On the problem of symmetrizability for hyperbolic systems of first order [1]

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Abstract

It is shown that a general nonstrictly hyperbolic first order $n \times n$-system with m spatial variables can be always symmetrized only in the cases when $n = 2$ or $m = 1$. Connection with symmetrizability (in Friedrich's sense) of scalar hyperbolic equations is considered. Some algebraic criterion of symmetrizability is also given.

It is well-known that a (nonstrictly) hyperbolic quasilinear first-order $n \times n$ system with only one spatial variable $x \in \mathbb{R}$

$$u_t + Au_x = 0, \quad u = (u_1, \ldots, u_n)^\top, \quad A = A(t, x, u) \in \mathrm{Mat}(n, \mathbb{R})$$

can be rewrited in symmetric form. Really, because of hyperbolicity, the matrix A has a real simple spectrum that is for each fixed arguments (t, x, u) it can be transformed to diagonal form: $A = Q^{-1}\Lambda Q$ where Q is a nondegenerate matrix, and $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, n$. Multiplying our system to the matrix $B = Q^\top Q$ from the left we obtain the following symmetric form

$$Bu_t + Cu_x = 0,$$

with symmetric matrices $B > 0$ (B is positively definite) and $C = Q^\top \Lambda Q$. For the case of a hyperbolic system with many spatial variables

$$u_t + \sum_{k=1}^{m} A_k u_{x_k} = 0, \quad A_k = A_k(t, x, u) \in \mathrm{Mat}(n, \mathbb{R}), \quad k = 1, \ldots, m \tag{1}$$

the problem of symmetrizability is more complicated. By hyperbolicity of (1) any matrix A in the real linear span M of the matrices A_k, $k = 1, \ldots, m$ has a real simple spectrum (i.e. it has a basis of eigenvectors in $\mathbb{R}^n$ with corresponding real eigenvalues). Symmetrizability of the system (1) means that all matrices in M can be transformed to symmetric (or Hermitian) form simultaneously by choice of a basis or, the same, by choice of an appropriate scalar multiplication (Bu, v), with a positively definite symmetric matrix B. Multiplying the system (1) to B we obtain the symmetric form of (1), in the same way as for one spatial variable. It is clear that the hyperbolicity condition

$$A \text{ has a simple real spectrum } \forall A \in M \tag{2}$$

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is necessary for symmetrizability of M. For the case $n = 2$ this condition seems to be also sufficient. This is rather well-known but, for completeness, we give below the proof for the general case of complex matrices.

Proposition. *Let $M \subset \mathrm{Mat}(2, k)$, $k = \mathbb{R}, \mathbb{C}$ be a real subspace of (generally complex) matrices that satysfies (2). Then all matrices in M can be reduced to symmetric (Hermitian) form.*

Proof. Consider the more complicated case of complex matrices. Firstly, without loss of generality, we can assume that the identity matrix E belongs to M. Indeed, we can replace M to the space $\{\ A + \lambda E \mid A \in M,\ \lambda \in \mathbb{R}\ \}$, with conservation of the property (2). If M doesn't contain a matrix with different eigenvalues then it consists of matrices λE, $\lambda \in \mathbb{R}$, which are Hermitian. So we can suppose that M contains a matrix A with different eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$. Choose a basis e_1, e_2 in $\mathbb{C}^2$ so that this matrix becomes diagonal $A = \mathrm{diag}\{\lambda_1, \lambda_2\}$. Then M contains matrices $xA + yE$, $x, y \in \mathbb{R}$ and, therefore, all real diagonal matrices. Let M_0 be subspace of M consisting of matrices $A = \begin{pmatrix} 0 & z \\ w & 0 \end{pmatrix}$, $z, w \in \mathbb{C}$ with zero diagonal elements. It is clear that $M = M_0 \oplus D$ where D is the space of diagonal matrices. If $M_0 = \{0\}$ then M consists of real diagonal matrices, which are Hermitian, and the proof in this case is complete. So, suppose that $M_0 \neq \{0\}$. If $A \in M_0$, $A \neq 0$ then by (2) the product zw is real and positive. Therefore, $w = c\bar{z}$ where $c > 0$, and $\bar{z}$ is a complex conjugation of z. Further, suppose that $A_1, A_2 \in M_0$, $A_1, A_2 \neq 0$. Let $A_1 = \begin{pmatrix} 0 & z_1 \\ c_1\overline{z_1} & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & z_2 \\ c_2\overline{z_2} & 0 \end{pmatrix}$. Show that $c_1 = c_2$. In the opposite case $c_1 \neq c_2$ consider the matrix $B = \alpha A_1 + \beta A_2 \in M_0$, $\alpha, \beta \in \mathbb{R}$. Then $B = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$ where $p = \alpha z_1 + \beta z_2$, $q = c_1\alpha\overline{z_1} + c_2\beta\overline{z_2}$. Therefore $pq = c_1|p|^2 + (c_2 - c_1)\beta p\overline{z_2}$ and, by the condition $pq \in \mathbb{R}$ and arbitrarity of $\alpha, \beta \in \mathbb{R}$, we conclude that $p\overline{z_2} \in \mathbb{R}$, which implies the inclusion $z_1\overline{z_2} \in \mathbb{R}$, i.e. $z_2 = kz_1$ for some real k. Taking $\alpha = k$, $\beta = -1$ we obtain that $p = 0$, and, by (2), necessarily $q = (c_1 - c_2)\overline{z_2}$ must be equal zero, which contradicts to the conditions $c_1 \neq c_2$ and $z_2 \neq 0$. Thus, we establish existence of the constant $c > 0$ such that all matrices in M_0 have the form $A = \begin{pmatrix} 0 & z \\ c\bar{z} & 0 \end{pmatrix}$. Replace the vector e_1 to $e_1' = e_1/\sqrt{c}$. Then all matrices in M_0 becomes Hermitian, and also diagonal matrices remain diagonal. Thus, all matrices in M are Hermitian in our new basis e_1', e_2. For the case of real matrices the proof is similar (and simplier). The proposition is proved.

In the case $n > 2$ the condition (2) and even the stronger condition (strict hyperbolicity)

$$A \text{ has a different real eigenvalues } \forall A \in M,\ A \neq 0 \tag{3}$$

isn't sufficient for symmetrizability of a matrix family M with $dim M \geq 2$. Confirm it by the following

Example. Consider a 2-dimensional linear matrix family

$$A = \begin{pmatrix} 0 & 0 & a-b \\ 0 & 0 & b \\ a-b & a & 0 \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

The eigenvalues of A are easily computed: $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{(a-b)^2 + ab}$, and they are real and different for $A \neq 0$ since the quadratic form $(a-b)^2 + ab$ is positively definite. Thus, the condition (3) is satisfied. Let's prove that this "strictly hyperbolic" family can not be symmetrized. Otherwise, there exists a scalar multiplication (Px, y) with a positively definite matrix $P = \begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{pmatrix}$ such that all matrices A are symmetric. Then $(PAx, y) = (x, PAy)$ and PA is symmetric with respect to original scalar multiplication. Revealing the equalities $(PA)_{12} = (PA)_{21}$, $(PA)_{13} = (PA)_{31}$, $(PA)_{23} = (PA)_{32}$ we obtain relations $ap_3 = (a-b)p_5$, $(a-b)p_1 + bp_2 = (a-b)p_6$, $(a-b)p_2 + bp_4 = ap_6$, for all $a, b \in \mathbb{R}$ which easily imply that $p_i = 0$, $i = 1, \ldots, 6$, i.e. $P = 0$. But it contradicts to the condition $P > 0$. This completes the proof.

Taking two basis matrices corresponded to values $a = 1, b = 0$, and $a = b = 1$ we obtain the following 2-dimensional strictly hyperbolic system

$$\begin{cases} u_t = w_x \\ v_t = w_y \\ w_t = (u+v)_x + v_y \end{cases}, \tag{4}$$

which is not symmetrizable.

Observe that the system (4) corresponds to a second order hyperbolic equation

$$P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f = f_{tt} - (f_{xx} + f_{xy} + f_{yy}) = 0$$

if to set $u = f_x$, $v = f_y$, $w = f_t$.

Such equation can be always symmetrized in Friedrich's sense ([1]), even for two reasons, firstly, because it has second order (see [1]), and, secondly, because it contains only two space variables (see [2]). This means that there exist symmetric 3×3-matrices B, A_1, A_2 with $B > 0$, and a vector $k \in \mathbb{R}^3$ such that

$$(\tau B + \xi A_1 + \eta A_2) \cdot (\xi, \eta, \tau)^\top = P(\tau, \xi, \eta)k = (\tau^2 - \xi^2 - \xi\eta - \eta^2)k.$$

We see that problems of symmetrizability of a hyperbolic equation and the corresponding first order system are different ones. Remark also that a singular hyperbolic equation is not always simmetrized. Thus, in paper [3] was found that some equations of 4-th order with 4 spatial variables could not be transformed to symmetric first-order system (in Friedrich's sense).

As follows from the results above, it is usefull to find conditions of symmetrizability for a general linear matrix family M. To the end of this paper we prove a simple purely algebraic criterion of symmetrizability. Let M be a subset of the matrix space $\mathrm{Mat}(n,k)$ under the field $k=\mathbb{R}$ or $k=\mathbb{C}$, and $L=L(M)\subset \mathrm{Mat}(n,\mathbb{C})$ be the minimal real Lie subalgebra (with commutator $[A,B]=AB-BA$ as a multiplication), which contains the set iM, $i^2=-1$.

Theorem. *The family M can be reduced to symmetric (Hermitian) form if and only if all matrices in the Lie algebra $L(M)$ have simple imaginary spectrum.*

Proof. In the case of real field $k=\mathbb{R}$ the family M can be considered as the set in $\mathrm{Mat}(n,\mathbb{C})$, after the complexification. If all matrices in M are Hermitian with respect to some complex scalar multiplication $(\cdot,\cdot)$ in $\mathbb{C}^n$ then they are symmetric with respect to the real scalar multiplication $Re(\cdot,\cdot)$ in $\mathbb{R}^n$. Thus, the case $k=\mathbb{R}$ reduces to the case of complex field k, that will be assumed below.

If matrices in M are Hermitian (w.r.t some scalar multiplication in $\mathbb{C}^n$) then iM is embedded in the Lie algebra $u(n)$ of scew-Hermitian matrices. This implies the inclusion $L(M)\subset u(n)$ and therefore $L(M)$ consists of matrices with imaginary simple spectrum.

Conversely, suppose that all matrices in $L=L(M)$ have simple imaginary spectrum. In particular, this implies that M satisfies (2). Consider a bilinear form $Q(X,Y)=-Tr(XY)$. Let $X\in L$, $X\neq 0$ and $i\lambda_k\in i\mathbb{R}$, $k=1,\dots,n$ be eigenvalues of X taken with account of multiplicity. Then $Q(X,X)=-Tr(X^2)=\sum\limits_{k=1}^{n}\lambda_k^2>0$. Thus, the form Q determines a scalar multiplication on L, and evidently operators $ad_X(Y)=[X,Y]$ are scewsymmetric with respect to this scalar multiplication. This means that the algebra L is compact. Therefore (see, for example, [4]) L is a direct sum $L=S+Z$, where Z is a center of L, and S is a semisimple compact Lie algebra. The space $\mathbb{C}^n$ can be decomposed into a direct sum of indecomposable under the acting of matrices from L invariant subspaces $\mathbb{C}^n=V_1+...+V_r$. Then any matrix $A\in Z$ acts trivially on V_i, $i=1,...,r$: $A=ia_iE$ on V_i, $a_i\in\mathbb{R}$, E is the identity operator. Indeed, in the opposite case V_i can be decomposed into the direct sum of nontrivial eigensubspaces of A (because A has a simple spectrum) $V_i=H_1+H_2$. Since A commutates with all matrices from L these subspaces H_1, H_2 are invariant for L that contradicts to indecomposability of V_i. Further, as is known in the theory of Lie algebras (see [4]), S is a Lie algebra of some compact simply connected Lie group G, and the embedding $S\in \mathrm{Mat}(n,\mathbb{C})$ induces a homomorphism of Lie groups $F:G\mapsto GL(n,\mathbb{C})$, i.e. the group G acts on $\mathbb{C}^n$: $gx=F(g)x$, $g\in G$, $x\in\mathbb{C}^n$. Since ImF are generated by $\exp(A)$, $A\in S$ the subspaces V_i are invariant under G-acting. Define the invariant scalar product on V_i setting $(x,y)_i=\int_G(gx,gy)d\mu(g)$, where $\mu(g)$ is the Haar measure on G, choose orthogonal basises in all V_i with respect to these scalar multiplications, and take the union of the constructed basises as a basis in $\mathbb{C}^n$. Then ImF becomes embedded in the group $U(n)$ of unitary matrices, and, therefore, matrices in S are scew-Hermitian. Besides, matrices in Z are also scew-Hermitian (and even diagonal). Since $iM\subset L$, M consists of Hermitian matrices. This completes the proof.

Remark that the fact that the system (4) is not symmetrized easily follows from our Theorem. Indeed, consider two matrices A_1, A_2 corresponded to (4):

$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then, as it is easily verified, the commutator $[A_2, A_1] = [iA_1, iA_2]$ has not all imaginary eigenvalues.

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