathrm{d}y\mathrm{d}x\\
&+\mathcal{O}(h^3)+\mathcal{O}(h^3)+\mathcal{O}(h^6)+\mathcal{O}(\Delta t^2).
\end{aligned}
$$

It follows from the proof of (5.4) that

$$
|\phi^{ij} - D^{ij}\phi| = \mathcal{O}(h^3),\;\; |u^{ij} - D^{ij}u| = \mathcal{O}(h^3),\;\; |v^{ij} - D^{ij}v| = \mathcal{O}(h^3),
$$

for any $i, j = 0, 1, 2$, see [11]. Therefore, we can write

$$
\begin{aligned}
\phi_P \;=\; & \frac{1}{2\pi h^2}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\int_{y_{l-\frac{1}{2}}}^{y_{l+\frac{1}{2}}}\int_0^{2\pi}\left(\left(\sum_{i=0}^{2}\sum_{j=0}^{2}\frac{D^{ij}\phi}{i!j!}(x-x_k)^i(y-y_l)^j\right)_Q\right.\\
&-2\cos\theta\left(\sum_{i=0}^{2}\sum_{j=0}^{2}\frac{D^{ij}u}{i!j!}(x-x_k)^i(y-y_l)^j\right)_Q\\
&\left.-2\sin\theta\left(\sum_{i=0}^{2}\sum_{j=0}^{2}\frac{D^{ij}v}{i!j!}(x-x_k)^i(y-y_l)^j\right)_Q\right)\mathrm{d}\theta\mathrm{d}y\mathrm{d}x\\
&+\mathcal{O}(h^3)+\mathcal{O}(\Delta t^2).
\end{aligned}
$$

Hence $|\phi_P - \phi_{kl}^{n+1}| = \mathcal{O}(h^3) + \mathcal{O}(\Delta t^2)$. This concludes the proof of the lemma. □

Analogously we have the following theorem.

Theorem 5.10 *If we use Simpson's rule as quadrature in (3.7) and if the recovery is of the type (5.1) such that* $(R_h\mathbf{U}^n(x,y) - \mathbf{U}(x,y,t_n)) = \mathcal{O}(h^3)$ *holds for each fixed time* t_n *and* $(x,y) \in \Omega_{kl}$*, then the finite volume approximation (3.6) and (3.7) is of third order accurate in space, provided that exact solution* $\mathbf{U}$ *is from* C^3*.*

Proof: First let us consider exact integration in (3.7). Basically the integral averages over the cell Ω_{kl} are replaced by averages along each edge of the cell boundary with one variable frozen. It is easily seen that one obtains the same order of approximation as before. When Simpson's rule is taken, we have an order $\mathcal{O}(h^5)$ for the integral and therefore $\mathcal{O}(h^4)$ for the average. So the spatial order is maintained. □

Remark 5.11 The results of Theorems 5.6 and 5.10 hold also for any of the approximate evolution operators EG1, EG2, EG3 and the approximate evolution operator of Ostkamp. See, e.g.,[6], [10] for the precise definition of the operators.

6 Numerical Tests

Example 6.1

Consider the two dimensional wave equation system together with the initial data

$$\begin{aligned} \phi(x,y,0) &= -(\sin(2\pi x) + \sin(2\pi y)), \\ u(x,y,0) &= v(x,y,0) = 0, \end{aligned} \tag{6.2}$$

where $(x,y) \in [-1,1] \times [-1,1]$. The exact solution is

$$\begin{aligned} \phi(x,y,t) &= -\cos(2\pi t)(\sin(2\pi x) + \sin(2\pi y)), \\ u(x,y,t) &= \sin(2\pi t)\cos(2\pi x), \\ v(x,y,t) &= \sin(2\pi t)\cos(2\pi y). \end{aligned} \tag{6.3}$$

Let $\mathbf{U}_N(T)$ and $\mathbf{U}_N^n$ denote, respectively, the exact and the approximate solution, evaluated on a mesh with $N \times N$ cells. The following two tables show the L^2-error on the subdomain $[0,1] \times [0,1]$ and the experimental order of convergence (EOC), which is defined using two solutions compute on meshes with $N_1 \times N_1$ and $N_2 \times N_2$ cells, as follows

$$\mathrm{EOC} = \ln \frac{\|\mathbf{U}_{N_1}(T) - \mathbf{U}_{N_1}^n\|}{\|\mathbf{U}_{N_2}(T) - \mathbf{U}_{N_2}^n\|} / \ln\left(\frac{N_2}{N_1}\right).$$

The numerical experiments are carried out with the finite volume EG4 scheme (FVEG4), cf. (3.6), (3.7) for the definition of the FVEG scheme and (4.7) - (4.9) for the EG4 approximate evolution operator, which is particularly used here. The integrals in (3.6), (3.7) along time interval $[0, \Delta t]$ and along cell interfaces of Ω_{kl} are approximated by the Simpson rule. Note that in our computations the Mach cone integrals with respect to θ, which appears in approximate evolution operator (4.7)-(4.9) are evaluated exactly. We take the absolute time $T = 0.2$ and $T = 0.4$, respectively, and set the constant c for the wave equation system to 1. The last columns of Tables 1 and 2 demonstrate that the experimental order of convergence is 3. This confirms our theoretical results that the method is of third order, cf. Theorem 5.10.

N	$\|\phi(T) - \phi^n\|$	$\|u(T) - u^n\|$	$\|\mathbf{U}(T) - \mathbf{U}^n\|$	EOC
20	0.00371611457	0.01967501454	0.02807172781	
40	0.00080037813	0.00235135064	0.00342027847	3.0369
80	0.00012733030	0.00027906337	0.00041468751	3.0440
160	0.00001930516	0.00003311038	0.00005064863	3.0334
320	0.00000305631	0.00000384377	0.00000623620	3.0218

Table 1: EG4 scheme, T=0.2, CFL=0.10

Note that the CFL number, CFL= $\frac{c\Delta t}{h}$, is taken relatively very small in order to guarantee stability of this third order scheme. This is a well-known feature of higher order schemes which are based on an approximate operator which is not stable up to CFL= 1. It was shown in [14] that the EG4 approximate evolution operator is stable only up to the CFL= 0.72. As a result also the stability region of the higher order FVEG4 scheme is reduced considerably. Note that in our forthcoming paper [4] we have derived a new approximate EG operator,

N	$\|\phi(T)-\phi^n\|$	$\|u(T)-u^n\|$	$\|\mathbf{U}(T)-\mathbf{U}^n\|$	EOC
20	0.04642912334	0.01739759142	0.05254536962	
40	0.00557834321	0.00257589619	0.00666246166	2.9794
80	0.00066333851	0.00034783539	0.00082583103	3.0121
160	0.00007838179	0.00004674909	0.00010254102	3.0096
320	0.00000897168	0.00000651191	0.00001285695	2.9956

Table 2: EG4 scheme, T=0.4, CFL=0.10

which is stable up to a natural stability limit CFL= 1. The results of this paper extend naturally also to the new EG operator and such a third order scheme is stable up to CFL= 1. Third order FVEG scheme derived in this paper for the wave equation system can be also easily generalized to nonlinear hyperbolic problems, e.g. the Euler equations of gas dynamics or the shallow water equations.

Example 6.4

This test contains a discontinuity in the initial data. It is set to be

$$\begin{aligned} \phi(x,y,0) &= 0, \\ u(x,y,0) &= v(x,y,0) = \frac{1}{\sqrt{2}} \begin{cases} 1, & |y| < |x|, \\ -1, & \text{elsewhere.} \end{cases} \end{aligned}$$

In Figures 2 and 3 the isolines of the computed approximate solutions for the first, the second and the third order FVEG4 schemes are shown. The computational domain $[-1,1]\times[-1,1]$ was divided into 400×400 cells, the absolute time is set to be 0.4 and the CFL number for the first, the second and the third order schemes is 0.55, 0.45 and 0.1, respectively. A cross-section plot along the line $y=0$ is shown in Figure 4.
In [2] structure of the exact solution to the above initial-value problem was studied. Particularly for the cross-section $y=0$ the exact solution at time $t=0.4$ reads:

$$(\phi,u,v) = \begin{cases} (0,\ 1/\sqrt{2},\ 1/\sqrt{2}) & x \in [-1,-0.4\sqrt{2}], \\ (1,\ 0,\ 0) & x \in (-0.4\sqrt{2},-0.4] \\ (-1,\ 0,\ 0) & x \in [0.4, 0.4\sqrt{2}) \\ (0,\ 1/\sqrt{2},\ 1/\sqrt{2}) & x \in [0.4\sqrt{2},1] \end{cases}$$

In the so-called subsonic region $[-0.4,0.4]$ all components are continuous and the first component ϕ changes monotonously from the value at $x=-0.4$ to its value at $x=0.4$.
It can be observed from the isolines as well as from the 1D plot and its comparison with the analytical values that the first order FVEG4 scheme has a considerable numerical dissipation and discontinuities are smeared out. The second order FVEG4 scheme resolves shocks much better but it produces some oscillations. The third order FVEG4 is clearly the best. It obtains less dissipation and it resolves better the shocks.

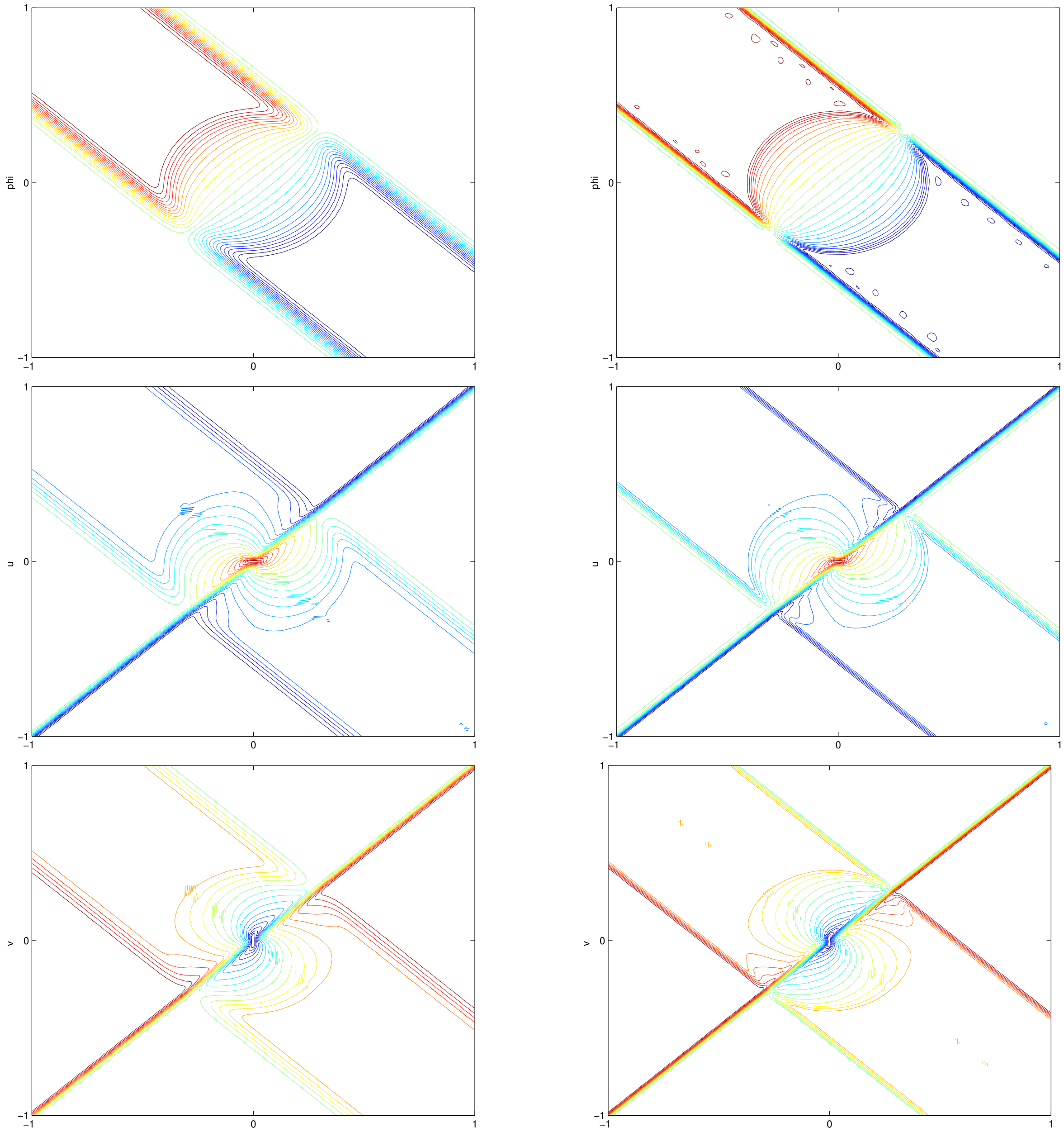

Figure 2: Isolines of the approximate solution obtained by FVEG4 first order and FVEG4 second order schemes.

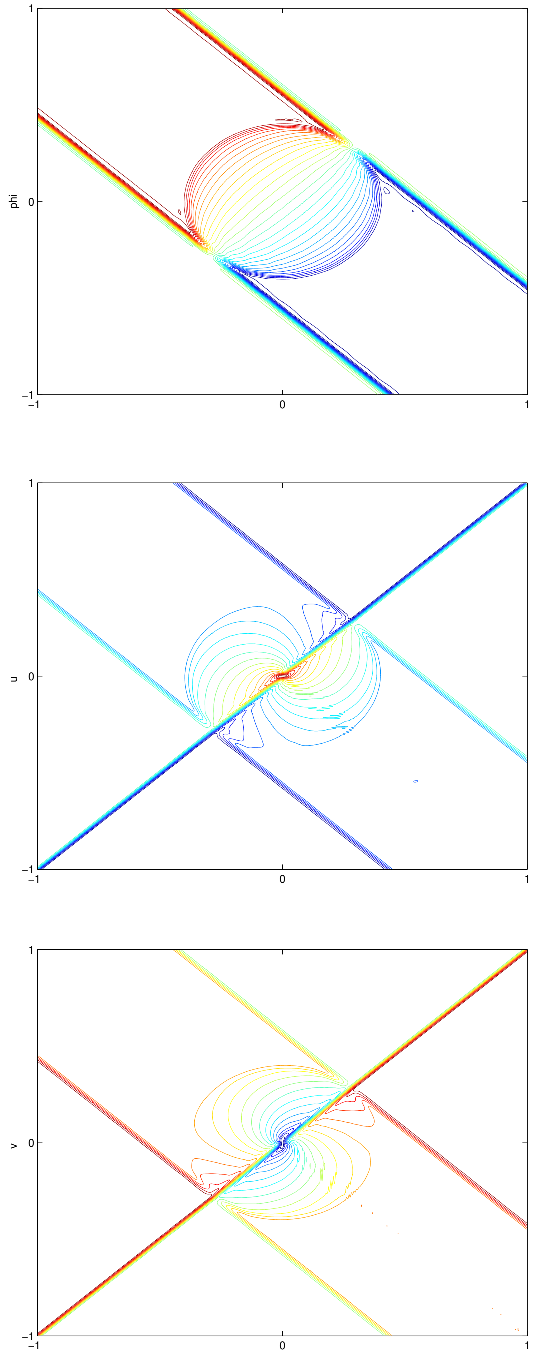

Figure 3: Isolines of the approximate solution obtained by FVEG4 third order scheme.

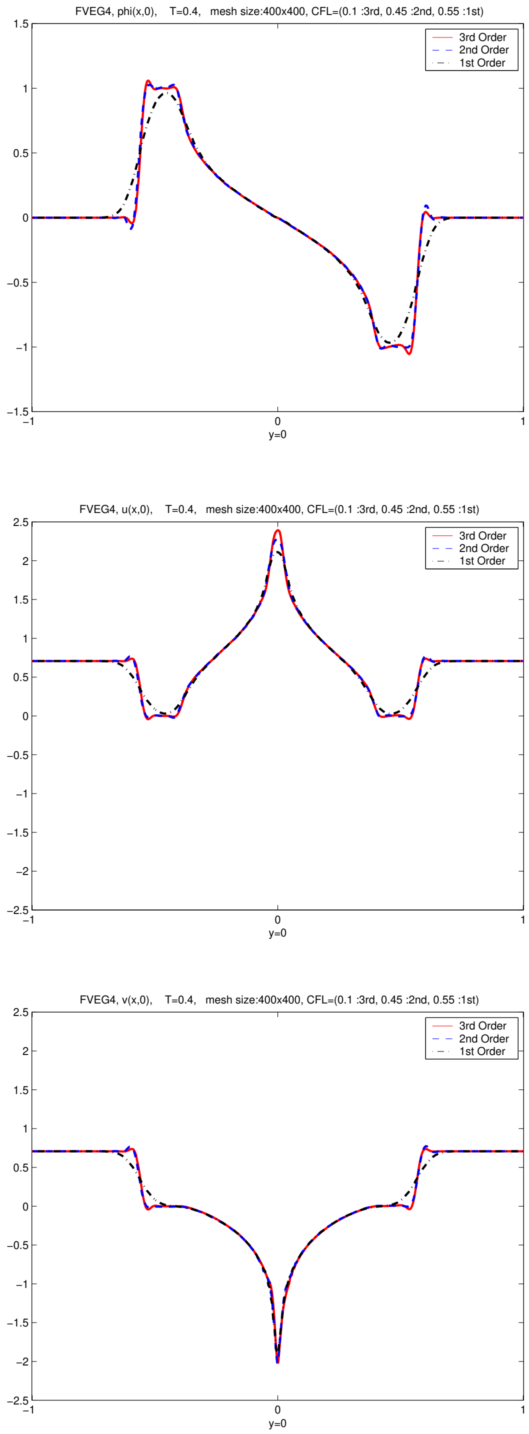

Figure 4: ϕ, u and v along the cross section $y = 0$.

Acknowledgements
This research was supported under the DFG grant No. Wa 633/6-2 of Deutsche Forschungsgemeinschaft, by the grants GACR 201/00/0557 and MSM 26210001 of the Czech Grant Agency as well as by the VolkswagenStiftung and DAAD Agencies.

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