Delta-shock Wave Type Solution of Hyperbolic Systems of Conservation Laws

V. G. Danilov and V. M. Shelkovich

Abstract. For some classes of hyperbolic systems of conservation laws we introduce a *new definition of a δ-shock wave type solution.* This definition gives natural generalizations of the classical definition of the weak solutions. It is *relevant* to the notion of δ-shocks. The *weak asymptotics method* developed by the authors is used to describe the propagation of δ-shock waves.

1. Introduction

1. One of the approaches to solving the problems related to singular solutions of quasilinear equations was developed by the authors in [**3**], [**4**]– [**8**] (see also [**2**], [**25**]). In these papers, the authors developed a new asymptotics method – the *weak asymptotics method* – for studying the *dynamics of propagation and interaction* of different singularities of quasilinear differential equations and first-order hyperbolic systems (infinitely narrow δ-solitons, shocks, δ-shocks). This method is based on the ideas of V. P. Maslov's approach that permits deriving the Rankine–Hugoniot conditions directly from the differential equations *considered in the weak sense* [**18**], [**21**] [**2**] (see also [**29**, 2.7]). Maslov's *algebras of singularities* [**19**], [**20**] [**2**] are essentially used in the *weak asymptotics method*.

In this paper in the framework of the *weak asymptotics method* for some classes of hyperbolic systems of conservation laws we introduce a *new definition of a δ-shock wave type solution* by integral identities.

2000 *Mathematics Subject Classification.* Primary 35L65; Secondary 35L67, 76L05.

Key words and phrases. Hyperbolic systems of conservation laws, generalized solutions, δ-shock waves, the weak asymptotics method.

The second author (V. S.) was supported in part by DFG Project 436 RUS 113/593/3 and Grant 02-01-00483 of Russian Foundation for Basic Research.

This definition gives *natural* generalizations of the classical definition of the weak L^∞-solutions and specifies the definition of measure-solutions given in [**1**], [**27**], [**30**].

By using the *weak asymptotics method* we describe the propagation of δ-*shock waves* to three types systems of conservation laws. Among them are well-known system (1.7), which was studied by B. Lee Keyfitz and H. C. Kranzer, and zero-pressure gas dynamics system (1.8).

For all these systems we construct *approximating solutions* in the weak sense and prove that the weak limits of these approximating solutions satisfy our definition.

Consider the system of equations

$$
\begin{aligned}
L_1[u,v] = u_t + \Big(F(u,v)\Big)_x &= 0,\\
L_2[u,v] = v_t + \Big(G(u,v)\Big)_x &= 0,
\end{aligned}
\tag{1.1}
$$

where $F(u,v)$ and $G(u,v)$ are smooth functions, $u = u(x,t)$, $v = v(x,t) \in \mathbb{R}$, and $x \in \mathbb{R}$. As is well known, such a system, even in the case of smooth (and, moreover, in the case of discontinuous) initial data $(u^0(x), v^0(x))$, can have a discontinuous *shock wave* type solution. In this case, it is said that the vector function $(u(x,t), v(x,t)) \in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}^2\big)$ is a generalized solution of the Cauchy problem (1.1) with the initial data $(u^0(x), v^0(x))$ if the integral identities

$$
\begin{aligned}
\int_0^\infty\int \Big(u\varphi_t + F(u,v)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty\int \Big(v\varphi_t + G(u,v)\varphi_x\Big)\,dx\,dt + \int v^0(x)\varphi(x,0)\,dx &= 0
\end{aligned}
\tag{1.2}
$$

hold for all compactly supported test functions $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\,\infty))$, where $\int\,\cdot\,dx$ denotes an improper integral $\int_{-\infty}^{\infty}\,\cdot\,dx$.

Consider the Cauchy problem for system (1.1), where functions $F(u,v)$, $G(u,v)$ are linear with respect to v, with initial data

$$
u^0(x) = u_0 + u_1H(-x), \quad v^0(x) = v_0 + v_1H(-x),
\tag{1.3}
$$

where u_0, u_1, v_0, and v_1 are constants and $H(\xi)$ is the Heaviside function. As was shown in [**1**], [**6**]– [**15**], [**27**], [**30**], in order to solve this problem for some "nonclassical cases", it is necessary to introduce new elementary singularities called δ-*shock waves*. These are generalized solutions of hyperbolic systems of conservation laws of the form

$$
\begin{aligned}
u(x,t) &= u_0 + u_1H(-x+ct),\\
v(x,t) &= v_0 + v_1H(-x+ct) + e(t)\delta(-x+ct),
\end{aligned}
\tag{1.4}
$$

where $e(0) = 0$ and $\delta(\xi)$ is the Dirac delta function.

At present several approaches to constructing such solutions are known. The actual difficulty in defining such solutions arises because of the fact that (as follows from (1.1), (1.4)), to introduce a definition of the δ-shock wave type solution, one need to define the *product of the Heaviside function and the δ-function.*

To obtain a δ-shock wave type solution of the system

$$\begin{aligned} u_t + (u^2)_x &= 0, \\ v_t + (uv)_x &= 0, \end{aligned} \tag{1.5}$$

(here $F(u,v) = u^2$, $G(u,v) = vu$), the following parabolic regularization is used in [**13**]:

$$u_t + (u^2)_x = \varepsilon u_{xx}, \quad v_t + (uv)_x = \varepsilon v_{xx}.$$

In [**12**], to obtain a δ-shock wave type solution of the system

$$\begin{aligned} L_{11}[u,v] = u_t + \big(f(u)\big)_x &= 0, \\ L_{12}[u,v] = v_t + \big(g(u)v\big)_x &= 0, \end{aligned} \tag{1.6}$$

(here $F(u,v) = f(u)$, $G(u,v) = vg(u)$), this system is reduced to a system of Hamilton–Jacobi equations, and then the Lax formula is used. In [**15**] in order to construct δ-shock solutions, the problem of multiplication of distributions is solved by using the definition of Volpert's averaged superposition [**28**]. In [**22**] the nonconservative product of singular functions is defined as a generalization of Volpert's ideas. In [**14**], the Colombeau theory approach, as well as the Dafermos–DiPerna regularization were used in a specific case of the system

$$\begin{aligned} L_{21}[u,v] &= u_t + (u^2 - v)_x = 0, \\ L_{22}[u,v] &= v_t + (\tfrac{1}{3}u^3 - u)_x = 0, \end{aligned} \tag{1.7}$$

(here $F(u,v) = u^2 - v$, $G(u,v) = u^3 - u$). In [**14**] *approximate solutions* was constructed, but the notion of a *singular solution* of system (1.7) has *not* been defined. In [**23**] in the framework of the Colombeau theory approach, for some classes of systems *approximate solutions* were constructed.

In [**27**] for system (1.5), in [**1**] for the system

$$\begin{aligned} L_{31}[u,v] &= v_t + \big(vu\big)_x = 0, \\ L_{32}[u,v] &= (vu)_t + \big(vu^2\big)_x = 0, \end{aligned} \tag{1.8}$$

(here $v \ge 0$ is the density, u is the velocity), and in [**30**] for the system

$$\begin{aligned} v_t + \big(vf(u)\big)_x &= 0, \\ (vu)_t + \big(vuf(u)\big)_x &= 0, \end{aligned} \tag{1.9}$$

with the initial data (1.3), the δ-shock wave type solution is defined as a measure-valued solution (see also [**26**]).

Let $BM(\mathbb{R})$ be the space of bounded Borel measures, $v(x,t) \in C\big([0,\infty), BM(\mathbb{R})\big)$, $u(x,t) \in L^\infty\big([0,\,\infty), L^\infty(\mathbb{R})\big)$. A pair (u,v) is said to be a measure-valued solution of the Cauchy problem (1.9), (1.3) if

$$
\begin{aligned}
\int_0^\infty \int \Big(\varphi_t + f(u)\varphi_x\Big)\, v(dx,t) &= 0, \\
\int_0^\infty \int u\Big(\varphi_t + f(u)\varphi_x\Big)\, v(dx,t) &= 0,
\end{aligned}
\tag{1.10}
$$

for all $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\,\infty))$.

Within the framework of this definition in [**27**] for system (1.5), in [**1**] for system (1.8), and in [**30**] for system (1.9), the following formulas for δ-shock waves where derived

$$
(u(x,t),\ v(x,t)) = \begin{cases} \big(u^-,\ v^-\big), & x < \phi(t), \\ \big(u_\delta,\ w(t)\delta(x-\phi(t))\big), & x = \phi(t), \\ \big(u^+,\ v^+\big), & x > \phi(t). \end{cases}
\tag{1.11}
$$

Here u^-, u^+ and u_δ are the velocities before the discontinuity, after the discontinuity, and at the point of discontinuity, respectively, and $\phi(t) = \sigma_\delta t$ is the equation for the discontinuity line.

In [**9**], the global δ-shock wave type solution was obtained for system (1.8). By using the Colombeau theory approach, in [**11**], an approximating solution of the Cauchy problem is constructed for this system.

The study of systems, which admit δ-shock wave type solutions is very important in applications, because systems of this type often arise in modelling of physical processes in gas dynamics, magnetohydrodynamics, filtration theory, and cosmogony. System (1.8) is called the "zero-pressure gas dynamics system". In multidimensional case this system was used to describe the motion of free particles which stick under collision and thus also is connected with the formation of large-scale structures in the universe [**31**].

2. In the framework of the *weak asymptotics method* for some classes of hyperbolic systems of conservation laws we present below the *definition of a δ-shock wave type solution*. This definition is *natural* generalizations of the classical definition (1.2) for L^∞ solutions and specifies the measure-solutions definition (1.10) (see in [**1**], [**27**], [**30**] for the case in which the support of the singular part of the measure v is a union of pieces of smooth curves. According to our Definitions 1.2, 1.3 a *generalized δ-shock wave type solution* is a pair of *distributions* $(u(x,t), v(x,t))$ unlike the Definition (1.10), where $v(dx,t)$ is a *measure* and $u(x,t)$ is understood as a *measurable function which is defined* $v(dx,t)$ *a.e.*.

We apply the *weak asymptotics method* in order to solve the Cauchy problem for systems (1.6) (see Sec. 2), (1.7) (see Sec. 3), and (1.8) (see

Sec. 4) with a special form of the initial data

$$
\begin{aligned}
u^0(x) &= u_0^0(x) + u_1^0(x)H(-x), \\
v^0(x) &= v_0^0(x) + v_1^0(x)H(-x) + e^0\delta(-x),
\end{aligned}
\tag{1.12}
$$

where $u_k^0(\mathrm{x})$, $v_k^0(x)$, $k = 0, 1$ are given smooth functions, e^0 is given constant. Here, in contrast to the well-known works about δ-shock waves (except for [**1**]), the initial data *can contain* δ-function.

The solutions of the Cauchy problems mentioned above are given by Theorems 2.6, 3.3, 4.3, 4.4 and Corollaries 2.7– 4.5. Thus, we construct *δ-shock wave type solutions* and study *dynamics of propagation* of δ-shocks. The fifth and sixth equations of systems in Theorems 2.3, 3.1, 4.1 are *the Rankine–Hugoniot conditions of delta-shocks.*

Remark 1.1. For the Cauchy problem (1.8), (1.12) the fact that the initial conditions contain $e^0 \neq 0$ implies that the system of ordinary differential equations (two last Eq. (4.12)) that determines the trajectory of a singularity and the coefficient of the δ-function is a system of second-order equations. For the unique solvability of the Cauchy problem posed for this system, it is necessary to specify the *initial velocity* along the trajectory of singularity. We study this in detail at the end of Sec. 4. Here we only note that the value of the *initial velocity* is not determined by the initial data (1.12), when $e^0 \neq 0$. Hence the Cauchy problem (1.8), (1.12) has a family solutions *parameterized* by the parameter of the *initial velocity*.

Therefore, as it will be shown in Sec. 4), the Cauchy problem for system (1.8) is well-posed if we use the initial data (4.1) instead of the initial data (1.12). Here in the initial data (4.1) we introduce the *initial velocity*.

The results of Theorem 4.4 and Corollary 4.5 coincide with the analogous statement from [**1**], [**16**], [**26**] if we *identify* the velocity on the discontinuity line $x = \phi(t)$ in formula (1.11) with the phase velocity of nonlinear wave:

$$u_\delta(t) = \dot{\phi}(t).$$

If $e^0 = 0$, and the initial data is piecewise-constants, our results about system (1.7) coincide with the main statements of [**14**]. In [**14**] particular case of an approximate solution (1.17), (3.2) of the Cauchy problem (1.7), (1.12) with piecewise constant initial data was constructed (see Sec. 3).

In conclusion, we note that the weak asymptotic method allows one to study the initial conditions having the form of the linear combination of functions contained in (1.12) and hence to solve the problem of δ-shock waves interaction [**6**]– [**8**].

3. In what follows, we introduce definitions of a *δ-shock wave type solution* for systems (1.1) and (1.8). Here we assume that the functions $F(u,v)$, $G(u,v)$ are *linear* with respect to v.

Suppose that $\Gamma=\{\gamma_i : i\in I\}$ is a connected graph in the upper half-plane $\{(x,t) : x\in\mathbb{R},\ t\in[0,\infty)\}\in\mathbb{R}^2$ containing smooth arcs γ_i, $i\in I$, and I is a finite set. By I_0 we denote a subset of I such that an arc γ_k for $k\in I_0$ starting from the points of the x-axis; $\Gamma_0=\{x_k^0 : k\in I_0\}$ is the set of initial points of arcs γ_k, $k\in I_0$.

Consider the initial data of the form $(u^0(x),v^0(x))$, where

$$v^0(x)=V^0(x)+E^0\delta(\Gamma_0),$$

$E^0\delta(\Gamma_0)=\sum_{k\in I_0}e_k^0\delta(x-x_k^0)$, $u^0,V^0\in L^\infty\big(\mathbb{R};\mathbb{R}\big)$, e_k^0 are constants, $k\in I_0$.

Definition 1.2. A pair of distributions $(u(x,t),v(x,t))$ and graph Γ, where $v(x,t)$ is represented in form of the sum

$$v(x,t)=V(x,t)+E(x,t)\delta(\Gamma),$$

$u,V\in L^\infty\big(\mathbb{R}\times(0,\,\infty);\mathbb{R}\big)$, $E(x,t)\delta(\Gamma)=\sum_{i\in I}e_i(x,t)\delta(\gamma_i)$, $e_i(x,t)\in C(\Gamma)$, $i\in I$, is called a *generalized δ-shock wave type solution* of system (1.1) with the initial data $(u^0(x),v^0(x))$ if the integral identities

$$\begin{aligned}
\int_0^\infty\int\Big(u\varphi_t+F(u,V)\varphi_x\Big)\,dx\,dt+\int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty\int\Big(V\varphi_t+G(u,V)\varphi_x\Big)\,dx\,dt\qquad\qquad&\\
+\sum_{i\in I}\int_{\gamma_i}e_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl&\\
+\int V^0(x)\varphi(x,0)\,dx+\sum_{k\in I_0}e_k^0\varphi(x_k^0,0) &= 0,
\end{aligned}\tag{1.13}$$

hold for all test functions $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\,\infty))$, where $\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}$ is the tangential derivative on the graph Γ, $\int_{\gamma_i}\cdot\,dl$ is a line integral over the arc γ_i.

For instance, the graph Γ containing only one arc $\{(x,t) : x=ct\}$, $\phi(0)=0$ corresponds to solution (1.4).

Now we introduce definitions of a *δ-shock wave type solution* for systems (1.8). Suppose that arcs of the graph $\Gamma=\{\gamma_i : i\in I\}$ have the form $\gamma_i=\{(x,t) : x=\phi_i(t)\}$, $i\in I$.

Definition 1.3. A pair of distributions $(u(x,t),v(x,t))$ and graph Γ from Definition 1.2 is called a *generalized δ-shock wave type solution* of system (1.8) with the initial data $(u^0(x), v^0(x); \dot{\phi}_i(0), i \in I_0)$ if the integral identities

$$
\begin{aligned}
&\int_0^\infty \int \Big(V\varphi_t + uV\varphi_x\Big)\,dx\,dt \\
&\qquad\qquad + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl \\
&\qquad + \int V^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} e_k^0\varphi(x_k^0,0) = 0, \\
&\int_0^\infty \int \Big(uV\varphi_t + u^2V\varphi_x\Big)\,dx\,dt \\
&\qquad\qquad + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\dot{\phi}_i(t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl \\
&+ \int u^0(x)V^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} e_k^0\dot{\phi}_k(0)\varphi(x_k^0,0) = 0,
\end{aligned}
\tag{1.14}
$$

hold for all $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\,\infty))$.

According to Remark 1.1, in Definition 1.3 we use the initial data $(u^0(x), v^0(x); \dot{\phi}_i(0), i \in I_0)$ instead of the initial data $(u^0(x), v^0(x))$, where $\dot{\phi}_i(0)$ is the *initial velocity*, $i \in I_0$. Thus, in addition to an initial data (1.12) we add the *initial velocity* $\dot{\phi}(0)$ to the initial data for system (1.8) (see (4.1)).

Next, we introduce a notion of a *weak asymptotic solution*, which is one of the most important in the *weak asymptotics method*.

We shall write $f(x,t,\varepsilon) = O_{\mathcal{D}'}(\varepsilon^\alpha)$, if $f(x,t,\varepsilon) \in \mathcal{D}'(\mathbb{R})$ is a distribution such that for any test function $\psi(x) \in \mathcal{D}(\mathbb{R}_x)$ we have

$$\langle f(x,t,\varepsilon), \psi(x)\rangle = O(\varepsilon^\alpha),$$

where $O(\varepsilon^\alpha)$ denotes a function continuous in t that admits the usual estimate $|O(\varepsilon^\alpha)| \le \text{const}\varepsilon^\alpha$ uniform in t.

Definition 1.4. A pair of functions $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ smooth as $\varepsilon > 0$ and graph Γ is called a *weak asymptotic solution* of the Cauchy problem for system (1.1) if the relations

$$
\begin{aligned}
L_1[u(x,t,\varepsilon), v(x,t,\varepsilon)] &= O_{\mathcal{D}'}(\varepsilon), \\
L_2[u(x,t,\varepsilon), v(x,t,\varepsilon)] &= O_{\mathcal{D}'}(\varepsilon), \\
u(x,0,\varepsilon) &= u^0(x) + O_{\mathcal{D}'}(\varepsilon), \\
v(x,0,\varepsilon) &= v^0(x) + O_{\mathcal{D}'}(\varepsilon).
\end{aligned}
\tag{1.15}
$$

hold.

Within the framework of the *weak asymptotics method*, we find the *generalized solution* $(u(x,t), v(x,t))$ of the Cauchy problem as the limit

$$
\begin{array}{rcl}
u(x,t) & = & \lim_{\varepsilon\to+0} u(x,t,\varepsilon), \\
v(x,t) & = & \lim_{\varepsilon\to+0} v(x,t,\varepsilon),
\end{array}
\tag{1.16}
$$

of the *weak asymptotic solution* $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ of this problem, where limits are understood in the weak sense (in the sense of the space of distributions $\mathcal{D}'$).

Multiplying relations (1.15) by a test function $\varphi(x,t)$, integrating by parts and then passing to the limit as $\varepsilon \to +0$, we obtain that the limits (1.16) of *weak asymptotic solutions* satisfy (1.13). Hence, the system of integral identities (1.13) generalizes the usual integral identities (1.2) to the case of δ-shock wave type solutions.

In the framework of the *weak asymptotics method* by (2.7), (3.11), (3.12), (4.10), (4.11) we define the *superposition of the Heaviside function and the delta function.* In the background of these formulas there is the *construction of asymptotic subalgebras of distributions*, but in the description of our technique we omit the algebraic aspects which are given in detail in [**2**],[**3**], [**25**].

4. In the framework of our approach, in order to solve the Cauchy problems (1.6), (1.12), or (1.7), (1.12), or (1.8), (1.12), we will seek a *weak asymptotic solution* in the form of the smooth ansatz

$$
\begin{array}{rcl}
u(x,t,\varepsilon) & = & u_0(x,t) + u_1(x,t)H_u(-x+\phi(t),\varepsilon) \\
 & & \qquad\qquad\qquad\qquad + R_u(x,t,\varepsilon), \\
v(x,t,\varepsilon) & = & v_0(x,t) + v_1(x,t)H_v\big(-x+\phi(t),\varepsilon\big) \\
 & & \quad + e(t)\delta_v\big(-x+\phi(t),\varepsilon\big) + R_v(x,t,\varepsilon),
\end{array}
\tag{1.17}
$$

where $u_k(x,t)$, $v_k(x,t)$, $k=0,1$, $e(t)$, $\phi(t)$, $R_u(x,t,\varepsilon)$, $R_v(x,t,\varepsilon)$ are the desired functions. Here $\delta_v(x,\varepsilon) = \varepsilon^{-1}\omega_\delta\big(x/\varepsilon\big)$ is a regularization of the δ-function,

$$
H_j(x,\varepsilon) = \omega_{0j}\Big(\frac{x}{\varepsilon}\Big) = \int_{-\infty}^{x/\varepsilon} \omega_j(\eta)\,d\eta, \quad j = u, v,
$$

are regularizations of the Heaviside function $H(x)$. The mollifiers $\omega_u(\eta)$, $\omega_v(\eta)$, $\omega_\delta(\eta)$ have the following properties: (a) $\omega(\eta) \in C^\infty(\mathbb{R})$, (b) $\omega(\eta)$ has a compact support or decreases sufficiently rapidly as $|\eta| \to \infty$, (c) $\int \omega(\eta)\,d\eta = 1$, (d) $\omega(\eta) \ge 0$, (e) $\omega(-\eta) = \omega(\eta)$.

The *corrections* $R_j(x,t,\varepsilon)$ are functions such that

$$
R_j(x,t,\varepsilon) = o_{\mathcal{D}'}(1), \quad \frac{\partial R_j(x,t,\varepsilon)}{\partial t} = o_{\mathcal{D}'}(1), \quad \varepsilon \to +0,
\tag{1.18}
$$

$j = u, v.$

Since the *generalized solution* of the Cauchy problem is defined as a weak limit (1.16), taking into account (1.18) we can see that the corrections $R_u(x,t,\varepsilon)$, $R_v(x,t,\varepsilon)$ make no contribution to the *generalized solution*. Otherwise, setting $R_u(x,t,\varepsilon) = R_v(x,t,\varepsilon) = 0$, i.e. without introducing these terms, we cannot solve the Cauchy problem with an arbitrary initial data (see Remarks 2.5, 3.5 below).

5. Let $\lambda_1(u,v)$, $\lambda_2(u,v)$ be the eigenvalues of the characteristic matrix of system (1.1). We assume that the "overcompression" conditions are satisfied, which serve as the stability conditions for the δ-shock waves:

$$\begin{array}{rcl} \lambda_1(u_+) \leq & \dot{\phi}(t) & \leq \lambda_1(u_-), \\ \lambda_2(u_+) \leq & \dot{\phi}(t) & \leq \lambda_2(u_-), \end{array} \tag{1.19}$$

where $\dot{\phi}(t)$ is the speed of propagation of δ-shock waves, and u_- and u_+ are the respective left- and right-hand values of u on the discontinuity curve.

2. Propagation of δ-shocks of system (1.6)

1. Let us consider the propagation of a single δ-shock wave of system (1.6), i.e. consider the Cauchy problem (1.6), (1.12), where $u_1^0(0) > 0$.

The eigenvalues of the characteristic matrix of system (1.6) are $\lambda_1(u) = f'(u)$, $\lambda_2(u) = g(u)$. We shall assume that the following conditions are satisfied (see [**6**]– [**10**], [**15**])

$$f''(u) > 0, \quad g'(u) > 0, \quad f'(u) \leq g(u). \tag{2.1}$$

We choose corrections in the form

$$\begin{array}{rcl} R_u(x,t,\varepsilon) & = & 0, \\ R_v(x,t,\varepsilon) & = & R(t)\frac{1}{\varepsilon}\Omega''\Big(\frac{-x+\phi(t)}{\varepsilon}\Big), \end{array} \tag{2.2}$$

where $R(t)$ is a continuous function, $\varepsilon^{-3}\Omega''\big(x/\varepsilon\big)$ is a regularization of the distribution $\delta''(x)$, $\Omega(\eta)$ has the properties (a)–(c) (see Sec. 1). Since for any test function $\psi(x) \in \mathcal{D}(\mathbb{R}_x)$ we have

$$\begin{array}{rcl} \displaystyle\int \frac{1}{\varepsilon}\Omega''\Big(\frac{x}{\varepsilon}\Big)\psi(x)\,dx & = & \varepsilon^2\psi''(0)\int\Omega(\xi)\,d\xi + O(\varepsilon^3), \\ \displaystyle\int \frac{\partial}{\partial x}\bigg(\frac{1}{\varepsilon}\Omega''\Big(\frac{x}{\varepsilon}\Big)\bigg)\psi(x)\,dx & = & -\varepsilon^2\psi'''(0)\int\Omega(\xi)\,d\xi + O(\varepsilon^3), \end{array}$$

relations (1.18) hold.

In order to construct a *weak asymptotic solution* of the Cauchy problem (1.6), (1.12) we will use lemmas on asymptotic expansions.

LEMMA 2.1. ([**5**, Corollary 1.1.]) *Let $f(u)$ be a smooth function, let $u_0(x,t)$, $u_1(x,t)$ be bounded functions. If $u(x,t,\varepsilon)$ is defined by* (1.17), (2.2) *then*

$$f\big(u(x,t,\varepsilon)\big)=f(u_0)+\big[f(u)\big]H(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon),\quad \varepsilon\to+0,$$

where $\big[f(u(x,t))\big]=f(u_0(x,t)+u_1(x,t))-f(u_0(x,t))$ is a jump in function $f(u(x,t))$ across the discontinuity curve $x=\phi(t)$.

LEMMA 2.2. *Let $g(u)$ be a smooth function, let $u_k(x,t)$, $v_k(x,t)$, $k=0,1$, $e(t)$ be bounded functions. If $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ are defined by* (1.17), (2.2) *then*

$$g\big(u(x,t,\varepsilon)\big)\Big(v(x,t,\varepsilon)\Big)=g(u_0)v_0+\big[g(u)v\big]H(-x+\phi(t))$$

$$+\Big\{e(t)a(t)+R(t)c(t)\Big\}\delta(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon),\quad \varepsilon\to+0,$$

where $\big[h(u(x,t),v(x,t))\big]=h(u_0(x,t)+u_1(x,t),v_0(x,t)+v_1(x,t))-h(u_0(x,t),v_0(x,t))$ is a jump in function $h(u(x,t),v(x,t))$ across the discontinuity curve $x=\phi(t)$,

$$\begin{array}{rcl} a(t) & = & \int g\big(u_0(\phi(t),t)+u_1(\phi(t),t)\omega_{0u}(\eta)\big)\omega_\delta(\eta)\,d\eta,\\ c(t) & = & \int g\big(u_0(\phi(t),t)+u_1(\phi(t),t)\omega_{0u}(\eta)\big)\Omega''(\eta)\,d\eta. \end{array}\tag{2.3}$$

PROOF. Using Lemma 2.1, it is easy to obtain the weak asymptotics

$$g\big(u(x,t,\varepsilon)\big)\Big(v_0(x,t)+v_1(x,t)H_v(-x+\phi(t),\varepsilon)\Big)$$

$$=g(u_0)v_0+\big[g(u)v\big]H(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon),\quad \varepsilon\to+0.$$

Next, after the change of variables $x=-\varepsilon\eta$, we have

$$J(\varepsilon)=\Big\langle g\big(u(x,t,\varepsilon)\big)\Big(e(t)\delta_v(-x+\phi(t),\varepsilon)$$

$$+R(t)\frac{1}{\varepsilon}\Omega''\Big(\frac{-x+\phi(t)}{\varepsilon}\Big)\Big),\psi(x)\Big\rangle$$

$$=\psi(\phi(t))\Big(e(t)a(t)+R(t)c(t)\Big)+O(\varepsilon),\quad \varepsilon\to+0,$$

for all $\psi(x)\in\mathcal{D}(\mathbb{R})$. □

THEOREM 2.3. *Let $u_1^0(0)>0$ and conditions* (2.1) *are satisfied. Then there exists $T>0$ such that, for $t\in[0,\,T)$, the Cauchy problem*

(1.6), (1.12), *has a weak asymptotic solution* (1.17), (2.2) *if and only if*

$$
\begin{array}{rcl}
L_{11}[u_0] & = & 0, \quad x > \phi(t), \\
L_{11}[u_0+u_1] & = & 0, \quad x < \phi(t), \\
L_{12}[u_0,v_0] & = & 0, \quad x > \phi(t), \\
L_{12}[u_0+u_1,v_0+v_1] & = & 0, \quad x < \phi(t), \\
\dot\phi(t) & = & \frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}, \\
\dot e(t) & = & \Big([vg(u)] - [v]\frac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)}, \\
R(t) & = & \frac{e(t)}{c(t)}\Bigg(\frac{[f(u)]}{[u]}\Big|_{x=\phi(t)} - a(t)\Bigg),
\end{array}
\tag{2.4}
$$

where $\dot{} = \frac{d}{dt}$, $a(t)$, $c(t)$ *are defined by* (2.3). *The initial data for system* (2.4) *are defined from* (1.12), *and*

$$
\begin{array}{rcl}
\phi(0) & = & 0, \\
R(0) & = & \frac{e^0}{c(0)}\Bigg(\frac{[f(u^0)]}{[u^0]}\Big|_{x=0} - a(0)\Bigg).
\end{array}
$$

Proof. Substitute into system (1.6) smooth ansatzs (1.17) and asymptotics $f(u(x,t,\varepsilon))$, $g(u(x,t,\varepsilon))v(x,t,\varepsilon)$ which are given by Lemmas 2.1, 2.2, we obtain up to $O_{\mathcal{D}'}(\varepsilon)$ the following relations

$$
\begin{aligned}
L_{11}[u(x,t,\varepsilon)] = {} & L_{11}[u_0] \\
& + \Big\{\frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x}\Big(f(u_0+u_1) - f(u_0)\Big)\Big\}H(-x+\phi(t)) \\
& + \Big\{u_1\dot\phi(t) - \Big(f(u_0+u_1) - f(u_0)\Big)\Big\}\delta(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon),
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
L_{12}[u(x,t,\varepsilon), v(x,t,\varepsilon)] = {} & L_{12}[u_0,v_0] \\
& + \Big\{\frac{\partial v_1}{\partial t} + \frac{\partial}{\partial x}\Big((v_0+v_1)g(u_0+u_1) - v_0 g(u_0)\Big)\Big\}H(-x+\phi(t)) \\
& + \Big\{v_1\dot\phi(t) + \dot e(t) - \Big((v_0+v_1)g(u_0+u_1) - v_0 g(u_0)\Big)\Big\}\delta(-x+\phi(t)) \\
& + \Big\{e(t)\dot\phi(t) - e(t)a(t) - c(t)R(t)\Big\}\delta'(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon),
\end{aligned}
\tag{2.6}
$$

where $a(t)$, $c(t) \neq 0$ are defined by (2.3). Here we take into account the estimates (1.18).

Setting of the right-hand side of (2.5), (2.6) equal to zero, we obtain the necessary and sufficient conditions for the relations

$$
L_{11}[u(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon), \quad L_{12}[u(x,t,\varepsilon), v(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon),
$$

i.e. system (2.4).

Now we consider the Cauchy problem $L_{11}[u] = u_t + \big(f(u)\big)_x = 0$, $u(x,0) = u^0(x)$. Since $f(u)$ is convex and $u_1^0(0) > 0$, according to the results [**17**, Ch.4.2.], we extend $u_+^0(x) = u_0^0(x)$ ($u_-^0(x) = u_0^0(x) + u_1^0(x)$) to $x \le 0$ ($x \ge 0$) in a bounded C^1 fashion and continue to denote the extended functions by $u_\pm^0(x)$. By $u_\pm(x,t)$ we denote the C^1 solutions of the problems

$$L_{11}[u] = u_t + \big(f(u)\big)_x = 0, \quad u_\pm(x,0) = u_\pm^0(x)$$

which exist for small enough time interval $[0,\ T_1]$ and are determined by integration along characteristics. The functions $u_\pm(x,t)$ determine a two-sheeted covering of the plane (x,t). Next, we define the function $x = \phi(t)$ as a solution of the problem

$$\dot\phi(t) = \frac{f\big(u_+(x,t)\big) - f\big(u_-(x,t)\big)}{u_+(x,t) - u_-(x,t)}\Big|_{x=\phi(t)}, \quad \phi(0) = 0.$$

It is clear that there exists unique function $\phi(t)$ for sufficiently short times $[0,\ T_2]$. To this end, for $T = \min(T_1, T_2)$ we define the shock solution by

$$u(x,t) = \begin{cases} u_+(x,t), & x \ > \ \phi(t), \\ u_-(x,t), & x \ < \ \phi(t). \end{cases}$$

Thus the first, second and fifth equations of system (2.4) define a unique solution of the Cauchy problem $L_{11}[u] = u_t + \big(f(u)\big)_x = 0$, $u(x,0) = u^0(x)$ for $t \in [0,\ T)$.

Solving this problem, we obtain $u(x,t)$, $\phi(t)$. Then substituting these functions into (2.4), we obtain $V(x,t) = v_0(x,t) + v_1(x,t)H(-x+\phi(t))$, $e(t)$, and $v(x,t) = V(x,t) + e(t)\delta(-x+\phi(t))$.

It is clear that functions $\omega_{0u}(\xi)$, $\Omega''(\xi)$ can be chosen such that $\int \omega_u(\eta)\Omega'(\eta)\,d\eta > 0$. Taking into account that $g'(u) > 0$, $u_1^0(x) > 0$ and integrating by parts, we have

$$c(t) = -\int g'\big(u_0 + u_1\omega_{0u}(\eta)\big)u_1\Big|_{x=\phi(t)}\omega_u(\eta)\Omega'(\eta)\,d\eta \ne 0.$$

So for any functions $u_0(x,t)$, $u_1(x,t)$, $e(t)$, $\phi(t)$, $t \in [0,\ T)$, there exists the function $R(t)$, which is defined by the last relation of (2.4). $\Box$

Remark 2.4. Using the last relation (2.4) can one obtain from Lemma 2.2 the following relation:

$$v(x,t,\varepsilon)g\big(u(x,t,\varepsilon)\big) = v_0 g(u_0) + \big[vg(u)\big]H(-x+\phi(t))$$

$$+e(t)\frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}\delta(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0, \tag{2.7}$$

REMARK 2.5. Without introducing correction (2.2), i.e. setting $R(t) = 0$, we derive from (2.4) the relation

$$\frac{\big[f(u(x,t))\big]}{\big[u(x,t)\big]}\Big|_{x=\phi(t)}$$

$$= \int g\big(u_0(\phi(t),t) + u_1(\phi(t),t)\omega_{0u}(\eta)\big)\omega_\delta(\eta)\,d\eta, \tag{2.8}$$

which shows that we cannot solve the Cauchy problem with an arbitrary jump.

2. Using a *weak asymptotic solution* constructed by Theorem 2.3 we obtain a *generalized solution* of the Cauchy problem (1.6), (1.12) as a weak limit (1.16).

THEOREM 2.6. *Let* $u_1^0(0) > 0$ *and conditions* (2.1) *are satisfied. Then, for* $t \in [0, T)$*, where* $T > 0$ *is given by Theorem* 2.3*, the Cauchy problem* (1.6), (1.12), (2.1)*, has a unique generalized solution*

$$\begin{array}{rcl} u(x,t) & = & u_0(x,t) + u_1(x,t)H(-x+\phi(t)), \\ v(x,t) & = & v_0(x,t) + v_1(x,t)H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{array}$$

satisfies the integral identities cf. (1.13)*:*

$$\begin{array}{rcl} \displaystyle\int_0^T\!\!\int \Big(u\varphi_t + f(u)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx & = & 0, \\ \displaystyle\int_0^T\!\!\int \Big(\varphi_t + g(u)\varphi_x\Big)V\,dx\,dt + \int V^0(x)\varphi(x,0)\,dx & & \\ \displaystyle + \int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl + e^0\varphi(0,0) & = & 0, \end{array} \tag{2.9}$$

where $\Gamma = \{(x,t) : x = \phi(t), \quad t \in [0,\ T)\}$*, and*

$$\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl = \int_0^T e(t)\Big(\varphi_t(\phi(t),t) + \dot\phi(t)\varphi_x(\phi(t),t)\Big)\,dt,$$

$V(x,t) = v_0 + v_1H(-x+\phi(t))$ *and functions* $u_k(x,t)$, $v_k(x,t)$, $\phi(t)$, $e(t)$ *are defined by Theorem* 2.3*.*

PROOF. By Theorem 2.3 we have the following estimates:

$$L_{11}[u(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon), \quad L_{12}[u(x,t,\varepsilon), v(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon).$$

Let us apply the left-hand and right-hand sides of these relations to an arbitrary test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\,T))$. Then integrating by parts, we obtain

$$\int_0^T\!\!\int \Big(u(x,t,\varepsilon)\varphi_t(x,t) + f(u(x,t,\varepsilon)\varphi_x(x,t)\Big)\,dxdt$$

$$+\int u(x,0,\varepsilon)\varphi(x,0)\,dx = O(\varepsilon),$$

$$\int_0^T\int\Big(\varphi_t(x,t)+g(u(x,t,\varepsilon))\varphi_x(x,t)\Big)v(x,t,\varepsilon)\,dxdt$$
$$+\int v(x,0,\varepsilon)\varphi(x,0)\,dx = O(\varepsilon), \quad \varepsilon\to+0.$$

Passing to the limit as $\varepsilon \to +0$ and taking into account Lemmas 2.1, 2.2, (2.7), and the fact that

$$\lim_{\varepsilon\to+0}\int_0^T\int e(t)\delta_v\big(-x+\phi(t),\varepsilon\big)\varphi(x,t)\,dxdt$$

$$= \int_0^T e(t)\varphi(\phi(t),t)\,dt, \tag{2.10}$$

$$\lim_{\varepsilon\to+0}\int e(0)\delta_v\big(-x,\varepsilon\big)\varphi(x,0)\,dx = e(0)\varphi(0,0), \tag{2.11}$$

we obtain the integral identities (2.9).

In view of the above remark in Theorem 2.6, the Cauchy problem has a unique generalized solution. □

Here the right-hand sides of the sixth equation of system (2.4)

$$\dot e(t) = \Big([vg(u)] - [v]\frac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)}$$

is the so-called *Rankine–Hugoniot deficit.*

If initial data (1.12) are piecewise-constants, i.e $u_0^0=u_0,\ u_1^0=u_1>0,\ v_0^0=v_0,\ v_1^0=v_1$, then we have from Theorem 2.6.

Corollary 2.7. *For* $t\in[0,\infty)$*, the Cauchy problem* (1.6), (1.12), (2.1)*, with piecewise-constants initial data has a unique generalized solution*

$$\begin{array}{rcl} u(x,t) &=& u_0+u_1H(-x+\phi(t)),\\ v(x,t) &=& v_0+v_1H(-x+\phi(t))+e(t)\delta(-x+\phi(t)), \end{array}$$

where

$$\begin{array}{rcl} \phi(t) &=& \frac{[f(u)]}{[u]}t,\\ e(t) &=& e^0+\Big([g(u)v]-\frac{[f(u)]}{[u]}[v]\Big)t. \end{array}$$

3. Propagation of δ-shocks of system (1.7)

1. Let us consider the Cauchy problem (1.7), (1.12). The "over-compression" condition is

$$\lambda_1(u_+) \le \lambda_2(u_+) \le \dot{\phi}(t) \le \lambda_1(u_-) \le \lambda_2(u_-), \tag{3.1}$$

where $\lambda_1(u) = u - 1$, $\lambda_2(u) = u + 1$ are the eigenvalues of the characteristic matrix of system (1.7).

We choose corrections $R(x, t, \varepsilon)$ in the form

$$\begin{aligned} R_u(x,t,\varepsilon) &= P(t)\tfrac{1}{\sqrt{\varepsilon}}\Omega_P\Big(\tfrac{-x+\phi(t)}{\varepsilon}\Big), \\ R_v(x,t,\varepsilon) &= 0, \end{aligned} \tag{3.2}$$

where $P(t)$ is a smooth function, $\frac{1}{\varepsilon}\Omega_P^2\big(x/\varepsilon\big)$ is a regularization of the delta function. Moreover, we assume that

$$\int \Omega_P^2(\eta)\,d\eta \ne 0, \qquad \int \Omega_P^3(\eta)\,d\eta = 0. \tag{3.3}$$

For example, we can choose $\Omega_P(-\eta) = -\Omega_P(\eta)$. It is clear that relations (1.18) hold.

Theorem 3.1. *Let*

$$u_0^0(0) + 1 \le \frac{[(u^0)^2] - [v^0]}{[u^0]}\bigg|_{x=0} \le u_0^0(0) + u_1^0(0) - 1, \tag{3.4}$$

then there exists $T > 0$ such that, for $t \in [0, T)$, the Cauchy problem (1.7), (1.12), *has a weak asymptotic solution* (1.17), (3.2), (3.3) *if and only if*

$$\begin{aligned} L_{21}[u_0, v_0] &= 0, \quad x > \phi(t), \\ L_{21}[u_0 + u_1, v_0 + v_1] &= 0, \quad x < \phi(t), \\ L_{22}[u_0, v_0] &= 0, \quad x > \phi(t), \\ L_{22}[u_0 + u_1, v_0 + v_1] &= 0, \quad x < \phi(t), \\ \dot{\phi}(t) &= \tfrac{[u^2]-[v]}{[u]}\Big|_{x=\phi(t)}, \\ \dot{e}(t) &= \Big(\tfrac{[u^3]}{3} - [u] - [v]\tfrac{[u^2]-[v]}{[u]}\Big)\Big|_{x=\phi(t)}, \\ P(t) &= \sqrt{\tfrac{e(t)}{a}}, \\ \frac{u_0 + u_1 - \frac{v_1}{u_1}}{u_1}\bigg|_{x=\phi(t)} &= \tfrac{b}{a}, \end{aligned} \tag{3.5}$$

where

$$a = \int \Omega_P^2(\eta)\,d\eta > 0, \quad b = \int \omega_{0u}(\eta)\Omega_P^2(\eta)\,d\eta. \tag{3.6}$$

The initial data for system (3.5) *are defined from* (1.12), *and*

$$\begin{array}{rcl}\phi(0) &=& 0,\\ P(0) &=& \sqrt{\frac{e^0}{a}}.\end{array}$$

Proof. Let $H(x,\varepsilon)=\omega_0\big(x/\varepsilon\big)$ be a regularization of the Heaviside function, and let $\delta_k(x,\varepsilon)=\varepsilon^{-1}\omega_k\big(x/\varepsilon\big)$, $k=1,2$ be regularizations of the delta function. Using Lemmas 2.1, 2.2, we obtain the following relations

$$\begin{array}{rcl}\Big(H(x,\varepsilon)\Big)^r &=& H(x)+O_{\mathcal{D}'}(\varepsilon),\\ \delta_1(x,\varepsilon)\Big(\omega_2\Big(\dfrac{x}{\varepsilon}\Big)\Big)^r &=& A_r\delta(x)+O_{\mathcal{D}'}(\varepsilon),\\ \delta_1(x,\varepsilon)\Big(H(x,\varepsilon)\Big)^r &=& B_r\delta(x)+O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon\to+0,\end{array} \tag{3.7}$$

where $A_r=\int\omega_1(\eta)\omega_2^r(\eta)\,d\eta$, $B_r=\int\omega_0^r(\eta)\omega(\eta)\,d\eta$, $r=1,2,\dots$.

Using (3.7), (3.3), one can calculate

$$\begin{array}{rcl}\Big(u(x,t,\varepsilon)\Big)^2 &=& u_0^2+\Big((u_0+u_1)^2-u_0^2\Big)H(-x+\phi(t))\\ && \qquad +aP^2(t)\delta(-x+\phi(t))+o_{\mathcal{D}'}(1),\\ \Big(u(x,t,\varepsilon)\Big)^3 &=& u_0^3+\Big((u_0+u_1)^3-u_0^3\Big)H(-x+\phi(t))\\ && +\Big((3au_0+3bu_1)P^2(t)\Big)\delta(-x+\phi(t))\\ && \qquad +o_{\mathcal{D}'}(1), \quad \varepsilon\to+0,\end{array} \tag{3.8}$$

where a, b are defined by (3.6).

Substituting (1.17), (3.2), (3.8) into the left-hand side of system (1.7), we obtain, up to $o_{\mathcal{D}'}(1)$,

$$\begin{aligned} L_{21}[u(x,t,\varepsilon),v(x,t,\varepsilon)] &= L_{21}[u_0,v_0]\\ &+\Big\{\frac{\partial u_1}{\partial t}+\frac{\partial}{\partial x}\big[u^2-v\big]\Big\}H(-x+\phi(t))\\ &+\Big\{u_1\dot\phi(t)-\big[u^2-v\big]\Big\}\delta(-x+\phi(t))\\ &+\Big\{e(t)-aP^2(t)\Big\}\delta'(-x+\phi(t))+o_{\mathcal{D}'}(1),\end{aligned} \tag{3.9}$$

$$\begin{aligned} L_{22}[u(x,t,\varepsilon),v(x,t,\varepsilon)] &= L_{22}[u_0,v_0]\\ &+\Big\{\frac{\partial v_1}{\partial t}+\frac{\partial}{\partial x}\Big[\frac{u^3}{3}-u\Big]\Big\}H(-x+\phi(t))\\ &+\Big\{v_1\dot\phi(t)+\dot e(t)-\Big[\frac{u^3}{3}-u\Big]\Big\}\delta(-x+\phi(t))\\ &+\Big\{e(t)\dot\phi(t)-\big(au_0+bu_1\big)P^2(t)\Big\}\delta'(-x+\phi(t))+o_{\mathcal{D}'}(1).\end{aligned} \tag{3.10}$$

Setting the left-hand side of (3.9), (3.10) equal to zero, we obtain systems (3.5).

According to the above arguments (see the proof of Theorem 2.3) and in view of the overcompression condition (3.1), (3.4), the first five equations of system (3.5) define a unique solution of the Cauchy problem

$$\begin{array}{rcl} L_{21}[u,V] & = & 0, \\ L_{22}[u,V] & = & 0, \end{array}$$

with initial data $u(x,0)=u^0(x)$, $V(x,0)=v_0^0(x)+v_1^0(x)H(-x)$, (3.4) for small enough time interval $[0,\ T]$. Solving this problem, we obtain $u(x,t)$, $V(x,t)$, $\phi(t)$ (see also [**24**, Ch.I,§8.]). Then substituting these functions into (3.5), we obtain $e(t)$, $P(t)$, $v(x,t)=V(x,t)+e(t)\delta(-x+\phi(t))$. □

Remark 3.2. Using (1.6), (3.5), (3.8), we obtain the following relations:

$$\big(u(x,t,\varepsilon)\big)^2-v(x,t,\varepsilon)=u_0^2-v_0+\big[u^2-v\big]H(-x+\phi(t))+o_{\mathcal{D}'}(1), \tag{3.11}$$

$$\begin{aligned} \frac{1}{3}\big(u(x,t,\varepsilon)\big)^3-u(x,t,\varepsilon) &= \frac{1}{3}u_0^3-u_0+\Big[\frac{1}{3}u^3-u\Big]H(-x+\phi(t)) \\ &+e(t)\frac{[u^2]}{[u]}\delta(-x+\phi(t))+o_{\mathcal{D}'}(1), \quad \varepsilon\to+0. \end{aligned} \tag{3.12}$$

2. Using a *weak asymptotic solution* and (3.11), (3.12), just as above in Subsec. 2.2., we can prove the following theorem.

Theorem 3.3. *There exists $T>0$ such that the Cauchy problem* (1.7), (1.12), (3.4) *for $t\in[0,\ T)$ has a unique generalized solution*

$$\begin{array}{rcl} u(x,t) & = & u_0(x,t)+u_1(x,t)H(-x+\phi(t)), \\ v(x,t) & = & v_0(x,t)+v_1(x,t)H(-x+\phi(t))+e(t)\delta(-x+\phi(t)), \end{array}$$

satisfies the integral identities (1.13)*:*

$$\begin{array}{rcl} \displaystyle\int_0^T\!\!\int\Big(u\varphi_t+\big(u^2-V\big)\varphi_x\Big)\,dx\,dt+\int u^0(x)\varphi(x,0)\,dx & = & 0, \\ \displaystyle\int_0^T\!\!\int\Big(V\varphi_t+\big(\frac{1}{3}u^3-u\big)\varphi_x\Big)\,dx\,dt+\int V^0(x)\varphi(x,0)\,dx & & \\ \displaystyle +\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl+e^0\varphi(0,0) & = & 0, \end{array}$$

where $\Gamma=\{(x,t): x=\phi(t),\quad t\in[0,\ T)\}$,

$$\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl=\int_0^T e(t)\Big(\varphi_t(\phi(t),t)+\dot\phi(t)\varphi_x(\phi(t),t)\Big)\,dt,$$

$$V(x,t) = v_0(x,t) + v_1(x,t)H(-x+\phi(t))$$

and functions $u_k(x,t)$, $v_k(x,t)$, $\phi(t)$, $e(t)$ *are defined by Theorem* 3.1.

The right-hand sides of the sixth equation of system (3.5)

$$\dot{e}(t) = \left(\frac{[u^3]}{3} - [u] - [v]\frac{[u^2]-[v]}{[u]}\right)\Big|_{x=\phi(t)}$$

is the so-called *Rankine–Hugoniot deficit*.

Remark 3.4. We can solve the Cauchy for initial data determined by the fourth relation (3.5):

$$\frac{u_0 + u_1 - \frac{v_1}{u_1}}{u_1}\Big|_{x=\phi(t)} = \frac{b}{a},$$

where $a = \int \Omega_P^2(\eta)\,d\eta$, $b = \int \omega_{0u}(\eta)\Omega_P^2(\eta)\,d\eta$. This relation can be rewritten as

$$\frac{u_0 - \frac{v_1}{u_1}}{u_1} = \frac{\dot{\phi}(t) - u_-}{u_1} = \frac{b-a}{a}, \tag{3.13}$$

where $u_- = u_0 + u_1$. In [**14**] the parameter $a = \int \Omega_P^2(\eta)\,d\eta$ was set to be 1. Hence

$$\frac{b-a}{a} = \int \Big(\omega_{0u}(\eta) - 1\Big)\Omega_P^2(\eta)\,d\eta < 1.$$

Here relation (3.13) coincides with the second relation [**14**, Proposition 2] and the last inequality coincides with the statement of [**14**, Lemma 1]. However in this case relation (3.13) still leaves one degree of freedom, to connect $u_- = u_0 + u_1$ and $u_+ = u_0$ (see [**14**, Proposition 2]).

Remark 3.5. Without introducing correction (3.2), i.e. setting $P(t) = 0$, according to (3.5), we can solve the Cauchy problem only if the following relation holds:

$$\left(\frac{[u^3]}{3} - [u] - [v]\frac{[u^2]-[v]}{[u]}\right)\Big|_{x=\phi(t)} = 0. \tag{3.14}$$

Corollary 3.6. *The Cauchy problem* (1.7), (1.12), (3.4), *where initial data is piecewise-constants, for* $t \in [0,\infty)$ *has a unique generalized solution*

$$\begin{aligned} u(x,t) &= u_0 + u_1 H(-x+\phi(t)), \\ v(x,t) &= v_0 + v_1 H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{aligned}$$

where

$$\begin{aligned} \phi(t) &= \tfrac{[u^2]-[v]}{[u]}t, \\ e(t) &= e^0 + \Big(\tfrac{[u^3]}{3} - [u] - [v]\tfrac{[u^2]-[v]}{[u]}\Big)t. \end{aligned}$$

If $e^0 = 0$, and the initial data is piecewise-constants, according to the seventh equation (3.5) the *Rankine–Hugoniot deficit* is positive:

$$\dot{e}(t) = \frac{[u^3]}{3} - [u] - [v]\frac{[u^2] - [v]}{[u]} > 0$$

as in [**14**].

4. Propagation of δ-shocks of zero-pressure gas dynamics system

1. According to Remark 1.1, we consider the Cauchy problem (1.8) which initial data of the form

$$\begin{array}{rcl} u^0(x) & = & u^0_0(x) + u^0_1(x)H(-x), \\ v^0(x) & = & v^0_0(x) + v^0_1(x)H(-x) + e^0\delta(-x). \\ \dot{\phi}(t)\Big|_{t=0} & = & \phi^1, \end{array} \tag{4.1}$$

where $u^0_k(\mathrm{x})$, $v^0_k(x)$, $k = 0, 1$ are given smooth functions, e^0, ϕ^1 are given constant, and $u^0_1(0) > 0$.

System (1.8) has a double eigenvalue $\lambda_1(u) = \lambda_2(u) = u$. In this case the entropy condition is

$$u_+ \le \dot{\phi}(t) \le u_-. \tag{4.2}$$

We choose corrections in the form

$$\begin{array}{rcl} R_u(x,t,\varepsilon) & = & Q(t)\Omega'\Big(\frac{-x+\phi(t)}{\varepsilon}\Big), \\ R_v(x,t,\varepsilon) & = & R(t)\frac{1}{\varepsilon}\widetilde{\Omega}''\Big(\frac{-x+\phi(t)}{\varepsilon}\Big), \end{array} \tag{4.3}$$

where $Q(t)$, $R(t)$ are continuous functions, $\varepsilon^{-1}\Omega\big(x/\varepsilon\big)$ and $\varepsilon^{-1}\widetilde{\Omega}\big(x/\varepsilon\big)$ are a regularization of the delta function, $\Omega(\eta)$ and $\widetilde{\Omega}(\eta)$ have the properties (a)–(c) (see Sec. 1). Thus, relations (1.18) hold.

Moreover, we assume that $\Omega(\eta)$, $\widetilde{\Omega}(\eta)$ are functions such that

$$\begin{array}{rcl} b_1 = \displaystyle\int \omega_\delta(\eta)\Omega'(\eta)\,d\eta & \ne & 0, \\ \displaystyle\int \omega_{0u}(\eta)\widetilde{\Omega}''(\eta)\,d\eta & = & 0, \\ c_2 = \displaystyle\int \omega_{0u}^2(\eta)\widetilde{\Omega}''(\eta)\,d\eta & \ne & 0, \\ \displaystyle\int \omega_{0u}(\eta)\Omega'(\eta)\widetilde{\Omega}''(\eta)\,d\eta & = & 0, \\ \displaystyle\int \Omega'(\eta)\widetilde{\Omega}''(\eta)\,d\eta & = & 0. \end{array} \tag{4.4}$$

It is clear that for any mollifiers $\omega_\delta(\eta)$, $\omega_{0u}(\eta)$ there are mollifiers $\Omega(\eta)$, $\widetilde{\Omega}(\eta)$ such that system (4.4) is solvable. For example, we can choose an even mollifier $\widetilde{\Omega}(\eta)$ and a mollifier $\Omega(\eta)$ such that the second and third relations (4.4) hold. Moreover, we assume that $\Omega(\eta)$, $\widetilde{\Omega}(\eta)$ such that $\operatorname{supp}\big(\Omega'\big) \cap \operatorname{supp}\big(\widetilde{\Omega}''\big) = \emptyset$. Hence the fifth and fourth relations (4.4) hold. In addition, we assume that $\Omega(\eta)$ such that $\Omega'(\eta)$ is "closed" to an odd function. Then the first relation (4.4) holds.

THEOREM 4.1. *There exists $T > 0$ such that the Cauchy problem* (1.8), (4.1) *for $t \in [0, T)$ has a weak asymptotic solution* (1.17), (4.3) *if and only if*

(4.5)
$$
\begin{array}{rcl}
L_{31}[u_0, v_0] &=& 0, \quad x > \phi(t), \\
L_{31}[u_0+u_1, v_0+v_1] &=& 0, \quad x < \phi(t), \\
L_{32}[u_0, v_0] &=& 0, \quad x > \phi(t), \\
L_{32}[u_0+u_1, v_0+v_1] &=& 0, \quad x < \phi(t), \\
\dot e(t) &=& \Big([uv] - [v]\dot\phi(t)\Big)\Big|_{x=\phi(t)}, \\
\dfrac{d\big(e(t)\dot\phi(t)\big)}{dt} &=& \Big([u^2v] - [uv]\dot\phi(t)\Big)\Big|_{x=\phi(t)}, \\
Q(t) &=& \frac{\dot\phi(t)-u_0-a_1u_1}{b_1}\Big|_{x=\phi(t)}, \\
R(t) &=& \frac{e(t)}{u_1^2c_2}\Big\{\big(\dot\phi(t)\big)^2 - \Big(u_0^2 + 2u_0u_1a_1 + u_1^2a_2 \\
&& +Q^2(t)b_2 + 2Q(t)\big(u_0b_1 + u_1c_1\big)\Big)\Big\}\Big|_{x=\phi(t)}.
\end{array}
$$

where the initial data for system (4.5) *are defined from* (4.1), *and* $\phi(0) = 0$,

$$
\begin{array}{rcl}
Q(0) &=& \frac{\dot\phi(0)-u_0^0-a_1u_1^0}{b_1}\Big|_{x=0}, \\
R(0) &=& \frac{e^0}{(u_1^0(0))^2c_2}\Big\{\big(\dot\phi(0)\big)^2 - \Big((u_0^0)^2 + 2u_0^0u_1^0a_1 + (u_1^0)^2a_2 \\
&& +Q^2(0)b_2 + 2Q(0)\big(u_0^0b_1 + u_1^0c_1\big)\Big)\Big\}\Big|_{x=0}.
\end{array}
$$

Here

$$
\begin{array}{rcl}
a_k &=& \int \big(\omega_{0u}(\eta)\big)^k\omega_\delta(\eta)\,d\eta, \quad k = 1, 2, \\
b_2 &=& \int \big(\Omega'(\eta)\big)^2\omega_\delta(\eta)\,d\eta, \\
c_1 &=& \int \omega_{0u}(\eta)\Omega'(\eta)\omega_\delta(\eta)\,d\eta, \\
c_2 &=& \int \omega_{0u}^2(\eta)\widetilde{\Omega}''(\eta)\,d\eta.
\end{array}
$$

PROOF. Just as above, using Lemmas 2.1, 2.2, and taking into account the relations $\int \omega_{0u}(\eta)\Omega'(\eta)\widetilde{\Omega}''(\eta)\,d\eta = 0$, $\int \omega_{0u}(\eta)\widetilde{\Omega}''(\eta)\,d\eta = 0$, $\int \Omega'(\eta)\widetilde{\Omega}''(\eta)\,d\eta = 0$, one can calculate

$$u(x,t,\varepsilon)v(x,t,\varepsilon) = u_0v_0 + \Big[uv\Big]H(-x+\phi(t))$$

$$+e(t)\Big(u_0 + u_1\int \omega_{0u}(\eta)\omega_\delta(\eta)\,d\eta$$

$$+Q(t)\int \omega_\delta(\eta)\Omega'(\eta)\,d\eta\Big)\delta(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon), \tag{4.6}$$

$$u^2(x,t,\varepsilon)v(x,t,\varepsilon) = u_0^2v_0 + \Big[u^2v\Big]H(-x+\phi(t))$$

$$+\Big\{e(t)\Big(u_0^2 + 2u_0u_1\int \omega_{0u}(\eta)\omega_\delta(\eta)\,d\eta + u_1^2\int \omega_{0u}^2(\eta)\omega_\delta(\eta)\,d\eta$$

$$+Q^2(t)\int (\Omega')^2(\eta)\omega_\delta(\eta)\,d\eta + 2Q(t)\big(u_0\int \Omega'(\eta)\omega_\delta(\eta)\,d\eta$$

$$+u_1\int \omega_{0u}(\eta)\Omega'(\eta)\omega_\delta(\eta)\,d\eta\big)\Big)$$

$$+R(t)u_1^2\int \omega_{0u}^2(\eta)\widetilde{\Omega}''(\eta)\,d\eta\Big\}\delta(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon). \tag{4.7}$$

Substitute into the first equation of system (1.8) smooth ansatzs (1.17), asymptotics (4.6), (4.7), and setting of the left-hand side equal to zero, we obtain the necessary and sufficient conditions for the equality $L_{31}[u(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon)$:

$$\begin{aligned} L_{31}[u_0,v_0] &= 0, \quad x > \phi(t), \\ L_{31}[u_0+u_1,v_0+v_1] &= 0, \quad x < \phi(t), \\ \dot{e}(t) &= \Big([uv] - [v]\dot{\phi}(t)\Big)\Big|_{x=\phi(t)}, \\ \dot{\phi}(t) &= \Big(u_0 + u_1a_1 + Q(t)b_1\Big)\Big|_{x=\phi(t)}. \end{aligned} \tag{4.8}$$

Substitute into the second equation of system (1.8) smooth ansatzs (1.17), asymptotics (4.6), (4.7), and taking into account the third relation of system (4.8), we obtain the necessary and sufficient conditions for the equality $L_{32}[u(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon)$:

$$
\begin{array}{rcl}
L_{32}[u_0, v_0] & = & 0, \quad x > \phi(t), \\
L_{32}[u_0 + u_1, v_0 + v_1] & = & 0, \quad x < \phi(t), \\
\dfrac{d\big(e(t)\dot\phi(t)\big)}{dt} & = & \Big([u^2 v] - [uv]\dot\phi(t)\Big)\Big|_{x=\phi(t)}, \\
e(t)\big(\dot\phi(t)\big)^2 & = & \Big\{e(t)\Big(u_0^2 + 2u_0u_1a_1 + u_1^2a_2 \\
 & & +Q^2(t)b_2 + 2Q(t)\big(u_0b_1 + u_1c_1\big)\Big) \\
 & & +R(t)u_1^2c_2\Big\}\Big|_{x=\phi(t)}.
\end{array}
\tag{4.9}
$$

From (4.8), (4.9) we have system (4.5). □

Remark 4.2. Using the two last equations of system (4.5), from (4.6), (4.7) we have the following relations:

$$
u(x,t,\varepsilon)v(x,t,\varepsilon) = u_0(x,t)v_0(x,t) + \Big[u(x,t)v(x,t)\Big]H(-x+\phi(t))
$$

$$
+e(t)\dot\phi(t)\delta(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon), \tag{4.10}
$$

$$
u^2(x,t,\varepsilon)v(x,t,\varepsilon) = u_0^2(x,t)v_0(x,t) + \Big[u^2(x,t)v(x,t)\Big]H(-x+\phi(t))
$$

$$
+e(t)\big(\dot\phi(t)\big)^2\delta(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0. \tag{4.11}
$$

2. We obtain a *generalized solution* of the Cauchy problem as a weak limit of a *weak asymptotic solution* constructed in Theorem 4.1.

Theorem 4.3. *There exists $T > 0$ such that the Cauchy problem (1.8), (4.1) for $t \in [0, T)$ has a unique generalized solution*

$$
\begin{array}{rcl}
u(x,t) & = & u_0(x,t) + u_1(x,t)H(-x+\phi(t)), \\
v(x,t) & = & v_0(x,t) + v_1(x,t)H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)),
\end{array}
$$

satisfies the integral identities (1.14), where

$$
V(x,t) = v_0(x,t) + v_1(x,t)H(-x+\phi(t)),
$$

and functions $u_k(x,t)$, $v_k(x,t)$, $\phi(t)$, $e(t)$ are defined by the system

$$
\begin{array}{rcl}
L_{31}[u_0, v_0] & = & 0, \quad x > \phi(t), \\
L_{31}[u_0 + u_1, v_0 + v_1] & = & 0, \quad x < \phi(t), \\
L_{32}[u_0, v_0] & = & 0, \quad x > \phi(t), \\
L_{32}[u_0 + u_1, v_0 + v_1] & = & 0, \quad x < \phi(t), \\
\dot e(t) & = & \Big([uv] - [v]\dot\phi(t)\Big)\Big|_{x=\phi(t)}, \\
\dfrac{d\big(e(t)\dot\phi(t)\big)}{dt} & = & \Big([u^2 v] - [uv]\dot\phi(t)\Big)\Big|_{x=\phi(t)},
\end{array}
\tag{4.12}
$$

where initial data are defined from (4.1).

PROOF. According to Theorem 4.1 we have the following relations $L_{31}[u(x,t,\varepsilon), v(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon)$, $L_{32}[u(x,t,\varepsilon), v(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon)$.

The proof of the first integral identity (1.14) is based on (4.10) and the same calculations as those carried out above in Subsec. 2.2., and we omit them here.

Let us apply the left-hand and right-hand sides of the relation $L_{32}[u(x,t,\varepsilon), v(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon)$ to an arbitrary test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0,\, T))$. Then integrating by parts, we obtain

$$\int_0^T \int \Big(u(x,t,\varepsilon)v(x,t,\varepsilon)\varphi_t(x,t) + u^2(x,t,\varepsilon)v(x,t,\varepsilon)\varphi_x(x,t) \Big)\, dxdt$$

$$+ \int u(x,0,\varepsilon)v(x,0,\varepsilon)\varphi(x,0)\, dx = O(\varepsilon).$$

Passing to the limit as $\varepsilon \to +0$ and taking into account (4.10), (4.11), and (2.10), (2.11), we obtain integral identities (2.9):

$$\int_0^T \int \Big(uV\varphi_t + u^2V\varphi_x \Big)\, dx\, dt + \int u^0(x)V^0(x)\varphi(x,0)\, dx$$

$$+ \int_0^T e(t)\dot{\phi}(t)\Big(\varphi_t(\phi(t),t) + \dot{\phi}(t)\varphi_x(\phi(t),t) \Big)\, dt + e^0\dot{\phi}(0)\varphi(0,0) = 0.$$

Here $\Gamma = \{(x,t) : x = \phi(t), \quad t \in [0,\ T)\}$, and

$$\int_\Gamma e\dot{\phi}(t)\frac{\partial \varphi}{\partial \mathbf{l}}\, dl = \int_0^T e(t)\dot{\phi}(t)\Big(\varphi_t(\phi(t),t) + \dot{\phi}(t)\varphi_x(\phi(t),t) \Big)\, dt.$$

□

If initial data (4.1) are piecewise-constants, i.e $u_0^0 = u_0$, $u_1^0 = u_1 > 0$, $v_0^0 = v_0$, $v_1^0 = v_1$, then we have the following statement from Theorem 4.3.

THEOREM 4.4. *The Cauchy problem* (1.8) *with piecewise-constants initial data* (4.1) *for* $t \in [0,\, \infty)$ *has a unique generalized solution*

$$\begin{aligned} u(x,t) &= u_0 + u_1 H(-x + \phi(t)), \\ v(x,t) &= v_0 + v_1 H(-x + \phi(t)) + e(t)\delta(-x + \phi(t)), \end{aligned}$$

where

(i) *if* $[v] \neq 0$, *then*

$$\begin{aligned} e(t) &= \sqrt{(e^0)^2 + 2e^0\dot{e}(0)t + \big([uv]^2 - [v][u^2v]\big)t^2}, \\ \phi(t) &= \frac{e^0 + [uv]t - \sqrt{(e^0)^2 + 2e^0\dot{e}(0)t + \big([uv]^2 - [v][u^2v]\big)t^2}}{[v]}, \end{aligned}$$

and $\dot{e}(0) = [uv] - [v]\dot{\phi}(0)$,

(ii) *if* $[v] = 0$, *then*

$$\begin{aligned} e(t) &= e^0 + [u]v_0 t, \\ \phi(t) &= \tfrac{e^0\dot{\phi}(0) + tv_0[u^2]/2}{e^0 + tv_0[u]} t. \end{aligned}$$

PROOF. In this case from Theorem 4.3 we have

$$\begin{aligned} \dot{e}(t) &= [uv] - [v]\dot{\phi}(t), \\ \frac{d\big(e(t)\dot{\phi}(t)\big)}{dt} &= [u^2 v] - [uv]\dot{\phi}(t). \end{aligned} \tag{4.13}$$

Let $[v] \neq 0$. Substituting $\dot{\phi}(t)$ from the first equation into the second one, we obtain

$$e\ddot{e}(t) + \big(\dot{e}(t)\big)^2 = [uv]^2 - [v][u^2 v].$$

Integrating the last expression, we obtain

$$e(t)\dot{e}(t) = \frac{1}{2}\frac{d\big(e^2(t)\big)}{dt} = A_1 + At,$$

where

$$\begin{aligned} A &= [uv]^2 - [v][u^2 v] \\ &= \big(u_- v_- - u_+ v_+\big)^2 - (v_- - v_+)\big(u_-^2 v_- - u_+^2 v_+\big) \\ &\qquad\qquad = v_- v_+ \big(u_- - u_+\big)^2 \geq 0, \end{aligned}$$

$u_+ = u_0$, $u_- = u_0 + u_1$, $v_+ = v_0$, $v_- = v_0 + v_1$, A_1 is a constant. Thus, we have

$$e(t) = \sqrt{A_0 + 2A_1 t + At^2}, \quad A_0 = (e^0)^2. \tag{4.14}$$

Hence

$$\dot{e}(t) = \frac{A_1 + At}{\sqrt{(e^0)^2 + 2A_1 t + At^2}}, \quad A_1 = e^0\dot{e}(0), \tag{4.15}$$

and

$$\begin{aligned} \dot{\phi}(t) &= \tfrac{[uv] - \dot{e}(t)}{[v]}, \\ \phi(t) &= \tfrac{e^0 + [uv]t - \sqrt{(e^0)^2 + 2e^0\dot{e}(0)t + At^2}}{[v]}. \end{aligned} \tag{4.16}$$

If $[v] = 0$ then $v_- = v_+ = v_0$. In this case $e(t) = e^0 + [u]v_0 t$. Substituting the last relation into the second equation of system (4.13), we have

$$e(t)\ddot{\phi}(t) + 2[u]v_0\dot{\phi}(t) = [u^2]v_0.$$

Integrating this differential equation, after elementary calculations, we obtain

$$\dot{\phi}(t) = \frac{[u^2]}{2[u]} + \Big(\dot{\phi}(0) - \frac{[u^2]}{2[u]}\Big)\frac{(e^0)^2}{(e^0 + tv_0[u])^2}.$$

Hence, $\phi(t) = \frac{e^0\dot{\phi}(0)+tv_0[u^2]/2}{e^0+tv_0[u]}t$.

Now we prove that constructed solution is entropy solution. Consider the case $[v] \neq 0$. In view of the fact that $A_1 + At = 0$ for $t = -\frac{A_1}{A} = \frac{e^0\dot{e}(0)}{v_-v_+[u]^2} < 0$, the function $(e^0)^2 + 2e^0\dot{e}(0)t + At^2 > 0$ is defined for all $t \geq 0$. From (4.15), (4.16) we obtain

$$\ddot{\phi}(t) = (e^0)^2\frac{(\dot{e}(0))^2 - v_-v_+[u]^2}{[v]\big((e^0)^2 + 2e^0\dot{e}(0)t + v_-v_+[u]^2t^2\big)^{3/2}}.$$

Thus, $\dot{\phi}(t)$ and $\dot{e}(t)$ are monotonous functions for all $t \geq 0$.

Consequently, taking into account that

$$\begin{array}{lll} \dot{e}(t) & \to & \sqrt{A} = [u]\sqrt{v_-v_+}, \\ \dot{\phi}(t) & \to & \frac{u_-\sqrt{v_-}+u_+\sqrt{v_+}}{\sqrt{v_-}+\sqrt{v_+}}, \qquad t \to \infty, \end{array}$$

and relations

$$\begin{array}{lllll} u_+ & \leq & \dot{\phi}(0) & \leq & u_-, \\ u_+ & \leq & \frac{u_-\sqrt{v_-}+u_+\sqrt{v_+}}{\sqrt{v_-}+\sqrt{v_+}} & \leq & u_-, \end{array}$$

one can easily see that entropy condition (4.2) holds.

The case $[v] = 0$ can be comsidered in the same way.

The proof of the theorem is complete. □

Corollary 4.5. *Let* $\dot{\phi}(0) = \frac{u_-\sqrt{v_-}+u_+\sqrt{v_+}}{\sqrt{v_-}+\sqrt{v_+}}$ (*i.e.* $\dot{e}(0) = \sqrt{v_-v_+}[u]$). *The the Cauchy problem* (1.8) *with piecewise-constants initial data* (4.1) *for* $t \in [0, \infty)$ *has a unique generalized solution*

$$\begin{array}{lll} u(x,t) & = & u_0 + u_1H(-x + \phi(t)), \\ v(x,t) & = & v_0 + v_1H(-x + \phi(t)) + e(t)\delta(-x + \phi(t)), \end{array}$$

where

$$\begin{array}{lll} \phi(t) & = & \frac{u_-\sqrt{v_-}+u_+\sqrt{v_+}}{\sqrt{v_-}+\sqrt{v_+}}t, \\ e(t) & = & e^0 + \sqrt{v_-v_+}\big(u_- - u_+\big)t, \end{array}$$

where $u_+ = u_0$, $u_- = u_0 + u_1$, $v_+ = v_0$, $v_- = v_0 + v_1$.

According to the results of Theorem 4.4 and Corollary 4.5, if $\dot{e}(0) = \sqrt{v_-v_+}[u]$, the trajectory of singularity has the same form as for $e(0) = 0$.

Acknowledgements

The authors are greatly indebted to Y. Belopolskaya for fruitful discussions.

References

[1] F. Bouchut, *On zero pressure gas dynamics*, Advances in Math. for Appl. Sci., World Scientific, **22**, (1994), 171-190.

[2] V. G. Danilov, V. P. Maslov, V. M. Shelkovich, *Algebra of singularities of singular solutions to first-order quasilinear strictly hyperbolic systems*, Theor. Math. Phys., **114**, no 1, (1998), 1-42.

[3] V. G. Danilov, G. A. Omel′yanov, V. M. Shelkovich, *Weak Asymptotics Method and Interaction of Nonlinear Waves*, in Mikhail Karasev (ed.), "Asymptotic Methods for Wave and Quantum Problems", Amer. Math. Soc. Transl., Ser. 2, **208**, 2003, 33-165.

[4] V. G. Danilov, V. M. Shelkovich, *Propagation and interaction of nonlinear waves to quasilinear equations*, Hyperbolic prorblems: Theory, Numerics, Applications (Eighth International Conference in Magdeburg, February/March 2000, v.I). International Series of Numerical Mathematics, v. 140, Birkhäuser Verlag Basel/Switzerland, 2001, 267-276.

[5] V. G. Danilov and V. M. Shelkovich, *Propagation and interaction of shock waves of quasilinear equation*, Nonlinear Studies, **8**, no 1, (2001), 135-169.

[6] V. G. Danilov, V. M. Shelkovich, *Propagation and interaction of delta-shock waves*, The Ninth International Conference on Hyperbolic prorblems. Theory, Numerics, and Applications. Abstracts. California Institute of Technology, California USA, March 25–29, 2002, 106-110.

[7] V. G. Danilov, V. M. Shelkovich, *Dynamics of propagation and interaction of δ-shock waves in hyperbolic systems*, In print.

[8] V. G. Danilov, V. M. Shelkovich, *Propagation and interaction of δ-shock waves to hyperbolic systems of coservation laws*, (To appear in Russian Acad. Sci. Dokl. Math., (2003)).

[9] Weinan E., Yu. Rykov, Ya. G. Sinai, *Generalized variational principles, global weak solutions and behavior with random initial data for dystems of conservation laws arising in adhesion particlae dynamics*, Comm. Math. Phys., **177**, (1996), 349-380.

[10] G. Ercole, *Delta-shock waves as self-similar viscosity limits*, Quart. Appl. Math., **LVIII**, no 1, (2000), 177-199.

[11] Jiaxin Hu, *The Rieman problem for pressureless fluid dynamics with distribution solutions in Colombeau's sence*, Comm. Math. Phys., **194**, (1998), 191-205.

[12] Feiming Huang, *Existence and uniqueness of discontinuous solutions for a class nonstrictly hyperbolic systems*, In Chen, Gui-Qiang (ed.) et al. Advances in nonlinear partial differential equations and related areas. Proceeding of conf. dedicated to prof. Xiaqi Ding, China, 1997, 187-208.

[13] K. T. Joseph, *A Rieman problem whose viscosity solutions contain δ-measures*, Asymptotic Analysis, **7**, (1993), 105-120.

[14] B. Lee Keyfitz and H. C. Kranzer, *Spaces of weighted measures for conservation laws with singular shock solutions*, Journal of Differential Equations, **118**, (1995), 420-451.

[15] P. Le Floch, *An existence and uniqueness result for two nonstrictly hyperbolic systems*, Nonlinear Evolution Equations That Change Type, Springer-Verlag, 1990, 126-138.

[16] J. Li and Tong Zhang, *On the initial-value problem for zero-pressure gas dynamics*, Hyperbolic prorblems: Theory, Numerics, Applications. Seventh International Conference in Zürich, February 1998, Birkhäuser Verlag, Basel, Boston, Berlin, 1999, 629-640.

[17] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Springer-Verlag New York, Berlin, Heidelberg, Tokyo, 1984.

[18] V. P. Maslov, *Propagation of shock waves in isoentropi nonviscous gas*, Itogi Nauki i Tekhn.: Sovremennye Probl. Mat., vol. 8, VINITI, Moscow, 1977, pp. 199–271; English transl., J. Soviet Math. **13** (1980), 119–163.

[19] V. P. Maslov, *Three algebras corresponding to nonsmooth solutions of systems of quasilinear hyperbolic equations*, Uspekhi Mat. Nauk **35** (1980) no. 2, 252–253. (Russian).

[20] V. P. Maslov, *Non-standard characteristics in asymptotical problems*, In: Proceeding of the International Congress of Mathematicians, August 16-24, 1983, Warszawa, vol. I, Amsterdam–New York–Oxford: North-Holland, 1984, 139–185

[21] V. P. Maslov and G. A. Omel′yanov *Asymptotic soliton-form solutions of equations with small dispersion*, Uspekhi Mat. Nauk. **36** (1981), no. 3, 63–126; English transl., Russian Math. Surveys **36** (1981), no. 3, 73-149.

[22] G. Dal Maso, P. G. Le Floch, and F. Murat, *Definition and weak stability of nonconservative products*, J. Math. Pures Appl., **74**, (1995), 483-548.

[23] M. Nedeljkov, *Delta and singular delta locus for one dimensional systems of conservation laws*, Preprint ESI 837, Vienna, 2000.

[24] B. L. Rozhdestvenskii and N. N. Yanenko, *Systems of Quasilinear Equations*, Moscow, Nauka, 1978 (in Russian) B. L. Rozhdestvenskii and N. N. Janenko, *Systems of Quasilinear Equations and Their Applications to Gas Dynamics*, New York, Am. Math., 1983.

[25] V. M. Shelkovich, *An associative-commutative algebra of distributions that includes multiplicators, generalized solutions of nonlinear equations*, Mathematical Notices, **57**, no 5, (1995), 765-783.

[26] Wancheg Shen, Tong Zhang, *The Rieman problem for the transportaion equations in gas dynamics*, Memoirs of the Amer. Math. Soc., **137**, no 654, (1999), 1-77.

[27] Dechun Tan, Tong Zhang and Yuxi Zheng, *Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws*, Journal of Differential Equations, **112**, (1994), 1-32.

[28] A. I. Volpert, *The space BV and quasilinear equations*, Math. USSR Sb. **2**, (1967), 225-267.

[29] G. B. Whitham, *Linear and Nonlinear Waves*, New York–London–Sydney–Toronto, Wiley, 1974

[30] Hanchun Yang, *Riemann problems for class of coupled hyperbolic systems of conservation laws*, Journal of Differential Equations, **159**, (1999), 447-484.

[31] Ya. B. Zeldovich, *Gravitationnal instability: An approximate theory for large density perturbations*, Astron. Astrophys., **5**, (1970), 84-89.

Department of Mathematics, Moscow Technical University of Communication and Informatics, Aviamotornaya, 8a, 111024, Moscow, Russia.

E-mail address: danilov@miem.edu.ru

Department of Mathematics, St.-Petersburg State Architecture and Civil Engineering University, 2 Krasnoarmeiskaya 4, 198005, St. Petersburg, Russia.

E-mail address: shelkv@svm.abu.spb.ru