NEUMANN BOUNDARY CONDITION FOR A HAMILTON JACOBI EQUATION IN A QUARTER PLANE

ADIMURTHI AND G.D. VEERAPPA GOWDA
T.I.F.R Centre, P.B.No.1234, Indian Institute of Science Campus,
Bangalore - 560012, INDIA.
(aditi@math.tifrbng.res.in, gowda@math.tifrbng.res.in)

Abstract. We consider Hamilton-Jacobi equation $u_t + H(u, u_x) = 0$ in the quarter plane and study initial boundary value problems with Neumann boundary condition on the line $x = 0$. We obtain explicit formulas for the viscosity solution, when $p \to H(u,p)$ is convex,positively homogeneous of degree $m \geq 1$. If $m = 1$ and $s \to H(s,p)$ is non increasing ,in general,this problem need not admits a continuous viscosity solution. Even in this case we obtain explicit formula for discontinuous viscosity solutions.

Key words. Hamilton-Jacobi equation, viscosity solution, Neumann boundary condition.

AMS subject classification. 35K22, 35L60

1. INTRODUCTION

LET $I \subset I\!R$ be an open interval and let $\Omega = I \times (0,\infty)$. Let $H \in C(I\!R \times I\!R)$, $u_0 \in W^{1,\infty}(I)$ and $\lambda \in C(\partial I \times (0,\infty))$. Consider the following initial boundary value problem for the Hamilton Jacobi equation:

$$\begin{array}{rcll} u_t + H(u, u_x) & = & 0 & \text{in} \quad \Omega \\ u(x,0) & = & u_0(x) & \text{for} \quad x \in I \\ \frac{\partial u}{\partial x} & = & \lambda & \text{on} \quad \partial I \times (0,\infty). \end{array} \tag{1.1}$$

In the case of a pure initial value problem i.e.,$I = I\!R$, when $\frac{\partial H}{\partial u} \geq 0$,then the existence and uniqueness of (1.1) is well understood see Lions[10],Crandall and Lions[4]. In the case of a initial boundary value problem (1.1),then it is not always possible to prescribe boundary condition on the boundary and hope to have a solution and hence these boundary conditions has to be understood in a relaxed sense. Lions ([10,11]) has discussed this problem in detail and showed the existence and uniqueness of the solutions.

When $H(s,p) = H(p)$,H is super linear growth and convex in p,infact one can derive the explicit formula for the solution of (1.1). In the case of a pure initial value problem i.e., $I = I\!R$, Hopf[5] has obtained an explicit formula for a viscosity solution as a minimization over controlled paths of a certain functional with controlled paths being straight lines. In the case of a quarter plane i.e., $I = I\!R_+$,problem was studied by Joseph and Gowda [7] and obtained an explicit formula for a viscosity solution of (1.1) similar to Hopf's formula. Here the controlled paths being certain piecewise linear curves having atmost three linear curves. Using explicit formula Lax-Olenik in the case of pure initialvalue problem and Joseph-Gowda[7] in the case of a quarter plane, entropy solution $\vartheta = u_x$ of the following scalar conservation law has been derived.

$$\begin{array}{rcll} \vartheta_t + H(\vartheta)_x &=& 0 & \text{in} \quad \Omega \\ \vartheta(x,0) &=& \frac{\partial u_0}{\partial x}(x) & \text{for} \quad x \in I \\ \vartheta &=& \lambda & \text{on} \quad \partial I \times (0,\infty). \end{array} \tag{1.2}$$

The problem (1.2) has also been studied by Lefloch[9] where a formula was derived which contains a solution of a variational inequality which may not be solvable explicitly. Furthermore using the numerical scheme, Joseph and Gowda [8] obtained a similar formula for a solution of (1.1) in the quarter plane when $H(p) = |p|$.

Now what happens when H depends on "u" and $p \to H(u,p)$ is convex?. In general obtaining an explicit formula for a solution is quite difficult. Recently,for the pure initial vale problem, under the assumption $\frac{\partial H}{\partial u} \geq 0, p \to H(u,p)$ is convex ,positively homogeneous of degree 1,Barron,Jensen and Liu[3] have obtained an explicit formula for a viscosoity solution.

If $\frac{\partial H}{\partial u}$ is not non negetive,this problem is not well studied in the literature. In general this problem admits discontinuous solution. Recently this problem is studied by Adimurthi and Gowda[1] under the assumption either $p \to H(u,p)$ is convex and positively homogeneous of degree $m > 1$ or $p \to H(u,p)$ is convex,positively homogeneous of degree $m = 1$ and $\frac{\partial H}{\partial u}(u,p) \leq 0$ and obtained an explicit formula for the viscosity solution in the case of a pure initial value problem.

Now the question is under the same assumption as above,does there exists an explicit formula for a solution of initial boundary vale problem (1.1). In this paper,for all the above cases we obtain an explicit formula for a solution of (1.1) in the qurter plane i.e., $I = I\!R_+$.

2. MAIN RESULTS

Hereafter assume that H satisfies:

(H_0) : $p \to H(u,p)$ is convex,positively homogeneous of degree $m \geq 1$ and $H(u,p) > 0$ for all $p \neq 0$.

Let $u_0 \in W^{1,\infty}(\mathbb{R}_+),\ \lambda \in C(\mathbb{R}_+)$. Let

$$\eta(u) = m \int_0^u \frac{d\theta}{H(\theta,1)}\ . \tag{2.1}$$

For $|w| = 1$ and $m > 1$ define,

$$G_w(u) = \int_0^u H(\theta,w)^{1/m-1} d\theta \tag{2.2}$$

$$L_w = \eta o G_w^{-1} \tag{2.3}$$

Hypothesis : Assume that η and G_w are diffeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$.

Difinition 2.1 : (Admissible curves). Let $0 \leq s < t$ and $\beta \in C([s,t], \overline{\mathbb{R}}_+)$. β is called an admissible curve if the following holds.

1. β consists of atmost three linear curves(Fig.1a.,Fig.1b.).
2. Let $s = t_3 \leq t_2 \leq t_1 \leq t_0 = t$ be such that for $i = 1,2,3,\quad \beta_i = \beta|_{[t_i,t_{i-1}]}$ be the linear parts of β. Then $\beta_2 = 0$.

Represent an admisible curve $\beta = \{\beta_1, \beta_2, \beta_3\}$ and $\dot{\beta} = (\dot{\beta}_1, \dot{\beta}_2, \dot{\beta}_3)$, where $\dot{\beta}_i = \frac{d\beta_i}{d\theta}$ and $|\dot{\beta}| = \max_{1 \leq i \leq 3} \left\{ |\dot{\beta}_i| \right\}$. Let $M > 0,\ x \in \overline{\mathbb{R}}_+,\ 0 \leq s < t$ and define

$$c(x,s,t) \ = \ \left\{ \beta \in C([s,t], \overline{\mathbb{R}}_+); \beta(t) = x, \beta \text{ is an admissible curve} \right\} \tag{2.4}$$

$$c_M(x,s,t) = \left\{ \beta \in c(x,s,t); \quad |\dot{\beta}| \leq M \right\}, \tag{2.5}$$

$$c(x,t) = c(x,0,t),\ c_M(x,t) = c_M(x,0,t). \tag{2.6}$$

For $a\,, b\, \in \mathbb{R}$, denote $a \bigvee b \ = \ \max(a,b)$.

The main results are divided into three parts.

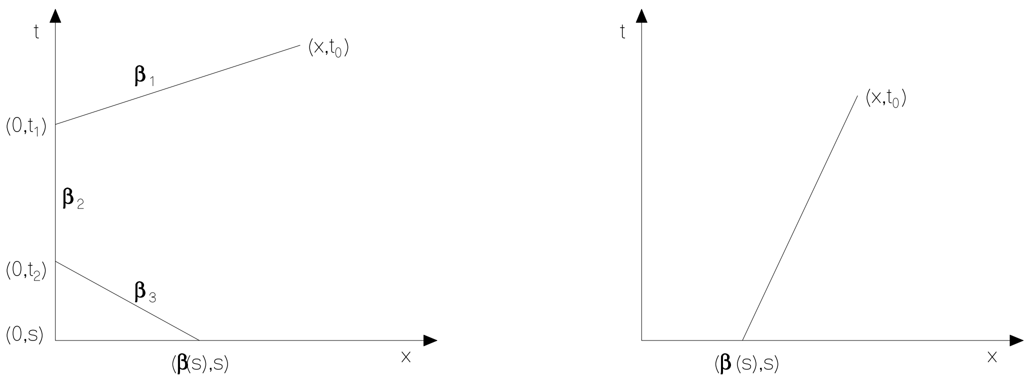

Fig.1a. with **β** 2 is nonempty

Fig.1b. with **β** 2 is empty

Part 2.1 : Assume that H satisfies the following assumption.
(H_1) $\quad m=1$ and $u \mapsto H(u,p)$ is non decreasing for each p.
Associated to H, let h be its quasi convex dual defined by

$$h(q) = \inf \{\gamma; H(\gamma,p) \geq pq \quad \forall |p| \leq 1\} \ . \tag{2.7}$$

For $(x,t) \in \overline{I\!R}_+ \times I\!R_+$ and $0 \leq s < t,\ \beta = (\beta_1, \beta_2, \beta_3) \in c(x,s,t),$ define

$$\rho_3(\beta) \ = \ \eta(h(\dot{\beta}_3)) \tag{2.8}$$

$$\rho_1(\beta) \ = \ \eta(h(\dot{\beta}_1)) \tag{2.9}$$

$$\int_\beta \lambda^+ \ = \ \int_{\{\beta=0\}} \lambda^+(\theta) d\theta \tag{2.10}$$

with $\rho_i(\beta) \ = \ -\infty$ if $\beta_i \ = \ \phi$. Then we have the following :

Theorem 2.1 : Let $(x,t) \in \overline{I\!R}_+ \times I\!R_+$ and define

$$\eta(u(x,t)) = \inf_{\beta \in c(x,t)} \left\{ \eta(u_0(\beta(0))) \bigvee \rho_3(\beta) - \int_\beta \lambda^+ \right\} \bigvee \rho_1(\beta) \ . \tag{2.11}$$

Then for any $T > 0,\ u \in W^{1,\infty}(\overline{I\!R}_+ \times [0,T])$ and u is a viscosity solution of (1.1) with $I = I\!R_+$. Furthermore minimizer in (2.11) exist.

Part 2.2 : Assume that H satisfies the following assumption.
(H_2) $\quad m=1 \quad$ and $\quad u \mapsto H(u,p)$ is non increasing for each fixed p.
Associated to H, let h be the quasi concave dual defined by

$$h(q) = \sup \{\nu; H(\nu,p) \geq pq \quad \forall |p| \leq 1\} \ . \tag{2.12}$$

Let $M > 0,\ 0 \le s < t$ and $(x,t) \in \overline{I\!R}_+ \times I\!R_+$. For a function ϑ on $\overline{I\!R_+ \times I\!R_+}$,define

$$\begin{aligned} \underline{A}(\vartheta, x, s, t) &= \Big\{\beta \in c(x,s,t); \vartheta(\beta(s), s) \le h(\dot\beta_3), \\ &\qquad \eta(\vartheta(\beta(s),s)) - \textstyle\int_\beta \lambda^+ \le \eta(h(\dot\beta_1))\Big\} \end{aligned} \tag{2.13}$$

$$\begin{aligned} \overline{A}(\vartheta, x, s, t) &= \Big\{\beta \in c(x,s,t); \vartheta(\beta(s), s) < h(\dot\beta_3), \\ &\qquad \eta(\vartheta(\beta(s),s)) - \textstyle\int_\beta \lambda^+ < \eta(h(\dot\beta_1))\Big\} \end{aligned} \tag{2.14}$$

$$\begin{aligned} \underline{A}_M(\vartheta, x, s, t) &= \Big\{\beta \in \underline{A}(\vartheta, x, s, t) : |\dot\beta| \le M\Big\} \\ \overline{A}_M(\vartheta, x, s, t) &= \Big\{\beta \in \overline{A}(\vartheta, x, s, t) : |\dot\beta| \le M\Big\}, \end{aligned}$$

$$\begin{cases} \underline{A}(\vartheta, x, t) &= \underline{A}(\vartheta, x, 0, t),\ \overline{A}(\vartheta, x, t) = \overline{A}(\vartheta, x, 0, t), \\ \underline{A}_M(\vartheta, x, t) &= \underline{A}_M(\vartheta, x, 0, t),\ \overline{A}_M(\vartheta, x, t) = \overline{A}_M(\vartheta, x, 0, t), \end{cases} \tag{2.15}$$

with $h(\dot\beta_i) = \infty$ if $\beta_i = \phi$. Then we have the following

Theorem 2.2 : Define

$$\underline{u}(x,t) = \inf \eta^{-1}\left\{\eta(u_0(\beta(0))) - \int_\beta \lambda^+; \beta \in \underline{A}(u_0, x, t)\right\} , \tag{2.16}$$

$$\overline{u}(x,t) = \inf \eta^{-1}\left\{\eta(u_0(\beta(0))) - \int_\beta \lambda^+, \beta \in \overline{A}(u_0, x, t)\right\} . \tag{2.17}$$

Let $I = I\!R_+$.Then $\underline{u}$ is a lower semicontinuous function , $\overline{u}$ is an upper semicontinuous function, $\underline{u}^* = \overline{u}$ and $\overline{u}_* = \underline{u}$. Furthermore $\underline{u}$ and $\overline{u}$ are viscosity solutions of (1.1).

Part 2.3 : Assume that $m > 1$. Let $(x,t) \in (\overline{I\!R}_+ \times I\!R_+),\ 0 \le s < t,\ \beta = (\beta_1, \beta_2, \beta_3) \in c(x,s,t)$ defined on the partition $s = t_3 \le t_2 \le t_1 \le t_0 = t$ and $|w_1| = |w_2| = 1$. Let

$$D(x,s,t) = \left\{ \begin{array}{lllll} (\beta, w_1, w_2); & (a) & \dot\beta \ne 0, & \dot\beta_1 w_1 \ge 0, & \dot\beta_3 w_2 \ge 0. \\ & (b) & \dot\beta = 0, & \multicolumn{2}{l}{\text{no condition on } w_i, i = 1,2} \end{array} \right\} \tag{2.18}$$

$$D(x,t) = D(x,0,t) \tag{2.19}$$

$$l_+(\beta) = \frac{m-1}{m}\int_{t_1}^{t} |\dot\beta_1|^{m/m-1} \tag{2.20}$$

$$l_-(\beta) = \frac{m-1}{m}\int_{s}^{t_2} |\dot\beta_3|^{m/m-1} \tag{2.21}$$

$$l_0(\beta) = \int_{\{\beta=0\}} \lambda^+(\theta)^m d\theta\ . \tag{2.22}$$

Then we have the following

Theorem 2.3 : For $|w| = 1$, let G_w and L_w are given by (2.2) and (2.3) respectively. Define

$$u(x,t) = \inf_{(\beta, w_1, w_2) \in D(x,t)} G_{w_1}^{-1}\Big\{L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u_0(\beta(0))) + l_-(\beta)\right\} - l_0(\beta)\right\} + l_+(\beta)\Big\}. \tag{2.23}$$

Then u is a Lipschitz continuous function in $\overline{I\!R}_+ \times [0,T]$ for any $T > 0$ and is a viscosity solution of

$$\begin{array}{rcll} u_t + \frac{1}{m}H(u,u_x) &=& 0 & \text{in} \quad I\!R_+ \times I\!R_+ \\ u(x,0) &=& u_0(x) & x \in I\!R_+ \\ \frac{\partial u}{\partial x}(0,t) &=& \lambda(t) & t \in I\!R_+ \ . \end{array}$$

Remark 2.1: The assumption on η and L_w being diffeomorphism $I\!R$ onto $I\!R$ is not required. With a slight modification,all the theorems still valid except for the fact that in general,minimizer may not exit.

3. PRELIMINARIES

In this section we recall the definition and some known facts from [2], [3],[6] and [11].

Definition 3.1 : Let $\Omega \subset I\!R^n$ be a domain and V be a locally bounded function. For $x \in \Omega$ define

$$\begin{array}{rcl} V^*(x) &=& \lim_{r\to 0} \sup \{V(z) : |x-z| \leq r\} \\ V_*(x) &=& \lim_{r\to 0} \inf \{V(z) : |x-z| \leq r\} \ . \end{array}$$

Then it follows easily that V^* and V_* are upper and lower semicontinuous functions respectively. Also $V_* \leq V^*$.

Definition 3.2 : Let U be a locally bounded function in $\overline{I\!R_+ \times I\!R_+}$.

1. U is said to be a subsolution of (1.1) if for any $(x_0,t_0) \in \overline{I\!R}_+ \times I\!R_+, \varphi \in C^1(\overline{I\!R}_+ \times I\!R_+)$ such that (x_0,t_0) is a local maximum for $U^* - \varphi$ with $U^*(x_0,t_0) = \varphi(x_0,t_0)$. Then at (x_0,t_0) either $\varphi_t + H(\varphi,\varphi_x) \leq 0$ or if $x_0 = 0$ and $\varphi_t + H(\varphi,\varphi_x) > 0$ then $\varphi_x(x_0,t_0) \geq \lambda(t_0)$. Further more $\lim_{t\to 0} U^*(x,t) \leq u_0(x)$.

2. U is said to be a super solution of (1.1) if for any $(x_0,t_0) \in \overline{I\!R}_+ \times I\!R_+, \varphi \in C^1(\overline{I\!R}_+ \times I\!R_+)$ such that (x_0,t_0) is a local minimum for $U_* - \varphi$ with $U_*(x_0,t_0) = \varphi(x_0,t_0)$. Then at (x_0,t_0), either $\varphi_t + H(\varphi,\varphi_x) \geq 0$ or if $x_0 = 0$ and $\varphi_t + H(\varphi,\varphi_x) < 0$, then $\varphi_x(x_0,t_0) \leq \lambda(t_0)$. Further more $\lim_{t\to 0} U_*(x,t) \geq u_0(x)$.

3. U is said to be a viscosity solution of (1.1) if U is both sub and super solution of (1.1).

Now recall some properties of quasiconvex(concave) dual of H.

Lemma 3.2 : Let H be as in (1.1) and is homogeneous of degree 1 in $p \mapsto H(u,p)$. Then

(a) Suppose $u \mapsto H(u,p)$ is non decreasing and h be its quasi convex dual defined by (2.7). Then h satisfies

(A_1) h is a lower semicontinuous function. For any $q_1, q_2 \in I\!R$, $t \in [0,1]$, $h(tq_1 + (1-t)q_2) \leq \max\{h(q_1), h(q_2)\}$

(A_2) $h(0) = -\infty, \lim_{|q|\to\infty} h(q) = \infty,$

(A_3) $H(s,p) = \sup\{pq; h(q) \leq s\}.$

(b) Suppose $u \mapsto H(u,p)$ is non increasing and h be its quasi concave dual defined by (2.12). Then

(A_4) h is an upper semicontinuous function. For any $q_1, q_2 \in I\!R$, $t \in [0,1]$, $h(tq_1 + (1-t)q_2) \geq \min\{h(q_1), h(q_2)\}$,

(A_5) $h(0) = +\infty, \lim_{|q|\to\infty} h(q) = -\infty,$

(A_6) $H(s,p) = \sup\{pq;\ s \leq h(q)\}.$

Proofs of (A_1) to (A_3) follows from Lemma 2.1 and Lemma 2.2 of [3]. (A_4) to (A_6) follows from (A_1) to (A_3) applied to the Hamiltonian $\tilde{H}(s,p) = H(-s,p)$.

4. PROOF OF THEOREMS

In order to prove Theorem 2.1,first we prove the following Lemmas.

Lemma 4.1 :(Existence of a minimizer) Let $T > 0$, $0 \leq s < t \leq T$, $x \in \overline{I\!R}_+$. Let $\vartheta : \overline{I\!R_+ \times I\!R_+} \to I\!R$ be a function such that $|\vartheta|_T = \sup_{\overline{I\!R}_+\times[0,T]} |\vartheta(x,t)| < \infty$. Define

$$\eta(V(x,t)) = \inf_{\beta\in c(x,s,t)} \left\{\eta(\vartheta(\beta(s),s)) \bigvee \rho_3(\beta) - \int_\beta \lambda^+\right\} \bigvee \rho_1(\beta). \tag{4.1}$$

Then

(1) $|V|_T < \infty$ and there exist $M = M(\vartheta, \lambda, T) > 0$ such that

$$\eta(V(x,t)) = \inf_{C_M(x,s,t)} \left\{\eta(\vartheta(\beta(s),s)) \bigvee \rho_3(\beta) - \int_\beta \lambda^+\right\} \bigvee \rho_1(\beta). \tag{4.2}$$

(2) Suppose $x \mapsto \vartheta(x,\theta)$ is lower semicontinuous for each θ, then there exist a minimizer β of (4.2).

Proof : Since $\eta(\pm\infty) = \pm\infty$ and $h(0) = -\infty$, by taking $\beta(\theta) = x$ for $\theta \in [s,t]$ to obtain $\eta(V(x,t)) \leq \eta(\vartheta(x,s)) - \int_\beta \lambda^+ \leq \eta(\vartheta(x,s))$. Hence $V(x,t) \leq |\vartheta|_T$. Also for any $\beta \in c(x,s,t)$

$$\left\{\eta(\vartheta(\beta(s))) \bigvee \rho_3(\beta) - \int_\beta \lambda^+\right\} \bigvee \rho_1(\beta) \geq \eta(\vartheta(\beta(s),s)) - \int_\beta \lambda^+ \geq \eta(-|\vartheta|_T) - T|\lambda^+|_T,$$

and hence $|V|_T < \infty$. Since $h(p) \to \infty$ as $|p| \to \infty$ and hence can choose a $M > 0$ such that whenever $|\dot{\beta}| \geq M$ then either $\rho_1(\beta) > \eta(|V|_T)$ or $\rho_3(\beta) > \eta(|\vartheta|_T) + \eta(|V|_T) + T|\lambda^+|_T$. In this case

$$(\eta(\vartheta(\beta(s),s))\bigvee \rho_3(\beta) - \int_\beta \lambda^+)\bigvee \rho_1(\beta) > \eta(|V|_T)$$

and hence (4.2) holds. This proves (1) .

Let $\{\beta_k\}$ be a minimizing sequence for (4.2). Since $|\dot{\beta}_k| \leq M$ and hence there exist a subsequence still denoted by $\{\beta_k\}$ converging to β in $W^{1,\infty}$. Since $x \mapsto \vartheta(x,\theta)$ and $p \mapsto h(p)$ are lower semicontinuous functions, it follows that

$$\begin{array}{rcl}(\eta(\vartheta(\beta(s),s))\vee\rho_3(\beta) - \int_\beta \lambda^+)\vee\rho_1(\beta) & \geq & \eta(V(x,t)) \\ & = & \lim_{k\to\infty}\left\{\eta(\vartheta(\beta_k(s),s))\vee\rho_3(\beta_k) - \int_{\beta_k}\lambda^+\right\}\vee\rho_1(\beta_k) \\ & \geq & (\eta(\vartheta(\beta(s),s))\vee\rho_3(\beta) - \int_\beta \lambda^+)\vee\rho_1(\beta),\end{array}$$

and hence β is a minimizer. This proves the Lemma.

Lemma 4.2 :(dynamic programming principle)Let u be as in (2.11). Let $x \in \overline{I\!R}_+$ and $0 \leq s < t$, then

$$\eta(u(x,t)) = \inf_{\beta\in c(x,s,t)}(\eta(u(\beta(s),s))\bigvee\rho_3(\beta) - \int_\beta \lambda^+)\bigvee\rho_1(\beta). \tag{4.3}$$

Proof : Let $\eta(\vartheta(x,t))$ denote the right hand side of (4.3). Let $\beta = (\beta_1,\beta_2,\beta_3) \in c(x,t)$ defined on the partition $0 = t_3 \leq t_2 \leq t_1 \leq t_0 = t$. Let $\beta^{(1)} = \beta|_{[0,s]} \in c(\beta(s),s)$ and $\beta^{(2)} = \beta|_{[s,t]} \in c(x,s,t)$.

Suppose $t_1 < s < t$, then $\rho_3(\beta^{(2)}) = -\infty$, $\rho_1(\beta^{(2)}) = \rho_1(\beta)$, $\rho_3(\beta^{(1)}) = \rho_3(\beta)$ and $\rho_1(\beta^{(1)}) = \rho_1(\beta)$. Hence

$$\begin{array}{rcl}\eta(\vartheta(x,t)) & \leq & \eta(u(\beta^{(2)}(s),s))\vee\rho_1(\beta^{(2)}) \\ & \leq & (\eta(u_0(\beta^{(1)}(0)))\vee\rho_3(\beta^{(1)}) - \int_{\beta^{(1)}}\lambda^+)\vee\rho_1(\beta^{(2)}) \\ & = & (\eta(u_0(\beta(0)))\vee\rho_3(\beta) - \int_\beta\lambda^+)\vee\rho_1(\beta).\end{array}$$

If $t_2 \leq s \leq t_1$, then $\rho_3(\beta^{(2)}) = -\infty, \rho_1(\beta^{(2)}) = \rho_1(\beta), \rho_1(\beta^{(1)}) = -\infty,\ \rho_3(\beta^{(1)}) = \rho_3(\beta), \int_{\beta^{(1)}}\lambda^+ + \int_{\beta^{(2)}}\lambda^+ = \int_\beta\lambda^+$. Hence

$$\begin{array}{rcl}\eta(\vartheta(x,t)) & \leq & (\eta(u(\beta^{(2)}(s),s)) - \int_{\beta^{(2)}}\lambda^+)\vee\rho_1(\beta^{(2)}) \\ & \leq & (\eta(u_0(\beta^{(1)}(0)))\vee\rho_3(\beta^{(1)}) - \int_{\beta^{(1)}}\lambda^+ - \int_{\beta^{(2)}}\lambda^+)\vee\rho_1(\beta^{(2)}) \\ & = & (\eta(u_0(\beta(0)))\vee\rho_3(\beta) - \int_\beta\lambda^+)\vee\rho_1(\beta).\end{array}$$

If $0 \leq s < t_2$, then $\rho_1(\beta^{(2)}) = \rho_1(\beta),\quad \rho_3(\beta^{(1)}) = \rho_3(\beta), \rho_1(\beta^{(1)}) = -\infty,\quad \rho_3(\beta^{(2)}) = \rho_3(\beta)$. Hence

$$
\begin{aligned}
\eta(\vartheta(x,t)) &\le (\eta(u(\beta^{(2)}(s),s))\bigvee\rho_3(\beta^{(2)}) - \textstyle\int_{\beta^{(2)}}\lambda^+)\bigvee\rho_1(\beta^{(2)})\\
&\le (\eta(u_0(\beta^{(1)}(0)))\bigvee\rho_3(\beta^{(1)})\bigvee\rho_3(\beta^{(2)}) - \textstyle\int_{\beta^{(2)}}\lambda^+)\bigvee\rho_1(\beta^{(2)})\\
&= (\eta(u_0(\beta(0)))\bigvee\rho_3(\beta) - \textstyle\int_{\beta}\lambda^+)\bigvee\rho_1(\beta).
\end{aligned}
$$

Now taking infinun over β to obtain $\eta(\vartheta(x,t)) \le \eta(u(x,t))$ and hence $\vartheta(x,t) \le u(x,t)$.

From Lemma 4.1, $|\vartheta|_T < \infty$ for any $T > 0$. Hence for every $\epsilon > 0$, there exist $\beta \in c(x,s,t)$ and $\alpha \in c(\beta(s),s)$ such that

$$
\begin{aligned}
\eta(\vartheta(x,t)) &\ge (\eta(u(\beta(s),s))\bigvee\rho_3(\beta) - \textstyle\int_{\beta}\lambda^+)\bigvee\rho_1(\beta) - \epsilon\\
\eta(u(\beta(s),s)) &= (\eta(u_0(\alpha(0)))\bigvee\rho_3(\alpha) - \textstyle\int_{\alpha}\lambda^+)\bigvee\rho_1(\alpha).
\end{aligned}
$$

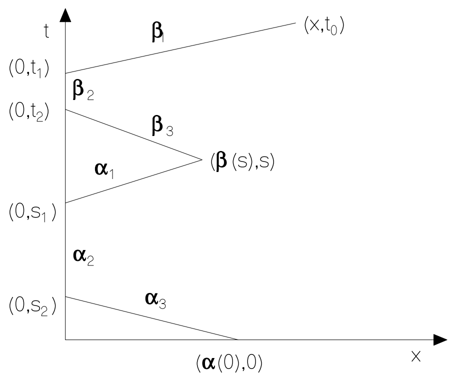

Fig.2a.

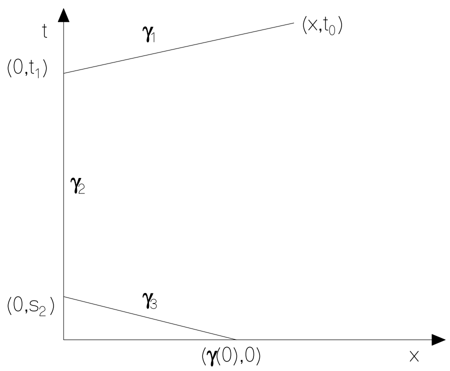

Fig.2b. with $\gamma_1 = \beta_1$, $\gamma_3 = \alpha_3$ and γ_2 is the line joining the points $(0,t_1)$ and $(0,s_2)$.

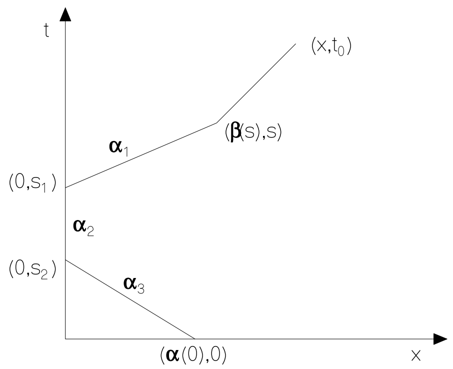

Fig.3a. with β_2 is empty .

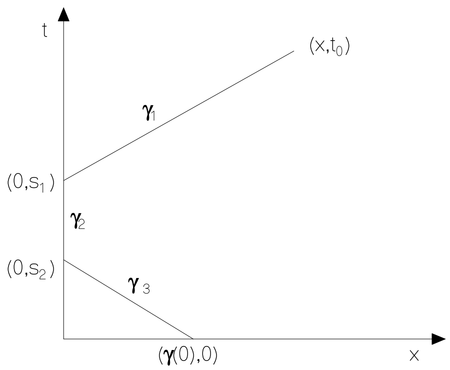

Fig.3b. with γ_1 is the line joining the points (x,t_0) and $(0,s_1)$, $\gamma_2 = \alpha_2$ and $\gamma_3 = \alpha_3$.

Suppose $[t_2,t_1]\neq\phi,[s_2,s_1]\neq\phi$ (Fig.2a.),then define $\gamma=(\gamma_1,\gamma_2,\gamma_3)$ by $\gamma_1=\beta_1,\gamma_2|_{[s_2,t_1]}=0,\gamma_3=\alpha_3$ (Fig.2b.). Then $\rho_1(\gamma)=\rho_1(\beta),\rho_3(\gamma)=\rho_3(\alpha)$ and $\int_\beta\lambda^+ + \int_\alpha\lambda^+\leq\int_\gamma\lambda^+$. Hence

$$\begin{aligned}
\eta(\vartheta(x,t)) &\geq (\eta(u(\beta(s),s))\vee\rho_3(\beta)-\textstyle\int_\beta\lambda^+)\vee\rho_1(\beta)-\epsilon\\
&\geq (\eta(u(\beta(s),s))-\textstyle\int_\beta\lambda^+)\vee\rho_1(\beta)-\epsilon\\
&\geq (\eta(u_0(\alpha(0)))\vee\rho_3(\alpha)-\textstyle\int_\alpha\lambda^+-\int_\beta\lambda^+)\vee\rho_1(\beta)-\epsilon\\
&\geq (\eta(u_0(\gamma(0)))\vee\rho_3(\gamma)-\textstyle\int_\gamma\lambda^+)\vee\rho_1(\gamma)-\epsilon\\
&\geq \eta(u(x,t))-\epsilon
\end{aligned}$$

Suppose $[t_2,t_1]=\phi,[s_2,s_1]\neq\phi$ (Fig.3a.),then define $\gamma=(\gamma_1,\gamma_2,\gamma_3)$ by $\gamma_1(\theta)=\frac{x}{t-s_1}(\theta-s_1)$ for $\theta\in[s_1,t],\gamma_2=\alpha_2,\gamma=\alpha_3$ (Fig.3b.). Since $\dot\gamma_1$ is a convex combination of $\dot\beta$ and $\dot\alpha_1$ and hence $\rho_1(\gamma)\leq\rho_1(\beta)\vee\rho_1(\alpha)$ if $\dot\beta\geq 0$. Also $\rho_1(\gamma)\leq\rho_1(\alpha)$ if $\dot\beta<0$. Therefore

$$\begin{aligned}
\eta(\vartheta(x,t)) &\geq \eta(u(\beta(s),s))\vee\eta(h(\dot\beta))-\epsilon\\
&= (\eta(u_0(\alpha(0)))\vee\rho_3(\alpha)-\textstyle\int_\alpha\lambda^+)\vee\rho_1(\alpha)\vee\eta(h(\dot\beta))-\epsilon\\
&\geq (\eta(u_0(\gamma(0)))\vee\rho_3(\gamma)-\textstyle\int_\gamma\lambda^+)\vee\rho_1(\gamma)-\epsilon\\
&\geq \eta(u(x,t))-\epsilon.
\end{aligned}$$

Similarly, if $[t_2,t_1]\neq\phi,[s_2,s_1]=\phi$ and $[t_2,t_1]=[s_2,s_1]=\phi$ it follows that $\eta(\vartheta(x,t))\geq\eta(u(x,t))-\epsilon$. Since ϵ is arbitrary, it follows that $\vartheta(x,t)\geq u(x,t)$ and hence the Lemma.

Lemma 4.3 :(Lipschitz continuity)Let u be as in (2.11). Then u is a Lipschitz continuous function on $\overline{I\!R}_+\times[0,T]$ for every $T>0$.

Proof : Let $0\leq x_2\leq x_1$ and $0<t\leq T$. From (2) of Lemma 4.1, there exist a $\beta=(\beta_1,\beta_2,\beta_3)\in c(x_1,t)$ defined on the partition $0=t_3\leq t_2\leq t_1\leq t_0=t$ such that

$$\eta(u(x_1,t))=(\eta(u_0(\beta(0)))\bigvee\rho_3(\beta)-\int_\beta\lambda^+)\bigvee\rho_1(\beta).$$

Suppose β is a line segment with $\dot\beta\leq 0$. Then define $\gamma(\theta)=x_2-x_1+\beta(\theta)\in c(x_2,t)$. Hence

$$\begin{aligned}
\eta(u(x_2,t))-\eta(u(x_1,t)) &\leq (\eta(u_0(\gamma(0)))\vee\rho_3(\gamma)-\textstyle\int_\gamma\lambda^+)-\eta(u_0(\beta(0)))\vee\rho_3(\beta)+\int_\beta\lambda^+\\
&\leq |\eta(u_0(\gamma(0)))-\eta(u_0(\beta(0)))|\\
&\leq M_1M_2|\gamma(0)-\beta(0)|=M_1M_2|x_1-x_2|\ ,
\end{aligned}$$

where M_1 and M_2 are Lipschitz constants for η on $[-|u_0|_\infty,|u_0|_\infty]$ and u_0 respectively.

Suppose β is not like above. Let $\tilde t_1=t-(x_2/\dot\beta_1)$ and define $\gamma\in c(x_2,t)$ as follows :

(i) Let $\beta_2\neq\phi$, then

$$\gamma(\theta)\ =\ \begin{cases}(\theta-t)\dot\beta_1+x_2 & \theta\in[\tilde t_1,t]\\ 0 & \theta\in[t_2,\tilde t_1]\\ \beta_3(\theta) & \theta\in[0,t_2].\end{cases}$$

(ii) Let $\beta_2 = \phi$, then

$$\gamma(\theta) \;=\; \begin{cases} (\theta - t)\dot\beta_1 + x_2 & \text{if} \quad \theta \in [\tilde t_1 \vee 0, t] \\ 0 & \text{if} \quad \theta \in [0, \tilde t_1 \vee 0]. \end{cases}$$

Clearly $\dot\gamma_1 = \dot\beta_1,\ \dot\gamma_3 = \dot\beta_3, \quad \{\beta = 0\} \subset \{\gamma = 0\}$ and $|\gamma(0) - \beta(0)| \le |x_1 - x_2|$. Hence

$$\begin{aligned} \eta(u(x_1,t)) - \eta(u(x_2,t)) \;&\le\; (\eta(u_0(\gamma(0))) \vee \rho_3(\gamma) - \textstyle\int_\gamma \lambda^+) \vee \rho_1(\gamma) \\ &\quad\; -(\eta(u_0(\beta(0))) \vee \rho_3(\beta) - \textstyle\int_\gamma \lambda^+) \vee \rho_1(\beta) \\ &\le\; |\eta(u_0(\gamma(0))) - \eta(u_0(\beta(0)))| \\ &\le\; M_1 M_2 |\gamma(0) - \beta(0)| \le M_1 M_2 |x_1 - x_2|. \end{aligned}$$

In order to get the opposite inequality, let $\beta = (\beta_1, \beta_2, \beta_3) \in c_M(x_2, t)$ defined on the partition $0 = t_3 \le t_2 \le t_1 \le t_0 = t$ such that

$$\begin{aligned} \eta(u(x_2,t)) \;&=\; (\eta(u_0(\beta(0))) \vee \rho_3(\beta) - \textstyle\int_\beta \lambda^+) \vee \rho_1(\beta) \\ &\ge\; \eta(u_0(\beta(0))) \vee \rho_3(\beta) - \textstyle\int_\beta \lambda^+. \end{aligned}$$

Now choose a $p_0 > 0$ such that

$$\eta(h(p_0)) \;\le\; \eta(-|u_0|_\infty) - T|\lambda^+|_T\,. \tag{4.4}$$

case (i) : $\dot\beta_1 \le p_0$.

Let $\tilde t_1 = t - \frac{x_1}{p_0}$ and define $\gamma \in c(x_1, t)$ by

$$\gamma(\theta) \;=\; \begin{cases} \frac{\theta - \tilde t_1}{t - \tilde t_1} x_1 & \text{if} \quad \tilde t_1 \ge t_1, \quad \theta \in [\tilde t_1, t] \text{ or if } \quad \tilde t_1 \le 0, \theta \in [0, t] \\ 0 & \text{if} \quad \tilde t_1 \ge t_1,\ \theta \in [t_1, \tilde t_1] \\ \beta(\theta) & \text{if} \quad \tilde t_1 \ge t_1,\ \theta \in [0, t_1] \text{ or if } \quad \tilde t_1 \in [t_2, t_1],\ \theta \in [0, \tilde t_1] \\ (\theta - \tilde t_1)\dot\beta_3 & \text{if} \quad \tilde t_1 \in [0, t_2],\ \theta \in [0, \tilde t_1]. \end{cases}$$

Clearly $\dot\gamma_1 = p_0, \dot\gamma_3 = \dot\beta_3$ if $\tilde t_1 > 0$. Hence from (4.4)

$$\begin{aligned} \eta(u_0(\gamma(0))) \vee \rho_3(\gamma) - \textstyle\int_\gamma \lambda^+ \;&\ge\; \eta(-|u_0|_\infty) - T|\lambda^+|_T \\ &\ge\; \eta(h(p_0)) = \rho_1(\gamma)\,, \end{aligned}$$

and hence

$$\begin{aligned} \eta(u(x_1,t)) \;&\le\; (\eta(u_0(\gamma(0))) \vee \rho_3(\gamma) - \textstyle\int_\gamma \lambda^+) \vee \rho_1(\gamma) \\ &=\; \eta(u_0(\gamma(0))) \vee \rho_3(\gamma) - \textstyle\int_\gamma \lambda^+\,. \end{aligned} \tag{4.5}$$

Let $\tilde t_1 \ge t_1$, then $\beta(0) = \gamma(0),\ \int_\beta \lambda^+ \le \int_\gamma \lambda^+, \rho_3(\beta) = \rho_3(\gamma)$. Therefore from (4.5),

$$\begin{aligned} \eta(u(x_2,t)) - \eta(u(x_1,t)) \;&\ge\; \eta(u_0(\beta(0))) \vee \rho_3(\beta) - \textstyle\int_\beta \lambda^+ - \eta(u_0(\gamma(0))) \vee \rho_3(\gamma) + \textstyle\int_\gamma \lambda^+ \\ &\ge\; 0 \ge -|x_1 - x_2|\,. \end{aligned}$$

Let $\tilde{t}_1 \in [t_1, t_2]$, then $\beta(0) = \gamma(0)$, $\rho_3(\beta) = \rho_3(\gamma)$. Now $\frac{x_2}{t-t_1} = \dot{\beta}_1 \leq p_0$, hence $\frac{x_2}{p_0} \leq t - t_1$. Therefore

$$\begin{array}{rcl} \eta(u(x_2,t)) - \eta(u(x_1,t)) & \geq & -\int_\beta \lambda^+ + \int_\gamma \lambda^+ = -\int_{\tilde{t}_1}^{t_1} \lambda^+ \geq -|\lambda^+|_T(t_1 - \tilde{t}_1) \\ & = & -|\lambda^+|_T(t_1 - t + \frac{x_1}{p_0}) \geq -\frac{|\lambda^+|_T(x_1-x_2)}{p_0}. \end{array}$$

Let $\tilde{t}_1 \in [0, t_2]$, then $t - t_2 \leq \frac{x_1}{p_0}$, $\dot{\beta}_3 = \dot{\gamma}_3$, $\gamma(0) = -\tilde{t}_1\dot{\beta}_3 = \frac{\tilde{t}_1}{t_2}\beta(0)$. Since $\beta \in c_M(x_2, t)$ implies that $\frac{\beta(0)}{t_2} = |\dot{\beta}_3| \leq M$. Hence

$$\begin{array}{rcl} |\gamma(0) - \beta(0)| & = & \frac{\beta(0)}{t_2}(t_2 - \tilde{t}_1) \leq M(t_2 - t + \frac{x_1}{p_0}) \\ & \leq & M(t_1 - t + \frac{x_1}{p_0}) \leq \frac{M}{p_0}(x_1 - x_2), \\ & & \\ t_1 - t_2 & = & t_1 - t + t - t_2 \leq -\frac{x_2}{p_0} + t - \tilde{t}_1 = \frac{x_1 - x_2}{p_0}. \\ & & \\ \eta(u(x_2,t)) - \eta(u(x_1,t)) & \geq & \eta(u_0(\beta(0)))\bigvee\rho_3(\beta) - \int_\beta \lambda^+ - \eta(u_0(\gamma(0)))\bigvee\rho_3(\gamma) \\ & \geq & -M_1M_2|\beta(0) - \gamma(0)| - |\lambda^+|_T(t_1 - t_2) \\ & \geq & -[M_1M_2M + |\lambda^+|_T]\frac{(x_1-x_2)}{p_0}. \end{array}$$

Let $\tilde{t}_1 \leq 0$. Then $t \leq \frac{x_1}{p_0}, \beta(0) \leq Mt_2 \leq M(t_2 - \tilde{t}_1) \leq \frac{M}{p_0}(x_1 - x_2), \gamma(0) = \frac{-\tilde{t}_1}{t-\tilde{t}_1}x_1 \leq x_1 - x_2$, $\quad t_1 - t_2 \leq \frac{M}{p_0}(x_1 - x_2)$. Hence

$$\begin{array}{rcl} \eta(u(x_2,t)) - \eta(u(x_1,t)) & \geq & \eta(u_0(\beta(0)))\bigvee\rho_3(\beta) - \int_\beta \lambda^+ - \eta(u_0(\gamma(0)))\bigvee\rho_3(\gamma) \\ & \geq & -|\eta(u_0(\beta(0))) - \eta(u_0(\gamma(0)))| - |\lambda^+|_T(t_1 - t_2) \\ & \geq & -M_1M_2(|\beta(0)| + |\gamma(0)|) - |\lambda^+|_T\frac{(x_1-x_2)}{p_0} \\ & \geq & -(MM_1M_2 + |\lambda^+|_T)\frac{x_1-x_2}{p_0}. \end{array}$$

case (ii) : $\dot{\beta}_1 \geq p_0$.

Then $p_0 \leq \dot{\beta}_1 = \frac{x_2}{t-t_1}$ and hence $\frac{t-t_1}{x_2} \leq \frac{1}{p_0}$. Let $\tilde{t}_1 = (1 - \frac{x_1}{x_2})t + \frac{x_1}{x_2}t_1$ and define $\gamma \in c(x_1, t)$ by

$$\gamma(\theta) = \left\{ \begin{array}{cl} \frac{\theta-\tilde{t}_1}{t-\tilde{t}_1}x_1 & \text{if } \theta \in [\tilde{t}_1\bigvee 0, t] \\ 0 & \text{if } \theta \in [t_2, \tilde{t}_1] \text{ and } \tilde{t}_1 \geq t_2 \\ \beta(\theta) & \text{if } \theta \in [0, t_2] \text{ and } \tilde{t}_1 \geq t_2 \\ (\theta - \tilde{t}_1)\dot{\beta}_3 & \text{if } 0 \leq \tilde{t}_1 \leq t_2\,. \end{array} \right.$$

Clearly $\dot{\gamma}_1 = \dot{\beta}_1$, $\dot{\gamma}_3 = \dot{\beta}_3$ if $\dot{\gamma}_3 \neq 0$ and $t_1 - \tilde{t}_1 = (\frac{x_1}{x_2} - 1)(t - t_1) = (x_1 - x_2))\frac{t-t_1}{x_2} \leq \frac{x_1-x_2}{p_0}$.

Let $\tilde{t}_1 \in [t_2, t_1]$, then $\beta(0) = \gamma(0), \quad \dot{\beta}_1 = \dot{\gamma}_1, \quad \dot{\beta}_3 = \dot{\gamma}_3$. Hence

$$\begin{aligned}\eta(u(x_2,t)) - \eta(u(x_1,t)) &\geq (\eta(u_0(\beta(0)))\vee\rho_3(\beta) - \textstyle\int_\beta \lambda^+)\vee\rho_1(\beta) \\ &\quad -(\eta(u_0(\gamma(0)))\vee\rho_3(\gamma) - \textstyle\int_\gamma \lambda^+)\vee\rho_1(\gamma) \\ &\geq -\textstyle\int_{\tilde{t}_1}^{t_1} \lambda^+ \geq -|\lambda^+|_T(t_1 - \tilde{t}_1) \geq \frac{-|\lambda^+|_T(x_1-x_2)}{p_0}.\end{aligned}$$

Let $0 \leq \tilde{t}_1 \leq t_2$, then $\dot{\beta}_1 = \dot{\gamma}_1, \quad \dot{\beta}_3 = \dot{\gamma}_3, \quad \gamma(0) = -\tilde{t}_1\dot{\beta}_3 = \frac{\tilde{t}_1}{t_2}\beta(0)$. Since $\beta \in c_M(x_2,t)$ and hence $\frac{\beta(0)}{t_2} = |\dot{\beta}_3| \leq M$. Hence

$$|\gamma(0) - \beta(0)| = \frac{\beta(0)}{t_2}(t_2 - \tilde{t}_1) \leq M(t_1 - \tilde{t}_1) = \frac{M}{p_0}(x_1 - x_2).$$

Hence

$$\begin{aligned}\eta(u(x_2,t)) - \eta(u(x_1,t)) &\geq (\eta(u_0(\beta(0)))\vee\rho_3(\beta) - \textstyle\int_\beta \lambda^+)\vee\rho_1(\beta) \\ &\quad -(\eta(u_0(\gamma(0)))\vee\rho_3(\gamma) - \textstyle\int_\gamma \lambda^+)\vee\rho_1(\gamma) \\ &\geq -|\eta(u_0(\beta(0))) - \eta(u_0(\gamma(0)))| - |\lambda^+|_T(t_1 - t_2) \\ &\geq -M_1M_2|\beta(0) - \gamma(0)| - \frac{|\lambda^+|_T}{p_0}(x_1 - x_2) \\ &\geq -[M_1M_2M + |\lambda^+|_T]\frac{x_1-x_2}{p_0}.\end{aligned}$$

Let $\tilde{t}_1 \leq 0$, then $\dot{\beta}_1 = \dot{\gamma}_1, \quad \gamma(0) = \frac{-\tilde{t}_1}{t-\tilde{t}_1}x_1 \leq x_1 - x_2, \quad \beta(0) \leq Mt_2 \leq M(t_1 - \tilde{t}_1) \leq \frac{M}{p_0}(x_1 - x_2)$. Hence

$$\begin{aligned}\eta(u(x_2,t)) - \eta(u(x_1,t)) &\geq \eta(u_0(\beta(0))) - \textstyle\int_\beta \lambda^+ - \eta(u_0(\gamma(0))) \\ &\geq -M_1M_2(\beta(0) + \gamma(0)) - |\lambda^+|_T(t_1 - t_2) \\ &\geq -(M_1M_2M + p_0 + |\lambda^+|_T)\frac{x_1-x_2}{p_0}.\end{aligned}$$

Combining all the above inequalities to obtain a constant $M(T)$ such that for $x_1, x_2 \in \overline{I\!R}_+, \quad 0 \leq t \leq T$,

$$|u(x_1,t) - u(x_2,t)| \leq M(T)|x_1 - x_2|. \tag{4.6}$$

From (4.6), $x \mapsto u(x,t)$ is continuous and hence from (2) of Lemma 4.1, for any $0 \leq s < t \leq T$ and $x \in \overline{I\!R}_+$, there eixst a $\beta \in c_M(x,s,t)$ such that

$$\eta(u(x,t)) = (\eta(u(\beta(s),s))\bigvee\rho_3(\beta) - \int_\beta \lambda^+)\bigvee\rho_1(\beta). \tag{4.7}$$

Since $|\dot{\beta}| \leq M$ and hence $|\beta(s) - x| \leq M(t-s)$. Therefore

$$\begin{aligned}\eta(u(x,t)) - \eta(u(x,s)) &\geq \eta(u(\beta(s),s)) - \eta(u(x,s)) - |\lambda^+|_T(t-s) \\ &\geq -M_1M(T)|\beta(s) - x| - |\lambda^+|_T(t-s) \\ &\geq -[MM_1M(T) + |\lambda^+|_T](t-s).\end{aligned}$$

Now let $\gamma(\theta) = x$ for $\theta \in [s,t]$. Then

$$\eta(u(x,t)) \leq \eta(u(x,s)) - \int_\gamma \lambda^+ \leq \eta(u(x,s)).$$

Hence there exist a constant $\tilde{M}(T)$ such that $|u(x,t)-u(x,s)| \leq \tilde{M}(T)(t-s)$. This together with (4.6) proves the Lemma.

Proof of Theorem 2.1 : From Lemma 4.3, $u \in W^{1,\infty}(\overline{I\!R}_+ \times [0,T])$ for any $T > 0$.

Sub soulution : Suppose u is not a subsolution, then there exist $(x_0,t_0) \in \overline{I\!R}_+ \times I\!R_+$, $\mu > 0, \varphi \in C^1(\overline{I\!R_+ \times I\!R_+})$ such that $u - \varphi$ has a local maximum at (x_0,t_0), $\quad u(x_0,t_0) = \varphi(x_0,t_0)$ and at (x_0,t_0), $\varphi_t + H(\varphi,\varphi_x) \geq 4\mu$ and $\varphi_x(x_0,t_0) < \lambda(t_0)$ if $x_0 = 0$. By continuity, there exist a $\delta > 0$ such that at (x_0,t_0), $\varphi_t + H(\varphi - \delta, \varphi_x) \geq 3\mu$ and $\varphi_x(x_0,t_0) < \lambda(t_0)$ if $x_0 = 0$. Hence from (A_3) there exist a p such that at $(x_0,t_0), \varphi_t + p\varphi_x \geq 2\mu$, $h(p) \leq \varphi(x_0,t_0) - \delta$, $\varphi_x(x_0,t_0) < \lambda(t_0)$ if $x_0 = 0$. Hence there exist a ball B centred at (x_0,t_0) such that in B

$$\varphi_t + p\varphi_x \geq \mu, \quad h(p) \leq \varphi, \quad \varphi_x < \lambda \text{ if } x_0 = 0. \tag{4.8}$$

Suppose $x_0 > 0$. By shrinking B if necessary, we can assume that $B \subset I\!R_+ \times I\!R_+$. Now choose s_0 such that for $\theta \in [s_0,t_0], (\beta(\theta),\theta) \in B$ where $\beta(\theta) = p(\theta - t_0) + x_0 \in c(x_0,s_0,t_0)$. Then from (4.3) and (4.8) we have

$$\begin{aligned}
\eta(\varphi(x_0,t_0)) = \eta(u(x_0,t_0)) &\leq \eta(u(\beta(s_0),s_0)) \bigvee \rho_3(\beta) \bigvee \rho_1(\beta) \\
&= \eta(u(\beta(s_0),s_0) \bigvee h(p)) \\
&\leq \eta(\varphi(\beta(s_0),s_0) \bigvee h(p)) \\
&\leq \eta(\varphi(\beta(s_0),s_0)) \,.
\end{aligned}$$

This implies that $\varphi(x_0,t_0) \leq \varphi(\beta(s_0),s_0)$. On the other hand, from (4.8)

$$\begin{aligned}
\varphi(x_0,t_0) - \varphi(\beta(s_0),s_0) &= \int_{s_0}^{t_0} \frac{d}{d\theta}\varphi(\beta(\theta),\theta)d\theta \\
&= \int_{s_0}^{t_0} (\varphi_t + p\varphi_x)d\theta \geq \mu(t_0 - s_0),
\end{aligned}$$

which is a contradiction.

Suppose $x_0 = 0$. If $p < 0$, like in the above proof, choose $\beta(\theta) = p(\theta - t_0) \in c(x_0,s_0,t_0)$ such that $\beta(\theta) \in B \cap (\overline{I\!R}_+ \times I\!R_+)$ and the contradiction follows. Therefore assume that $p > 0$. Suppose $\varphi_x(x_0,t_0) \leq 0$, then $2\mu \leq (\varphi_t + p\varphi_x)(x_0,t_0) \leq \varphi_t(x_0,t_0)$ and hence in B, $\quad \mu \leq \varphi_t$. This implies that if $s_0 < t_0$ such that $(x_0,s_0) \in B$, then $\mu(t_0 - s_0) \leq \varphi(x_0,t_0) - \varphi(x_0,s_0)$. But from (4.3), taking $\beta(\theta) = 0$ for $\theta \in [s_0,t_0]$ to obtain $\varphi(x_0,t_0) \leq \varphi(x_0,s_0)$ which is a contradiction. Now let $\varphi_x(x_0,t_0) > 0$ and hence $\lambda(t_0) > 0$. By shrinking B if necessary, we can assume that $\lambda > 0$. Choose s_0, t_0 such that $\{0\} \times [s_0,t_0] \subset B$. Let $\beta(\theta) = x_0 \quad$ for $\quad \theta \in [s_0,t_0]$, then from (4.3) $\eta(\varphi(x_0,t_0)) = \eta(u(x_0,t_0)) \leq \eta(\varphi(x_0,s_0)) - \int_{s_0}^{t_0} \lambda^+$ and hence

$$\begin{aligned}
(\eta'(\varphi)\varphi_t)(x_0,t_0) &= \lim_{s_0 \to t_0} \frac{\eta(\varphi(x_0,t_0)) - \eta(\varphi(x_0,s_0))}{t_0 - s_0} \\
&\leq -\lim_{s_0 \to t_0} \frac{1}{t_0 - s_0} \int_{s_0}^{t_0} \lambda^+ = -\lambda^+(t_0) = -\lambda(t_0),
\end{aligned}$$

that is, $(\varphi_t + \lambda(t_0)H(\varphi,1))(x_0,t_0) \leq 0$ and hence $\mu \leq (\varphi_t + H(\varphi,\varphi_x))(x_0,t_0) = (\varphi_t + \varphi_x H(\varphi,1))(x_0,t_0) \leq (\varphi_t + \lambda H(\varphi,1))(x_0,t_0) \leq 0$, which is a contradiction. This proves that u is a subsolution.

Super Solution : Suppose u is not a super solution,then there exist $(x_0,t_0) \in \overline{I\!R}_+ \times I\!R_+, \mu > 0, \varphi \in C^1(\overline{I\!R}_+ \times I\!R_+)$ such that $u-\varphi$ has local minimum at $(x_0,t_0), u(x_0,t_0) = \varphi(x_0,t_0)$ and $(\varphi_t + H(\varphi,\varphi_x))(x_0,t_0) \leq -4\mu, \quad \varphi_x(x_0,t_0) > \lambda(t_0)$ if $x_0 = 0$. Hence in a Ball B centred at $(x_0,t_0), \varphi_t + H(\varphi,\varphi_x) \leq -3\mu, \varphi_x > \lambda$ if $x_0 = 0$ holds. Hence form (A_6), in B,

$$\left\{\begin{array}{lll} h(p) & \leq & \varphi \text{ implies that } \varphi_t + p\varphi_x \leq -3\mu \\ \varphi_x & > & \lambda \quad \text{if } x_0 = 0\ . \end{array}\right. \tag{4.9}$$

From (2) of Lemma 4.1 and 4.3, for each $s < t_0$, choose a minimizing curve $\beta_s \in c_M(x_0,s,t_0)$. For $\theta \in [s_0,t_0], |\beta_s(\theta) - x_0| \leq (t_0 - s)|\dot{\beta}_s| \leq M(t_0 - s)$, hence choose $s_1 < t_0$ such that for any $s_1 \leq s < t_0, \quad (\beta_s(\theta),\theta) \in B$ for $\theta \in [s,t_0]$.

Claim : There exist an $s_0 \in [s_1,t_0]$ such that for almost all $\theta \in [s_0,t_0]$

$$(\varphi_t + \dot{\beta}_{s_0}\varphi_x)(\beta_{s_0}(\theta),\theta) \leq -\mu. \tag{4.10}$$

Suppose not, then there exist a sequence $s_k \to t_0, \quad \theta_k \in [s_k,t_0]$ with $\beta_k = \beta_{s_k}, p_k = \dot{\beta}_k(\theta_k)$ such that $-\mu \leq (\varphi_t + p_k\varphi_x)(\beta_k(\theta_k),\theta_k)$ holds. By going to a sub sequence assume that $p_k \to p_0$. Hence by lower semi continuity

$$\begin{array}{rcl} \eta(\varphi(x_0,t_0)) = \eta(u(x_0,t_0)) & = & \lim_{k\to\infty}(\eta(u(\beta_k(s_k),s_k))\bigvee\rho_3(\beta_k) - \int_{\beta_k}\lambda^+)\bigvee\rho_1(\beta_k) \\ & \geq & \lim_{k\to\infty}[\eta(\varphi(\beta_k(s_k),s_k))\bigvee\rho_3(\beta_k) - \int_{\beta_k}\lambda^+)\bigvee\rho_1(\beta_k)] \\ & \geq & \eta(\varphi(x_0,t_0)\bigvee h(p))\ , \end{array}$$

and hence $\varphi(x_0,t_0) \geq h(p)$. Hence from (4.9) we have $-\mu \leq \lim_{k\to\infty}(\varphi_t + p_k\varphi_x)(\beta_k(\theta_k),\theta_k) = (\varphi_t + p\varphi_x)(x_0,t_0) \leq -3\mu$, which is a contradiction and hence the claim.

Let $x_0 > 0$. Then by shrinking B if necessary such that $B \subset I\!R_+ \times I\!R_+$. Hence β_{s_0} is a line segment. Hence

$$\begin{array}{rcl} \eta(\varphi(x_0,t_0)) = \eta(u(x_0,t_0)) & = & \eta(u(\beta_{s_0}(s_0),s_0)\bigvee h(\dot{\beta}_{s_0})) \\ & \geq & \eta(\varphi(\beta_{s_0}(s_0),s_0)\bigvee h(\dot{\beta}_{s_0})), \end{array}$$

hence $\varphi(x_0,t_0) \geq \varphi(\beta_{s_0}(s_0),s_0)$. Integrating (4.10) from s_0 to t_0 to obtain $\varphi(x_0,t_0) - \varphi(\beta_{s_0}(s_0),s_0) \leq -\mu(t_0 - s_0)$ which is a contradiction.

Let $x_0 = 0$ and $\lambda(t_0) < 0$. Shrinking B if necessary, assume that $\lambda < 0$ in B and hence $\lambda^+ = 0$ in B. Hence by (4.3) it follows that $\varphi(x_0, t_0) \geq \varphi(\beta_{s_0}(s_0), s_0)$ and a contradiction is obtained from integrating (4.10) from s_0 to t_0.

Let $x_0 = 0$ and $\lambda(t_0) \geq 0$. Then $\varphi_x(x_0, t_0) > \lambda(t_0) \geq 0$. Shrinking B if necessary, can assume that $\varphi_x > 0$ in $B \cap (\overline{I\!R}_+ \times I\!R_+)$. Hence $\varphi(x, s) \geq \varphi(0, s)$. Therefore

$$\begin{array}{rcl} \eta(\varphi(x_0, t_0)) = \eta(u(x_0, t_0)) & = & \eta(u(\beta_s(s), s)) \bigvee \rho_3(\beta_s) - \int_{\beta_s} \lambda^+ \\ & \geq & \eta(\varphi(\beta_s(s), s)) - \int_{\beta_s} \lambda^+ \\ & \geq & \eta(\varphi(0, s)) - \int_s^{t_0} \lambda^+ . \end{array}$$

Hence

$$\begin{array}{rcl} (\eta'(\varphi)\varphi_t)(x_0, t_0) & = & \lim_{s \to t} \frac{\eta(\varphi(x_0, t_0)) - \eta(\varphi(0, s))}{t_0 - s} \\ & \geq & -\lambda^+(t_0) = -\lambda(t_0) . \end{array}$$

That is $0 \leq (\varphi_t + \lambda(t_0) H(\varphi, 1))(x_0, t_0) \leq (\varphi_t + \varphi_x H(\varphi, 1))(x_0, t_0) = (\varphi_t + H(\varphi, \varphi_x))(x_0, t_0) \leq -4\mu$ which is a contradiction. This proves the theorem.

In order to prove Theorem 2.2, we need following Lemmas.

Lemma 4.4 : Let $T > 0$ and ϑ be a real valued function defined on $\overline{I\!R_+ \times I\!R_+}$ such that $|\vartheta|_T = \sup\{|\vartheta(x, t)|; x \in I\!R, 0 \leq t \leq T\} < \infty$. Let $x \in \overline{I\!R}_+$ and $0 \leq s < t \leq T$. Define

$$\eta(\underline{V}(x, t)) = \inf \left\{ \eta(\vartheta(\beta(s), s)) - \int_\beta \lambda^+; \beta \in \underline{A}(\vartheta, x, s, t) \right\} \tag{4.11}$$

$$\eta(\overline{V}(x, t)) = \inf \left\{ \eta(\vartheta(\beta(s), s)) - \int_\beta \lambda^+; \beta \in \overline{A}(\vartheta, x, s, t) \right\} . \tag{4.12}$$

Then there exist a constant $M = M(T, |\vartheta|_T, |\lambda|_\infty)$ such that $|\underline{V}_T| + |\overline{V}_T| < \infty$,

$$\eta(\underline{V}(x, t)) = \inf \left\{ \eta(\vartheta(\beta(s), s)) - \int_\beta \lambda^+; \beta \in \underline{A}_M(\vartheta, x, s, t) \right\} , \tag{4.13}$$

$$\eta(\overline{V}(x, t)) = \inf \left\{ \eta(\vartheta(\beta(s), s)) - \int_\beta \lambda^+; \beta \in \overline{A}_M(\vartheta, x, s, t) \right\} . \tag{4.14}$$

(1) If $x \mapsto \vartheta(x, \theta)$ is lower semicontinous for each θ, then a minimizer $\beta \in \underline{A}_M(\vartheta, x, s, t)$ exist in (4.13) .

(2) If $x \mapsto \vartheta(x, \theta)$ is continuous, then there exist a sequence $\beta_k \in \overline{A}_M(\vartheta, x, s, t)$ converging to $\beta \in \underline{A}_M(\vartheta, x, s, t)$ in $W^{1,\infty}$ such that

$$\eta(\overline{V}(x, t)) = \eta(\vartheta(\beta(s), s)) - \int_\beta \lambda^+ \tag{4.15}$$

Proof : From $(A_5), h(0) = +\infty, h(q) \to -\infty$, as $|q| \to \infty$. Let $\beta(\theta) = x\ \forall\theta \in [s,t]$, then $h(\dot\beta) = h(0) = \infty$. Hence $\beta \in \underline{A}(\vartheta, x, s, t) \cap \overline{A}(\vartheta, x, s, t)$. Therefore $\eta(\underline{V}(x,t)) \leq \eta(\vartheta(\beta(s), s)) - \int_\beta \lambda^+ \leq \eta(|\vartheta|_T)$ and similarly $\eta(\overline{V}(x,t)) \leq \eta(|\vartheta|_T)$. Now for any $\beta \in \underline{A}(\vartheta, x, s, t), \eta(\vartheta(\beta(s), s)) - \int_\beta \lambda^+ \geq \eta(-|\vartheta|_T) - T|\lambda^+|$ and hence by taking infimum over β, $\eta(\underline{V}(x,t)) \geq \eta(-|\vartheta|_T) - T|\lambda^+|_T$. This proves $|\underline{V}|_T < \infty$ and similarly $|\overline{V}|_T < \infty$. Since $h(q) \to -\infty$ as $|q| \to \infty$, we can choose a $M > 0$ such that on $|q| \geq M$, $h(q) < -|\vartheta|_T$. Hence if $\beta \in c(x,s,t)$ such that $|\dot\beta| \geq M$, then either $h(\dot\beta_1) < -|\vartheta|_T$ or $h(\dot\beta_3) < -|\vartheta|_T$. In either case $\beta \notin \underline{A}(\vartheta, x, s, t)$ and $\beta \notin \overline{A}(\vartheta, x, s, t)$ .This proves (4.13) and (4,14).

Let $\{\beta_k\} \subset \underline{A}_M(\vartheta, x, s, t)$ be a minimizing sequence in (4.13). Extract a subsequence converging to $\beta \in c_M(\vartheta, x, s, t)$ in $W^{1,\infty}$. From (A_4) h is an upper semicontinuous and $x \mapsto \vartheta(x,\theta)$ is lower semicontinous, it follows that $\beta \in \underline{A}_M(\vartheta, x, s, t)$ and $\eta(\underline{V}(x,t)) = \eta(\vartheta(\beta(s), s)) - \int_\beta \lambda^+$. This proves (1). Apply the similar argument for $\overline{V}(x,t)$ to conclude (4.15). The only difference is that in this case β need not be in $\overline{A}_M(\vartheta, x, s, t)$. This proves the Lemma.

Lemma 4.5 : Let $\underline{u}$ be as in (2.16). Let $T > 0$, $x \in \overline{I\!R}_+$, $0 \leq s < t \leq T$. Then $|\underline{u}|_T < \infty$, $x \mapsto \underline{u}(x,\theta)$ is lower semicontinuous and there exist an $M = M(|u|_T, |\lambda^+|_T, T)$ such that

$$\underline{u}(x,t) \leq \inf \eta^{-1}\left\{\eta(\underline{u}(\beta(s),s)) - \int_\beta \lambda^+; \beta \in \underline{A}_M(\underline{u}^*, x, s, t)\right\},$$

$$\eta(\underline{u}^*(x,t)) \leq \inf\left\{\eta(\underline{u}^*(\beta(s),s)) - \int_\beta \lambda^+; \beta \in \overline{A}_M(\underline{u}^*, x, s, t)\right\}. \tag{4.16}$$

Proof : Since $|u_0|_T < \infty$ then from Lemma 4.4, $|\underline{u}_T| < \infty$. Let $(x_m, t_m) \to (x_0, t_0)$ as $m \to \infty$. Since u_0 is continuous, from (1) of Lemma 4.4 there exist a $M_1 > 0$ and a minimizer $\beta_m \in \underline{A}_{M_1}(u_0, x_m, t_m)$ of (2.16). Hence by going to a subsequence, let $\{\beta_m\}$ converges to $\beta \in \underline{A}_{M_1}(u_0, x_0, t_0)$ in $W^{1,\infty}$. Hence

$$\begin{array}{rcl}\eta(\underline{u}(x_0,t_0)) & \leq & \eta(u_0(\beta(0))) - \int_\beta \lambda^+ \\ & = & \underline{\lim}_{m\to\infty}\left\{\eta(u_0(\beta_m(0))) - \int_{\beta_m} \lambda^+\right\} \\ & = & \underline{\lim}_{m\to\infty}\eta(\underline{u}(x_m,t_m)),\end{array}$$

and this proves $\underline{u}$ is lower semicontinuous. Next define ϑ_1 by

$$\eta(\vartheta_1(x,t)) = \inf\left\{\eta(\underline{u}(\beta(s),s)) - \int_\beta \lambda^+; \beta \in \underline{A}(\underline{u}, x, s, t)\right\} \tag{4.17}$$

Step-1 : $\vartheta_1(x,t) \geq \underline{u}(x,t)$.

Since $|\underline{u}|_T < \infty$ and $x \mapsto \underline{u}(x,\theta)$ is lower semicontinuous, hence from (1) of Lemma 4.4, there exist $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \underline{A}(\underline{u}, x, s, t)$ defined on the partition $s = t_3 \leq t_2 \leq t_1 \leq t_0 = t$

and $\beta = (\beta_1, \beta_2, \beta_3) \in \underline{A}(u_0, \alpha(s), s)$ defined on the partition $0 = s_3 \le s_2 \le s_1 \le s_0 = s$ such that $\eta(\vartheta_1(x,t)) = \eta(\underline{u}(\alpha(s),s)) - \int_\alpha \lambda^+, \quad \eta(\underline{u}(\alpha(s),s)) = \eta(u_0(\beta(0))) - \int_\beta \lambda^+$. Hence

$$\eta(\vartheta_1(x,t)) = \eta(u_0(\beta(0))) - \int_\alpha \lambda^+ - \int_\beta \lambda^+. \tag{4.18}$$

Suppose $[t_2, t_1] \neq \phi, [s_2, s_1] \neq \phi$. Define $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in c(x,t)$ by $\gamma_1 = \alpha_1, \quad \gamma_2|_{[s_2,t_1]} = 0, \quad \gamma_3 = \beta_3$. Then $u_0(\gamma(0)) = u_0(\beta(0)) \le h(\dot{\beta}_3) = h(\dot{\gamma}_3), \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ = \eta(u_0(\beta(0))) - \int_\gamma \lambda^+ \le \eta(u_0(\beta(0))) - \int_\beta \lambda^+ - \int_\alpha \lambda^+ = \eta(\underline{u}(\alpha(s),s)) - \int_\alpha \lambda^+ \le \eta(h(\dot{\alpha}_1)) = \eta(h(\dot{\gamma}_1))$. Hence $\gamma \in \underline{A}(u_0, x, t)$ and from (4.18) we have $\eta(\vartheta_1(x,t)) = \eta(u_0(\gamma(0))) - \int_\alpha \lambda^+ - \int_\beta \lambda^+ \ge \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ \ge \eta(\underline{u}(x,t))$. Hence $\vartheta_1 \ge \underline{u}$.

Suppose $[t_2, t_1] \neq \phi, [s_2, s_1] = \phi$. Then β is a line segment and $\int_\beta \lambda^+ = 0$. Hence $\underline{u}(\alpha(s), s) = u_0(\beta(0))$ and $u_0(\beta(0)) \le h(\dot{\beta})$. Now define $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in c(x,t)$ by $\gamma_1 = \alpha_1, \gamma_2 = \alpha_2, \gamma_3(\theta) = \frac{t_2 - \theta}{t_2}\beta(0)$ for $\theta \in [0, t_2]$. Suppose $\dot{\alpha_3} \le \dot{\beta}$, then by construction $h(\dot{\alpha_3}) \le h(\dot{\gamma_3})$ and hence $u_0(\gamma(0)) = u_0(\beta(0)) = \underline{u}(\alpha(s), s) \le h(\dot{\alpha}_3) \le h(\dot{\gamma}_3)$ and $\eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ = \eta(\underline{u}(\alpha(s), s)) - \int_\alpha \lambda^+ \le \eta(h(\dot{\alpha_1})) = \eta(h(\dot{\gamma_1}))$. Hence $\gamma \in \underline{A}(u_0, x, t)$. If $\dot{\beta} \le \dot{\alpha}_3 \le 0$, then clearly $h(\dot{\beta}) \le h(\dot{\gamma}_3)$ and hence $(\gamma(0))) - \int_\gamma \lambda^+ = \eta(\underline{u}(\alpha(s), s)) - \int_\alpha \lambda^+ \le \eta(h(\dot{\alpha}_1)) = \eta(h(\dot{\gamma}_1))$. Hence $\gamma \in \underline{A}(u_0, x, t)$. This implies that $\eta(\vartheta_1(x,t)) = \eta(u_0(\beta(0))) - \int_\alpha \lambda^+ = \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ \ge \eta(\underline{u}(x,t))$.Hence $\vartheta_1(x,t) \ge \underline{u}(x,t)$.

Suppose $[t_2, t_1] = \phi, [s_2, s_1] = \phi$, then α and β are line segments with $\int_\alpha \lambda^+ = \int_\beta \lambda^+ = 0, \quad \vartheta_1(x,t) = \underline{u}(\alpha(s), s) = u_0(\beta(0)) \le \min(h(\dot{\alpha}), h(\dot{\beta}))$. Let $\gamma(\theta) = \frac{\theta}{t}x + (1 - \frac{\theta}{t})\beta(0)$ for $\theta \in [0, t]$, clearly $\gamma \in c(x,t)$ and by quasi concavity of h, $\min(h(\dot{\alpha}), h(\dot{\beta})) \le h(\dot{\gamma})$ and hence $\gamma \in \underline{A}(u_0, x, t)$ with $\int_\gamma \lambda^+ = 0$. Hence $\eta(\vartheta_1(x,t)) = \eta(u_0(\beta(0))) = \eta(u_0(\gamma(0))) \ge \eta(\underline{u}(x,t))$. This implies that $\vartheta_1(x,t) \ge \underline{u}(x,t)$. Suppose $[t_2, t_1] = \phi, \ [s_2, s_1] \neq \phi$. Then $\vartheta_1(x,t) = \underline{u}(\alpha(s), s)$. Define $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in c(x,t)$ by $\gamma_1(\theta) = \frac{\theta - s_1}{t - s_1}x_1$ if $\theta \in [s_1, t], \gamma_2 = \beta_2, \gamma_3 = \beta_3$. Hence $u_0(\gamma(0)) = u_0(\beta(0)) \le h(\dot{\beta}_3) = h(\dot{\gamma}_3)$. Now $\underline{u}(\alpha(s), s) \le h(\dot{\alpha})$ and $\eta(\underline{u}(\alpha(s), s) - \int_\beta \lambda^+ \le \eta(h(\beta_1))$. Hence $\eta(\underline{u}(\alpha(s), s)) - \int_\beta \lambda^+ \le \min(\eta(h(\dot{\beta}_1)), \eta(h(\dot{\alpha}))) \le \eta(h(\dot{\gamma}))$ since $\dot{\gamma}$ is a convex combination of $\dot{\alpha}$ and $\dot{\beta}_1$. Hence $\gamma \in \underline{A}(u_0, x, t)$ and $\eta(\vartheta_1(x,t)) = \eta(u_0(\beta(0))) - \int_\beta \lambda^+ = \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ \ge \eta(\underline{u}(x,t))$. Hence $\vartheta_1(x,t) \ge \underline{u}(x,t)$ and this proves the step-1.

Step-2. Let $(x_m, t_m) \to (x,t)$ such that $\underline{u}^*(x,t) = \lim_{m\to\infty} \underline{u}(x_m, t_m)$. Let $\beta = (\beta_1, \beta_2, \beta_3) \in \overline{A}(\underline{u}^*, x, t, s)$ defined on the partition $s = t_3 \le t_2 \le t_1 \le t_0 = t$. Now for each m , define $\beta^{(m)} = (\beta_1^{(m)}, \beta_2^{(m)}, \beta_3^{(m)})$ by $\beta_1^{(m)}(\theta) = \frac{\theta - t_1}{t_m - t_1}x_m$ if $\theta \in [t_1, t]$ and $\beta^{(m)}|_{[0,t_1]} = \beta$. Then $\beta^{(m)} \in c(x_m, s, t_m)$ and $\beta^{(m)} \to \beta$ in $W^{1,\infty}$. Hence by upper semicontinuity of $\underline{u}^*$ and h it follows that there exist an $m_0 > 0$ such that for all $m \ge m_0, \quad \beta_m \in \overline{A}(\underline{u}, x_m, t_m, s)$. Hence by the step-1,

$$
\begin{aligned}
\underline{u}^*(x,t) &= \lim_{m\to\infty} \underline{u}(x_m,t_m) \le \lim_{m\to\infty} \vartheta_1(x_m,t_m) \\
&\le \lim_{m\to\infty} \eta^{-1}\left\{\eta(\underline{u}(\beta^m(s),s)) - \textstyle\int_{\beta^m}\lambda^+\right\} \\
&\le \lim_{m\to\infty} \eta^{-1}\left\{\eta(\underline{u}^*(\beta^m(s),s)) - \textstyle\int_{\beta^m}\lambda^+\right\} \\
&\le \eta^{-1}\left\{\eta(\underline{u}^*(\beta(s),s)) - \textstyle\int_{\beta}\lambda^+\right\}.
\end{aligned}
$$

Hence by taking infimum over all β and from (2) of Lemma 4.4 to obtain (4.16). This proves the Lemma.

Lemma 4.6 : Let $\overline{u}$ be as in (2.17). Let $T > 0, x \in I\!\!R_+, 0 \le s < t \le T$. Then $|\overline{u}|_T < \infty, x \mapsto \overline{u}(x,\theta)$ is upper semicontinuous and there exist an $M = M(|\overline{u}|_T, |\lambda^+|_T, T)$ such that

$$\eta(\overline{u}(x,t)) \ge \inf\left\{\eta(\overline{u}(\beta(s),s)) - \int_\beta \lambda^+; \beta \in \overline{A}_M(\overline{u},x,s,t)\right\},$$

$$\eta(\overline{u}_*(x,t)) \ge \inf\left\{\eta(\overline{u}_*(\beta(s),s)) - \int_\beta \lambda^+; \beta \in \underline{A}_M(\overline{u}_*,x,s,t)\right\}.$$

Proof : Since $|u_0|_T < \infty$ and hence Lemma 4.4 implies that $|\overline{u}|_T < \infty$. Let $(x_m,t_m) \to (x,t)$ as $m \to \infty$. Since u_0 is continuous, from (2) of Lemma 4.4, there exist $M_1 > 0$ and a sequence $\beta^{(k)} \in \overline{A}_{M_1}(u_0,x,t)$ such that $\beta^{(k)} \to \beta$ in $W^{1,\infty}$ and $\eta(\overline{u}(x,t)) = \eta(u_0(\beta(0))) - \int_\beta \lambda^+$. Let $\beta^{(k)} = (\beta_1^{(k)}, \beta_2^{(k)}, \beta_3^{(k)})$ be defined on the partition $s = t_{3k} \le t_{2k} \le t_{1k} \le t_{0k} = t$. For m large, define $\beta^{m,k}$ as follows :

case (i) : $x > 0$. If $[t_{2k}, t_{1k}] = \phi$, then $\beta^{(k)}$ is a line segment. Define $\beta^{m,k} \in c(x_m,t_m)$ the line segment parallel to $\beta^{(k)}$. In this case $\beta^{m,k} \to \beta^{(k)}$ as $m \to \infty$ in $W^{1,\infty}$ and $\dot\beta^{m,k} = \dot\beta^{(k)}$ for all large m. If $[t_{2k}, t_{1k}] \ne \phi$, let $\beta^{m,k} = (\beta_1^{m,k}, \beta_2^{m,k}, \beta_3^{m,k})$ such that $\beta_1^{m,k}$ is the line segment starting at (x_m,t_m) and parallel to $\beta_1^{(k)}$. Let $(0, t_{1,m,k})$ be the end point of $\beta_1^{m,k}$. Then $\beta^{m,k} \to \beta^{(k)}$ in $W^{1,\infty}, \dot\beta_i^{m,k} = \dot\beta_i^{(k)}$ for $i = 1,2,3$ and $\int_{\beta^{m,k}}\lambda^+ \to \int_{\beta^k}\lambda^+$ as $m \to \infty$. $\beta_2^{m,k} = 0$ in $[t_{2k}, t_{1,m,k}]$ if $t_{1,m,k} \ge t_{2k}$ and $\beta^{m.k} = \beta^{(k)}$ in $[0, t_{2k}]$. If $t_{1,m,k} \in [t_{2k}, t_{1k}]$,then $\beta^{m,k} = \beta^{(k)}$ in $[0, t_{1,m,k}]$. If $t_{1,m,k} < t_{2k}$ then $\beta_3^{m,k}$ is the line segment parallel to $\beta_3^{(k)}$ in $[0, t_{1,m,k}]$.

case (ii) : $x = 0$. Choose $p_0 > 0$ such that $h(p_0) > \eta(|u_0|_T)$. If $x_m = 0$, define $\beta^{m,k} \in c(x_m,t_m)$ as follows. If $\beta^{(k)}$ is a line segment, then $\beta^{m,k}$ is the line segment parallel to β^k. If $\beta^{(k)}$ is not a line segment, then if $x_m > 0, \beta_1^{m,k}(\theta) = \frac{\theta - t_{2k}}{t_m - t_{2k}} x_m$ for $\theta \in [t_{2k}, t_m]$ and $\beta^{m,k} = \beta^{(k)}$ in $[0, t_{2k}]$. If $x_m = 0$, then $\beta_2^{m,k}(\theta) = 0$ in $[t_{2k}, t_m]$ and $\beta^{m,k} = \beta^{(k)}$ in $[0, t_{2k}]$. Now for m large and $x_m > 0, \dot\beta_1^{m,k} = \frac{x_m}{t_m - t_{2k}} < p_0$ and hence $h(\dot\beta_1^{m,k}) \ge h(p_0) > \eta(|u_0|_T) > \eta(u_0(\beta^{m,k}(0)))$. Also $\beta^{m,k} \to \beta^{(k)}$ in $W^{1,\infty}$.

Now by construction, upper semicontinuity of h and by continuity of u_0, it follows easily that there exist $m_k > 0$ such that for all $m \geq m_k, \beta^{m,k} \in \overline{A}(u_0, x_m, t_m)$. Hence

$$\begin{array}{rcl}\overline{\lim}_{m\to\infty}\eta(\overline{u}(x_m,t_m)) & \leq & \lim_{k\to\infty}\overline{\lim}_{m\to\infty}\left\{\eta(u_0(\beta^{m,k}(0))) - \int_{\beta^{m,k}}\lambda^+\right\} \\ & = & \lim_{k\to\infty}\eta(u_0(\beta^{(k)}(0))) - \int_{\beta^k}\lambda^+ \\ & = & \eta(u_0(\beta(0))) - \int_\beta \lambda^+ \\ & = & \eta(\overline{u}(x,t)).\end{array}$$

This proves that $x \mapsto \overline{u}(x,t)$ is upper semicontinuous.

Claim 1 : Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \overline{A}(u_0, x, t)$ defined on the partition $0 = t_3 \leq t_2 \leq t_1 \leq t_0 = t$ and $\alpha = \gamma|_{[s,t]}$ and $\beta = \gamma|_{[0,s]}$. Then $\alpha \in \overline{A}(\overline{u}, x, s, t)$ and $\beta \in \overline{A}(u_0, \alpha(s), s)$.

Since $\gamma \in \overline{A}(u_0, x, t)$, hence

$$u_0(\gamma(0)) < h(\dot{\gamma}_3), \quad \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ < \eta(h(\dot{\gamma}_1)). \tag{4.19}$$

Case (i) : $[t_2, t_1] \neq \phi$.

Let $s \in (t_1, t)$ then $\dot{\gamma}_1 = \dot{\alpha} = \dot{\beta}_1,\ \dot{\gamma}_3 = \dot{\beta}_3, \quad \int_\gamma \lambda^+ = \int_\beta \lambda^+$. Hence from (4.19) $\eta(u_0(\beta(0))) - \int_\beta \lambda^+ < \eta(h(\dot{\beta}_1)), \quad u_0(\beta(0)) < h(\dot{\beta}_3)$ and hence $\beta \in \overline{A}(u_0, \alpha(s), s)$. $\eta(\overline{u}(\beta(s), s)) \leq \eta(u_0(\beta(0))) - \int_\beta \lambda^+ < \eta(h(\dot{\beta}_1)) = \eta(h(\dot{\alpha}))$, implies that $\overline{u}(\alpha(s), s) = \overline{u}(\beta(s), s) < h(\dot{\alpha}_1)$. Hence $\alpha \in \overline{A}(\overline{u}, x, s, t)$. Let $s \in [t_2, t_1]$, then $\dot{\gamma}_1 = \dot{\alpha}_1, \dot{\gamma}_3 = \dot{\beta}_3, \int_\alpha \lambda^+ + \int_\beta \lambda^+ = \int_\gamma \lambda^+,\ \beta_1 = \phi, \alpha_3 = \phi$. Hence from (4.19) $u_0(\beta(0)) < h(\dot{\beta}_3)$ implies that $\beta \in \overline{A}(u_0, \alpha(s), s)$. Hence $\eta(\overline{u}(\alpha(s), s)) - \int_\alpha \lambda^+ \leq \eta(u_0(\beta(0))) - \int_\beta \lambda^+ - \int_\alpha \lambda^+ = \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ < h(\dot{\alpha}_1)$. Hence $\alpha \in \overline{A}(\overline{u}, x, s, t)$. Let $s \in [0, t_2)$. Then $\dot{\gamma}_1 = \dot{\alpha}_1,\ \dot{\gamma}_3 = \dot{\alpha}_3 = \dot{\beta},\ \beta_1 = \beta_2 = \phi$. Hence from (4.19) $\beta \in \overline{A}(u_0, \alpha(s), s)$ and hence $\eta(\overline{u}(\alpha(s), s)) \leq \eta(u_0(\beta(0))$. This implies that $\eta(\overline{u}(\alpha(s), s)) - \int_\alpha \lambda^+ \leq \eta(u_0(\beta(0))) - \int_\alpha \lambda^+ = \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ < \eta(h(\dot{\gamma}_1)) = \eta(h(\dot{\alpha}_1))$. Hence $\alpha \in \overline{A}(\overline{u}, x, s, t)$.

Case (ii) $[t_2, t_1] = \phi$

Then α, β, γ are line segments. If $\dot{\gamma} = 0$ then $\dot{\alpha} = \dot{\beta} = 0$. Hence $\alpha \in \overline{A}(u_0, x, s, t), \beta \in \overline{A}(\overline{u}, \alpha(s), s)$ since $h(0) = \infty$. Suppose $\dot{\gamma} < 0$, then $\int_\alpha \lambda^+ = 0$, $\dot{\alpha} = \dot{\beta} = \dot{\gamma} < 0$ and $u_0(\beta(0)) = u_0(\gamma(0))) < h(\dot{\gamma}) = h(\dot{\beta})$. Hence $\beta \in \overline{A}(u_0, \alpha(s), s)$ and this implies that $\eta(\overline{u}(\alpha(s), s)) \leq \eta(u_0(\beta(0)))$. Therefore $\overline{u}(\alpha(s), s) \leq u_0(\beta(0)) < h(\dot{\beta}) = h(\dot{\alpha})$. Hence $\alpha \in \overline{A}(\overline{u}, x, s, t)$

Suppose $\dot{\gamma} > 0$, then $\dot{\alpha} = \dot{\beta} = \dot{\gamma}$, $\int_\gamma \lambda^+ = 0$ and $\eta(u_0(\beta(0))) < \eta(h(\dot{\gamma})) = \eta(h(\dot{\beta}))$. Hence $\beta \in \overline{A}(u_0, \alpha(s), s)$ and $\eta(\overline{u}(\alpha(s), s)) \leq \eta(u_0(\beta(0))) < \eta(h(\dot{\alpha}))$. Hence $\alpha \in \overline{A}(\overline{u}, x, s, t)$. This proves the claim.

From the claim it follows that the set

$$D = \left\{(\alpha, \beta); \alpha = \gamma|_{[s,t]}, \beta = \gamma|_{[0,s]}, \gamma \in \overline{A}(u_0, x, t)\right\} \subset \left\{(\alpha, \beta); \alpha \in \overline{A}(\overline{u}, x, s, t), \beta \in \overline{A}(u_0, \alpha(s), s)\right\}.$$

Hence if we define $\vartheta_1(x,t)$ as follows,then from (4.14)

$$\begin{array}{rcl}
\eta(\vartheta_1(x,t)) & = & \inf\left\{\eta(\overline{u}(\alpha(s),s)) - \int_\alpha \lambda^+; \alpha \in \overline{A}(\overline{u},x,s,t)\right\} \\
& = & \inf_\alpha \left\{\inf_\beta \left\{\eta(u_0(\beta(0))) - \int_\beta \lambda^+; \beta \in \overline{A}(u_0,\alpha(s),s)\right\} - \int_\alpha \lambda^+; \alpha \in \overline{A}(\overline{u},x,s,t)\ \right\} \\
& \leq & \inf\left\{\eta(u_0(\beta(0))) - \int_\alpha \lambda^+ - \int_\beta \lambda^+; (\alpha,\beta) \in D\right\} \\
& = & \inf\left\{\eta(u_0(\gamma(0))) - \int_\gamma \lambda^+; \gamma \in \overline{A}(u_0,x,t)\right\} \\
& = & \eta(\overline{u}(x,t)).
\end{array} \tag{4.20}$$

Hence the first inequaliy of the Lemma.

Let $(x_k,t_k) \to (x,t)$ such that $\overline{u}_*(x,t) = \lim_{k\to\infty} \overline{u}(x_k,t_k)$. Let $\epsilon > 0$, from (4.20) choose $\beta^{(k)} \in \overline{A}_M(\overline{u},x_k,s,t_k)$ and $k_0 > 0$ such that for all $k \geq k_0$

$$\eta(\overline{u}_*(x,t)) \geq \eta(\overline{u}(x_k,t_k)) - \epsilon/2 \tag{4.21}$$

$$\eta(\overline{u}(x_k,t_k)) > \eta(\overline{u}(\beta^{(k)}(s),s)) - \int_{\beta^{(k)}} \lambda^+ - \epsilon/2. \tag{4.22}$$

Extract a subsequence still denoted by $\beta^{(k)}$ converging to β in $W^{1,\infty}$. Then

$$\begin{array}{rcl}
\overline{u}_*(\beta(s),s) & \leq & \underline{\lim}_{k\to\infty}\overline{u}_*(\beta^{(k)}(s),s) \\
& \leq & \lim_{k\to\infty}\overline{u}(\beta^{(k)}(s),s) \\
& \leq & \overline{\lim}_{k\to\infty}h(\dot{\beta}_3^{(k)}). \\
& \leq & h(\dot{\beta}_3) \\
\eta(\overline{u}_*(\beta(s),s)) - \int_\beta \lambda^+ & \leq & \underline{\lim}_{k\to\infty}\left\{\eta(\overline{u}_*(\beta^{(k)}(s),s) - \int_{\beta^{(k)}} \lambda^+\right\} \\
& \leq & \underline{lim}_{k\to\infty}\{\eta(\overline{u}(\beta^k(s),s)) - \int_{\beta^{(k)}} \lambda^+\} \\
& \leq & \overline{\lim}_{k\to\infty}\eta(h(\dot{\beta}_1{}^{(k)})) \\
& \leq & \eta(h(\dot{\beta}_1)).
\end{array}$$

Hence $\beta \in \underline{A}(\overline{u}_*,x,s,t)$ and from (4.21) and (4.22)

$$\begin{array}{rcl}
\eta(\overline{u}_*(x,t)) & \geq & \lim_{k\to\infty}\eta(\overline{u}(x_k,t_k)) - \epsilon/2 \\
& \geq & \lim_{k\to\infty}(\eta(\overline{u}(\beta^{(k)}(s),s)) - \int_{\beta^k} \lambda^+) - \epsilon \\
& \geq & \lim_{k\to\infty}(\eta(\overline{u}_*(\beta^{(k)}(s),s)) - \int_{\beta^k} \lambda^+) - \epsilon \\
& \geq & (\eta(\overline{u}_*(\beta(s),s)) - \int_\beta \lambda^+) - \epsilon \\
& \geq & \inf\left\{\eta(\overline{u}_*(\gamma(s),s)) - \int_\gamma \lambda^+; \gamma \in \underline{A}_M(\overline{u}_*,x,s,t)\right\} - \epsilon.
\end{array}$$

Since ϵ is arbitrary and hence the Lemma follows.

Lemma 4.7 : Let $\underline{u}$ and $\overline{u}$ be as in (2.16) and (2.17) respectivly. Then $\underline{u}^* = \overline{u}$ and $\overline{u}_* = \underline{u}$.

Proof: Proof is divided into three steps.

Step 1 : Let $\alpha > 1$ and $\{q_k\}$ be a bounded sequence. Then $\underline{\lim}_{k\to\infty}\, h(\alpha q_k) < \underline{\lim}_{k\to\infty}h(q_k)$.

Suppose not, then let for a subsequence still denoted by $\{q_k\}$ such that

$$q_k \to q_0, \quad \lim_{k\to\infty} h(\alpha q_k) = \lim_{k\to\infty} h(q_k) = \eta$$

. Choose $|p_k| = |\tilde{p}_k| = 1$ such that for all $|p| = 1$

$$\begin{array}{lcll} (i) \quad H(h(q_k), p_k) & = & < q_k, p_k >, & H(h(q_k), p) \geq < q_k, p > \\ (ii) \quad H(h(\alpha q_k), \tilde{p}_k) & = & \alpha < q_k, \tilde{p}_k > . & H(h(\alpha q_k), p) \geq \alpha < q_k, p > . \end{array}$$

Again going to a subsequence, can assume that $p_k \to p_0$ $\tilde{p}_k \to \tilde{p}_0$ as $k \to \infty$. Then by continuity of H, $< q_0,\ p_0 >= H(\eta, p_0) = \lim_{k\to\infty} H(h(\alpha q_k), p_0) \geq \alpha < q_0, p_0 >$. Since $H(\eta, p_0) > 0$, it follows that $\alpha \leq 1$ which is a contradiction. This proves step 1.

Step 2: Let $\tau > t$. Then $\underline{u}(x,t) \geq \overline{u}(x,\tau)$.

Let $\beta = (\beta_1, \beta_2, \beta_3) \in \underline{A}(u_0, x, t)$ defined on the partition $0 = t_3 \leq t_2 \leq t_1 \leq t_0 = t$ be a minimiser for $\underline{u}$. Then $\eta(\underline{u}(x,t)) = \eta(u_0(\beta(0))) - \int_\beta \lambda^+$, $\eta(u_0(\beta(0))) - \int_\beta \lambda^+ \leq \eta(h(\dot{\beta}_1))$ and $u_0(\beta(0)) \leq h(\dot{\beta}_3)$. Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \overline{A}(u_0, x, \tau)$ be defined as follows.

Case(i): Suppose $u_0(\beta(0)) < h(\dot{\beta}_3)$. Then $\gamma \in \overline{A}(u_0, x, \tau)$ is given by

$$\begin{array}{lcl} \gamma_1(\theta) & = & x + (\theta - \tau)\frac{x}{\tau - t_1} \quad \text{if} \quad \theta \in [t_1, \tau] \\ \gamma_2(\theta) & = & \beta_2(\theta) \quad \text{if} \quad \theta \in [t_2, t_1] \\ \gamma_3(\theta) & = & \beta_3(\theta) \quad \text{if} \quad \theta \in [0, t_2]. \end{array}$$

This implies $u_0(\gamma(0)) = u_0(\beta(0)) < h(\dot{\beta}_3) = h(\dot{\gamma}_3)$, $\eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ = \eta(u_0(\beta(0)) - \int_\beta \lambda^+ \leq \eta(h(\dot{\beta}_1)) = \eta(h(\frac{x}{t-t_1})) = \eta(h(\frac{x}{\tau-t_1}\frac{\tau-t_1}{t-t_1})) < \eta(h(\frac{x}{\tau-t_1}))$,by step-1. Hence $\gamma \in \overline{A}(u_0, x, \tau)$ and $\underline{u}(x,t) \geq \overline{u}(x,\tau)$.

Case(ii): Suppose $u_0((\beta(0)) = h(\dot{\beta}_3)$. Then

$$\begin{array}{lcl} \eta(h(\frac{-\beta(0)}{t_2})) - \int_{t_2}^{t_1} \lambda^+ & = & \eta(u_0(\beta(0)) - \int_\beta \lambda^+ \\ & \leq & \eta(h(\dot{\beta}_1)) \\ & < & \eta(h(\frac{x}{\tau-t_1})). \end{array}$$

Let $A = \left\{\mu; \mu < t_1, \eta(h(\frac{-\beta(0)}{\mu})) - \int_\mu^{t_1} \lambda^+ < \eta(h(\frac{x}{\tau-t_1}))\right\}$. h is an upper semicontinuous function implies A is open and hence there exists ε_0 such that $\forall \mu \in [t_2, t_2+\varepsilon_0)$, $\eta(h(-\frac{\beta(0)}{\mu})) - \int_\mu^{t_1} \lambda^+ < \eta(h(\frac{x}{\tau-t_1}))$. Let $\mu \in (t_2, t_2+\varepsilon_0)$. Define now γ by

$$\begin{array}{lcl} \gamma_1(\theta) & = & x + (\theta - \tau)\frac{x}{\tau - t_1} \quad \text{if} \quad \theta \in [t_1, \tau] \\ \gamma_2(\theta) & = & 0 \quad \text{if} \quad \theta \in [\mu, t_1] \\ \gamma_3(\theta) & = & \beta(0) + \frac{-\beta(0)}{\mu}\theta \quad \text{if} \quad \theta \in [0, \mu]. \end{array}$$

Then $u_0(\gamma(0)) = u_0(\beta(0)) = h(-\frac{\beta(0)}{t_2}) = h(-\frac{\beta(0)}{\mu}\frac{\mu}{t_2}) < h(-\frac{\beta(0)}{\mu}) = h(\dot{\gamma}_3)$ by step-1. Hence $\eta(u_0(\gamma(0)) - \int_\gamma \lambda^+ = \eta(u_0(\beta(0)) - \int_\mu^{t_1} \lambda^+ < h(-\frac{\beta(0)}{\mu}) - \int_\mu^{t_1} \lambda^+ < h(\dot{\gamma}_3)$. This implies $\gamma \in \overline{A}(u_0, x, \tau)$. Hence $\underline{u}(x,t) \geq \overline{u}(x,\tau)$.

Step 3: Let $B_r(x,t)$ be a ball centered at (x,t) with radius r. Let $t>0$ and let $t_k<t$ and $t_k\to t$. Then from step (2)

$$\begin{array}{rcl}\underline{u}^*(x,t) & = & \lim_{r\to 0}\sup_{B_r}\underline{u}(z)\\ & \geq & \lim_{k\to\infty}\underline{u}(x,t_k)\geq \overline{u}(x,t).\end{array}$$

On the other hand $\underline{u}\leq\overline{u}$ and hence $\underline{u}^*(x,t)\leq\overline{u}(x,t)$,implies that $\underline{u}^*(x,t)=\overline{u}(x,t)$. Similarly $\overline{u}_*=\underline{u}$. This proves the Lemma.

Lemma 4.8 :(dynamic programming principle) Let $\underline{u}$ and $\overline{u}$ be defined as in (2.16) and (2.17) respectively. Then

$$\begin{array}{rcl}\underline{u}(x,t) & = & \inf\eta^{-1}\left\{\eta(\underline{u}(\beta(s),s))-\int_\beta\lambda^+;\beta\in\underline{A}_M(\underline{u},x,s,t)\right\},\\ \overline{u}(x,t) & = & \inf\eta^{-1}\left\{\eta(\overline{u}(\beta(s),s))-\int_\beta\lambda^+;\beta\in\overline{A}_M(\overline{u},x,s,t)\right\}.\end{array}$$

Proof: Proof follows from Lemma 4.5,Lemma 4.6 and Lemma 4.7.

Proof of theorem 2.2 : Let $0<t\leq T$ and $x\in\overline{I\!R}_+$. From (1) and (2) of Lemma 4.4, there exist $M(T)>0$, $\alpha^{(t)},\beta^{(t)}\in C_{M(T)}(x,t)$ such that $\eta(\underline{u}(x,t))=\eta(u_0(\beta^{(t)}(0)))-\int_{\beta^{(t)}}\lambda^+$, $\eta(\overline{u}(x,t))=\eta(u_0(\alpha^{(t)}(0)))-\int_{\alpha^{(t)}}\lambda^+$. Now $|(x,x)-(\alpha^{(t)},\beta^{(t)})|=|\int_0^t(\dot{\alpha}^{(t)}(\theta),\dot{\beta}^{(t)}(\theta))\,d\theta|\leq M(T)t$. This implies that $\lim_{t\to 0}(\overline{u}(x,t),\underline{u}(x,t))=(u_0(x),u_0(x))$.

Sub solution : Suppose $\underline{u}$ is not a sub solution. Then there exist $(x_0,t_0)\in(\overline{I\!R}_+\times I\!R_+),\mu>0,\varphi\in C^1(\overline{I\!R_+\times I\!R_+})$ such that $\underline{u}^*(x_0,t_0)=\varphi(x_0,t_0)$, $\underline{u}^*-\varphi$ has local maximum at (x_0,t_0) and at (x_0,t_0), $\varphi_t+H(\varphi,\varphi_x)\geq 4\mu$ and $\varphi_x(x_0,t_0)<\lambda(t_0)$ if $x_0=0$. By continuity, there exist $\delta>0$ such that at (x_0,t_0), $\varphi_t+H(\varphi+\delta,\varphi_x)\geq 3\mu,\varphi_x(x_0,t_0)<\lambda(t_0)$ if $x_0=0$. Hence from (A_6) there exist a p such that at $(x_0,t_0),\varphi_t+p\varphi_x\geq 2\mu,\varphi(x_0,t_0)+\delta\leq h(p)$ and $\varphi_x(x_0,t_0)<\lambda(t_0)$ if $x_0=0$. Hence there exist a ball B around (x_0,t_0) such that for all $(x,t)\in B\cap(\overline{I\!R}_+\times I\!R_+)$,

$$\varphi_t+p\varphi_x\geq\mu,\ \varphi\leq h(p),\ \varphi_x<\lambda\text{ if }x_0=0. \tag{4.23}$$

Suppose $x_0>0$. Then by shrinking B if necessary, can assume that $B\subset I\!R_+\times I\!R_+$. Let $\beta(\theta)=p(\theta-t_0)+x_0$ and choose $s_0<t_0$ such that $(\beta(\theta),\theta)\in B$ for $\theta\in[s_0,t_0]$. Since $\underline{u}^*\leq\varphi\leq h(p)=h(\dot{\beta})$ and $\int_\beta\lambda^+=0$, it follows from (4.16) that $\varphi(x_0,t_0)=\underline{u}^*(x_0,t_0)\leq\underline{u}^*(\beta(s_0),s_0)\leq\varphi(\beta(s_0),s_0)$. On the other hand from (4.23)

$$\begin{array}{rcl}\varphi(x_0,t_0)-\varphi(\beta(s_0),s_0) & = & \int_{s_0}^{t_0}\frac{d}{d\theta}\varphi(\beta(\theta),\theta)\,d\theta\\ & = & \int_{s_0}^{t_0}(\varphi_t+p\varphi_x)(\beta(\theta),\theta)\,d\theta\\ & \geq & \mu(t_0-s_0)>0,\end{array}$$

which is a contradiction.

Suppose $x_0 = 0$. If $p < 0$, let $\beta(\theta) = p(\theta - t_0) + x_0$ and choose $s_0 < t_0$ such that $(\beta(\theta), \theta) \in B \cap (\overline{I\!R}_+ \times I\!R_+)$ for $\theta \in [s_0, t_0]$. Since $\{\beta = 0\} = \{x_0\}$ and hence $\int_\beta \lambda^+ = 0$. Since $\underline{u}^* < \varphi \le h(p) = h(\dot{\beta})$ and hence as in earlier situation we obtain a contradiction .

Let $p \ge 0$. Suppose $\varphi_x(x_0, t_0) \le 0$, then $2\mu \le (\varphi_t + p\varphi_x)(x_0, t_0) \le \varphi_t(x_0, t_0)$. By shrinking B if necessary it follows that $\varphi_t \ge \mu$ in $B \cap (\overline{I\!R}_+ \times I\!R_+)$ and hence $\varphi(x_0, t_0) \ge \varphi(x_0, s_0) + \mu(t_0 - s_0)$ for $s_0 < t_0$ sufficiently close to t_0. Let $p_1 < 0$ with $\varphi < h(p_1)$ in $B \cap (\overline{I\!R}_+ \times I\!R_+)$ and $\beta(\theta) = p_1(\theta - t_0) + x_0$, then $\int_\beta \lambda^+ = 0$ and $\underline{u}^* \le \varphi \le h(p_1)$. Hence from (4.16), $\phi(x_0, t_0) = \underline{u}^*(x_0, t_0) \le \underline{u}^*(\beta(s_0), s_0) \le \varphi(\beta(s_0), s_0)$. Letting $p_1 \to 0$ to obtain $\varphi(x_0, t_0) \le \varphi(x_0, s_0)$ which is a contradiction . Suppose $\varphi_x(x_0, t_0) > 0$ then $\lambda(t_0) > 0$. Let $\beta(\theta) = x_0$ for $\theta \in [s_0, t_0]$. Since $h(\dot{\beta}) = h(0) = \infty$, hence $\beta \in \overline{A}(\underline{u}^*, x_0, s_0, t_0)$ and hence from (4.16)

$$\begin{aligned} \eta(\varphi(x_0, t_0)) &= \eta(\underline{u}^*(x_0, t_0)) \le \eta(\underline{u}^*(x_0, s_0)) - \textstyle\int_{s_0}^{t_0} \lambda^+ \\ &\le \eta(\varphi(x_0, s_0)) - \textstyle\int_{s_0}^{t_0} \lambda^+. \end{aligned}$$

This implies that

$$\begin{aligned} \eta'(\varphi(x_0, t_0))\varphi_t(x_0, t_0) &= \lim_{s_0 \to t_0} \tfrac{\eta(\varphi(x_0, t_0)) - \eta(\varphi(x_0, s_0))}{t_0 - s_0} \\ &\le -\lambda^+(t_0) = -\lambda(t_0). \end{aligned}$$

Since $\eta'(u) = \frac{1}{H(u,1)}$ it follows from the above inequality that at (x_0, t_0), $4\mu \le \phi_t + H(\varphi, \varphi_x) = \varphi_t + \varphi_x H(\varphi, 1) \le (\varphi_t + \lambda H(\varphi, 1))(x_0, t_0) \le 0$, which is a contradiction . This proves that $\underline{u}$ is a subsolution.

Super solution : Suppose $\overline{u}$ is not a super solution. Then there exist $(x_0, t_0) \in (\overline{I\!R}_+ \times I\!R_+), \mu > 0, \varphi \in C^1(\overline{I\!R_+ \times I\!R_+})$ such that $\overline{u}_*(x_0, t_0) = \varphi(x_0, t_0), \overline{u}_* - \varphi$ has local minima at (x_0, t_0) and at $(x_0, t_0), \varphi_t + H(\varphi, \varphi_x) \le -4\mu, \varphi_x(x_0, t_0) > \lambda(t_0)$ if $x_0 = 0$. Hence in a ball B around (x_0, t_0), $\varphi_t + H(\varphi, \varphi_x) \le -3\mu$ and $\varphi_x > \lambda$ if $x_0 = 0$. Hence from (A_6), for $(x, t) \in B \cap (\overline{I\!R}_+ \times I\!R_+)$,

$$\text{if } \varphi(x, t) \le h(p), \text{ then } \varphi_t + p\varphi_x \le -3\mu \quad \text{and} \quad \varphi_x > \lambda \quad \text{if} \quad x_0 = 0. \tag{4.24}$$

From (1) of Lemma 4.4, let for each $s < t_0, \beta_s$ be a minimizer for

$$\eta(W(x, t)) = \inf\left\{\eta(\overline{u}_*(\beta(s), s)) - \int_\beta \lambda^+; \quad \beta \in \underline{A}_M(\overline{u}_*, x, s, t)\right\}, \tag{4.25}$$

then from Lemma 4.6 we have

$$\begin{aligned} \eta(\varphi(x_0, t_0)) &= \eta(\overline{u}_*(x_0, t_0)) \\ &\ge \eta(\overline{u}_*(\beta_s(s), s)) - \textstyle\int_{\beta_s} \lambda^+ \\ &\ge \eta(\varphi(\beta_s(s), s)) - \textstyle\int_{\beta_s} \lambda^+, \end{aligned}$$

$$\eta(\varphi(x_0,t_0)) - \eta(\varphi(x_0,s)) \geq \eta(\varphi(\beta_s(s),s)) - \eta(\varphi(x_0,s)) - \int_{\beta_s} \lambda^+. \tag{4.26}$$

Let $x_0 > 0$. Without loss of generality assume that $B \subset (I\!R_+ \times I\!R_+)$ and there exist a $s_0 < t_0$ such that $(\beta_s(\theta),\theta) \in B$ for all $s \in [s_0,t_0]$, $\theta \in [s,t_0]$. Further more $\beta_s(\theta) = p_s(\theta - t_0) + x_0$ and $\{\beta_s = 0\} = \phi$. As $s \to t_0$, let $p_s \to p_0$. Divide $(t_0 - s)$ in (4.26) and letting $s \to t_0$, to obtain $\eta'(\varphi(x_0,t_0))(\varphi_t + p_0\varphi_x)(x_0,t_0) \geq 0$, $\varphi(x_0,t_0) = \lim_{s\to t_0} \varphi(\beta_s(s),s) \leq \lim_{s\to t_0} \overline{u}_*(\beta_s(s),s) \leq \lim_{s\to\infty} h(p_s) \leq h(p_0)$. This contradicts (4.24).

Let $x_0 = 0$. If $\beta_s = x_0 + p_s(\theta - t_0)$, then $p_s \leq 0$. Let $p_s \to p_0$ as $s \to t_0$. Divide by $(t_0 - s)$ and letting $s \to t_0$ in (4.26) to obtain $\eta'(\varphi(x_0,t_0))(\varphi_t + p_0\varphi_x)(x_0,t_0) \geq -\lambda^+(t_0)$ and $\varphi(x_0,t_0) \leq h(p_0)$. If $\lambda(t_0) \leq 0$, then $(\varphi_t + p_0\varphi_x)(x_0,t_0) \geq 0$ which contradicts (4.24). If $\lambda(t_0) \geq 0$, then $\varphi_x(x_0,t_0) > \lambda(t_0) \geq 0$ hence $0 \leq (H(\varphi,1)\lambda(t_0) + (\varphi_t + p_0\varphi_x))(x_0,t_0) \leq (H(\varphi,\varphi_x) + \varphi_t)(x_0,t_0) \leq -4\mu$ which is a contradiction.

Suppose β_s is not a line segment, let $\beta_s = (\phi, \beta_{2s}, \beta_{3s})$ defined on the partition $s = s_3 < s_2 < s_1 = s_0 = t_0$. Let $\beta_{3s}(\theta) = p_s(\theta - s_1)$ for $\theta \in [s,s_2]$ with $\varphi(\beta_s(s),s) \leq h(\dot{\beta}_{3s}) = h(p_s)$ and $p_s < 0$. Let $p_s \to p_0$ as $s \to t_0$. Suppose $\lambda(t_0) \geq 0$, then $\varphi_x(x_0,t_0) > 0$ and hence in $B \cap (\overline{I\!R}_+ \times I\!R_+)$, $\varphi_x \geq 0$. This implies tht $\varphi(\beta_s(s),s) \geq \varphi(x_0,s)$. Hence from Lemma 4.6, $\eta(\varphi(x_0,t_0)) = \eta(\overline{u}_*(x_0,t_0)) \geq \eta(\varphi(\beta_s(s),s)) - \int_{s_1}^{t_0} \lambda^+ \geq \eta(\varphi(x_0,s)) - \int_{s_1}^{t_0} \lambda^+$. Divide by $t_0 - s$ and letting $s \to t_0$ to obtain $\eta'(\varphi(x_0,t_0))\varphi_t(x_0,t_0) \geq -\lambda^+(t_0) = -\lambda(t_0)$. That is $0 \leq (\varphi_t + \lambda(t_0)H(\varphi,1))(x_0,t_0) \geq (\varphi_t + \varphi_x H(\varphi,1))(x_0,t_0) \leq -4\mu$ which is a contradiction. Suppose $\lambda(t_0) < 0$, then $\lambda^+ = 0$ hence from Lemma 4.6, $\varphi(x_0,t_0) = \overline{u}_*(x_0,t_0) \geq \overline{u}_*(\beta_s(s),s) \geq \varphi(\beta_s(s),s)$. Hence $0 \leq \lim_{s\to t_0} \frac{\varphi(x_0,t_0)-\varphi(\beta_s(s),s)}{t_0-s} = \varphi_t - p_0\varphi_x$ where $r = \lim_{s\to t_0} \frac{s-s_1}{t_0-s}$. Suppose $\varphi_x \leq 0$, since $p_0 \leq 0$ to obtain $0 \leq (\varphi_t - p_0 r\varphi_x)(x_0,t_0) \leq \varphi_t(x_0,t_0) \leq (\varphi_t + H(\varphi,\varphi_x))(x_0,t_0) \leq -4\mu$, which is a contradiction. If $\varphi_x > 0$, then $\varphi(x_0,t_0) > \varphi(\beta_s(s),s) \geq \varphi(x_0,s)$ and hence $\varphi_t(x_0,t_0) \geq 0$. But $\varphi_t(x_0,t_0) \leq (\varphi_t + H(\varphi,\varphi_x))(x_0,t_0) \leq -4\mu$ which is a contradiction. This proves that $\overline{u}$ is a super solution. Since $\underline{u}^* = \overline{u}$ and $\overline{u}_* = \underline{u}$ and hence $\underline{u}, \overline{u}$ are viscosity solutions of (1.1).

In order to prove Theorem 2.3 , we need following Lemmas.

Let $T > 0$, $0 \leq s < t \leq T$, $x \in \overline{I\!R}_+$, $D(x,s,t)$, $D(x,t)$, l_+, l_-, l_0 are as in (2.18) to (2.22) respectively. Let $\mu > 0$, $\delta = \frac{m}{m-1}$ be the conjugate of m and for $\beta \in c(x,s,t)$

$$||\beta|| = (\int_s^t |\dot{\beta}(\theta)|^\delta \, d\theta)^{1/\delta} \tag{4.27}$$

$$D_M(x,s,t) = \left\{(\beta, w_1, w_2) \in D(x,s,t); ||\beta||^\delta \leq M\right\}, \tag{4.28}$$

$$D_M(x,t) = D_M(x,0,t) \tag{4.29}$$

Lemma 4.9 : Let ϑ be a function defined on $(\overline{I\!R_+ \times I\!R_+})$ such that $|\vartheta|_T < \infty$. Let η, G_w, L_w be as in (2.1) to (2.3). Assume tht η and G_w are diffeomorphisms from $I\!R \to I\!R$. Define

$$V(x,t) = \inf_{(\beta,w_1,w_2)\in D(x,s,t)} G_{w_1}^{-1}\left\{L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(\vartheta(\beta(s),s)) + l_-(\beta)\right\} - l_0(\beta)\right\} + l_+(\beta)\right\}, \tag{4.30}$$

then

(1) $D_M(x,s,t)$ is weakly compact .
(2) $|V|_T < \infty$ and there exist $M = M(|\vartheta|_T, T)$ such that

$$V(x,t) = \inf_{(\beta,w_1,w_2)\in D_M(x,s,t)} G_{w_1}^{-1}\left\{L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(\vartheta(\beta(s),s)) + l_-(\beta)\right\} - l_0(\beta)\right\} + l_+(\beta)\right\} \tag{4.31}$$

(3) Assume that $x \mapsto \vartheta(x,\theta)$ is lower semicontinuous for each θ, then there exist a minimizer (β, w_1, w_2) for (4.31).

Proof : Let $\left\{(\beta^{(k)}, w_1^{(k)}, w_2^{(k)}\right\}$ be a sequence in $D_M(x,s,t)$ and for a subsequence let $\beta^{(k)} \rightharpoonup \beta$ weakly in $W^{1,\delta}$, $(w_1^{(k)}, w_2^{(k)}) \to (w_1, w_2)$. Hence $\dot\beta_1 \geq 0$ and $\dot\beta_3 \leq 0$. Since $w_1\dot\beta_1^{(k)} \geq 0, w_2\dot\beta_3^{(k)} \geq 0$, it follows that $w_1\dot\beta_1 \geq 0, w_2\dot\beta_3 \geq 0$. Therefore $(\beta_1, w_1, w_2) \in D_M(x,s,t)$ and this proves (1). For (β, w_1, w_2),

$$\begin{aligned} G_{w_1}^{-1}\left\{L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(\vartheta(\beta(s),s))\right.\right.\right. &+ \left.\left.l_-(\beta)\right\} - l_0(\beta)\right\} + l_+(\beta)\} \\ &\geq G_{w_1}^{-1}L_{w_1}^{-1}\left\{G_{w_2}\, L_{w_2}(\vartheta(\beta(s),s)) - l_0(\beta)\right\} \\ &= \eta^{-1}\left\{\eta(\vartheta(\beta(s),s)) - l_0(\beta)\right\} \\ &\geq \eta^{-1}\left\{\eta(-|\vartheta|_T) - T|\lambda^+|_T{}^m\right\} \end{aligned}$$

and hence $V(x,t) \geq \eta^{-1}\left\{\eta(-|\vartheta|_T) - T|\lambda^+|_T{}^m\right\}$. Let $\beta(\theta) = x$ for $\theta \in [s,t]$, then $(\beta, w, w) \in D(x,s,t)$ and $l_+(\beta) = l_-(\beta) = 0$. Hence $V(x,t) \leq \vartheta(x,s) \leq |\vartheta|_T$. Hence $|V|_T < \infty$. Let $(\beta^{(k)}, w_1^{(k)}, w_2^{(k)}) \in D(x,s,t)$ such that $\lim_{k\to\infty} \|\beta^{(k)}\| = \infty$. Since $l_+(\beta^{(k)}) + l_-(\beta^(k)) = \delta\|\beta^{(k)}\|^\delta$ and for a subsequence either $l_+(\beta^{(k)}) \to \infty$ or $l_-(\beta^{(k)}) \to \infty$. This implies that $G_{w^{(k)}}^{-1}\left\{L_{w^{(k)}}^{-1}\left\{L_{w^{(k)}}\left\{G_{w^{(k)}}(\vartheta(\beta^{(k)}(s),s)) + l_-(\beta^{(k)})\right\} - l_0(\beta^{(k)})\right\} + l_+(\beta^{(k)})\right\} \to \infty$ or $k \to \infty$. Since $|V|_T < \infty$ and hence there exist an $M > 0$ such that (4.31) holds. This proves (2).

Let $(\beta^{(k)}, w_1^{(k)}, w_2^{(k)})$ be a minimizing sequence of (4. 31). From (1) let $(\beta^{(k)}, w_1^{(k)}, w_2^{(k)}) \to (\beta, w_1, w_2,)$ weakly in $W^{1,\delta} \times I\!R \times I\!R$ and $\beta^{(k)} \to \beta$ in C^0 . Hence by lower semicontinuity of $x \mapsto \vartheta(x,\theta)$, (β, w_1, w_2) is a minimizer.

Lemma 4.10 :(dynamic programming principle) Let u be as in (2.23). Then for any $0 \leq s < t,\ x \in \overline{I\!R}_+$,

$$u(x,t) = \inf_{(\beta,w_1,w_2)\in D(x,s,t)} G_{w_1}^{-1}\left\{L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u(\beta(s),s)) + l_-(\beta)\right\} - l_0(\beta)\right\} + l_+(\beta)\right\}. \tag{4.32}$$

Proof : Denote $\vartheta(x,t)$ the right hand side of (4.32). Since $u_0 \in W^{1,\infty}$, hence from (1) and (3) of Lemma 4.9 , $|u|_T < \infty$ and there exist a minimizer $(\gamma, w_1, w_2) \in D(x,t)$. Let $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ given on the partition $0 \leq t_3 \leq t_2 \leq t_1 \leq t_0 = t$. Let $\alpha = \gamma|_{[0,s]}$ and $\beta = \gamma|_{[s,t]}$

Step 1 $\vartheta(x,t) \leq u(x,t)$.

Let $t_1 \leq s < t$. Then $(\beta, w_1, w_1) \in D(x,s,t)$, $(\alpha, w_1, w_2) \in D(\beta(s), s)$, $l_-(\beta) = l_0(\beta) = 0$ and $l_0(\gamma) = l_0(\alpha)$. Hence

$$\begin{aligned}
\vartheta(x,t) &\leq G_{w_1}^{-1}\left\{G_{w_1}(u(\beta(s),s)) + l_+(\beta)\right\}\\
&\leq G_{w_1}^{-1}\left\{L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u(\alpha(0))) + l_-(\alpha)\right\} - l_0(\alpha)\right\} + l_+(\alpha) + l_+(\beta)\right\}\\
&= G_{w_1}^{-1}\left\{L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u(\gamma(0))) + l_-(\gamma)\right\} - l_0(\gamma)\right\} + l_+(\gamma)\right\}\\
&= u(x,t).
\end{aligned}$$

Let $t_2 \leq s < t_1$, then $(\beta, w_1, w_2) \in D(x,s,t), (\alpha, w_2, w_2) \in D(\beta(s), s)$, $\quad l_0(\gamma) = l_0(\alpha) + l_0(\beta), l_+(\alpha) = l_-(\beta) = 0$ and $l_+(\beta) = l_+(\gamma)$,

$$\begin{aligned}
G_{w_1}(\vartheta(x,t)) &\leq L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u(\beta(s),s)) + l_-(\beta)\right\} - l_o(\beta)\right\} + l_+(\beta)\\
&= L_{w_1}^{-1}\left\{L_{w_2}\ G_{w_2}(u(\beta(s),s)) - l_0(\beta)\right\} + l_+(\gamma)\\
&\leq L_{w_1}^{-1}\left\{L_{w_2}\left\{L_{w_2}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u_0(\alpha(0))) + l_-(\alpha)\right\} - l_0(\alpha) - l_0(\beta)\right\} + l_+(\alpha)\right\} + l_+(\gamma)\\
&= L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u_0(\gamma(0))) + l_-(\gamma)\right\} - l_0(\gamma)\right\} + l_+(\gamma)\\
&= G_{w_1}(u(x,t)).
\end{aligned}$$

Let $0 \leq s < t_2$. Then $(\beta, w_1, w_2) \in D(x,s,t), (\alpha, w_2, w_2) \in D(\beta(s), s), l_0(\beta) = l_0(\gamma), l_+(\beta) = l_+(\gamma), l_0(\alpha) = l_+(\alpha) = 0$ and $l_-(\alpha) + l_-(\beta) = l_-(\gamma)$. Hence

$$\begin{aligned}
G_{w_1}(\vartheta(x,t)) &\leq L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u(\beta(s),s)) + l_-(\beta)\right\} - l_0(\beta)\right\} + l_+(\beta)\\
&\leq L_{w_1}^{-1}\left\{L_{w_2}\left\{L_{w_2}^{-1}\left\{L_{w2}\left\{G_{w_2}(u_0(\alpha(0))) + l_-(\alpha)\right\} + l_-(\beta)\right\} - l_0(\gamma)\right\} + l_+(\gamma)\\
&= L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u_0(\gamma(0))) + l_-(\gamma)\right\} - l_0(\gamma)\right\} + l_+(\gamma)\\
&= G_{w_1}(u(x,t)).
\end{aligned}$$

Combining all the inequalities to give $\vartheta(x,t) \leq u(x,t)$ and this proves step (1) .

Step 2 : $\vartheta(x,t) \geq u(x,t)$.

From (2) and (3) of Lemma 4.9, for every $\epsilon > 0$, there exist $(\beta, w_1, w_2) \in D(x, s, t)$ and $(\alpha, \mu_1, \mu_2) \in D(\beta(s), s)$ such that

$$G_{w_1}(\vartheta(x,t) + \epsilon_0) \geq L_{w_1}^{-1}\{L_{w_2}\{G_{w_2}(u(\beta(s), s)) + l_-(\beta)\} - l_0(\beta)\} + l_+(\beta) \tag{4.33}$$

$$G_{\mu_1}(u(\beta(s), s)) = L_{\mu_1}^{-1}\{L_{\mu_2}\{G_{\mu_2}(u_0(\alpha(0))) + l_-(\alpha)\} - l_0(\alpha)\} + l_+(\alpha). \tag{4.34}$$

Let $\beta = (\beta_1, \beta_2, \beta_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be defined on the partition $s = t_3 \leq t_2 \leq t_1 \leq t_0 = t$ and $0 = s_3 \leq s_2 \leq s_1 \leq s_0 = s$ respectively .

Suppose $[t_2, t_1] \neq \phi,\ [s_2, s_1] \neq \phi$. Define $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ by $\gamma_1 = \beta_1,\ \gamma_2|_{[s_2,t_1]} = 0,\ \gamma_3 = \beta_3$, then $(w_1, w_2) = (\mu_1, \mu_2),\ (\gamma, w_1, w_2,) \in D(x,t),\ l_0(\gamma) \geq l_0(\alpha) + l_0(\beta),\ l_-(\alpha) = l_-(\gamma),\ l_+(\beta) = l_+(\gamma)$. Hence from (4.33), (4.34)

$$\begin{array}{rcl}
G_{w_1}(\vartheta(x,t)+\epsilon) & \geq & L_{w_1}^{-1}\{L_{w_2}\ G_{w_2}(u(\beta(s),s)) - l_0(\beta)\} + l_+(\gamma) \\
& \geq & L_{w_1}^{-1}\left\{\eta\ o\ G_{\mu_1}^{-1}\ L_{\mu_1}^{-1}\ \{L_{\mu_2}\{G_{\mu_2}(u_0(\alpha(0))) + l_-(\alpha)\} - l_0(\alpha) - l_0(\beta)\} + l_+(\gamma)\right. \\
& \geq & L_{w_1}^{-1}\{L_{w_2}\ \{G_{w_2}(u_0(\gamma(0)) + l_-(\gamma)\} - l_0(\gamma)\} + l_+(\gamma) \\
& = & G_{w_1}(u(x,t)).
\end{array}$$

Suppose $[t_2, t_1] = \phi, [s_2, s_1] \neq \phi$. Define $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ by $\gamma_1(\theta) = \frac{\theta - s_1}{t - s_1}x_1$ if $\theta \in [s_1, t], \gamma_2 = \alpha_2, \gamma_3 = \alpha_3$. Let $\dot{\beta} \geq 0$, then $w_1 = \mu_1,\ (\gamma, \mu_1, \mu_2) \in D(x,t), l_-(\beta) = l_0(\beta) = 0, l_+(\gamma) = l_+(\beta), l_-(\gamma) = l_-(\alpha), l_0(\gamma) = l_0(\alpha)$. Hence from (4.33), (4.34) and from (2.3)

$$\begin{array}{rcl}
G_{w_1}(\vartheta(x,t)+\epsilon) & \geq & G_{w_1}(u(\beta(s),s)) + l_+(\beta) \\
& \geq & L_{\mu_1}^{-1}\{L_{\mu_2}\{G_{\mu_2}(u_0(\alpha(0))) + l_-(\alpha)\} - l_0(\alpha)\} + l_+(\beta) \\
& = & L_{\mu_1}^{-1}\{L_{\mu_2}\{G_{\mu_2}(u_0(\gamma(0))) + l_-(\gamma)\} - l_0(\gamma) + l_+(\gamma) \\
& \geq & G_{w_1}(u(x,t)).
\end{array}$$

Let $\dot{\beta} \leq 0$, then $w_2 = \mu_2,\ \dot{\gamma}_1 > 0, (\gamma, \mu_1, \mu_2) \in D(x,t), l_+(\beta) = l_0(\beta) = 0,\ l_0(\gamma) = l_0(\alpha),\ l_-(\gamma) = l_-(\alpha)$. Since $\dot{\beta} \leq 0$ and hence $x \leq \dot{\beta}(s)$, therefore $\delta l_+(\gamma) = (\frac{d\gamma_1}{d\theta})^\delta (t - s_1) = \frac{x^\delta}{(t-s_1)^{\delta-1}} \leq \frac{\beta(s)^\delta}{(t-s_1)^{\delta-1}} \leq \frac{\beta(s)^\delta}{(s-s_1)^{\delta-1}} = \delta l_+(\alpha)$. Hence

$$\begin{array}{rcl}
u(\beta(s),s) & \geq & G_{\mu_1}^{-1}\left\{L_{\mu_1}^{-1}\{L_{\mu_2}\{G_{\mu_2}(u_0(\alpha(0))) + l_-(\alpha)\} - l_0(\alpha)\} + l_+(\alpha)\right\} \\
& \geq & G_{\mu_1}^{-1}\left\{L_{\mu_1}^{-1}\{L_{\mu_2}\{G_{\mu_2}(u_0(\gamma(0))) + l_-(\gamma)\} - l_0(\gamma)\} + l_+(\gamma)\right\} \\
& \geq & u(x,t)
\end{array}$$

Therefore

$$\begin{array}{rcl}
\vartheta(x,t) + \epsilon & \geq & G_{w_1}^{-1}\left\{L_{w_1}^{-1}\{L_{w_2}\{G_{w_2}(u(\beta(s),s)) + l_-(\beta)\}\right. \\
& \geq & u(\beta(s),s) \geq u(x,t).
\end{array}$$

Suppose $[t_2, t_1] \neq \phi,\ [s_2, s_1] = \phi$. Let $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ be defined by $\gamma_1 = \beta_1, \gamma_2 = \beta_2, \gamma_3(\theta) = \frac{t_2 - \theta}{t_2}\alpha(0)$. Then $(\gamma, w_1, w_2) \in D(x,t), l_+(\gamma) = l_+(\beta), l_0(\gamma) = l_0(\beta)$. Let $\dot{\alpha} \geq 0$, then $\gamma(0)=\alpha(0)\leq\beta(s)$ and hence $\delta l_-(\gamma)=\frac{\alpha(0)^\delta}{t_2^{\delta-1}}\leq\frac{\beta(s)^\delta}{(t_2-s)^{\delta-1}}=\delta l_-(\beta)$. Also $u(\beta(s),s)=G_{w_2}^{-1}\left\{G_{w_2}(u_0(\alpha(0)))+l_-(\alpha)\right\}\geq u_0(\alpha(0))=u_0(\gamma(0))$. Hence

$$\begin{aligned}
G_{w_1}^{-1}(\vartheta(x,t)+\epsilon) &\geq L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u(\beta(s),s))+l_-(\beta)\right\}-l_0(\beta)\right\}+l_+(\beta)\\
&\geq L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u_0(\gamma(0)))+l_-(\gamma)\right\}-l_0(\gamma)\right\}+l_+(\gamma)\\
&\geq G_{w_1}^{-1}(u(x,t)).
\end{aligned}$$

Let $\dot\alpha<0$. Then $\beta(s)=\alpha(s)<\alpha(0), w_2=\mu_2$ and $\frac{\alpha(0)}{t_2}=(1-\frac{s}{t_2})\frac{\beta(s)}{t_2-s}+\frac{s}{t_2}\frac{(\alpha(0)-\beta(s))}{s}$. Hence by convexity $(\frac{\alpha(0)}{t_2})^\delta\leq(1-\frac{s}{t_2})(\frac{\beta(s)}{t_2-s})^\delta+\frac{s}{t_2}(\frac{\alpha(0)-\beta(s)}{s})^\delta$. This implies that $l_-(\gamma)\leq l_-(\beta)+l_-(\alpha)$. Therefore $G_{w_2}(u(\beta(s),s))+l_-(\beta)\geq G_{w_2}(u_0(\alpha(0)))+l_-(\beta)+l_-(\alpha)\geq G_{w_2}(u_0(\gamma(0)))+l_-(\gamma)$ and $G_{w_1}(V(x,t)+\epsilon)\geq L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u_0(\gamma(0)))+l_-(\gamma\right\}-l_0(\gamma)\right\}+l_+(\gamma)=G_{w_1}(u(x,t))$.

Suppose $[t_2,t_1]=\phi,\ [s_2,s_1]=\phi$. Let $\gamma(\theta)=\frac{\theta}{t}x+(1-\frac{\theta}{t})\alpha(0)$. Then $l_0(\alpha)=l_0(\beta)=l_0(\gamma)=0$. Suppose $\dot\alpha\geq 0,\dot\beta\geq 0$, then $\dot\gamma\geq 0,\ w_1=\mu_1,\ (\gamma,w_1,w_2)\in D(x,t), l_-(\alpha)=l_-(\beta)=l_-(\gamma)=0$. Since $\frac{x-\alpha(0)}{t}=(1-\frac{s}{t})\ \frac{x-\beta(0)}{t-s}+\frac{s}{t}\frac{(\beta(0)-\alpha(0))}{s}$, hence by convexity, $t|\frac{x-\alpha(0)}{t}|^\delta\leq(t-s)|\frac{x-\beta(0)}{t-s}|^\delta+s|\beta(0)-\alpha(0)|^\delta$. That is $l_+(\gamma)\leq l_+(\alpha)+l_+(\beta)$. Therefore

$$\begin{aligned}
G_{w_1}(\vartheta(x,t)+\epsilon) &\geq G_{w_1}(u(\beta(s),s))+l_+(\beta)\\
&= G_{w_1}(u_0(\alpha(0)))+l_+(\alpha)+l_+(\beta)\\
&\geq G_{w_1}(u_0(\gamma(0)))+l_+(\gamma)\\
&\geq G_{w_1}(u(x,t)).
\end{aligned}$$

Suppose $\dot\alpha<0,\dot\beta\geq 0$ and $\dot\gamma\geq 0$. Then $(\gamma,w_1,w_2)\in D(x,t),\quad \beta(s)<\alpha(0)$. Hence $\delta l_+(\beta)=\frac{(x-\beta(s))^\delta}{(t-s)^{\delta-1}}\geq\frac{(x-\alpha(0))^\delta}{t^{\delta-1}}=\delta l_+(\gamma)$. Since $u(\beta(s),s)=G_{\mu_2}^{-1}\left\{G_{\mu_2}(u_0(\alpha(0)))+l_-(\alpha)\right\}\geq u_0(\alpha(0))=u_0(\gamma(0))$, it follows that $G_{w_1}(\vartheta(x,t)+\epsilon)\geq G_{w_1}(u_0(\gamma(0)))+l_+(\beta)\geq G_{w_1}(u_0(\gamma(0)))+l_+(\gamma)\geq G_{w_1}(u(x,t))$. Suppose $\dot\gamma<0$, then $x\geq\beta(s),\gamma(0)=\alpha(0),(\gamma,w_2,w_1)\in D(x,t)$ and $\delta l_-(\beta)=\frac{(\alpha(0)-\beta(s))^\delta}{s^{\delta-1}}\geq\frac{(\gamma(0)-x)^\delta}{t^{\delta-1}}=\delta l_-(\gamma)$. Also $u(\beta(s),s)\geq u_0(\alpha(0))=u_0(\gamma(0))$. Hence $G_{w_2}(\vartheta(x,t)+\epsilon)\geq G_{w_2}(u(\beta(s),s))+l_-(\beta)\geq G_{w_2}(u_0(\gamma(0)))+l_-(\gamma)\geq G_{w_1}(u(x,t))$. Similarly if $\dot\beta<0,\dot\alpha<0$ and $\dot\beta<0,\dot\alpha\geq 0$ then by similar argument allows to show that $\vartheta(x,t)+\epsilon\geq u(x,t)$. Combining all the cases it follows that $\vartheta(x,t)+\epsilon\geq u(x,t)$ and ϵ is arbitrary and hence $\vartheta(x,t)\geq u(x,t)$. This proves the Lemma .

Let $T>0$. Since $|u_0|_\infty<\infty$ hence from Lemma 4.9, $|u|_T<\infty$ and there exist a $M=M(u_0,\lambda,T)>0$ such that

$$u(x,t)=\inf_{(\beta,w_1,w_2)\in D_M(x,t)} G_{w_1}^{-1}\left\{L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u_0(\beta(0)))+l_-(\beta)\right\}-l_0(\beta)\right\}+l_+(\beta)\right\}.$$

This implies that domain of definition of the maps $G_w, G_w^{-1}, L_w, L_w^{-1}$ are contained in a fixed compact interval $[-N, N]$ where $N = N(u_0, T)$ and hence they are uniformly Lipschitz continuous functions. Let

$$M_1 = \sup_{|w|=1} \left\{ (1 + \text{ Lip } G_w)\,(1 + \text{ Lip } G_w^{-1})\,(1 + \text{ Lip } L_w)\,(1 + \text{ Lip } L_w^{-1})\,(1 + \text{ Lip } u_0) \right\}. \tag{4.35}$$

Lemma 4.11 : Let $T > 0$, then there exist a constant $C = C(T, u_0, \lambda)$ such that for $x \in \overline{I\!R}^+, 0 \le t \le T$,

$$u(x,t) = \inf_{(\beta, w_1, w_2) \in D(x,t),\ |\dot\beta| \le C} G_{w_1}^{-1} \left\{ L_{w_1}^{-1} \left\{ L_{w_2} \left\{ G_{w_2}(u_0(\beta(0))) + l_-(\beta) \right\} - l_0(\beta) \right\} + l_+(\beta) \right\}. \tag{4.36}$$

Proof : From (3) of Lemma 4.9 , there exist a minimizer $(\beta, w_1, w_2) \in D_M(x,t)$ for $u(x,t)$. Let $\beta = (\beta_1, \beta_2, \beta_3)$ defined on the partition $0 \le t_2 \le t_1 \le t$ and M_1 be as in (4.35). Let $T_1 = t_2 + \beta(0)$ and define $(\gamma, w_1, w_2) \in D(x,t)$ by

$$\gamma(\theta) = \begin{cases} \beta(\theta) & \text{if} \quad t_2 \le T_1 \le t_1, \quad \theta \in [T_1, t] \\ \frac{T_1 - \theta}{T_1}\beta(0) & \text{if} \quad t_2 \le T_1 \le t_1, \quad \theta \in [0, t_1] \\ \beta(\theta) & \text{if} \quad T_1 > t_1, \quad \in [t_1, t] \\ 0 & \text{if} \quad T_1 > t_1, \quad \theta \in [0, t_1] \end{cases}.$$

Suppose $t_2 \le T_1 \le t_1$, then $l_+(\beta) = l_+(\gamma), l_-(\gamma) = \frac{m-1}{m}(\frac{\beta(0)}{T_1})^{1/m-1}\beta(0) \le \beta(0)$, $|l_0(\beta) - l_0(\gamma)| = \int_{t_2}^{T_1} (\lambda^+)^m \le |\lambda^+|_T^m (T_1 - t_2) = |\lambda^+|_T^m \beta(0)$. Since (β, w_1, w_2) is a minimizer, we obtain

$$\begin{aligned} G_{w_1}(u(x,t)) &= L_{w_1}^{-1} \left\{ L_{w_2} \left\{ G_{w_2}(u_0(\beta(0))) + l_-(\beta) \right\} - l_0(\beta) \right\} + l_+(\gamma) \\ &\le L_{w_1}^{-1} \left\{ L_{w_2} \left\{ G_{w_2}(u_0(\gamma))) + l_-(\gamma) \right\} - l_0(\gamma) \right\} + l_+(\gamma) \end{aligned}$$

Since $G_w, L_w, G_w{}^{-1}, L_w{}^{-1}$ are uniformly Lipschitz functions on the maximal domain of definition, hence there exist a $C_1 = C_1(u_0, \lambda^+, T)$ such that

$$|l_-(\beta) - l_-(\gamma)| \le C_1 |l_0(\beta) - l_0(\gamma)| \le C_1 |\lambda^+|_T^m \beta(0)$$

and therefore

$$(\frac{\beta(0)}{t_2})^{1/m-1} \le (\frac{\beta(0)}{T_1})^{1/m-1} + C_1|\lambda^+|_T^m \le (1 + C_1|\lambda^+|_T^m)$$

Suppose $t_1 < T_1$, then $t_1 - t_2 \le \beta(0), \quad l_+(\gamma) = l_+(\beta), l_0(\beta) = \int_{t_2}^{t_1} (\lambda^+)^m \le |\lambda^+|_T^m (t_1 - t_2) \le |\lambda^+|_T^m \beta(0)$ and $\gamma(0) = 0$. Again by minimizing property for u(x,t),

$$\begin{aligned} L_{w_1}^{-1} \left\{ L_{w_2} \left\{ G_{w_2}(u_0(\beta(0))) \right.\right. &+ \left. l_-(\beta) \right\} - l_0(\beta) \right\} + l_+(\beta) \\ &\le L_{w_1}^{-1} \left\{ L_{w_2} \left\{ G_{w_2}(u_0(\gamma(0))) + l_-(\gamma) \right\} - l_0(\gamma) \right\} + l_+(\gamma) \\ &\le L_{w_1}^{-1} \left\{ L_{w_2} \left\{ G_{w_2}(u_0(\gamma(0))) + l_-(\gamma) \right\} \right\} + l_+(\beta). \end{aligned}$$

Hence there exist a $C_2 = C_2(u_0, \lambda, T)$ such that

$$(\frac{\beta(0)}{t_2})^{1/m-1} \leq \frac{l_-(\beta)}{\beta(0)} \leq C_2 \frac{(\beta(0) + l_0(\beta))}{\beta(0)} \leq C_2(1 + |\lambda^+|_T^m).$$

Hence there exist a $C_3 = C_3(u_0, \lambda, T)$ such that $\frac{\beta(0)}{t_2} \leq C_3$.

Let $T_2 = t_1 - x$ and define $(\mu, w_1, w_2) \in D(x,t)$ by

$$\mu(\theta) = \begin{cases} \frac{\theta - T_2}{t - T_2} x & \text{if} \quad \theta \in [T_2 \bigvee 0, t] \\ \beta(\theta) & \text{if} \quad T_2 \in [t_2, t_1], \theta \in [0, T_2] \\ \frac{T_2 - \theta}{t_2} \beta(0) & \text{if} \quad T_2 < t_2, \theta \in [0, T_2 \bigvee 0]. \end{cases}$$

Suppose $T_2 \in [t_2, t_1]$. Then $\mu(0) = \beta(0), l_-(\mu) = l_-(\beta), |l_0(\mu) - l_0(\beta)| \leq |\lambda^+|_T^m (t_1 - T_2) = |\lambda^+|_T^m x$. Hence there exist a $C_4 = C_4(u_0, \lambda, T) > 0$ such that

$$\begin{aligned} \frac{m-1}{m}[(\frac{x}{t-t_1})^{1/m-1} x \quad &- \quad (\frac{x}{t-T_2})^{1/m-1} x] = l_+(\beta) - l_-(\mu) \\ &\leq \quad L_{w_1}^{-1}\{L_{w_2}\{G_{w_2}(u_0(\mu(0))) + l_-(\mu)\} - l_0(\mu)\} \\ &\quad\; -L_{w_1}^{-1}\{L_{w_2}\{G_{w_2}(u_0(\beta(0))) + l_-(\beta)\} - l_0(\beta)\} \\ &\leq \quad C_4 |l_0(\mu) - l_0(\beta)| \\ &\leq \quad C_4 |\lambda^+|_T^m x, \end{aligned}$$

and hence

$$(\frac{x}{t-t_1})^{1/m-1} \leq (\frac{x}{t-T_2})^{1/m-1} + C_4 |\lambda^+|_T^m \leq (1 + C_4 |\lambda^+|_T^m).$$

Suppose $0 \leq T_2 \leq t_2$. Then $t_1 - t_2 \leq x, \quad |\beta(0) - \mu(0)| = |\beta(0) - \frac{\beta(0)}{t_2} T_2| = \frac{\beta(0)}{t_2}|t_1 - t_2 + x| \leq 2C_4 x, \;\; |l_-(\beta) - l_-(\mu| \leq |\beta(0)(\frac{\beta(0)}{t_2})^{1/m-1} - \mu(0)(\frac{\mu(0)}{T_2})^{1/m-1}| = (\frac{\beta(0)}{t_2})^{1/m-1}|t_1 - t_2 + x| \leq 2C_4^{1/m-1} x$. Hence there exist a $C_5 = C_5(u_0, \lambda, T)$ such that

$$\begin{aligned} \frac{m-1}{m}(\frac{x}{t-t_1})^{1/m-1} = \frac{l_+(\beta)}{x} \quad &\leq \quad \frac{1}{x}[L_{w_1}^{-1}\{L_{w_2}\{G_{w_2}(u_0(\mu(0))) + l_-(\mu) - l_0(\mu)\} \\ &\qquad -L_{w_1}^{-1}\{L_{w_2}\{G_{w_2}(u_0(\beta(0))) + l_-(\beta)\} - l_0(\beta)\}] \\ &\leq \quad \frac{C_5}{x}\{|\mu(0) - \beta(0)| + |l_-(\mu) - l_-(\beta)| + l_+(\mu)\} \\ &\leq \quad C_5\left\{2C_4 + 2C_4^{1/m-1} + (\frac{x}{t-T_2})^{1/m-1}\right\} \\ &\leq \quad C_5(1 + 2C_4 + 2C_4^{1/m-1}). \end{aligned}$$

Suppose $T_2 < 0$, then $t_1 - t_2 \leq x, l_-(\mu) = 0, l_0(\beta) \leq |\lambda^+|_T^m (t_1 - t_2) \leq |\lambda^+|_T^m x, l_+(\mu) = \frac{m}{m-1}(\frac{\beta(0)}{x})^{1/m-1}\beta(0) \leq C_4^{m/m-1}\beta(0) \leq C_4^{m/m-1} x$. Hence there exist C_6 and C_7 such that

$$\begin{aligned} (\frac{m-1}{m})(\frac{x}{t-t_1})^{1/m-1} \quad &\leq \quad \frac{l_+(\beta)}{x} \\ &\leq \quad \frac{1}{x}[L_{w_1}^{-1}\{L_{w_2}\{G_{w_2}(u_0(\mu(0))) + l_-(\mu)\} - l_0(\mu)\} + l_+(\mu) \\ &\qquad -L_{w_1}^{-1}\{L_{w_2}\{G_{w_2}(u_0(\beta(0))) + l_-(\beta)\} - l_0(\beta)\}] \\ &\leq \quad \frac{C_6}{x}\{\beta(0) + l_-(\beta) + l_0(\beta) + l_+(\beta)\} \\ &\leq \quad C_7. \end{aligned}$$

Combining all the cases to obtain a constant $C = C(u_0, \lambda, T)$ such that $|\dot{\beta}| = \max(|\dot{\beta}_1|, |\dot{\beta}_3|) = \max(\frac{x}{t-t_1}, \frac{\beta(0)}{t_2}) \leq C$. This proves the Lemma .

Lemma 4.12 : Let $T > 0$. There exist a constant $M = M(u_0, \lambda, T) > 0$ such that for $0 < t \leq T,\ 0 \leq x_2 \leq x_1$,

$$|u(x_2, t) - u(x_1, t)| \leq M|x_1 - x_2| \tag{4.37}$$

Proof : Let $(\beta, w_1, w_2) \in D_{M_1}(x_1, t)$ be a minimizer for $u(x_1, t)$. Let $\beta = (\beta_1, \beta_2, \beta_3)$ be defined on $0 \leq t_2 \leq t_1 \leq t$. Suppose $[t_2, t_1] = \phi$, then β is a line segment and define $(\alpha, w_1, w_2) \in D(x_2, t)$ by

$$\alpha(\theta) = \begin{cases} \beta(\theta) + x_2 - x_1 & \text{if} \quad \dot{\beta} \leq 0 \\ (\theta - t)\dot{\beta} + x_2 & \text{if} \quad \dot{\beta} > 0, \theta \in [(t - \frac{x_2}{\dot{\beta}})\bigvee 0, t] \\ 0 & \text{if} \quad \dot{\beta} > 0, \theta \in [0, (t - \frac{x_2}{\dot{\beta}})\bigvee 0]. \end{cases}$$

Clearly $l_0(\beta) = 0, l_+(\alpha) \leq l_+(\beta), l_-(\alpha) = l_-(\beta)$ and $|\alpha(0) - \beta(0)| \leq |x_1 - x_2|$. Hence

$$\begin{aligned} G_{w_1}(u(x_2,t)) - G_{w_1}(u(x_1,t)) &\leq L_{w_1}^{-1}\{L_{w_2}\{G_{w_2}(u_0(\alpha(0))) + l_-(\alpha)\} - l_0(\alpha)\} + l_+(\alpha) \\ &\qquad -L_{w_1}^{-1}\{L_{w_2}\{G_{w_2}(u_0(\beta(0))) + l_-(\beta)\} - l_+(\beta)\} - l_+(\beta) \\ &\leq M_1|\alpha(0) - \beta(0)| \\ &\leq M_1|x_1 - x_2| \end{aligned}$$

Suppose $[t_2, t_1] \neq \phi$. Let $\tilde{t}_1 = (1 - \frac{x_2}{x_1})t + \frac{x_2}{x_1}t_1$ and define $(\alpha, w_1, w_2) \in D(x_2, t)$ by

$$\alpha(\theta) = \begin{cases} \frac{\theta - \tilde{t}_1}{t - \tilde{t}_1}x_2 & \text{if} \quad \theta \in [\tilde{t}_1, t] \\ 0 & \text{if} \quad \theta \in [t_1, \tilde{t}_1] \\ \beta(\theta) & \text{if} \quad \theta \in [0, t_1]. \end{cases}$$

Then $l_-(\beta) = l_-(\alpha), l_0(\beta) \leq l_0(\alpha), \alpha(0) = \beta(0),\ \dot{\alpha}_1 = \dot{\beta}_1$ and $(\frac{m-1}{m})l_+(\alpha) = (t - \tilde{t}_1)(\dot{\beta}_1)^{m/m-1} \leq (t - t_1)(\dot{\alpha}_1)^{m/m-1} = \frac{m-1}{m}l_+(\beta)$. This implies that $u(x_2, t) - u(x_1, t) \leq 0 \leq M_1\ |x_1 - x_2|$. Hence

$$u(x_2, t) - u(x_1, t) \leq M_1|x_1 - x_2|. \tag{4.38}$$

In order to obtain the other inequality, from lemma (4.9) choose a minimizer $(\beta, w_1, w_2) \in D(x_2, t)$ with $|\dot{\beta}| \leq C$, for $u(x_2, t)$. Let $\beta = (\beta_1, \beta_2, \beta_3)$ be defined on the partition $0 \leq t_2 \leq t_1 \leq t$.

Suppose $[t_2, t_1] = \phi$, then β is a line segment and define $(\alpha, w_1, w_2) \in D(x_1, t)$ by $\alpha = \beta + x_1 - x_2$. Then $l_\pm(\alpha) = l_\pm(\beta),\ l_0(\alpha) = l_0(\beta) = 0$ and $|\alpha(0) - \beta(0)| = |x_1 - x_2|$. Hence

$$\begin{array}{rcl} G_{w_1}(u(x_2,t)) - G_{w_1}(u(x_1,t)) & \geq & L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u_0(\beta(0))) + l_-(\beta)\right\} - l_0(\beta)\right\} + l_+(\beta) \\ & & \quad -L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u_0(\alpha(0))) + l_-(\alpha)\right\} - l_0(\alpha)\right\} - l_+(\alpha) \\ & \geq & -M_1|\alpha(0) - \beta(0)| \\ & = & -M_1|x_1 - x_2|. \end{array}$$

Suppose $[t_2, t_1] \neq \phi$. Let $\tilde{t_1} = t - Cx_1$ where $|\dot{\beta}| = \max(\dot{\beta}_1, |\dot{\beta}_3|) = \max(\frac{x_2}{t-t_1}, \frac{\beta(0)}{t_2}) \leq \frac{1}{C}$. Suppose $t_1 \leq \tilde{t}_1 = t - x_1$. Define $(\gamma, w_1, w_2) \in D(x_1, t)$ by

$$\gamma(\theta) = \begin{cases} \frac{\theta - t_1}{t - t_1}x_1 & \text{if} \quad \theta \in [t_1, t] \\ \beta(\theta) & \text{if} \quad \theta \in [0, t_1]. \end{cases}$$

Then $Cx_1 \leq t - t_1,\ l_0(\gamma) = l_0(\beta), l_-(\gamma) = l_-(\beta),\ \gamma(0) = \beta(0)$ and there exist $\theta \in (x_2, x_1)$ such that

$$\begin{array}{rcl} l_+(\beta) - l_+(\gamma) & = & \frac{(m-1}{m}\frac{(x_1^{m/m-1} - x_2^{m/m-1})}{(t-t_1)^{1/m-1}} \\ & = & \frac{\theta^{1/m-1}(x_2 - x_1)}{(t-t_1)^{1/m-1}} \\ & \geq & -(\frac{x_1}{t-t_1})^{1/m-1}(x_1 - x_2) \\ & \geq & -C(x_1 - x_2). \end{array}$$

Hence

$$G_{w_1}(u(x_2,t)) - G_{w_1}(u(x_1,t)) \geq l_+(\beta) - l_+(\gamma) \geq -C(x_1 - x_2).$$

Suppose $t_2 \leq \tilde{t}_1 \leq t_1$. Define $(\gamma, w_1, w_2) \in D(x_1, t)$ by

$$\gamma(\theta) = \begin{cases} \frac{\theta - \tilde{t}_1}{t - \tilde{t}_1}x_1 & \text{if} \quad \theta \in [\tilde{t}_1, t] \\ \beta(\theta) & \text{if} \quad \theta \in [0, \tilde{t}_1]. \end{cases}$$

then $\gamma(0) = \beta(0),\ l_-(\gamma) = l_-(\beta),\ t - t_1 \leq Cx_1,\ Cx_2 \leq (t - t_1), t_1 - \tilde{t}_1 = Cx_1 - (t - t_1) \leq C(x_1 - x_2),\ l_+(\beta)$

$$\begin{array}{rcl} l_+(\gamma) & = & \frac{m-1}{m}\left[\frac{x_2^{m/m-1}}{(t-t_1)^{1/m-1}} - \frac{x_1^{m/m-1}}{(t-\tilde{t}_1)^{1/m-1}}\right] \\ & \geq & (\frac{m-1}{m})\left[\frac{x_2^{m/m-1}}{(Cx_1)^{1/m-1}} - \frac{x_1^{m/m-1}}{(Cx_1)^{1/m-1}}\right] \\ & = & \frac{(m-1)}{m}\frac{(x_1^{m/m-1} - x_2^{m/m-1})}{C^{1/m-1}x_1^{1/m-1}} \geq -\frac{(x_1 - x_2)}{C^{1/m-1}}, \end{array}$$

$|l_0(\beta) - l_0(\gamma)| \leq |\lambda^+|_T^m(t_1 - \tilde{t}_1) \leq C|\lambda^+|_T^m(x_1 - x_2)$. Hence

$$\begin{array}{rcl} G_{w_1}(u(x_2,t)) - G_{w_1}(u(x_1,t)) & \geq & L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u_0(\beta(0))) + l_-(\beta)\right\} - l_0(\beta)\right\} + l_+(\beta) \\ & - & L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u_0(\gamma(0)) + l_-(\gamma)\right\} - l_0(\gamma)\right\} - l_+(\gamma) \\ & \geq & -M_1\left\{|l_0(\beta) - l_0(\gamma))\right\} + l_+(\beta) - l_+(\gamma) \\ & \geq & -[M_1C|\lambda^+|_T^m + \frac{1}{C^{1/m-1}}](x_1 - x_2). \end{array}$$

Suppose $0 \leq \tilde{t}_1 \leq t_2$. Define $(\gamma, w_1, w_2) \in D(x_1, t)$ by

$$\gamma(\theta) = \begin{cases} \frac{\theta - \tilde{t}_1}{t - \tilde{t}_1}x_1 & \text{if} \quad \theta \in [\tilde{t}_1, t] \\ (\tilde{t}_1 - \theta)\frac{\beta(0)}{t_2} & \text{if} \quad \theta \in [0, \tilde{t}_1]. \end{cases}$$

Since $\tilde{t}_1 \leq t_2 \leq t_1$ implies $t - t_1 \leq Cx_1$ and hence as in the previous case $l_+(\beta) - l_+(\gamma) \geq -\frac{(x_1-x_2)}{C^{1/m-1}}$. Also $t - t_2 \leq Cx_1,\ t_1 - t_2 = (t-t_2) - (t-t_1) \leq C(x_1 - x_2), l_0(\beta) \leq |\lambda^+|_T^m(t_1 - t_2) \leq C|\lambda^+|_T^m(x_1 - x_2),\ |\beta(0) - \gamma(0)| = |\beta(0) - frac\beta(0)t_2\tilde{t}_1| = \frac{\beta(0)}{t_2}(t_2 - \tilde{t}_1) \leq \frac{1}{C}(-t + t_2 + Cx_1) \leq (x_1 - x_2), |l_-(\beta) - l_-(\gamma)| = (\frac{m-1}{m})|(\frac{\beta(0)}{t_2})^{m/m-1}t_2 - (\frac{\beta(0)}{t_2})^{m/m-1}\tilde{t}_1| \leq \frac{x_1-x_2}{C^{1/m-1}}$. Hence

$$\begin{aligned} G_{w_1}(u(x_2,t)) - G_{w_1}(u(x_1,t) &\geq -M_1\left\{|\beta(0) - \gamma(0)| + l_0(\beta) + |l_-(\beta) - l_-(\gamma)|\right\} + l_+(\beta) - l_-(\gamma) \\ &\geq -M_1(1 + C|\lambda^+|_T^m + \frac{2}{C^{1/m-1}})(x_1 - x_2). \end{aligned}$$

Suppose $\tilde{t}_1 \leq 0$. Define $(\gamma, w_1, w_2) \in D(x_1, t)$ by $\gamma(\theta) = \frac{\theta - \tilde{t}_1}{t - \tilde{t}_1}x_1$ for $\theta \in [0, t]$. $\tilde{t}_1 \leq 0$ implies that $t \leq Cx_1,\ t_1 - t_2 = (t - t_2) - (t - t_1) \leq C(x_1 - x_2),\ t_2 \leq t_1 - \tilde{t}_1 \leq Cx_1 - (t - t_1) \leq C(x_1 - x_2),\quad \beta(0) \leq \frac{1}{C}t_2 \leq (x_1 - x_2), l_0(\beta) \leq |\lambda^+|_T^m(t_1 - t_2) \leq C|\lambda^+|_T^m(x_1 - x_2),\ l_-(\beta) = \frac{m-1}{m}(\frac{\beta(0)}{t_2})^{1/m-1}\beta(0) \leq \frac{1}{C^{1/m-1}}(x_1 - x_2),\ l_+(\beta) - l_+(\gamma) = \frac{m-1}{m}[\frac{x_2^{m/m-1}}{(t-t_1)^{1/m-1}} - \frac{t}{C^{1/m-1}}] \geq \frac{m-1}{m}[\frac{x_2^{m/m-1}}{(Cx_1)^{1/m-1}} - \frac{x_1}{C^{1/m-1}}] = -\frac{m-1}{m}(\frac{x_1^{m/m-1} - x_2^{m/m-1}}{c^{1/m-1}x_1^{1/m-1}}) \geq -\frac{1}{C^{1/m-1}}(x_1 - x_2)$ and $\gamma(0) \leq x_1 - x_2$. Hence

$$\begin{aligned} G_{w_1}(u(x_2,t)) - G_{w_1}(u(x_1,t)) &\geq -M_1\left\{\beta(0) + \gamma(0) + l_0(\beta) + l_-(\beta)\right\} \\ &\quad + l_+(\beta) - l_+(\gamma) \\ &\geq -M_1(2 + C|\lambda^+|_T^m + \frac{2}{C^{1/m-1}})(x_1 - x_2) \end{aligned}$$

Since G_w is a diffeomorphism, $|u|_T < \infty$ and hence by combining all the above inequalities, to obtain a constant $C_1 = C_1(u_0, \lambda, T)$ such that

$$u(x_2, t) - u(x_1, t) \geq -C_1(x_1 - x_2).$$

This together with (4.38) gives (4.37) with $M = \max(M_1, C_1)$ and hence the Lemma .

Lemma 4.13 : Let $T > 0$. Then there exist $M = M(u_0, \lambda, T)$ such that for $x_1, x_2 \in \overline{I\!R}_+,\quad 0 \leq s < t \leq T$,

$$|u(x_1, t) - u(x_2, s)| \leq M(|x_1 - x_2| + |t - s|). \tag{4.39}$$

Proof : In view of Lemma 4.12 , it is enough to prove that

$$|u(x,t) - u(x,s)| \leq M|t - s|.$$

Define $(\gamma, w, w) \in D(x, s, t)$ by $\gamma(\theta) = x$ for $\theta \in [s, t]$ then $l_\pm(\gamma) = 0$ and hence from (4.32)

$$u(x,t) \leq u(\gamma(s), s) = u(x, s).$$

From Lemma 4.12 , $x \mapsto u(x, s)$ is a uniformly Lipschitz continous function and hence from Lemma 4.9, (4.37) and following the same proof of Lemma 4.11, there exist a constant

$C = C(u_0, \lambda, T)$ $(\beta, w_1, w_2) \in D(x,s,t)$ with $|\dot\beta| \le C$ a minimizer for $u(x,t)$. Hence $|\beta(s) - x| \le |\dot\beta|(t-s) \le C(t-s), l_0(\beta) \le |\lambda^+|_T^m(t_1 - t_2) \le |\lambda^+|_T^m(t-s)$. Hence from Lemma 4.12

$$\begin{aligned}
G_{w_1}(u(x,t)) - G_{w_1}(u(x,s)) &= L_{w_1}^{-1}\left\{L_{w_2}\left\{G_{w_2}(u(\beta(s),s)) + l_-(\beta)\right\} - l_0(\beta)\right\} + l_+(\beta) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - G_{w_1}(u(x,s)) \\
&\ge L_{w_1}^{-1}\left\{L_{w_2}G_{w_2}(u(\beta(s),s)) - l_0(\beta)\right\} - L_{w_1}^{-1}(L_{w_1}G_{w_1}(u(x,s))) \\
&= L_{w_1}^{-1}\left\{\eta(u(\beta(s),s) - l_0(\beta)\right\} - L_{w_1}^{-1}(\eta(u(x,s)) \\
&\ge -M_1\left\{|u(\beta(s),s) - u(x,s)| + l_0(\beta)\right\} \\
&\ge -M_1\left\{M|\beta(s) - x| + |\lambda^+|_T^m(t-s)\right\} \\
&\ge -[CM_1M + |\lambda^+|_T^m](t-s).
\end{aligned}$$

This proves the Lemma.

Proof of theorem 2.3 : From Lemma 4.12, $\lim_{t\to\infty} u(x,t) = u_0(x)$.

Sub solution : Suppose u is not a subsolution. Then there exist an $(x_0, t_0) \in \overline{I\!R}_+ \times I\!R_+$, $\varphi \in C^1(\overline{I\!R_+ \times I\!R_+}), \mu > 0$ such that $u(x_0,t_0) = \varphi(x_0,t_0), u - \varphi$ has a local maximima at $(x_0, t_0,$ $\varphi_t + H(\varphi, \varphi_x) \ge 4\mu$ and if $x_0 = 0$ then$\varphi_x(x_0,t_0) < \lambda(t_0)$. Hence we can find a ball B centered at (x_0,t_0) such that in B, $u \le \varphi$, $\varphi_x < \lambda$ if $x_0 = 0$ and $\varphi_t + H(\varphi,\varphi_x) \ge 2\mu$.

Let $x_0 > 0$. Then shrinking B if necessary, can assume that $B \subset I\!R_+ \times I\!R_+$. Let $a > 0$, $w \in \{\pm 1\}$, $0 < s_0 < t$ and $\beta(\theta) = x_0 - a\,w(t_0 - \theta)$. Then $\dot\beta(\theta)w = a\,w^2 > 0$ and hence $(\beta, w, w) \in D(x, s_0, t)$ for $|s_0 - t_0|$ is sufficiently small. Choose w such that $w\varphi_x(x_0,t_0) \ge 0$. Let $l(q) = \frac{1}{\delta}|q|^\delta$. Then either $l_+(\beta) = 0$ or $l_-(\beta) = 0$. Suppose $l_-(\beta) = 0$, then $l_+(\beta) = (t_0 - s_0)l(\frac{x_0 - \beta(s_0)}{t_0 - s_0})$ and from (4.32)

$$\begin{aligned}
G_w(\varphi(x_0,t_0)) &= G_w(u(x_0,t_0)) \\
&\le G_w(u(\beta(s_0),s_0)) + l_+(\beta) \\
&\le G_w(\varphi(\beta(s_0),s_0) + (t_0 - s_0)l(\tfrac{x_0 - \beta(s_0)}{t_0 - s_0}),
\end{aligned}$$

that is ,

$$\frac{G_w(\varphi(x_0,t_0)) - G_w(\varphi(x_0 - aw(t_0 - s_0), s_0)}{t_0 - s_0} \le l(a)$$

letting $s_0 \to t_0$ to obtain at $(x_0,t_0), G'_w(\varphi)\varphi_t + G'_w(\varphi)\frac{\partial\varphi}{\partial x}\lambda w \le l(a)$. Hence $G'_{w_1}(\varphi)\varphi_t + \sup_{a>0}\left\{G'_w(\varphi)\frac{\partial\varphi}{\partial x}wa - l(a)\right\} \le 0$. Since $\frac{\partial\varphi}{\partial x}w \ge 0$, to obtain , $G'_w(\varphi)\varphi_t + \frac{1}{m}G'_w(\varphi)^m|\frac{\partial\varphi}{\partial x}|^m \le 0$, and hence $G_t + \frac{1}{m}H(\varphi,\varphi_x) \le 0$, which is a contradiction.

Let $x_0 = 0$. Suppose $\varphi_x(x_0,t_0) < 0$. Then in $B \cap \overline{I\!R}_+ \times I\!R_+, \varphi_x < 0$ and hence $\varphi_t \ge 2\mu$. This implies that $\varphi(x_0,t_0) \ge \varphi(x_0,s_0) + \mu(t_0 - s_0)$. Let $\beta(\theta) = x_0$ for $\theta \in [s_0,t_0]$. Then from (4.32) $\varphi(x_0,t_0) = u(x_0,t_0) \le u(x_0,s_0) \le \varphi(x_0,s_0)$ which is a contradiction. Now suppose

$\varphi_x(x_0,t_0) \geq 0$. Then $\lambda(t_0) > 0$ and hence $\lambda^+ = \lambda$. Let $\beta(\theta) = x$ for $\theta \in [s_0,t_0]$. Then $l_{\pm 1}(\beta) = 0$ and hence from (4.32)

$$\begin{aligned} \eta(\varphi(x_0,t_0)) &= \eta(u(x_0,t_0)) \leq \eta(u(\beta(s_0),s_0) - l_0(\beta) \\ &\leq \eta(\varphi(x_0,s_0)) - \textstyle\int_{s_0}^{t_0}(\lambda^+)^m \end{aligned}$$

Hence dividing by $t_0 - s_0$ and letting $s_0 \to t_0$ to obtain $(\eta'(\varphi)\varphi_t)(x_0,t_0) \leq -(\lambda^+)^m(t_0) = -\lambda^m(t_0)$. That is $(\varphi_t + \frac{1}{m}\lambda^m H(\varphi,1))(x_0,t_0) \leq 0$. On the other hand at (x_0,t_0) $2\mu \leq \varphi_t + \frac{1}{m}H(\varphi,\varphi_x) \leq \varphi_t + \frac{\varphi_x^m}{m}H(\varphi,1) \leq \varphi_t + \frac{\lambda^m}{m}H(\varphi,1) \leq 0$ which is a contradiction . This proves that u is a subsolution.

Super Solutions : Suppose u is not a super solution. Then there exist a $(x_0,t_0) \in \overline{I\!R}_+ \times I\!R_+$, $\varphi \in C^1(\overline{I\!R_+ \times I\!R_+})$, $\mu > 0$ such that $u(x_0,t_0) = \varphi(x_0,t_0)$, $u - \varphi$ has a local minima at (x_0,t_0) and at (x_0,t_0), $\varphi_t + \frac{1}{m}H(\varphi,\varphi_x) \leq -4\mu$, $\varphi_x(x_0,t_0) > \lambda(t_0)$ if $x_0 = 0$. Hence by continuity , there exist a ball B around (x_0,t_0) such that in B, $\varphi_t + \frac{1}{m}H(\varphi,\varphi_x) \leq -3\mu$ and $\varphi_x > \lambda$ if $x_0 = 0$.

Let for each $s < t_0, (\beta_s, w_{1s}, w_{2s}) \in D_M(x_0,s,t)$ be a minimizer for $u(x,t)$ given in (4.32) . Now for $\theta \in [s,t_0], |\beta_s(\theta) - x_0| \leq ||\beta_s||(t_0 - s)^{\frac{1}{m}} \leq M^{\frac{1}{\delta}}(t_0 - s)^{\frac{1}{m}}$ and hence $\beta_s \to x_0$ uniformly as $s \to t_0$.

Suppose $x_0 > 0$. By shrinking B if necessary,can assume that $B \subset I\!R_+ \times I\!R_+$. Hence choose $s_0 < t_0$ such that for all $s \in [s_0,t_0], (\beta_s(\theta),\theta) \in B$ for all $\theta \in [s,t_0]$. This implies that β_s is a linear curve with $l_0(\beta_s) = 0$. Also suppose $x_0 = 0$ and $l_0(\beta_s) = 0$,then by convexity of $l_-(\beta_s), \beta_s$ is a linear curve with $l_+(\beta_s) = 0$. Suppose $l_+(\beta_s) = 0$.Let $w = -1$ and from (4.32) $G_w(\varphi(x_0,t_0)) = G_w(u(x_0,t_0)) = G_w(u(\beta(s),s)) + l_-(\beta_s) \geq G_w(\varphi(\beta(s),s)) + l_-(\beta_s)$. Hence $\varphi(x_0,t_0) \geq \varphi(\beta(s),s))$. Suppose $\varphi_x(x_0,t_0) > 0$,then by shrinking B if necessary,can assume that $\varphi_x > 0, \varphi_t < -3\mu$ in B. Since $l_+(\beta_s) = 0$ and hence $x_0 \leq \beta_s(s)$. This implies that $\varphi(x_0,t_0) \leq \varphi(\beta_s(s),t_0) < \varphi(\beta_s(s),s)$ which is a contradiction. If $\varphi_x(x_0,t_0) \leq 0$,then at $(x_0,t_0), \varphi_t + \frac{1}{m}H(\varphi,\varphi_x) \leq -4\mu$ and hence $G'_w(\varphi)\varphi_t + \frac{1}{m}G'_w(\varphi)^m|\varphi_x)|^m \leq -4\mu G'_w(\varphi)$. Hence by shrinking B if necessary,can assume that$G'_w(\varphi)\varphi_t + \frac{1}{m}G'_w(\varphi)^m|\varphi_x)|^m \leq -3\mu G'_w(\varphi)$ in B. Therefore for any$\alpha \in I\!R, (x,t) \in B$

$$G'_w(\varphi)\varphi_t + \alpha G'_w(\varphi)\varphi_x \;\leq\; -3\mu G'_w(\varphi) + \tfrac{1}{\delta}|\alpha|^\delta. \tag{4.40}$$

Let $\alpha = \dot{\beta}_s$ and $\psi(x,t) = G_w(\varphi(x,t))$.Then from (4.40)

$$\begin{array}{rcl} l_-(\beta_s) \leq G_w(\varphi(x_0,t_0)) - G_w(\varphi(\beta_s(s),s)) & = & \int_0^1 \frac{d}{d\theta}\psi(\theta x_0 + (1-\theta)\beta_s(s), \theta t_0 + (1-\theta)s_0)d\theta \\ & = & (t_0 - s)\int_0^1 G_w'(\varphi)(\alpha\varphi_x + \varphi_t)\ d\theta \\ & \leq & \frac{(t_0-s)}{\delta}|\alpha|^\delta - 3\mu(t_0-s)\int_0^1 G_w'(\varphi)d\theta \\ & = & l_-(\beta_s) - 3\mu(t_0-s)\int_0^1 G_w'(\varphi)d\theta \end{array}$$

which is a contradiction. If $l_-(\beta_s) = 0$,then by similar arguement a contradiction follows. Let $x_0 = 0$. If $\lambda(t_0) < 0$, then $\lambda^+ = 0$ in B and hence $l_0(\beta_s) = 0$ and by above argument a contradiction follows. Therefore can assume that $\lambda(t_0) \geq 0$ and hence $\varphi_x(x_0,t_0) > \lambda(t_0) \geq 0$. This implies that by shrinking B ,$\varphi_x(x_0,t_0) > 0$ and $\varphi_t < 0$ in B. Let $\beta_s = (\phi, \beta_2, \beta_3)$ be defined on the partition $s = s_3 \leq s_2 \leq s_1 = t_0$. Then $\varphi(x_0,s) \leq \varphi(\beta_s(s),s)$. Now with $w = -1$

$$\begin{array}{rcl} \eta(\varphi(x_0,t_0)) - \eta(\varphi(x_0,s)) & = & \eta(u(x_0,t_0)) - \eta(\varphi(x_0,s)) \\ & = & L_w\left\{G_w(u(\beta(s),s))\right\} - l_0(\beta_s) - \eta(\varphi(x_0,s)) \\ & \geq & \eta(\varphi(\beta(s),s)) - \eta(\varphi(x_0,s)) - l_0(\beta_s) \\ & \geq & -\int_s^{t_0}(\lambda^+)^m. \end{array}$$

Dividing by $(t_0 - s)$ and letting $s \to t_0$ in the above inequality to obtain, at $(x_0,t_0), 0 \leq \varphi_t + \frac{1}{m}(\lambda^+)^m H(\varphi,1) \leq \varphi_t + \frac{1}{m}H(\varphi,\varphi_x) \leq -4\mu$ which is a contradiction. Hence u is a super solution and this completes the proof of the Theorem 2.3.

REFERENCES

1. Adimurthi and G. D. Veerappa gowda, Hopf-Lax type formula for sub and super solutions of a Hamilton-Jacobi equation, 1997 , to appear in Advances in Differential equations.

2. G. Barles and B. Perthame, Discontinous solutions of deterministic optimal stopping time problems, Math. Modeling Numer. Anal. 21 (1987), 557-579.

3. E. N. Barron, R. Jensen and W. Liu , Hopf-Lax type formula for $u_t + H(u, Du) = 0$, J. Diff. Eqn. 126 (1996), 48-64.

4. M.G.Crandall and P.L.Lions, Viscosity solution of Hamilton-Jacobi equations, Trans.Amer. Math.Soc.,277(1983),1-42.

5. L. C. Evans, Partial differential equations, Berkely mathematics Lecture notes, Vols 3A, 3B , (1994) .

6. Hitoshi Ishii, Perron's method for Hamilton-Jacobi equations., Duke. math. J., 55 (1987), 369-384.

7. K. T. Joseph and G. D. Veerappa gowda, Explicit formula for the solution of convex conservation laws with boundary conditions, Duke. Math. J., 62 (1991) , 401-416.

8. K. T. Joseph and G. D. Veerappa gowda, The Hamilton-Jacobi equation $V_t + |V_x| = 0$ in the quarter plane, Non linear Analysis TMA., 18 (1992) , 1147-1158 .

9. P. Le Floch, Explicit formula for scalar non linear conservation laws with boundary condition, Math, Methods in Appl. Sci., 10 (1988), 265-287.

10. P. L. Lions, Generalized solutions of Hamilton-Jacobi equations., Research notes in mthematics, Pitmann, 1982.

11. P. L. Lions, Neumann type boundary conditions for Hamilton-Jacobi equations., Duke. Math. J., 52 (1985), 893-820 .