Control Problems for Systems of Conservation Laws

Ph.D. Thesis

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"The privileged reader screwed
up his lamp, and solitary
above the billowy roofs of the
town, like a lighthouse-keeper
above the sea, he turned
to the pages of the story."

JOSEPH CONRAD, Lord Jim (1900)

To my father

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Contents

Chapter 1

Introduction

Consider an $n \times n$ system of conservation laws on a bounded interval:

$$u_t + f(u)_x = 0, \qquad t \geq 0, \quad x \in \,]a, b[\,. \tag{1.0.1}$$

The system is assumed to be strictly hyperbolic, each characteristic field being either linearly degenerate or genuinely nonlinear in the sense of Lax [34]. We shall also assume that all characteristic speeds are bounded away from zero. More precisely, let $f : U \to \mathbb{R}^n$ be a smooth map, defined on an open set $U \subset \mathbb{R}^n$. For each $u \in U$, call $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of the Jacobian matrix $Df(u)$. We assume that there exists a minimum speed $c_0 > 0$ and an integer $p \in \{1, \ldots, n\}$ such that

$$\begin{cases} \lambda_i(u) < 0, & \text{if} \quad i \leq p, \\ \lambda_i(u) > 0, & \text{if} \quad i \geq p, \end{cases} \tag{1.0.2}$$

$$\big|\lambda_i(u)\big| \geq c_0 > 0, \qquad u \in U. \tag{1.0.3}$$

By (1.0.2), for a solution defined on the strip $t \geq 0, \quad x \in \,]a, b[\,$, there will be $n - p$ characteristics entering at the boundary point $x = a$, and p characteristics entering at $x = b$. The initial-boundary value problem is thus well posed if we prescribe $n - p$ scalar conditions at $x = a$ and p scalar conditions at $x = b$. In a very general form, these can be written as

$$\begin{cases} \varphi_a\big(u(t, a+)\big) = 0, & \text{for} \quad t > 0, \\ \varphi_b\big(u(t, b-)\big) = 0, & \text{for} \quad t > 0, \end{cases} \tag{1.0.4}$$

for suitable functions $\varphi_a : \mathbb{R}^n \to \mathbb{R}^{n-p}$, $\varphi_b : \mathbb{R}^n \to \mathbb{R}^p$. See also [1, 2] for the case of general entropy-weak solutions taking values in the space BV of functions with bounded variation.

In the present thesis we study the effect of boundary conditions on the solution of (1.0.1) from the point of view of control theory. Namely, given an initial condition

$$u(0, x) = \phi(x), \qquad x \in \,]a, b[, \tag{1.0.5}$$

we assume that the evolution of the system can be affected by an external controller, acting through the boundary conditions. Instead of (1.0.4), the conditions will thus take the form

$$\begin{cases} \varphi_a\big(u(t,a+),\alpha(t)\big) = 0, & \text{for} \quad t>0, \\ \varphi_b\big(u(t,b-),\beta(t)\big) = 0, & \text{for} \quad t>0, \end{cases} \tag{1.0.6}$$

for suitable *control functions* $\alpha(\cdot)$, $\beta(\cdot)$. Given an initial data as in (1.0.5), one of our main concerns will be to describe the set of attainable configurations:

$$\mathcal{A}(T,\phi) \doteq \big\{u(T,\cdot)\text{solution of (1.0.1), (1.0.5), (1.0.6) for some } \alpha_a,\ \alpha_b\big\} \subset L^1\big([a,b]\,;\ \mathbb{R}^n\big) \tag{1.0.7}$$

which can be attained by the system at a given time $T>0$ (see [28, 33, 45] and references therein for several weak formulations of (1.0.6)).

Definition 1.0.1 *Given a family $\mathcal{F}$ of initial states and $T>0$, we say that the problem (1.0.1) is exactly controllable at time T to the state $v \in L^1\big([a,b];\mathbb{R}^n\big)$ if for every $\phi \in \mathcal{F}$ there exist control functions α, β such that the solution of (1.0.1), (1.0.5), (1.0.6) satisfies*

$$u(T,\cdot) \equiv v, \qquad a.\ e.\ in\ [a,b].$$

We say that the problem (1.0.1) is asymptotically stabilizable near the state $v \in L^1\big([a,b];\mathbb{R}^n\big)$ if for every $\phi \in \mathcal{F}$ there exist control functions α, β such that the solution of (1.0.1), (1.0.5), (1.0.6) satisfies

$$u(t,\cdot) \longrightarrow v, \qquad in\ L^1([a,b]) \qquad as\ t \longrightarrow +\infty.$$

To see the very simplest example, consider a strictly hyperbolic system with constant coefficients:

$$u_t + Au_x = 0, \tag{1.0.8}$$

where A is a $n\times n$ constant matrix, with real distinct eigenvalues

$$\lambda_1 < \cdots < \lambda_p < 0 < \lambda_{p+1} < \cdots < \lambda_n\,.$$

Assume that all incoming components at each boundary point can be assigned by the controller, namely consider the boundary conditions (1.0.6) with

$$\varphi_a(u,\alpha) = \pi_a(u) - \alpha, \qquad \varphi_b(u,\beta) = \pi_b(u) - \beta, \tag{1.0.9}$$

where π_a, π_b are suitable projections on $\mathbb{R}^n$. Call

$$\tau \doteq \max_i \frac{b-a}{|\lambda_i|}$$

the maximum time taken by waves to cross the interval $[a,b]$. In this case, it is easy to see that the reachable set in (1.0.7) is the entire space: $\mathcal{A}(T) = L^1$ for all $T \geq \tau$. In other words, the system is completely controllable after time τ. Indeed, for any $T\geq\tau$ and initial and terminal data $\phi, \psi \in L^1\big([a,b];\mathbb{R}^n\big)$, one can always find a solution of (1.0.5), defined on the rectangle $[0,T] \times [a,b]$ such that

$$u(0,x) = \phi(x), \qquad u(T,x) = \psi(x), \qquad x \in [a,b].$$

Such solution can be constructed as follows. Let $l_1, \ldots, l_n$ and $r_1, \ldots, r_n$ be dual bases of right and left eigenvectors of A so that $l_i \cdot r_j = \delta_{ij}$. For $i = 1, \ldots, n$, let $u_i(t,x)$ be a solution to the scalar Cauchy problem

$$u_{i,t} + \lambda_i u_{i,x} = 0,$$

$$u_i(0,x) = \begin{cases} l_i \cdot \phi(x), & \text{if} \quad x \in [a,b], \\ l_i \cdot \psi(x + \lambda_i T), & \text{if} \quad x \in [a - \lambda_i T,\ b - \lambda_i T], \\ 0, & \text{otherwise.} \end{cases}$$

Then the restriction of

$$u(t,x) = \sum_i u_i(t,x) r_i$$

to the interval $[0,T] \times [a,b]$ satisfies (1.0.8) and takes the required initial and terminal values. Of course, this corresponds to the solution of an initial-boundary value problem, determined by the n boundary conditions

$$\begin{cases} l_i \cdot u(t,a) = u_i(t,a), & \text{if} \quad i = p+1, \ldots, n, \\ l_i \cdot u(t,b) = u_i(t,b), & \text{if} \quad i = 1, \ldots, p. \end{cases}$$

This result on exact boundary controllability has been extended in [35, 36] to the case of general quasilinear systems of the form

$$u_t + A(u) u_x = 0.$$

In this case, the existence of a solution taking the prescribed initial and terminal values is obtained for all sufficiently small data $\phi, \psi \in \mathcal{C}^1$.

Aim of the present thesis is to study analogous controllability properties within the context of entropy weak solutions $t \mapsto u(t,\cdot) \in BV$, or L^∞. For the definitions and basic properties of weak solutions we refer to [12]. For general nonlinear systems, it is clear that a complete controllability result within the space BV or L^1 cannot hold. Indeed, already for a scalar conservation law with boundary conditions (1.0.6), (1.0.9), it was proved in [7] (see also [32] for the results on the Burgers equation) that the profiles $\psi \in BV$ which can be attained at a large time $T > 0$ are precisely those which satisfy the Oleinik-type conditions

$$\psi'(x) \leq \frac{f'\big(\psi(x)\big)}{(x-a) f''\big(\psi(x)\big)}, \qquad \text{for a.e.} \quad x \in [a,b].$$

For general $n \times n$ systems, a complete characterization of the reachable set $\mathcal{A}(T)$ does not seem possible, due to the complexity of repeated wave-front interactions.

The thesis is organized as follows. In Chapter 2 (see [4, 3]) we consider the case of Temple class systems [46] with boundary conditions (1.0.6), (1.0.9). We provide a description of the corresponding attainable set (1.0.7) in terms of suitable Oleinik-type estimates, which is a natural extension of the results in [7, 8, 32] concerning scalar conservation laws.

In Chapter 3 (see [14, 3]) we show that for general nonlinear systems, one cannot expect such an exact controllability result. We prove for a class of strictly hyperbolic, genuinely nonlinear 2×2 systems of the form (1.0.1) that constant states cannot be exactly reached in finite time, but only approached exponentially fast, as $t \to \infty$. A particular system belonging to that class is the one studied by DiPerna in [27]:

$$\begin{cases} \rho_t + (u\rho)_x & = 0, \\ u_t + \left(\dfrac{u^2}{2} + \dfrac{K^2}{\gamma - 1}\rho^{\gamma-1}\right)_x & = 0, \end{cases} \tag{1.0.10}$$

with $1 < \gamma < 3$. Here $\rho > 0$ and u denote the density and the velocity of a gas, respectively. The system describes the evolution of a particular gas.

In Chapter 4 (see [14, 3]) we look for the asymptotic stabilizability of the system (1.0.1) with boundary conditions (1.0.6), (1.0.9) near the constant states. We prove that if the initial condition ϕ has sufficiently small total variation, then the system is asymptotically stabilizable at any nearby constant state u^*. The rate of convergence is faster than exponential. Indeed it can be estimate by

$$\left\| u(t,\cdot) - u^* \right\|_{L^\infty} \leq C_0 \, e^{-2^{\kappa t}}, \qquad t > 0, \tag{1.0.11}$$

for some positive constants C_0, κ.

As in [35], all of the above results refer to the case where total control on the boundary values is available. As a consequence, the problem is reduced to proving the existence (or nonexistence) of an entropy weak solution defined on the open strip $t > 0$, $x \in]a, b[$, satisfying the required conditions. This is a first step toward the analysis of more general controllability problems, where the control acts only on some of the boundary conditions. We shall be mainly concerned the asymptotic stabilization near constant states. For this purpose we fix a constant state u^* such that

$$\varphi_a(u^*) = \varphi_b(u^*) = 0,$$

and we look at the corresponding linearized system at u^*. In this connection, the two following *conjectures* seem natural:

$$\begin{gathered} \text{exact controllability for the linearized system} \\ \Downarrow \\ \text{asymptotic stabilizability for the nonlinear system near } u^* \text{ with rate as in (1.0.11)} \end{gathered}$$

and

$$\begin{gathered} \text{exponential asymptotic stabilizability for the linearized system} \\ \Downarrow \\ \text{asymptotic stabilizability for the nonlinear system near } u^*. \end{gathered}$$

The analysis of these more general boundary controls should thus begin by looking at what happens in the linear case with constant coefficients. In Chapter 5 we thus consider the system

$$\begin{cases} u_i(t,a^+) = \sum_{j=1}^{p} C_{a,ij} \cdot u_j(t,a^+) + \sum_{l=1}^{\mu} \Gamma_{a,il} \cdot \alpha_l(t), & i = 1,\dots,p, \\ u_j(t,b^-) = \sum_{i=p+1}^{n} C_{b,ji} \cdot u_i(t,b^-) + \sum_{l=1}^{\nu} \Gamma_{b,jl} \cdot \beta_l(t), & j = p+1,\dots,n, \end{cases}$$

for $0 < t < T$, so the outgoing waves from the boundaries $x = a$ and $x = b$ depend on the incoming characteristics and on the controls. To simplify the notations and the computations we consider the following

$$\begin{cases} \omega_t + A \cdot \omega_x = 0, & a < x < b,\ t > 0, \\ \omega_i(t,a) = \sum_{j=1}^{p} C_{a,ij}\omega_j(t,a), & t > 0,\ \ i = p+1, ..., n, \\ \omega_j(t,b) = \sum_{i=p+1}^{n} C_{b,ji}\omega_i(t,b) + \sum_{l=1}^{\nu} \Gamma_{b,jl}\alpha_l(t), & t > 0,\ \ j = 1, ..., p, \\ \omega(0,x) = \omega_0(x), & a < x < b. \end{cases} \tag{1.0.12}$$

Observe that

i) all the controls act on the boundary $x = b$;

ii) we have only $\nu \leq p$ scalar controls, namely less or equal controls than outgoing characteristics from $x = b$;

iii) we work in the L^1-functional framework, instead of the L^2 one, that is the most commonly used for linear control problems like (1.0.12).

In Chapter 5 (see [5]) we prove that under orthogonality assumptions on the boundary and control matrices $\{C_{a,ij}\}_{ij}$, $\{C_{b,ji}\}_{ji}$, $\{\Gamma_{b,jl}\}_{jl}$, (1.0.12) is exactly controllable at time $T > 0$, for T sufficiently large. Moreover we show that some of these conditions are necessary.

In Chapter 6 (see [5, 42]) we look for two smallness assumptions on the boundary and control matrices $\{C_{a,ij}\}_{ij}$, $\{C_{b,ji}\}_{ji}$, $\{\Gamma_{b,jl}\}_{jl}$, that guarantee the exponential asymptotic stabilizability of (1.0.12) in the L^1 and L^2 norms. Finally, in the last section we show with some examples that the conditions in [5] are independent of the ones in [42]. Our conditions are sufficient but not necessary for the asymptotic stabilizability as is shown in Section 6.3.

Chapter 2

The Attainable Set for Temple Class Systems

Consider the initial-boundary value problem for a strictly hyperbolic, genuinely nonlinear, system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \tag{2.0.1}$$

$$u(0,x) = \overline{u}(x), \tag{2.0.2}$$

$$u(t,a) = \widetilde{u}_a(t), \tag{2.0.3}$$

$$u(t,b) = \widetilde{u}_b(t), \tag{2.0.4}$$

on the strip $\Omega = \{(t,x) \in \mathbb{R}^2 \ ; \ t \geq 0, x \in [a,b]\}$. Here, $u = u(t,x) \in \mathbb{R}^n$ is the vector of the conserved quantities, $\widetilde{u}_a$, $\widetilde{u}_b$ are measurable, bounded boundary data, and the flux function $f : U \to \mathbb{R}^n$ is a smooth vector field defined on some open set $U \subset \mathbb{R}^n$, that belongs to a class of fields introduced by Temple [46, 45] for which rarefaction and Hugoniot curves coincide. We recall that, for problems of this type, classical solutions may develop discontinuities in finite time, no matter of the regularity of the initial and boundary data. Hence, it is natural to consider weak solutions in the sense of distributions. Moreover, since, in general, the conditions (2.0.3)-(2.0.4) cannot be fulfilled pointwise a.e. (see [10, 28]), different weaker formulations of the boundary conditions have been considered in the literature (see [1, 33, 44] and references therein). Here, following F. Dubois, P. G. LeFloch [28], we will adopt a formulation of (2.0.3)-(2.0.4) based on the definition of a time-dependent set of *admissible boundary data*, that is related to the notion of Riemann problem.

In the present chapter, having in mind applications of Temple class systems to problems of oil reservoir simulation, multicomponent chromatography, as well as in traffic flow models (see [9, 21, 22, 23, 30, 31, 38, 40, 41]), we study the effect of the boundary conditions (2.0.3)-(2.0.4) on the solution of (2.0.1)-(2.0.2) from the point of view of control theory. Namely, following the same approach adopted in [7, 8] for scalar conservation laws, we fix an initial data $\overline{u} \in L^\infty([a,b])$

and we consider the family of configurations

$$\mathcal{A}(T;\mathcal{U}_a,\mathcal{U}_b) \doteq \big\{u(T,\cdot)\ ;\ \ u \ \text{ is a sol. to } \ (2.0.1)-(2.0.4), \quad \widetilde{u}_a \in \mathcal{U}_a,\ \ \widetilde{u}_b \in \mathcal{U}_b\big\} \tag{2.0.5}$$

that can be attained at a given time $T > 0$ by solutions to (2.0.1)-(2.0.4), with boundary data $\widetilde{u}_a, \widetilde{u}_b$ that vary in prescribed sets $\mathcal{U}_a, \mathcal{U}_b \subset L^\infty(\mathbb{R}^+)$ of *admissible boundary controls*. In the case of scalar, convex conservation laws, it was proved in [7], by using the theory of generalized characteristics [24], that the profiles $w(x)$ which can be attained at a fixed time $T > 0$ are only those for which the map $x \mapsto f'(w(x))/x$ is non increasing. Under the assumption that $f'(u) \geq 0$ for all u, and for solutions of the mixed problem (2.0.1)-(2.0.4) on the region Ω, this condition is equivalent to the Oleinik-type inequalities

$$D^+w(x) \leq \frac{f'(w(x))}{(x-a)\, f''(w(x))} \qquad \text{for a.e. } \ x \in [a,b]\,, \tag{2.0.6}$$

(D^+w denoting the upper Dini derivative of w). For general $n \times n$ systems, a complete characterization of the attainable set does not seem possible, due to the complexity of repeated wave-front interactions. However, in the particular case of Temple systems, wave interactions can only change the speed of wave-fronts, without modifying their amplitudes, due to the special geometric features of such systems. Therefore, the only restriction to boundary controllability is the decay due to genuine nonlinearity. We then consider here a convex, compact set $\Gamma \subset U$, and provide a description of the attainable set

$$\mathcal{A}(T) \doteq \mathcal{A}(T;\mathcal{U}^\infty,\mathcal{U}^\infty), \qquad\qquad \mathcal{U}^\infty \doteq L^\infty([0,T],\Gamma)\,,$$

in terms of certain Oleinik-type conditions. We also establish the compactness of $\mathcal{A}(T)$ in the L^1 topology. Applications to calculus of variations and problems of optimization (where the cost functional depends on the profile of the solution at a fixed time T) motivate the study of topological properties of $\mathcal{A}(T)$.

The chapter is organized as follows. Section 2.1 contains the basic definitions and the statement of the main results. We also provide in this section a review of the existence and well-posedness theory for the mixed problem (2.0.1)-(2.0.4), and a description of a front tracking algorithm that will be used throughout the chapter. In Section 2.2 we establish some preliminary estimates, and a regularity result concerning the global structure of solutions to the mixed problem (2.0.1)-(2.0.4) generated by a front tracking algorithm. The proof of the main results is contained in Section 2.3.

2.1 Preliminaries and Statement of the Main Results

2.1.1 Formulation of the Problem

Let $f : U \to \mathbb{R}^n$ be the flux function of the strictly hyperbolic system (2.0.1) defined on a neighborhood of the origin $U \subset \mathbb{R}^n$. Denote by $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of the Jacobian matrix $Df(u)$, and let $\{r_1(u), \ldots, r_n(u)\}$ be a basis of right eigenvectors of $Df(u)$. By possibly considering a sufficiently small restriction of the domain U, we may assume that the following *uniform* strict hyperbolicity condition holds.

(SH1) For every $u, v \in U$, the characteristic speeds at these points satisfy

$$\lambda_i(u) < \lambda_j(v), \qquad \forall\, 1 \le i < j \le n\,. \tag{2.1.1}$$

We also assume that there is a fixed set of characteristic lines entering the interior of the strip $[a,b] \times \mathbb{R}^+$ at the boundaries $x = a,\ x = b$, i.e. that, for some index $p \in \{1, \dots, n\}$, there holds

$$\lambda_p(u) < 0 < \lambda_{p+1}(u), \qquad \forall\, u \in U, \tag{2.1.2}$$

and we let $\lambda^{\min}$, $\lambda^{\max}$ denote the minimum and maximum characteristic speed so that there holds

$$0 < \lambda^{\min} \le |\lambda_i(u)| \le \lambda^{\max}, \qquad \forall\, u \in U. \tag{2.1.3}$$

Moreover, we assume that each i-th characteristic field r_i is *genuinely nonlinear* in the sense of Lax [34], and that system (2.0.1) is of Temple class according with the following.

Definition 2.1.1 *A system of conservation laws is of Temple class if there exists a system of coordinates $w = (w_1, \dots, w_n)$ consisting of Riemann invariants, and such that the level sets $\big\{u \in U;\ w_i(u) = constant\big\}$ are hyperplanes (see [45]).*

By possibly performing a translation of coordinates, it is not restrictive to assume that the Riemann invariants are chosen so that $\partial_i\, \lambda_i(w) > 0,\ i = 1, \dots n$, for all $w = w(u)$, $u \in U$. Throughout the chapter, we will often write $w_i(t,x) \doteq w_i\big(u(t,x)\big)$ to denote the i-th Riemann coordinate of a solution $u = u(t,x)$ to (2.0.1). We recall that, for a Temple class system, Hugoniot and rarefaction curves coincide [46]. Moreover, as observed in [6], thanks to the existence of Riemann coordinates one can show that the assumption (**SH1**) implies the invertibility of the map $f : U \to f(U)$.

We next introduce a definition of weak solution to (2.0.1)-(2.0.4) which includes an entropy admissibility condition of Oleinik type on the decay of positive waves, so to achieve uniqueness. The boundary conditions (2.0.3)-(2.0.4) are formulated in terms of the weak trace of the flux $f(u)$ at the the boundaries $x = a$, $x = b$, and are related to the notion of Riemann problem in the same spirit of [28]. To this purpose, letting $u(t,x) = W(\xi = x/t;\, u_L, u_R),\ \ u_L,\, u_R \in U$, denote the self-similar solution of the Riemann problem for (2.0.1) with initial data

$$u(0,x) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0, \end{cases}$$

for any given boundary state $\widetilde{u} \in U$, we define the set of *admissible states at the boundaries*

$$\begin{aligned} \mathcal{V}_a(\widetilde{u}) &\doteq \big\{W(0+;\, \widetilde{u}, u_R)\ ;\ u_R \in U\big\}, \\ \mathcal{V}_b(\widetilde{u}) &\doteq \big\{W(0-;\, u_L, \widetilde{u})\ ;\ u_L \in U\big\}. \end{aligned} \tag{2.1.4}$$

Definition 2.1.2 *A function $u : [0, T] \times [a, b] \to U$ is an entropy weak solution of the initial-boundary value problem (2.0.1)-(2.0.4) on $\Omega_T \doteq [0, T] \times [a, b]$, if it is continuous as a function from $]0, T]$ into L^1, and the following properties hold:*

(i) u is a distributional solution to the Cauchy problem (2.0.1)-(2.0.2) on Ω_T in the sense that, for every test function $\phi \in \mathcal{C}_c^1$ with compact support contained in the set $\{(t, x) \in \mathbb{R}^2 ;\ a < x < b,\ t < T\}$, there holds

$$\int_0^T \int_a^b \big(u(t,x) \cdot \phi_t(t,x) + f(u(t,x)) \cdot \phi_x(t,x)\big) dx\ dt + \int_a^b \overline{u}(x) \cdot \phi(0,x) dx = 0\,;$$

(ii) the flux $f(u)$ admits weak traces at the boundaries $x = a$, $x = b$, i.e. there exist two measurable functions $\Psi_a, \Psi_b : [0, T] \to \mathbb{R}^n$ such that*

$$f(u(\cdot, x)) \underset{x \to a^+}{\overset{*}{\rightharpoonup}} \Psi_a, \qquad f(u(\cdot, x)) \underset{x \to b^-}{\overset{*}{\rightharpoonup}} \Psi_b \qquad \text{in} \quad L^\infty([0, T]), \tag{2.1.5}$$

and the boundary conditions (2.0.3)-(2.0.4) are satisfied in the following sense

$$\Psi_a(t) \in f\big(\mathcal{V}_a(\widetilde{u}_a(t))\big), \qquad \Psi_b(t) \in f\big(\mathcal{V}_b(\widetilde{u}_b(t))\big) \qquad \text{for a.e.} \quad 0 \le t \le T; \tag{2.1.6}$$

(iii) u satisfies the following entropy conditions on the decay of positive waves in time and in space. There exists some constant $C > 0$, depending only on the system (2.0.1), so that

(a) For any $0 < t \le T$, and for a.e. $a < x < y < b$, there holds

$$w_i(t, y) - w_i(t, x) \le C \cdot \left\{ \frac{y - x}{t} + \log\left(\frac{y - b}{x - b}\right) \right\} \qquad \text{if} \qquad i \in \{1, \dots, p\}, \tag{2.1.7}$$

$$w_i(t, y) - w_i(t, x) \le C \cdot \left\{ \frac{y - x}{t} + \log\left(\frac{y - a}{x - a}\right) \right\} \qquad \text{if} \qquad i \in \{p + 1, \dots, n\}; \tag{2.1.8}$$

(b) For a.e. $a < x < b$, and for a.e. $0 < \tau_1 < \tau_2 \le T$, there holds

$$w_i(\tau_2, x) - w_i(\tau_1, x) \le C \cdot \left\{ \frac{\tau_2 - \tau_1}{x - b} + \log\left(\frac{\tau_2}{\tau_1}\right) \right\} \qquad \text{if} \qquad i \in \{1, \dots, p\}, \tag{2.1.9}$$

$$w_i(\tau_2, x) - w_i(\tau_1, x) \le C \cdot \left\{ \frac{\tau_2 - \tau_1}{x - a} + \log\left(\frac{\tau_2}{\tau_1}\right) \right\} \qquad \text{if} \qquad i \in \{p + 1, \dots, n\}. \tag{2.1.10}$$

Remark 2.1.1 The set of admissible flux values at the boundaries $x = a$, $x = b$, can be expressed in Riemann coordinates as

$$\begin{aligned} f\big(\mathcal{V}_a(\widetilde{u})\big) &= \Big\{ f(u)\ ;\ w_i(u) = w_i(\widetilde{u}) \qquad \forall\, i = p + 1, \dots, n \Big\}, \\ f\big(\mathcal{V}_b(\widetilde{u})\big) &= \Big\{ f(u)\ ;\ w_i(u) = w_i(\widetilde{u}) \qquad \forall\, i = 1, \dots, p \Big\}. \end{aligned} \tag{2.1.11}$$

Hence, by the invertibility of the map $f : U \to f(U)$, the above boundary conditions (2.1.6) are equivalent to the set of equalities

$$
\begin{aligned}
&w_i\big(f^{-1}(\Psi_a(t))\big) = w_i\big(\widetilde{u}_a(t)\big) \qquad \text{for a.e.} \quad 0 \le t \le T, \qquad i = p+1, \dots, n,\\
&w_i\big(f^{-1}(\Psi_b(t))\big) = w_i\big(\widetilde{u}_b(t)\big) \qquad \text{for a.e.} \quad 0 \le t \le T, \qquad i = 1, \dots, p.
\end{aligned}
\tag{2.1.12}
$$

This means that the boundary conditions (2.1.6) guarantee that, at almost every time $t \in [0, T]$, the solution to the Riemann problem for (2.0.1), having left and right initial states $u^L = \widetilde{u}_a(t)$, $u^R = f^{-1}(\Psi_a(t))$, contains only waves with negative speeds, while the solution to the Riemann problem with initial states $u^L = f^{-1}(\Psi_b(t))$, $u^R = \widetilde{u}_b(t)$, contains only waves with positive speeds. Thus, in particular, such solutions do not contain any front entering the domain $[t, +\infty[\, \times]a, b[$.

In the present chapter we regard the boundary data as admissible controls and, in connection with a fixed convex, compact set $\Gamma \subset U$ having the form

$$
\Gamma = \Big\{ u \in U; \quad w_i(u) \in [\alpha_i,\ \beta_i], \qquad i = 1, \dots, n \Big\},
\tag{2.1.13}
$$

we study the basic properties of the *attainable set* for (2.0.1)-(2.0.2), i.e. of the set

$$
\mathcal{A}(T) \doteq \big\{ u(T, \cdot)\ ; \quad u \ \text{ is a sol. to } \ (2.0.1) - (2.0.4), \quad \widetilde{u}_a, \widetilde{u}_b \in L^\infty([0, T], \Gamma) \big\}
\tag{2.1.14}
$$

which consists of all profiles that can be attained at a fixed time $T > 0$, by entropy weak solutions of (2.0.1)-(2.0.4) (according with Definition 2.1.2) with a fixed initial data $\overline{u} \in L^\infty([a, b], \Gamma)$, and boundary data $\widetilde{u}_a$, $\widetilde{u}_b$ that vary in

$$
\mathcal{U}_T^\infty \doteq L^\infty([0, T], \Gamma).
\tag{2.1.15}
$$

We will establish a characterization of (2.1.14) in terms of certain Oleinik type estimates on the decay of positive waves, and we will prove the compactness of (2.1.14) in the L^1 topology.

2.1.2 Statements of the Main Results

For any $\rho > 0$, consider the set of maps

$$
K^\rho \doteq \left\{ \varphi \in L^\infty([a, b],\ \Gamma)\ ; \quad
\begin{array}{ll}
\dfrac{w_i(\varphi(y)) - w_i(\varphi(x))}{y - x} \le \dfrac{\rho}{x - a} & \begin{cases} \text{for a.e.} \quad a < x < y < b, \\ \text{if} \quad i \in \{p+1, ..., n\} \end{cases} \\[4ex]
\dfrac{w_i(\varphi(y)) - w_i(\varphi(x))}{y - x} \le \dfrac{\rho}{b - y} & \begin{cases} \text{for a.e.} \quad a < x < y < b, \\ \text{if} \quad i \in \{1, ..., p\} \end{cases}
\end{array}
\right\}.
\tag{2.1.16}
$$

The inequalities in (2.1.16) reflect the fact that positive waves entering through the boundaries $x = a$, $x = b$ decay in time. Therefore, their density (expressed in terms of Riemann coordinates) is inversely proportional to their distance from their entrance point on the boundary.

The main results of this chapter are the following (see [4, 3]).

Theorem 2.1.1 *Let (2.0.1) be a system of Temple class with all characteristic fields genuinely nonlinear, and assume that the strict hyperbolicity condition* (**SH1**) *is verified. Then, for every fixed* $\overline{\tau} > 0$, *there exists* $\rho = \rho(\overline{\tau}) > 0$ *such that*

$$\mathcal{A}(\tau) \subset K^{\rho} \qquad \forall\, \tau \geq \overline{\tau}\,. \tag{2.1.17}$$

Moreover, there exist $T > 0$ *and* $\rho' < \rho(T)$, *such that*

$$K^{\rho'} \subset \mathcal{A}(\tau) \qquad \forall\, \tau > T\,. \tag{2.1.18}$$

Remark 2.1.2 Observe that, given $\varphi \in K^{\rho}$, any map $x \mapsto w_i(\varphi(x))$, $i \in \{1, \ldots, n\}$, is essentially bounded and has finite total increasing variation on subsets of $[a,\, b]$ bounded away from the end points a, b. Hence, any map $x \mapsto w_i(\varphi(x))$, $i \in \{1, \ldots, n\}$, has also finite total variation on such sets and, in particular, it admits left and right limits in any point $x \in]a,\, b[$. Moreover, since an element φ of K^{ρ} is defined up to L^1 equivalence, we may always assume that there is a right continuous representative of $w_i(\varphi)$, $i \in \{1, \ldots, n\}$, that satisfies the inequalities appearing in the definition of K^{ρ}.

Theorem 2.1.2 *Under the same assumptions of Theorem 1, the set* $\mathcal{A}(T)$ *is a compact subset of* $L^1([a, b], \Gamma)$ *for each* $T > 0$.

Indeed, we will prove in Section 2.3 that the compactness of the attainable set $\mathcal{A}(T)$ holds even in the case where $\mathcal{A}(T)$ is defined as the set of all configurations that can be reached at time T only by solutions of the mixed problem for (2.0.1) that admit a strong L^1 trace at the boundaries $x = a$, $x = b$ (as the ones generated by a front tracking algorithm).

2.1.3 Existence and Uniqueness of Solutions

We describe here a front tracking algorithm that generates approximate solutions to (2.0.1) on the strip $[a, b] \times \mathbb{R}^+$ continuously depending on the initial and boundary data, which represents a natural extension of [6, 16]. Fix an integer $\nu \geq 1$ and consider the discrete set of points in Γ whose coordinates are integer multiples of $2^{-\nu}$:

$$\Gamma^{\nu} \doteq \Big\{ u \in \Gamma\,;\; w_i(u) \in 2^{-\nu}\mathbb{Z}, \quad i = 1, \ldots, n \Big\}. \tag{2.1.19}$$

Moreover, consider the domain

$$\mathcal{D}^{\nu} \doteq \Big\{ (u, u', u'');\quad u \in L^{\infty}([a, b], \Gamma^{\nu}),\; u', u'' \in L^{\infty}(\mathbb{R}^+, \Gamma^{\nu}),\quad u,\, u',\, u'' \;\text{ are piecewise constant} \Big\}. \tag{2.1.20}$$

On $\mathcal{D}^\nu$ we now construct a flow map E^ν whose trajectories are front tracking approximate solutions of (2.0.1). To this end, we first describe how to solve a Riemann problem with left and right initial states $u^L, u^R \in \Gamma^\nu$. In Riemann coordinates, assume that

$$w(u^L) \doteq w^L = (w_1^L, \dots, w_n^L), \qquad w(u^R) \doteq w^R = (w_1^R, \dots, w_n^R).$$

Consider the intermediate states

$$z^0 \doteq u^L, \qquad \dots, \qquad z^i \doteq u(w_1^R, \dots, w_i^R, w_{i+1}^L, \dots, w_n^L), \qquad \dots, \qquad z^n \doteq u^R. \tag{2.1.21}$$

The solution to the Riemann problem (u^L, u^R) is constructed by piecing together the solutions to the simple Riemann problems (z^{i-1}, z^i), $i = 1, \dots, n$. If $w_i^R < w_i^L$, the solution of the Riemann problems (z^{i-1}, z^i) will contain a single i-shock, connecting the states z^{i-1}, z^i, and traveling with the Rankine-Hugoniot speed $\lambda_i(z^{i-1},\, z^i)$. Here and in the sequel, by $\lambda_i(u, u')$ we denote the i-th eigenvalue of the averaged matrix

$$A(u, u') \doteq \int_0^1 Df\big(\theta u + (1-\theta)u'\big)\, d\theta. \tag{2.1.22}$$

If $w_i^R > w_i^L$, the exact solution of the Riemann problem (z^{i-1}, z^i) would contain a centered rarefaction wave. This is approximated by a rarefaction fan as follows. If $w_i^R = w_i^L + p_i\, 2^{-\nu}$ we insert the states

$$z^{i,\ell} \doteq (w_1^R, \dots, w_i^L + \ell\, 2^{-\nu}, w_{i+1}^L, \dots, w_n^L), \qquad\qquad \ell = 0, \dots, p_i, \tag{2.1.23}$$

so that $z^{i,0} = z^{i-1}$, $z^{i,p_i} = z^i$. Our front tracking solution will then contain p_i fronts of the i-th family, each connecting a couple of states $z^{i,\ell-1}$, $z^{i,\ell}$ and traveling with speed $\lambda_i\big(z^{i,\ell-1},\ z^{i,\ell}\big)$.

For any given triple of (piecewise constant) initial and boundary data $(\,\overline{u},\, \widetilde{u}_a\,,\, \widetilde{u}_b\,) \in \mathcal{D}^\nu$, the approximate solution $u(t,\cdot) \doteq E_t^\nu(\,\overline{u},\, \widetilde{u}_a\,,\, \widetilde{u}_b\,)$ is now constructed as follows. At time $t = 0$, for $a < x < b$ we solve the initial Riemann problems determined by the jumps in $\overline{u}$ according to the above procedure, while at $x = a$ we construct the solution to the Riemann problem with left and right initial states $u^L = \widetilde{u}_a(0+)$, $u^R = \overline{u}(a+)$ and take its restriction to the interior of the domain Ω. In the same way, at $x = b$, we take the restriction to the interior of Ω of the solution to the Riemann problem with initial states $u^L = \overline{u}(b-)$, $u^R = \widetilde{u}_b(0+)$. This yields a piecewise constant function with finitely many fronts, traveling with constant speeds. The solution is then prolonged up to the first time t_1 at which one of the following events takes place:

a) two or more discontinuities interact in the interior of Ω;
b) one or more discontinuities hit the boundary of Ω;
c) the boundary data $\widetilde{u}_a$ has a jump;
d) the boundary data $\widetilde{u}_b$ has a jump.

If the case a) occurs, we then solve the resulting Riemann problems applying again the above procedure, while in the other three cases b)-c)-d) we construct the solution to the Riemann problem with left and right initial states $u^L = \widetilde{u}_a(t_1+)$, $u^R = u(t_1, a+)$, or $u^L = u(t_1, b-)$, $u^R = \widetilde{u}_b(t_1+)$, and take its restriction to the interior of the domain Ω. This determines the solution $u(t,\cdot)$ until the time $t_2 > t_1$ where one of the events a), b), c) again takes place, etc... Notice that at any time where case b) occurs but c) or d) do not take place, no new wave is generated. Therefore, waves entering the domain Ω at the boundaries $x = a$, $x = b$ are produced only by the jumps of the boundary data $\widetilde{u}_a$, $\widetilde{u}_b$.

As in [6, 16], one checks that the approximate solution u constructed with this algorithm is well defined for all times $t \geq 0$. Indeed, the following properties hold.

- The total variation of $u(t,\cdot)$, measured w.r.t. the Riemann coordinates $w_1(t,\cdot), \ldots, w_n(t,\cdot)$, is non-increasing in time.

- The number of wave-fronts in $u(t,\cdot)$ is non-increasing at each interaction. Hence, the total number of wave-fronts in $u(t,\cdot)$ remains finite.

It is then possible to define a flow map

$$\mathbf{p} \mapsto E_t^\nu \mathbf{p}, \qquad \mathbf{p} \doteq (\,\overline{u},\, \widetilde{u}_a\,,\, \widetilde{u}_b\,) \in \mathcal{D}^\nu, \quad t \geq 0 \tag{2.1.24}$$

of approximate solutions of (2.0.1). By construction, each trajectory $t \mapsto E_t^\nu \mathbf{p}$ is a weak solution of (2.0.1) (because all fronts of $u(t,\cdot) \doteq E_t^\nu \mathbf{p}$ satisfy the Rankine-Hugoniot conditions), but may contain discontinuities that do not satisfy the usual Lax stability conditions (due to the presence of rarefaction fronts). On the other hand, one can verify as in [6, Lemma 4.4] that, due to genuine nonlinearity, the amount of positive waves in $u(t,\cdot)$, measured w.r.t. the Riemann coordinates $w_1(t,\cdot), \ldots, w_n(t,\cdot)$, decays in time and in space. Hence, for a.e. $a < x < y < b$, one obtains the Oleinik type estimates

$$\begin{aligned}
w_i(t,y) - w_i(t,x) &\leq C \cdot \left\{ \frac{y-x}{t} + \log\left(\frac{y-b}{x-b}\right) \right\} + N_\nu\, 2^{-\nu} \quad \text{if} \quad i \in \{1, \ldots, p\}, \\
w_i(t,y) - w_i(t,x) &\leq C \cdot \left\{ \frac{y-x}{t} + \log\left(\frac{y-a}{x-a}\right) \right\} + N_\nu\, 2^{-\nu} \quad \text{if} \quad i \in \{p+1, \ldots, n\},
\end{aligned} \tag{2.1.25}$$

where N_ν denotes the maximum number of shocks of each family present in the initial data $\overline{u}$, and in the boundary data $\widetilde{u}_a$, $\widetilde{u}_b$. Similarly, one can check that along the x-sections, for a.e. $0 < \tau_1 < \tau_2$, there holds

$$\begin{aligned}
w_i(\tau_2,x) - w_i(\tau_1,x) &\leq C \cdot \left\{ \frac{\tau_2-\tau_1}{x-b} + \log\left(\frac{\tau_2}{\tau_1}\right) \right\} + N_\nu\, 2^{-\nu} \quad \text{if} \quad i \in \{1, \ldots, p\}, \\
w_i(\tau_2,x) - w_i(\tau_1,x) &\leq C \cdot \left\{ \frac{\tau_2-\tau_1}{x-a} + \log\left(\frac{\tau_2}{\tau_1}\right) \right\} + N_\nu\, 2^{-\nu} \quad \text{if} \quad i \in \{p+1, \ldots, n\}.
\end{aligned} \tag{2.1.26}$$

Remark 2.1.3 Observe that, if $u(t,x)$ is a front tracking solution of the Cauchy problem for (2.0.1) (with initial data $\overline{u}(x) \doteq u(0,x)$) constructed by the algorithm in [16] on the upper half plane $\mathbb{R}^+ \times \mathbb{R}$, then the restriction of $u(t,\cdot)$ to the interval $[a,b]$ coincides with the front tracking solution $E_t^\nu(\,\overline{u},\, \widetilde{u}_a\,,\, \widetilde{u}_b\,)$ of the mixed problem for (2.0.1), with boundary data $\widetilde{u}_a(t) \doteq u(t,a)$, $\widetilde{u}_b(t) \doteq u(t,b)$.

As $\nu \to \infty$, the domains $\mathcal{D}^\nu$ become dense in

$$\mathcal{D} \doteq \Big\{(\,\overline{u},\, \widetilde{u}_a\; \widetilde{u}_b\,)\ ;\ \overline{u} \in L^\infty([a,b],\Gamma),\ \ \widetilde{u}_a,\, \widetilde{u}_b \in L^\infty(\mathbb{R}^+,\Gamma)\Big\}. \tag{2.1.27}$$

Thus, following the same technique adopted in [6], one can define a flow map E_t on $\mathcal{D}$ as a suitable limit of the flows E_t^ν in (2.1.24), that depends Lipschitz continuously on the initial and boundary data. Namely, the following holds.

Theorem 2.1.3 *Let (2.0.1) be a system of Temple class with all characteristic fields genuinely nonlinear, and assume that the strict hyperbolicity condition* (**SH1**) *holds. Then, there exists a continuous map*

$$(t,\,\overline{u},\,\widetilde{u}_a\,,\,\widetilde{u}_b\,) \mapsto E_t(\,\overline{u},\,\widetilde{u}_a\,,\,\widetilde{u}_b\,) \qquad t \geq 0,\ \ (\,\overline{u},\,\widetilde{u}_a\,,\,\widetilde{u}_b\,) \in \mathcal{D}\,, \tag{2.1.28}$$

and some constant $C > 0$ depending only on the system (2.0.1) and on the domain Γ, so that, for every fixed $0 < \delta < (b-a)/2$, and for all $\mathbf{p}_1 \doteq (\,\overline{u},\,\widetilde{u}_a\,,\,\widetilde{u}_b\,)$, $\mathbf{p}_2 \doteq (\,\overline{v},\,\widetilde{v}_a\,,\,\widetilde{v}_b\,) \in \mathcal{D}$, letting $L_t \doteq L_t(\delta) = C(1+\log(t/\delta))$, there holds

$$\big\|E_t\mathbf{p}_1 - E_t\mathbf{p}_2\big\|_{L^1([a+\delta,\,b-\delta])} \leq L_t \cdot \Big\{\big\|\,\overline{u}-\overline{v}\,\big\|_{L^1([a,b])} + \big\|\widetilde{u}_a - \widetilde{v}_a\big\|_{L^1([0,\,t])} + \big\|\widetilde{u}_b - \widetilde{v}_b\big\|_{L^1([0,\,t])}\Big\} \tag{2.1.29}$$

for all $t \geq \delta$. Moreover, the map $(t,x) \mapsto E_t(\,\overline{u},\,\widetilde{u}_a\,,\,\widetilde{u}_b\,)(x)$ yields an entropy weak solution (in the sense of Definition 2.1.2) to the initial-boundary value problem (2.0.1)-(2.0.4) on Ω, that admits strong L^1 traces at the boundaries $x = a$ and $x = b$, i.e. there exist two measurable maps $\psi_a,\,\psi_b : \mathbb{R}^+ \to U$ such that

$$\begin{aligned} &\lim_{x\to a^+}\int_0^\tau \big|E_t(\,\overline{u},\,\widetilde{u}_a\,,\,\widetilde{u}_b\,)(x) - \psi_a(t)\big|\;dt\; = 0,\\ &\lim_{x\to b^-}\int_0^\tau \big|E_t(\,\overline{u},\,\widetilde{u}_a\,,\,\widetilde{u}_b\,)(x) - \psi_b(t)\big|\;dt\; = 0, \end{aligned} \qquad \forall\ \tau \geq 0. \tag{2.1.30}$$

The proof of Theorem 2.1.3 can be obtained with entirely similar arguments to those used to establish [6, Theorem 2.1], where a continuous flow of solutions to (2.0.1) is constructed in the case of a mixed problem on the quarter of plane $\{(t,x) \in \mathbb{R}^2\ ;\ t \geq 0,\, x \geq 0\}$, with a single boundary at $x = 0$.

Concerning uniqueness, with the same arguments in [6] one obtains the following result which is the extension of [6, Theorem 2.2] to the present case of a domain Ω with two boundaries at $x = a$ and at $x = b$.

Theorem 2.1.4 *Let (2.0.1) be a system of Temple class satisfying the same assumptions as in Theorem 2.1.3. Let $u = u(t,x)$ be an entropy weak solution to the mixed problem (2.0.1)-(2.0.4) on the region $\Omega_T \doteq [0,\,T] \times [a,b]$ (in the sense of Definition 2.1.2). Assume that the following conditions hold.*

(i) The map $(t,x)\mapsto\big(u(t,\cdot),\,u(\cdot,x)\big)$ *takes values within the domain*

$$\mathcal{D}_T \doteq \Big\{(\,\overline{u},\,\widetilde{u}_a,\,\widetilde{u}_b\,)\ ;\ \overline{u}\in L^\infty([a,b],\Gamma),\,\widetilde{u}_a,\,\widetilde{u}_b\in L^\infty([0,T],\Gamma)\Big\}. \tag{2.1.31}$$

(ii) There holds

$$ess\ \sup_{t\to 0+}\int_a^b \big|u(t,x)-\overline{u}(x)\big|\,dx = 0\,. \tag{2.1.32}$$

(iii) There holds

$$ess\ \sup_{x\to a+}\int_0^T \big|w_i\big(u(t,x)\big)-w_i(\widetilde{u}_a(t))\big|\,dt\ =\ 0 \qquad \forall\, i=p+1,\dots,n, \tag{2.1.33}$$

$$ess\ \sup_{x\to b-}\int_0^T \big|w_i\big(u(t,x)\big)-w_i(\widetilde{u}_b(t))\big|\,dt\ =\ 0 \qquad \forall\, i=1,\dots,p. \tag{2.1.34}$$

Then, u coincides with the corresponding trajectory of the flow map E_t provided by Theorem 2.1.3, namely one has

$$u(t,\cdot)=E_t(\,\overline{u},\,\widetilde{u}_a\,,\,\widetilde{u}_b)(\cdot), \qquad \forall\, 0\le t\le T. \tag{2.1.35}$$

The next result shows that the conditions (2.1.32)-(2.1.34) are certainly satisfied by entropy weak solutions to the mixed problem (2.0.1)-(2.0.4) obtained as limit of front tracking approximations.

Theorem 2.1.5 *Let (2.0.1) be a system of Temple class satisfying the same assumptions as in Theorem 2.1.3. Consider a sequence $u^\nu(t,\cdot):[a,b]\to\Gamma^\nu$ of wave-front tracking approximate solutions of the mixed problem for (2.0.1) (constructed with the above algorithm) that converges in L^1, as $\nu\to\infty$, to some function $u(t,\cdot):[a,b]\to\Gamma$, for every $t\in[0,T]$. Then, there exist the right limit at $x=a$, and the left limit at $x=b$, of the map $x\mapsto u(t,x)$ for every $t\in[0,T]$, and the right limit at $t=0$ of the map $t\mapsto u(t,x)$ for every $x\in[a,b]$. Moreover, there is a countable set $\mathcal{N}\subset\mathbb{R}$ such that $u(t,a)=u(t,a+)$, $u(t,b)=u(t,b-)$ for all $t\in[0,T]\setminus\mathcal{N}$, and $u(0,x)=u(0+,x)$ for all $x\in[a,b]\setminus\mathcal{N}$, and, setting $\overline{u}\doteq u(0,\cdot)$, $\widetilde{u}_a\doteq u(\cdot,a)$, $\widetilde{u}_b\doteq u(\cdot,b)$, there holds (2.1.35).*

Remark 2.1.4 It was shown in [6, Lemma 2.1] that an alternative way to prove the essential limits (2.1.33)-(2.1.34), is to employ the distributional entropy inequalities associated to the "boundary entropy pairs" for (2.0.1), introduced by G.-Q. Chen and H. Frid in [19, 20]. However, in order to apply [6, Lemma 2.1] to a function u obtained as a limit of approximate solutions u^ν, it is necessary to know the L^1 convergence of the sequence of the corresponding boundary data $\widetilde{u}_a^\nu$, $\widetilde{u}_b^\nu$. Instead, the result provided here by Theorem 2.1.5 allows to derive the limits (2.1.33)-(2.1.34) requiring only the L^1 convergence of the sequence of the approximate solutions $u^\nu(t,\cdot)$, for all t. This property will be crucial to establish the main result of the chapter stated in Theorems 2.1.1-2.1.2.

In order to prove Theorem 2.1.5, we will show in the next section that, for Temple systems, solutions of the mixed problem (2.0.1)-(2.0.4) with possibly unbounded variation enjoy the same regularity property (of being continuous outside a countable number of Lipschitz curves) possessed by solutions with small total variation of a general system, thus extending the regularity results obtained under the smallness assumption of the total variation by DiPerna [26] and Liu [39] (for solutions constructed by the Glimm scheme) and by Bressan and LeFloch [17] (for solutions generated by a front tracking algorithm).

Proposition 2.1.1 *In the same setting as Theorem 2.1.5, consider a sequence $u^\nu(t,\cdot):[a,b]\to\Gamma^\nu$ of wave-front tracking approximate solutions of the mixed problem for (2.0.1) (constructed with the above algorithm) that converges in L^1, as $\nu\to\infty$, to some function $u(t,\cdot):[a,b]\to\Gamma$, for every $t\in[0,T]$. Then, there exist a countable set of interaction points $\Theta\doteq\big\{(\tau_l,\,x_l);\ l\in\mathbb{N}\big\}\subset\Omega_T\doteq[0,\,T]\times[a,\,b]$, and a countable family of Lipschitz continuous shock curves $\Upsilon\doteq\big\{x=y_m(t);\ t\in\,]r_m,\,s_m[,\ m\in\mathbb{N}\big\}$, such that the following hold.*

(i) For each $m\in\mathbb{N}$, and for any $\tau\in\,]r_m,\,s_m[$ with $(\tau,y_m(\tau))\notin\Theta$, there exist the derivative $\dot y_m(\tau)$ and the left and right limits

$$\lim_{(s,y)\to(\tau,y_m(\tau)),\,y<y_m(\tau)} u(s,y)\doteq u^-,\qquad \lim_{(s,y)\to(\tau,y_m(\tau)),\,y>y_m(\tau)} u(s,y)\doteq u^+\,. \tag{2.1.36}$$

Moreover, these limits satisfy the Rankine Hugoniot relations

$$\dot y_m(\tau)\cdot(u^+-u^-)=f(u^+)-f(u^-) \tag{2.1.37}$$

and, for some $i\in\{1,...,n\}$, there hold the Lax entropy inequalities

$$\lambda_i(u^+)<\dot y_m(t)<\lambda_i(u^-)\,. \tag{2.1.38}$$

(ii) The map u is continuous outside the set $\Theta\cup\Upsilon$.

2.2 Preliminary Results

In this section we first provide some estimates on the distance between two rarefaction fronts of a front tracking solution (constructed by the algorithm described in Section 2.1.3) similar to [16, Lemma 4], [11, Prop. 4.5]. We next show how to approximate the profile $u(t,\cdot)$ of a solution of the mixed problem (2.0.1)-(2.0.4), with a function taking values in the discrete set Γ^ν defined at (2.1.19), which enjoys the same type of estimates on the positive waves as $u(t,\cdot)$. We conclude the section establishing the regularity result stated in Proposition 2.1.1 on the global structure of solutions to the mixed problem for (2.0.1), which in turn yields Theorem 2.1.5.

Lemma 2.2.1 *There exists some constant $C_1>0$ depending only on the system (2.0.1) such that the following holds. Consider a front tracking solution $u(t,x)$ with values in Γ^ν, constructed by the* *algorithm of Section 2.1.3 on the region* $[\tau, \tau'] \times [a, b]$. *Then, given any two adjacent rarefaction fronts of* u *located at* $x(t) \leq y(t)$, $t \in [\tau, \tau']$, *and belonging to the same family, there holds*

$$\left|y(\tau') - x(\tau')\right| \leq \left|y(\tau) - x(\tau)\right| + C_1(\tau' - \tau)\, 2^{-\nu}\,. \tag{2.2.1}$$

PROOF. Consider two adjacent rarefaction fronts of the k-th family $x(t) \leq y(t)$, $t \in [\tau, \tau']$, and let $\tau_1 < \ldots < \tau_N$ be the interaction times of $x(t)$ in the interval $[\tau, \tau']$. Set $\tau_0 \doteq \tau$, $\tau_{N+1} \doteq \tau'$, and fix $\alpha \in \{0, \ldots, N\}$. Let $t \mapsto z(t; s, x)$ be the characteristic curve of the k-th family starting at (s, x), i.e. the solution to the ODE

$$\dot{z} = \lambda_k(u(t, z)), \qquad z(s; s, x) = x.$$

Notice that, although the above ODE has discontinuous right hand-side (because of the discontinuities in the front tacking solution u), its solution $z(\cdot; s, x)$ is unique and depends Lipschitz continuously on the initial data x since it crosses only a finite number of jumps (see [13]). Choose $t_0 < t_1 < \tau_{\alpha+1}$ so that the characteristic curve $z(\cdot; t_0, x(t_0))$ does not cross any wave-front of the other families in the interval $[t_0, t_1]$, and then, by induction, define a sequence of times $\{t_i\}_{i\in\mathbb{Z}} \subset\ ]\tau_\alpha, \tau_{\alpha+1}[$ so that

$$\begin{gathered}\tau_\alpha < t_{-i-1} < t_{-i} \leq t_0 \leq t_i < t_{i+1} < \tau_{\alpha+1}, \quad i \in \mathbb{N}, \\ \lim_{i\to-\infty} t_i = \tau_\alpha, \qquad \lim_{i\to+\infty} t_i = \tau_{\alpha+1},\end{gathered} \tag{2.2.2}$$

with the properties that the characteristic curve of the k-th family starting at $(t_i, x(t_i))$, does not cross any wave-front of the other families in the interval $[t_i, t_{i+1}]$, for each $i \in \mathbb{Z}$. Thus, setting

$$u_i^+ \doteq u(t_i,\, x(t_i)+), \qquad u_i^- \doteq u(t_i,\, x(t_i)-),$$

and observing that, by construction, one has $|w(u_i^+) - w(u_i^-)| < 2^{-\nu}$, we derive

$$\begin{aligned}\left|z(t_{i+1};\, t_i,\, x(t_i)) - x(t_{i+1})\right| &\leq (t_{i+1} - t_i) \cdot \left|\lambda_k(u_i^+) - \lambda_k(u_i^+,\, u_i^-)\right| \\ &\leq c \cdot (t_{i+1} - t_i) \cdot \left|w(u_i^+) - w(u_i^-)\right| \\ &\leq c \cdot (t_{i+1} - t_i) \cdot 2^{-\nu}\end{aligned} \tag{2.2.3}$$

for some constant $c > 0$ depending only on the system. Relying on (2.2.3), and since $z(\tau';\, t_{i+1},\, x)$ depends Lipschitz continuously on the initial data x, we deduce that there exists some other constant $c' > 0$, depending only on the system and on the set Γ, so that there holds

$$\begin{aligned}\left|z(\tau';\, t_i,\, x(t_i)) - z(\tau';\, t_{i+1},\, x(t_{i+1}))\right| &\leq c' \cdot \left|z(t_{i+1};\, t_i,\, x(t_i)) - x(t_{i+1})\right| \\ &\leq c' \cdot c \cdot (t_{i+1} - t_i) \cdot 2^{-\nu}\end{aligned} \tag{2.2.4}$$

for any $i \in \mathbb{Z}$. Thus, by (2.2.2), and thanks to (2.2.4), we obtain

$$\begin{aligned}\left|z(\tau';\, \tau_\alpha,\, x(\tau_\alpha)) - z(\tau';\, \tau_{\alpha+1},\, x(\tau_\alpha))\right| &\leq \sum_{i\in\mathbb{Z}} \left|z(\tau';\, t_i, x(t_i)) - z(\tau';\, t_{i+1},\, , x(t_{i+1}))\right| \\ &\leq c' \cdot c \cdot (\tau_{\alpha+1} - \tau_\alpha) \cdot 2^{-\nu}\,.\end{aligned} \tag{2.2.5}$$

Repeating this computation for every interval $]\tau_\alpha, \tau_{\alpha+1}[$, $\alpha \in \{0, ..., N\}$, we get

$$\begin{aligned} \big|z(\tau';\, \tau,\, x(\tau)) - x(\tau')\big| &\leq \sum_{\alpha=0}^{N} \big|z(\tau';\, \tau_\alpha,\, x(\tau_\alpha)) - z(\tau';\, \tau_{\alpha+1},\, x(\tau_\alpha))\big| \\ &\leq c' \cdot c \cdot (\tau' - \tau) \cdot 2^{-\nu}\,. \end{aligned} \tag{2.2.6}$$

Clearly, one obtains the same type of estimate as (2.2.6) for the other rarefaction front $y(t)$, i.e. there holds

$$\big|z(\tau';\, \tau,\, y(\tau)) - y(\tau')\big| \leq c' \cdot c \cdot (\tau' - \tau) \cdot 2^{-\nu}\,. \tag{2.2.7}$$

On the other hand, by (2.1.3), we have

$$\big|z(\tau';\, \tau,\, x(\tau)) - z(\tau';\, \tau,\, y(\tau))\big| \leq \big|x(\tau) - y(\tau)\big| + 2\,\lambda^{\max} \cdot (\tau' - \tau). \tag{2.2.8}$$

Thus, (2.2.6)-(2.2.8) together yield (2.2.1), concluding the proof. □

In the following, in connection with any (right continuous) piecewise constant map $\psi : [a, b] \to 2^{-\nu}\,\mathbb{Z}$, we will let $\pi(\psi) = \{x_0 = a < x_1 < \cdots < x_{\overline{\ell}} = b\}$ denote the partition of $[a, b]$ induced by ψ, in the sense that $\psi(x)$ is constant on every interval $[x_\ell,\, x_{\ell+1}[\,,\ 0 \leq \ell < \overline{\ell}$. Then, given $\rho > 0$, for any $\nu \geq 1$, consider the set of piecewise constant maps

$$K^\rho_\nu \doteq \left\{ \varphi : [a,b] \to \Gamma^\nu;\quad \begin{array}{ll} \dfrac{w_i(\varphi(x_k)) - w_i(\varphi(x_h))}{x_k - x_h} \leq \dfrac{5\rho}{x_h - a} & \begin{cases} \text{for} & a < x_h < x_k < b, \\ & x_h,\, x_k \in \pi(w_i \circ \varphi), \\ \text{if} & i \in \{p+1, ..., n\} \end{cases} \\[4ex] \dfrac{w_i(\varphi(x_k)) - w_i(\varphi(x_h))}{x_k - x_h} \leq \dfrac{5\rho}{b - x_k} & \begin{cases} \text{for} & a < x_h < x_k < b, \\ & x_h,\, x_k \in \pi(w_i \circ \varphi), \\ \text{if} & i \in \{1, ..., p\} \end{cases} \end{array} \right\}. \tag{2.2.9}$$

The next lemma shows that we can approximate in L^1 any map $\varphi \in K^\rho$ with a piecewise constant function $\varphi_\nu \in K^\rho_\nu$.

Lemma 2.2.2 *For any given $\varphi \in K^\rho$, there exists a sequence of right continuous maps $\varphi_\nu \in K^\rho_\nu$, $\nu \geq 1$, such that:*

a) for every $i \in \{1, \ldots, n\}$, and for any $x_h \in \pi(w_i \circ \varphi_\nu)$, there holds

$$w_i(\varphi_\nu(x_{h+1})) > w_i(\varphi_\nu(x_h)) \qquad \Longrightarrow \qquad w_i(\varphi(x_{h+1})) = w_i(\varphi(x_h)) + 2^{-\nu}\,; \tag{2.2.10}$$

b) there holds

$$\varphi_\nu \to \varphi \qquad \text{in} \qquad L^1([a, b]). \tag{2.2.11}$$

1. First observe that, by Remark 2.1.2, any map $x \mapsto w_i(\varphi(x))$, $i \in \{1,\dots,n\}$ has finite total variation on $[a+\varepsilon,\, b-\varepsilon]$, $\varepsilon > 0$. Hence, we may assume that $w_i(\varphi(\cdot))$ admits left and right limits in any point $x \in]a,\, b[$, and that $w_i(\varphi(x)) = w_i(\varphi(x^+)) \doteq \lim_{\xi\to x^+} w_i(\varphi(\xi))$, for all $i \in \{1,\dots,n\}$. Let $\{y_{i,m}\ ;\ m \in \mathbb{N}\}$ be the countable set of discontinuities of $w_i(\varphi(\cdot))$, $i \in \{1,\dots,n\}$. Then, we can find a partition $\xi^1_{i,m} = y_{i,m} < \xi^2_{i,m} < \cdots < \xi^{\ell_{i,m}}_{i,m} = y_{i,m'}$ of each interval $[y_{i,m},\, y_{i,m'}[$ where $x \mapsto w_i(\varphi(x))$ is continuous, so that:

i) for every $1 < \ell < \ell_{i,m}$ there holds

$$w_i(\varphi(\xi^\ell_{i,m})) \in 2^{-\nu}\,\mathbb{Z}\,; \tag{2.2.12}$$

ii) for every $1 \le \ell < \ell_{i,m}$ one has

$$\big|w_i(\varphi(x)) - w_i(\varphi(\xi^\ell_{i,m}))\big| \le 2^{-\nu} \qquad \forall\, x \in [\xi^\ell_{i,m},\, \xi^{\ell+1}_{i,m}[\,. \tag{2.2.13}$$

Notice that the Oleinik type conditions stated in the definition of K^ρ imply that, at any discontinuity point $y_{i,m}$ of $w_i(\varphi(\cdot))$, one has

$$\lim_{\xi\to y^-_{i,m}} w_i(\varphi(\xi)) > w_i(\varphi(y_{i,m}))\,. \tag{2.2.14}$$

2. Let $\varphi_\nu : [a,b] \to \Gamma^\nu$ be the piecewise constant, right continuous map defined by setting, for every $i \in \{1,\dots,n\}$, and for any interval $[y_{i,m},\, y_{i,m'}[$ where $w_i(\varphi(\cdot))$ is continuous,

$$w_i(\varphi_\nu(x)) \doteq \begin{cases} 2^{-\nu}\lfloor 2^\nu\, w_i(\varphi(\xi^1_{i,m}))\rfloor & \text{if} \quad \begin{cases} x \in [\xi^1_{i,m},\, \xi^2_{i,m}[\,, \quad \text{and} \\ w_i(\varphi_\nu(\xi^1_{i,m})) \le 2^{-\nu}\big(\lfloor 2^\nu\, w_i(\varphi(\xi^1_{i,m}))\rfloor + 2^{-1}\big), \end{cases} \\[2ex] 2^{-\nu}\big(\lfloor 2^\nu\, w_i(\varphi(\xi^1_{i,m}))\rfloor + 1\big) & \text{if} \quad \begin{cases} x \in [\xi^1_{i,m},\, \xi^2_{i,m}[\,, \quad \text{and} \\ w_i(\varphi_\nu(\xi^1_{i,m})) > 2^{-\nu}\big(\lfloor 2^\nu\, w_i(\varphi(\xi^1_{i,m}))\rfloor + 2^{-1}\big), \end{cases} \\[2ex] w_i(\varphi(\xi^\ell_{i,m})) & \text{if} \quad x \in [\xi^\ell_{i,m},\, \xi^{\ell+1}_{i,m}[\,, \qquad 1 < \ell < \ell_{i,m}\,, \end{cases} \tag{2.2.15}$$

where $\lfloor\cdot\rfloor$ denotes the integer part. Notice that, by construction, and because of (2.2.12)-(2.2.13), (2.2.14), the map $\varphi_\nu : [a,b] \to \Gamma^\nu$ enjoys the following property

$$\left.\begin{aligned} & w_i(\varphi_\nu(x_k)) > w_i(\varphi_\nu(x_h)) \\ & x_h < x_k \in \pi(w_i\circ\varphi_\nu) \end{aligned}\right\} \qquad \Longrightarrow \qquad w_i(\varphi(x_k)) > w_i(\varphi(x_h)) + 2^{-(\nu+1)}\,. \tag{2.2.16}$$

Therefore, since $\varphi \in K^\rho$, relying on (2.2.13), (2.2.16), we deduce that, for every $w_i(\varphi_\nu(\cdot))$, $i \in \{p+1,\dots,n\}$, and for any $x_h < x_k \in \pi(w_i\circ\varphi_\nu)$ such that $w_i(\varphi_\nu(x_k)) > w_i(\varphi_\nu(x_h))$, there holds

$$\begin{aligned} \frac{w_i(\varphi_\nu(x_k)) - w_i(\varphi_\nu(x_h))}{x_k - x_h} &\le \frac{w_i(\varphi(x_k)) - w_i(\varphi(x_h)) + 2^{-(\nu-1)}}{x_k - x_h} \\ &\le \frac{5\,\big(w_i(\varphi(x_k)) - w_i(\varphi(x_h))\big)}{x_k - x_h} \\ &\le \frac{5\,\rho}{x_h - a}\,. \end{aligned} \tag{2.2.17}$$

Clearly, with the same computations, we can show that, for every $w_i(\varphi_\nu(\cdot))$, $i \in \{1, \dots, p\}$, and for any $x_h < x_k \in \pi(w_i \circ \varphi_\nu)$, there holds

$$\frac{w_i(\varphi_\nu(x_k)) - w_i(\varphi_\nu(x_h))}{x_k - x_h} \leq \frac{5\,\rho}{b - x_k}\,. \tag{2.2.18}$$

The estimates (2.2.17)-(2.2.18), together, imply that $\varphi_\nu \in K_\nu^\rho$, while (2.2.13) yields (2.2.11). On the other hand observe that, by construction, and because of (2.2.14), the map φ_ν satisfies condition (2.2.10), which completes the proof of the lemma. □

We now provide a further estimate on the distance between two rarefaction fronts of a front tracking solution that, at a fixed time τ, attains a profile belonging to the set (2.2.9).

Lemma 2.2.3 *Consider a front tracking solution $u(t,x)$ with values in Γ^ν, $\nu \geq 1$,constructed by the algorithm of Section 2.1.3 on the region $[\tau, \tau'] \times [a, b]$. Assume that $u(\tau', \cdot)$ is right-continuous, verifies condition a) of Lemma 2.2.2, and satisfies*

$$u(\tau', \cdot) \in K_\nu^{\rho'}\,, \qquad\qquad \rho' \doteq \frac{\lambda^{\min}}{6\,C_1}\,, \tag{2.2.19}$$

where $\lambda^{\min}$, C_1, are the minimum speed in (2.1.3), and the constant of Lemma 2.2.1. Then, given any two adjacent rarefaction fronts of u located at $x(t) \leq y(t)$, $t \in [\tau, \tau']$, and belonging to the same family, there holds

$$x(\tau) < y(\tau). \tag{2.2.20}$$

Proof. To fix the ideas, assume that $x(t) \leq y(t)$ are the locations of two adjacent rarefaction fronts of the $k \in \{p+1, \dots, n\}$ - th family, and hence, by (2.1.2), have positive speeds. Observe that, by condition a) of Lemma 2.2.2, one has

$$w_k(u(\tau', y(\tau'))) - w_k(u(\tau', x(\tau'))) = 2^{-\nu}\,. \tag{2.2.21}$$

Moreover, since u is a front tracking solution constructed by the algorithm of Section 2.1.3 on the region $[\tau, \tau'] \times [a, b]$, we can apply Lemma 2.2.1. Thus, using (2.1.3), (2.2.1), (2.2.21), and recalling the definition (2.2.9) of $K_\nu^{\rho'}$, we deduce

$$\begin{aligned}
y(\tau') - x(\tau') &\leq y(\tau) - x(\tau) + C_1(\tau' - \tau)\,2^{-\nu} \\
&\leq y(\tau) - x(\tau) + C_1 \frac{x(\tau') - x(\tau)}{\lambda^{\min}} \cdot \big(w_k(\varphi_\nu(y(\tau'))) - w_k(\varphi_\nu(x(\tau')))\big) \\
&\leq y(\tau) - x(\tau) + C_1 \frac{5\rho'}{\lambda^{\min}} \cdot \big(y(\tau') - x(\tau')\big)
\end{aligned}$$

which, because of (2.2.19), implies

$$y(\tau) - x(\tau) \geq \left(1 - C_1 \frac{5\rho'}{\lambda^{\min}}\right) \cdot \big(y(\tau') - x(\tau')\big) > 0\,,$$

proving (2.2.20). □

We next derive a regularity property enjoyed by general BV solutions of Temple systems defined as limit of front tracking approximations, which allows us to establish Proposition 2.1.1. This is an extension of the regularity results obtained in [26, 39, 17] for solution with small total variation of general systems. The arguments of the proof are quite similar as for the corresponding result in [17], but we will repeat some of them for completeness, referring to [17] (see also [12, Theorem 10.4]) for further details.

Lemma 2.2.4 *Let (2.0.1) be a system of Temple class satisfying the same assumptions as in Theorem 2.1.3. Consider a sequence $u^\nu(t,\cdot):[c,\,d]\to\Gamma^\nu$, $t\in[r,\,s]$, of front tracking approximate solutions of the mixed problem for (2.0.1) (constructed by the algorithm of Section 2.1.3), that converges in L^1, as $\nu\to\infty$, to some function $u(t,\cdot):[c,\,d]\to\Gamma$, for every $t\in[r,\,s]\subset\mathbb{R}^+$. Assume that*

$$\textit{Tot.Var.}(u^\nu(t,\cdot))\le M,\qquad \textit{Tot.Var.}(u^\nu(\cdot,x))\le M\qquad \forall\ t,x,\,\nu\,, \tag{2.2.22}$$

for some constant $M>0$. Then, there exist a countable set of interaction points $\Theta\doteq\big\{(\tau_l,\,x_l);\ l\in\mathbb{N}\big\}\subset D\doteq[r,\,s]\times[c,\,d]$, and a countable family of Lipschitz continuous shock curves $\Upsilon\doteq\big\{x=y_m(t);\ t\in]r_m,\,s_m[,\ m\in\mathbb{N}\big\}$, such that the following hold.

(i) For each $m\in\mathbb{N}$, and for any $\tau\in]r_m,\,s_m[$ with $(\tau,y_m(\tau))\notin\Theta$, there exist the left and right limits (2.1.36) of u at $(\tau,y_m(\tau))$ and the shock speed $\dot y_m(\tau)$. Moreover, these limits satisfy the Rankine Hugoniot relations (2.1.37) and the Lax entropy inequality (2.1.38), for some $i\in\{1,...,n\}$.

(ii) The map u is continuous outside the set $\Theta\cup\Upsilon$.

Proof.

1. To establish (i) we need to recall some technical tools introduced in [17] (see also [12, Theorem 10.4]). For every front tracking solution u^ν, we define the *interaction and cancellation measure* μ_ν^{IC} that is a positive, purely atomic measure on D, concentrated on the set of points P where two or more wave-fronts of u^ν interact. Namely, if the incoming fronts at P have size $\sigma_1,\ldots,\sigma_\ell$ (w.r.t. the Riemann coordinates), and belong to the families $i_1,\ldots,i_\ell$ respectively, we set

$$\mu_\nu^{IC}(P)\doteq\sum_{\alpha,\beta}\big|\sigma_\alpha\,\sigma_\beta\big|+\sum_i\left(\sum_{\{i_\alpha\,;\,i_\alpha=i\}}|\sigma_\alpha|-\Bigg|\sum_{\{i_\alpha\,;\,i_\alpha=i\}}\sigma_\alpha\Bigg|\right). \tag{2.2.23}$$

Since μ_ν^{IC} have a uniformly bounded total mass, by possibly taking a subsequence we can assume the weak convergence

$$\mu_\nu^{IC}\rightharpoonup\mu^{IC} \tag{2.2.24}$$

for some positive, purely atomic measure μ^{IC} on D. Call Θ the countable set of atoms of μ^{IC}, i.e. set

$$\Theta\doteq\{P\in D;\ \mu^{IC}(P)>0\}.$$

For every approximate solution u^ν taking values in Γ^ν, $\nu \geq 1$, and for any fixed $\varepsilon \geq 2^{-\nu}$, by an *$\varepsilon-$shock front of the $i-$th family* in u^ν we mean a polygonal line in D, with nodes $(\tau_0, x_0), ..., (\tau_N, x_N)$, having the following properties.

(I) The nodes (τ_h, x_h) are interaction points or lie on the boundary of D, and the sequence of times is increasing $\tau_0 < \tau_1 < \cdots < \tau_N$.

(II) Along each segment joining (τ_{h-1}, x_{h-1}) with (τ_h, x_h), the function u^ν has an $i-$shock with strength $|\sigma_h| \geq \varepsilon$.

(III) For $h < N$, if two (or more) incoming $i-$shocks of strength $\geq \varepsilon$ interact at the node (τ_h, x_h), then the shock coming from (τ_{h-1}, x_{h-1}) has the larger speed, i.e. is the one coming from the left.

An $\varepsilon-$shock front which is maximal with respect to the set theoretical inclusion will be called a *maximal $\varepsilon-$shock front.* Observe that, because of (III), two maximal $\varepsilon-$shock fronts of the same family either are disjoint or coincide. Moreover, by (2.2.22), the number of maximal $\varepsilon-$shock front that starts at the boundary of D is uniformly bounded by $3M/\varepsilon$. On the other hand, the special geometric features of Temple class systems guarantee that no new shock front can arise in the interior of D. Indeed, the coinciding shock and rarefaction assumption together with the existence of Riemann invariants prevents the creation of shocks of other families than the ones of the incoming fronts at any interaction point. Therefore, for fixed $\varepsilon > 0$, and $i \in \{1, \ldots, n\}$, the number of maximal $\varepsilon-$shock front of the i-th family remains uniformly bounded by $M_\varepsilon \doteq 3M/\varepsilon$ in all u^ν, $\nu \geq 1$. Denote such curves by

$$y^\varepsilon_{\nu,m} : [t^{\varepsilon,-}_{\nu,m},\ t^{\varepsilon,+}_{\nu,m}] \to \mathbb{R}, \qquad m = 1, ..., M_\varepsilon \,.$$

By possibly extracting a further subsequence, we can assume the convergence

$$y^\varepsilon_{\nu,m}(\cdot) \longrightarrow y^\varepsilon_m(\cdot), \qquad t^{\varepsilon,\pm}_{\nu,m} \longrightarrow t^{\varepsilon,\pm}_m, \qquad m = 1, ..., M_\varepsilon \,,$$

for some Lipschitz continuous paths $y^\varepsilon_m : [t^{\varepsilon,-}_m,\ t^{\varepsilon,+}_m] \to \mathbb{R}$, $m = 1, ..., M_\varepsilon$. Repeating this construction in connection with a sequence $\varepsilon_k \to 0$, and taking the union of all the paths thus obtained, we find, for each characteristic family $i \in \{1, \ldots, n\}$, a countable family of Lipschitz continuous curves $y_m : [t^-_m, t^+_m] \to \mathbb{R}$, $m \in \mathbb{N}$. Call Υ the union of all such curves.

2. Consider now a point $P = (\tau, y_m(\tau)) \notin \Theta$ along a curve $y_m \in \Upsilon$ of a family $i \in \{1, \ldots, n\}$. Notice that, by construction, and because of (2.2.24), no curve in Υ can cross y_m at P. Moreover, by (2.2.22), the function $u(\tau, \cdot)$ has bounded variation, and hence there exist the limits

$$\lim_{x \to y_m(\tau)-} u(\tau, x) \doteq u^-, \qquad \lim_{x \to y_m(\tau)+} u(\tau, x) \doteq u^+. \tag{2.2.25}$$

We claim that also the limits (2.1.36) exist, and thus coincide with those in (2.2.25). To this end observe that, by construction, there exist a sequence of shocks curves $y_{\nu,m}$ of the i-th family converging to y_m, along which each approximate solution u^ν has a jump of strength $\geq \varepsilon^*$, for some $\varepsilon^* > 0$. Then, relying on the assumption

$$\mu^{IC}(\{P\}) = 0\,, \tag{2.2.26}$$

and letting $B(P,r)$ denote the ball centered at P with radius r, one can establish the limits

$$\lim_{r\to 0+} \limsup_{\nu\to+\infty} \left(\sup_{\substack{(t,x)\in B(P,r)\\ x<y_{\nu,m}(t)}} \left| u^\nu(t,x) - u^- \right| \right) = 0, \tag{2.2.27}$$

$$\lim_{r\to 0+} \limsup_{\nu\to+\infty} \left(\sup_{\substack{(t,x)\in B(P,r)\\ x>y_{\nu,m}(t)}} \left| u^\nu(t,x) - u^- \right| \right) = 0, \tag{2.2.28}$$

which clearly yield (2.1.36). Indeed, if for example (2.2.27) do not hold, by possibly taking a subsequence we would find $\varepsilon > 0$ and points $P_\nu \doteq (t_\nu,\, \xi_\nu) \to P$ on the left of $y_{\nu,m}$ such that

$$\left| u^\nu(t_\nu,\, \xi_\nu) - u^- \right| \geq \varepsilon \qquad \forall\ \nu.$$

On the other hand, by the first limit in (2.2.25), and since $u^\nu(\tau,x) \to u(\tau,x)$ for a.e. $x \in [\alpha,\beta]$, we could also find points $Q_\nu \doteq (\tau,\, \xi'_\nu) \to P$ on the left of $y_{\nu,m}$ such that

$$u^\nu(\tau,\, \xi'_\nu) \to u^-, \qquad\qquad \frac{|\xi_\nu - \xi'_\nu|}{|t_\nu - \tau|} > \lambda^{\max} \qquad \forall\ \nu\,,$$

where $\lambda^{\max}$ denotes the maximum speed at (2.1.3). But then, for each solution u^ν, the segment $\overrightarrow{P_\nu\, Q_\nu}$ would be crossed by an amount of waves of strength $\geq \varepsilon$. Hence, by strict hyperbolicity and genuine nonlinearity, this would generate a uniformly positive amount of interaction and cancellation within an arbitrary small neighborhood of P (see. [12, Theorem 10.4-Step 5]) which, by the definition (2.2.23), and because of (2.2.24), contradicts the assumption (2.2.26).

To complete the proof of (i) observe that, by construction, the states $u^-_{\nu,m}(\tau)$, $u^+_{\nu,m}(\tau)$ to the left and to the right of the jump in u^ν at $y_{\nu,m}(\tau)$ satisfy the Rankine Hugoniot conditions. Thus, relying on (2.2.27)-(2.2.28), and on the convergence $y_{\nu,m} \to y_\nu$, one deduces (2.1.37). The proof of (ii) can be established with the same type of arguments (see. [12, Theorem 10.4-Step 8]). □

As an immediate consequence of Lemma 2.2.4, we derive Proposition 2.1.1 stated in Section 2.1.3.

Proof of Proposition 2.1.1. Consider a sequence $u^\nu(t,\cdot) : [a,b] \to \Gamma^\nu$ of front tracking approximate solutions of the mixed problem for (2.0.1) on the region $\Omega_T \doteq [0,T]\times[a,b]$, that converges in L^1, as $\nu\to\infty$, to some function $u(t,\cdot) : [a,b] \to \Gamma$, for every $t \in [0,T]$. Observe that, by Theorem 2.1.3 one can find another sequence $\{v^\nu\}_{\nu\geq 1}$ of approximate solutions of (2.0.1) on the region Ω_T, whose initial and boundary data have a number of shocks $N_\nu \leq \nu$ for each characteristic family, and such that

$$\left\| u^\nu(t,\cdot) - v^\nu(t,\cdot) \right\|_{L^1([a,b])} \leq 1/\nu \qquad \forall\ t \in [1/\nu,\, T]\,.$$

Then, thanks to the Oleinik estimates (2.1.25)-(2.1.26), and because all v^ν take values in the compact set (2.1.13), there will be, for every fixed $\varepsilon > 0$, some constant $M_\varepsilon > 0$ such that

$$\left.\begin{aligned} \text{Tot.Var.}\big\{v^\nu(t,\,\cdot)\ ;\ [a+\varepsilon,\,b-\varepsilon]\big\} &\le M_\varepsilon \qquad \forall\, t \in [\varepsilon,\,T]\,,\\ \text{Tot.Var.}\big\{v^\nu(\cdot,\,x)\ ;\ [\varepsilon,\,T]\big\} &\le M_\varepsilon \qquad \forall\, x \in [a+\varepsilon,\,b-\varepsilon]\,, \end{aligned}\right. \qquad \forall\, \nu \in \mathbb{N}. \tag{2.2.29}$$

Thus, writing Ω_T as the countable union

$$\Omega_T = \cup_k D_k, \qquad D_k \doteq [1/k,\,T] \times [a + (1/k),\, b - (1/k)],$$

and applying Lemma 2.2.4 to each sequence of maps $v_k^\nu \doteq v^\nu \restriction_{D_k}$, $\nu \ge 1$, defined as the restriction of v^ν to the domain D_k, we clearly reach the conclusion of Proposition 2.1.1. □

We are now in the position to establish Theorem 2.1.5, relying on Proposition 2.1.1 and on Theorem 2.1.4.

Proof of Theorem 2.1.5. Let $u^\nu(t,\cdot) : [a,b] \to \Gamma^\nu$ be a sequence of front tracking approximate solutions of the mixed problem for (2.0.1) on the region $\Omega_T \doteq [0,T] \times [a,b]$, that converges in L^1, as $\nu \to \infty$, to some function $u(t,\cdot) : [a,b] \to \Gamma$, for every $t \in [0,T]$. Since, by construction, each u^ν is a weak solution of (2.0.1), and because $u^\nu(0,\cdot) \to u(0,\cdot) = \overline{u}$, also the limit function u is a weak solution of the Cauchy problem (2.0.1)-(2.0.2) on the region Ω_T. Moreover, applying Proposition 2.1.1, we deduce that u admits at $t = 0$ and at $x = a$, $x = b$ the left and right limits stated in Theorem 2.1.5. On the other hand, by the same arguments used in the proof of Proposition 2.1.1, we may assume that the initial and boundary data of each approximate solution u^ν have at most $N_\nu \le \nu$ shocks for every characteristic family. Then, letting $\nu \to \infty$ in (2.1.25)-(2.1.26), by the lower semicontinuity of the total variation we find that u satisfies the entropy conditions (2.1.7)-(2.1.10) on the decay of positive waves. It follows that u is an entropy weak solution of the mixed problem (2.0.1)-(2.0.4) according with Definition 2.1.2. Hence, observing that by construction the map $(t,x) \mapsto \big(u(t,\cdot),\, u(\cdot,x)\big)$ takes values within the domain $\mathcal{D}_T$ defined in (2.1.31), and applying Theorem 2.1.4, we deduce that (2.1.35) is verified. □

2.3 Proof of Theorems 2.1.1-2.1.2

Proof of Theorem 2.1.1. We shall first prove that, for every fixed $\overline{\tau} > 0$, there exists some constant $\rho = \rho(\overline{\tau}) > 0$ so that (2.1.17) holds. Given $\widetilde{u}_a \in \mathcal{U}_\tau^\infty$, $\widetilde{u}_b \in \mathcal{U}_\tau^\infty$, $\tau \ge \overline{\tau}$, let $u = u(t,x)$ be an entropy weak solution of (2.0.1)-(2.0.4) on the region $[0,\tau] \times [a,b]$ according with Definition 2.1.2. Then, the Oleinik-type estimates (2.1.8) on the decay of positive waves imply that, for $i \in \{p+1, ..., n\}$, $\tau \geq \overline{\tau}$, and for a.e. $a < x < y < b$, there holds

$$\begin{aligned}
\frac{w_i(\tau,y) - w_i(\tau,x)}{y-x} &\leq C \cdot \left\{ \frac{y-x}{\tau} + \log\left(\frac{y-a}{x-a}\right) \right\} \\
&\leq (b-a)\, C \cdot \left\{ \frac{1}{\overline{\tau}} + \frac{1}{x-a} \right\} \\
&\leq \frac{C\,(b-a)\big((b-a)+\overline{\tau}\big)}{\overline{\tau}} \cdot \frac{1}{x-a}.
\end{aligned} \tag{2.3.1}$$

Clearly, with the same computations, relying on the Oleinik-type estimates (2.1.7), we deduce that, for $i \in \{1, ..., p\}$, $\tau \geq \overline{\tau}$, and for a.e. $a < x < y < b$, there holds

$$\frac{w_i(\tau,y) - w_i(\tau,x)}{y-x} \leq \frac{C\,(b-a)\big((b-a)+\overline{\tau}\big)}{\overline{\tau}} \cdot \frac{1}{b-y}. \tag{2.3.2}$$

Hence, taking

$$\rho \geq \frac{C\,(b-a)\big((b-a)+\overline{\tau}\big)}{\overline{\tau}} \tag{2.3.3}$$

from (2.3.1)-(2.3.2) we derive $u(\tau,\cdot) \in K^{\rho}$, which proves (2.1.17).

Concerning the second statement of the theorem, we will show that, letting $\lambda^{\min}$, ρ', be the minimum speed in (2.1.3), and the constant (2.2.19) of Lemma 2.2.1, and taking

$$T \doteq \frac{4\,(b-a)}{\lambda^{\min}} \tag{2.3.4}$$

the relation (2.1.18) is verified, i.e. that, given $\varphi \in K^{\rho'}$, and $\tau > T$, there exist $\widetilde{u}_a \in \mathcal{U}_\tau^\infty$, $\widetilde{u}_b \in \mathcal{U}_\tau^\infty$, and a solution $u(t,x)$ of (2.0.1)-(2.0.4) on $[0,\tau] \times [a,b]$ (according with Definition 2.1.2), such that $u(\tau,\cdot) \equiv \varphi$. Notice that, by Remark 2.1.2, we may assume that $w_i(\varphi(x))$ admits left and right limits in any point $x \in]a, b[$, and that $w_i(\varphi(x)) = w_i(\varphi(x^+)) \doteq \lim_{\xi \to x^+} w_i(\varphi(\xi))$, for all $i \in \{1, \ldots, n\}$. The proof is divided in two steps.

Step 1. Backward construction of front tracking approximations. Letting $\rho' > 0$ be the constant in (2.2.19), consider a sequence $\{\varphi_\nu\}_{\nu \geq 1}$ of (right continuous) piecewise constant maps in $K_\nu^{\rho'}$, satisfying the conditions a)-b) of Lemma 2.2.2, and take a piecewise constant approximation $\overline{u}^\nu : [a,b] \to \Gamma^\nu$ of the initial data $\overline{u}$, so that $\overline{u}^\nu \to \overline{u}$ in L^1. Given $\tau > T$ (T being the time defined in (2.3.4)), for each $\nu \geq 1$, we will construct here a front tracking solution $u^\nu(t,x)$ of (2.0.1) on the region $[0,\tau] \times [a,b]$, with initial data $u^\nu(0,\cdot) = \overline{u}^\nu$, so that

$$u^\nu(\tau, \cdot\,) = \varphi_\nu\,. \tag{2.3.5}$$

This goal is accomplished by proving the following two lemmas.

Lemma 2.3.1 *Let T, $\rho' > 0$ be the constants in (2.3.4) and (2.2.19). Then, for every (right continuous) $\varphi_\nu \in K_\nu^{\rho'}$, $\nu \geq 1$, satisfying the condition a) of Lemma 2.2.2, and for any $\tau > T$, there exists a front tracking solution $u^\nu(t,x)$ of (2.0.1) on the region $[(3/4)T,\, \tau] \times [a,b]$, with boundary data $\widetilde{u}_a^\nu \doteq u^\nu(\cdot,\, a)$, $\widetilde{u}_b^\nu \doteq u^\nu(\cdot,\, b) \in L^\infty([(3/4)T,\, \tau],\, \Gamma^\nu)$, so that*

$$u^\nu\big((3/4)T,\, x\big) \equiv \omega, \qquad u^\nu\big(\tau,\, x\big) = \varphi_\nu(x)\,, \qquad \forall\, x \in [a,b]\,, \tag{2.3.6}$$

for some constant state $\omega \in \Gamma^\nu$.

PROOF. Given $\tau > T$, and $\varphi_\nu \in K_\nu^{\rho'}$, $\nu \geq 1$, satisfying the condition a) of Lemma 2.2.2, we will use the algorithm described in Section 2.1.3 to construct backward in time a front tracking solution that takes value φ_ν at time τ. To this end, we first observe that according with the algorithm of Section 2.1.3 we can always construct the backward solution of a Riemann problem with terminal data

$$u(t,x) = \begin{cases} u^L & \text{if} \quad x < \xi\,, \\ u^R & \text{if} \quad x > \xi\,, \end{cases} \tag{2.3.7}$$

if the the terminal states u^L, $u^R \in \Gamma^\nu$ have Riemann coordinates

$$w(u^L) \doteq w^L = (w_1^L, \dots, w_n^L), \qquad w(u^R) \doteq w^R = (w_1^R, \dots, w_n^R)$$

that satisfy

$$w_i^L < w_i^R \qquad \Longrightarrow \qquad w_i^R = w_i^L + 2^{-\nu} \qquad \forall\, i\,. \tag{2.3.8}$$

Indeed, if we consider the intermediate states

$$z^i = \begin{cases} u^L & \text{if} \quad i = 0\,, \\ u(w_1^L, \dots, w_{n-i}^L\,, w_{n-i+1}^R, \dots, w_n^R) & \text{if} \quad 0 < i < n\,, \\ u^R & \text{if} \quad i = n\,, \end{cases} \tag{2.3.9}$$

we realize that, because of (2.3.8), the solution of every Riemann problem with initial states (z^{i-1}, z^i) (defined as in Section 2.1.3) contains only a single front. Thus, we can construct the solution to the Riemann problem with terminal data (2.3.7) in a backward neighborhood of (t, ξ) by piecing together the solutions to the simple Riemann problems (z^{i-1}, z^i), $i = 1, \dots, n$.

A front tracking solution u^ν can now be constructed backward in time starting at $t = \tau$, and piecing together the backward solutions of the Riemann problems determined by the jumps in φ_ν. The resulting piecewise constant function $u^\nu(\tau-, \cdot)$ is then prolonged for $t < \tau$ tracing backward the incoming fronts at $t = \tau$, up to the first time $\tau_1 < \tau$ at which two or more discontinuities cross in the interior of Ω. Observe that, since u^ν is a front tracking solution constructed by the algorithm of Section 2.1.3 on the region $[\tau_1, \tau] \times [a, b]$, we can apply Lemma 2.2.3. Hence, it follows that the left and right states of the jumps occurring in $u^\nu(\tau_1, \cdot)$ satisfy condition (2.3.8), because (2.2.20) guarantees that two (or more) adjacent rarefaction fronts of the same family cannot cross at time τ_1. We then solve backward the resulting Riemann problems applying again the above procedure. This determines the solution $u^\nu(t, \cdot)$ until the time $\tau_2 < \tau_1$ at which another intersection between its fronts takes place in the interior of Ω, and so on (see Figure 2.1a).

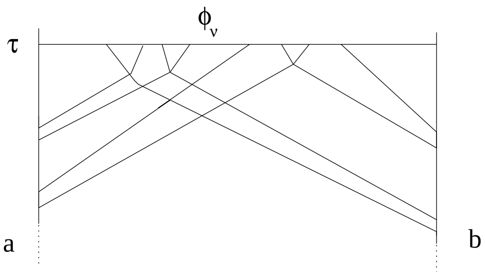

Figure 2.1a

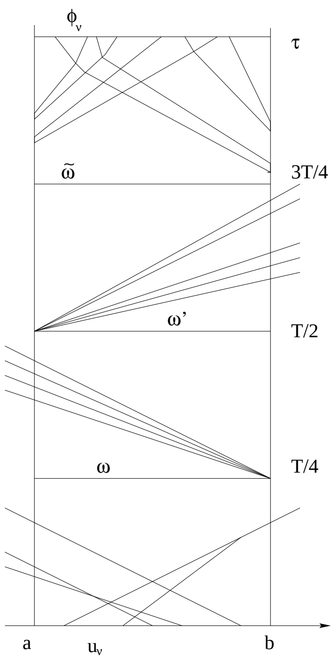

Figure 2.1b

With this construction we define a front tracking solution $u^\nu(t,x)$ on the whole region $[(3/4)T,\ \tau]\times[a,b]$, that verifies the first equality in (2.3.6), and corresponds to the boundary data $\widetilde{u}_a^\nu \doteq u^\nu(\cdot,a)$, $\widetilde{u}_b^\nu \doteq u^\nu(\cdot,b) \in L^\infty([(3/4)T,\ \tau],\ \Gamma^\nu)$. Clearly, the total number of wave-fronts in $u^\nu(t,\ \cdot)$ decreases, as $t \downarrow (3/4)T$, whenever a (backward) front crosses the boundary points $x=a$, $x=b$. Since (2.1.3) implies that the maximum time taken by fronts of u^ν to cross the interval $[a,b]$ is $(b-a)/\lambda^{\min}$, the definition (2.3.4) of T guarantees that all the (backward) fronts of u^ν will hit the boundaries $x=a$, $x=b$ within some time $\tau' \in](3/4)T,\ \tau[$, which shows that also the second equality in (2.3.6) is verified, thus completing the proof. □

Lemma 2.3.2 *Let $T>0$ be the constant in (2.3.4). Then, for any piecewise constant function $\overline{u}^\nu \in L^\infty([a,b],\Gamma^\nu)$, and for every state $\omega \in \Gamma^\nu$, there exists a front tracking solution $u^\nu(t,x)$ of (2.0.1) on the region $[0,\ (3/4)\,T] \times [a,b]$, corresponding to some boundary data $\widetilde{u}_a^\nu,\ \widetilde{u}_b^\nu \in L^\infty([0,\ (3/4)\,T],\Gamma^\nu)$, so that*

$$u^\nu\big(0,\ x\big) = \overline{u}^\nu(x), \qquad u^\nu\big((3/4)\,T,\ x\big) \equiv \omega\,, \qquad \forall\ x \in [a,b]\,. \tag{2.3.10}$$

Proof. The approximate solution u^ν is constructed as follows. By Remark 2.1.3, for $t \in [0, T/4]$, we can define $u^\nu(t,x)$ as the restriction to the region $[0,T/4]\times[a,b]$ of the front tracking solution to the Cauchy problem for (2.0.1), with initial data

$$\overline{u}(x) = \begin{cases} \overline{u}^\nu(a+) & \text{if} \quad x<a, \\ \overline{u}^\nu(x) & \text{if} \quad a \le x \le b, \\ \overline{u}^\nu(b-) & \text{if} \quad x>b, \end{cases}$$

(constructed as in [16] with the same type of algorithm described in Section 2.1.3). Observe that, since u^ν contains only fronts originated at the points of the segment $\{(0,x)\,;\, x\in[a,b]\}$, because of (2.1.3), (2.3.4) these wave-fronts cross the whole interval $[a,b]$ and exit from the boundaries $x=a$, $x=b$ before time $T/4$ (see Figure 2.1b). Hence, there will be some state $\omega'\in\Gamma^\nu$ such that

$$u^\nu\big(T/4,\,x\big)\equiv\omega' \qquad \forall\, x\in[a,b]\,. \tag{2.3.11}$$

Thus, introducing the intermediate state

$$\widetilde{\omega}\doteq\big(\omega_1,\dots,\omega_p,\,\omega'_{p+1},\dots,\omega'_n\big)$$

between ω' and ω, we will define $u^\nu(t,x)$, for $t\in[T/4,\,T/2]$, as the restriction to the region $[T/4,T/2]\times[a,b]$ of the approximate solution to the Riemann problem for (2.0.1), with initial data

$$u^\nu(T/4,x)=\begin{cases} u(\omega') & \text{if} \quad x<b\,,\\ u(\widetilde{\omega}) & \text{if} \quad x>b\,,\end{cases} \tag{2.3.12}$$

while, for $t\in[T/2,\,(3/4)T]$, we will let $u^\nu(t,x)$ be the restriction to the region $[T/2,\,(3/4)T]\times[a,b]$ of the approximate solution to the Riemann problem for (2.0.1), with initial data

$$u^\nu(T/2,x)=\begin{cases} u(\omega) & \text{if} \quad x<a\,,\\ u(\widetilde{\omega}) & \text{if} \quad x>a\,.\end{cases} \tag{2.3.13}$$

By the definition of $\widetilde{\omega}$, and because of (2.1.3), (2.3.4), on $[T/4,\,T/2]$ the solution of the Riemann problems with initial data (2.3.12) contains only wave-fronts originated at the point $(T/4,\,b)$, that cross the whole interval $[a,b]$ and exit from the boundary $x=a$ before time $T/2$. Similarly, still by (2.1.3), (2.3.4), for $t\in[T/2,\,(3/4)T]$ the solution of the Riemann problem with initial data (2.3.13), contains only wave-fronts originated at $(T/2,a)$, that cross the whole interval $[a,b]$, and exit from the boundary $x=b$ before time $(3/4)T$ (see Figure 2.1b). Hence, $u^\nu(t,x)$ is a front-tracking solution defined on the whole region $[0,\,(3/4)\,T]\times[a,b]$, that corresponds to the boundary data $\widetilde{u}^\nu_a\doteq u^\nu(\cdot,a)$, $\widetilde{u}^\nu_b\doteq u^\nu(\cdot,b)\in L^\infty([0,\,(3/4)\,T],\Gamma^\nu)$, and verifies the conditions (2.3.10). □

Step 2. Convergence of the approximate solutions. By Step 1, for a given $\varphi\in K^{\rho'}$ (with ρ' as in (2.2.19)), we have found a sequence of initial data $\overline{u}^\nu$, and of boundary data $\widetilde{u}^\nu_a,\,\widetilde{u}^\nu_b\in\mathcal{U}^\infty_\tau$, so that, letting $u^\nu(\tau,\cdot)\doteq E^\nu_\tau(\overline{u}^\nu,\,\widetilde{u}^\nu_a,\,\widetilde{u}^\nu_b)$ be the corresponding front tracking solution, there holds

$$\overline{u}^\nu\to\overline{u},\qquad\quad u^\nu(\tau,\cdot)\to\varphi\qquad\text{in}\qquad L^1([a,b]). \tag{2.3.14}$$

By the same arguments used in the proof of Proposition 2.1.1, we may assume that the initial and boundary data of each approximate solution u^ν have at most $N_\nu\le\nu$ shocks for every characteristic family. Then, thanks to the Oleinik-type estimates (2.1.25), and because u^ν are uniformly bounded since they take values in the compact set (2.1.13), for every fixed $\varepsilon>0$, there will be some constant $C_\varepsilon>0$ such that

$$\left.\begin{aligned} &\text{Tot.Var.}\big\{u^\nu(t,\cdot)\,;\,[a+\varepsilon,\,b-\varepsilon]\big\}\le C_\varepsilon \qquad \forall\, t\in[\varepsilon,\tau]\,,\\ &\int_{a+\varepsilon}^{b-\varepsilon}|u^\nu(t,x)-u^\nu(s,x)|\,dx\le C_\varepsilon|t-s| \qquad \forall\, t,s\in[\varepsilon,\tau]\,,\end{aligned}\right\}\quad \forall\,\nu\in\mathbb{N}. \tag{2.3.15}$$

Hence, applying Helly's Theorem, we deduce that there exists a subsequence $\{u^{\nu_j}\}_{j\geq 0}$ that converges in $L^1([a,b],\,\Gamma)$ to some function $u_\varepsilon(t,\cdot)$, for any $t\in[\varepsilon,\,\tau]$. Therefore, repeating the same construction in connection with a sequence $\varepsilon_k\to 0+$, and using a diagonal procedure, we obtain a subsequence $\{u^{\nu'}(t,\cdot)\}_{\nu'\geq 0}$ that converges in $L^1([a,b],\Gamma)$ to some function $u(t,\cdot)$, for any $t\in[0,\,\tau]$. Then, by Theorem 2.1.5, there holds (2.1.35), with $\widetilde{u}_a\doteq u(\cdot,a)$, $\widetilde{u}_b\doteq u(\cdot,b)\in\mathcal{U}^\infty_\tau$, while (2.3.14) implies $u(\tau,\cdot)=\varphi$, which shows $\varphi\in\mathcal{A}(\tau)$. This completes the proof of Theorem 2.1.1. □

We next establish the compactness of the attainable set (2.1.14) stated in Theorem 2.1.2. The proof is quite similar to that of [6, Theorem 2.3]. We repeat it for completeness.

Proof of Theorem 2.1.2. Fix $T>0$, and consider a sequence $\{u^\nu\}_{\nu\geq 0}$ of entropy weak solutions to the mixed problem for (2.0.1) on $\Omega_T\doteq[0,T]\times[a,b]$ (according with Definition 2.1.2), with a fixed initial data $\overline{u}\in L^\infty([a,b],\Gamma)$. Since all u^ν are uniformly bounded, and because of the Oleinik-type estimates (2.1.7)-(2.1.8), one can find, for every $\varepsilon>0$, some constant $C_\varepsilon>0$ so that (2.3.15) holds. Thus, with the same arguments used in **Step 2** of the previous proof, we can construct a subsequence $\{u^{\nu'}\}_{\nu'\geq 0}$ so that, for any $t\in[0,\,T]$, $u^{\nu'}(t,\cdot)$ converges in L^1 to some function $u(t,\cdot)$, which is continuous as a map from $]0,T]$ into $L^1([a,b],\Gamma)$, and satisfies the entropy conditions (2.1.7)-(2.1.10) on the decay of positive waves. On the other hand, the weak traces $\Psi_a^{\nu'}$, $\Psi_b^{\nu'}$ of the fluxes $f(u^{\nu'})$ at the boundaries $x=a$, $x=b$ are uniformly bounded, and hence are weak* relatively compact in $L^\infty([0,T])$. Thus, by possibly taking a further subsequence, we have

$$\Psi_a^{\nu'}\overset{*}{\rightharpoonup}\Psi_a\,,\qquad \Psi_b^{\nu'}\overset{*}{\rightharpoonup}\Psi_b\qquad\text{in}\qquad L^\infty([0,T]),\tag{2.3.16}$$

for some maps $\Psi_a,\ \Psi_b\ \in L^\infty([0,T])$. Notice that, by the properties of the Riemann invariants, the set $f(\Gamma)$ is closed and convex, and hence also the weak limits $\Psi_a,\ \Psi_b$ take values in $f(\Gamma)$. Moreover, since each u^ν is a distributional solution of (2.0.1)-(2.0.2) on Ω_T, also the limit function u is a distributional solution of the Cauchy problem (2.0.1)-(2.0.2) on the region Ω_T. Then, setting $\widetilde{u}_a\doteq f^{-1}\circ\Psi_a$, $\widetilde{u}_b\doteq f^{-1}\circ\Psi_b$, it follows that u is an entropy weak solution of the mixed problem (2.0.1)-(2.0.4) (with boundary data in $\mathcal{U}^\infty_T$) according with Definition 2.1.2, which shows that $u(T,\cdot)\in\mathcal{A}(T)$. This completes the proof of Theorem 2.1.2. □

If we take in consideration only solutions to the mixed problem (2.0.1)-(2.0.4) that are trajectories of the flow map E obtained in Theorem 2.1.3 (which, in particular, admit a strong L^1 trace at the boundaries $x=a,x=b$), we are lead to study the set of attainable profiles

$$\mathcal{A}_E(T)\doteq\big\{E_T(\overline{u},\,\widetilde{u}_a\,,\,\widetilde{u}_b\,)\ ;\qquad \widetilde{u}_a,\,\widetilde{u}_b\in L^\infty([0,T],\Gamma)\big\}\,.\tag{2.3.17}$$

Since $\mathcal{A}_E(T)\subset\mathcal{A}(T)$, and by the proof of Theorem 2.1.1, it clearly follows that the characterization of the set $\mathcal{A}(T)$ provided by the inclusions (2.1.17)-(2.1.18) of Theorem 2.1.1 holds also for $\mathcal{A}_E(T)$. Concerning the compactness of the set $\mathcal{A}_E(T)$, observe that, given any sequence of exact solutions $u^\nu(t,\cdot) \doteq E_t(\overline{u}^\nu, \widetilde{u}_a^\nu, \widetilde{u}_b^\nu)$, $\nu \geq 1$, by Theorem 2.1.3 one can find another sequence of approximate solutions $v^\nu(t,\cdot)$ constructed by the front tracking algorithm of Section 2.1.3, so that

$$\left\|u^\nu(t,\cdot) - v^\nu(t,\cdot)\right\|_{L^1([a,b])} \leq 1/\nu \qquad \forall\, t \in [1/\nu, T].$$

Therefore, relying on the regularity property of a solution obtained as limit of front tracking approximations provided by Theorem 2.1.5, with the same arguments used in the proof of Theorem 2.1.2 one can establish also the compactness of the set $\mathcal{A}_E(T)$.

Theorem 2.3.1 *Under the same assumptions of Theorem 1, the set $\mathcal{A}_E(T)$ is a compact subset of $L^1([a,b],\Gamma)$ for each $T > 0$.*

Chapter 3

A Counterexample to Exact Controllability

An interesting question is whether the constant states can be exactly reached, in finite time. By the results in [4, 3, 7, 35], this is indeed the case Temple class systems, scalar conservation laws with convex flows and general quasilinear systems when the initial data have small $\mathcal{C}^1$ norm, respectively. On the contrary, in this chapter, we show that exact controllability in finite time cannot be attained in general, if the initial data is only assumed to be small in BV.

Our counterexample is concerned with a class of strictly hyperbolic, genuinely nonlinear 2×2 systems of the form (1.0.1). More precisely, we assume the following.

(H) The eigenvalues $\lambda_1(u)$, $\lambda_2(u)$ of the Jacobian matrix $A(u) = Df(u)$ satisfy

$$-\lambda^* < \lambda_1(u) < -\lambda_* \; < 0 \; < \lambda_* < \lambda_2(u) < \lambda^*, \tag{3.0.1}$$

for some constants $0 < \lambda_* < \lambda^*$. Moreover, the right eigenvectors $r_1(u)$, $r_2(u)$ satisfy the inequalities

$$D\lambda_1 \cdot r_1 > 0, \qquad\qquad D\lambda_2 \cdot r_2 > 0, \tag{3.0.2}$$

$$r_1 \wedge r_2 < 0, \qquad r_1 \wedge (Dr_1 \cdot r_1) < 0, \qquad r_2 \wedge (Dr_2 \cdot r_2) < 0. \tag{3.0.3}$$

Here $D\lambda_i$, Dr_i denote the differentials of the functions $\lambda_i(u)$, $r_i(u)$, while $\wedge$ is the wedge product: if $v = (v_1, v_2)$, $w = (w_1, w_2)$, we define

$$v \wedge w \doteq v_1 w_2 - v_2 w_1 \,.$$

A particular system which satisfies the above assumptions is the one studied by DiPerna [27]:

$$\begin{cases} \rho_t + (u\rho)_x & = 0, \\ u_t + \left(\dfrac{u^2}{2} + \dfrac{K^2}{\gamma - 1}\rho^{\gamma-1}\right)_x & = 0, \end{cases}$$

with $1 < \gamma < 3$. Here $\rho > 0$ and u denote the density and the velocity of a gas, respectively and this system describes the evolution of a gas in presence of explosions.

The last two inequalities in (3.0.3) imply that the rarefaction curves (i.e. the integral curves of the vector fields r_1, r_2) in the (u_1, u_2) plane turn clockwise (Figure 3.1). In such case, the interaction of two shocks of the same family generates a shock in the other family.

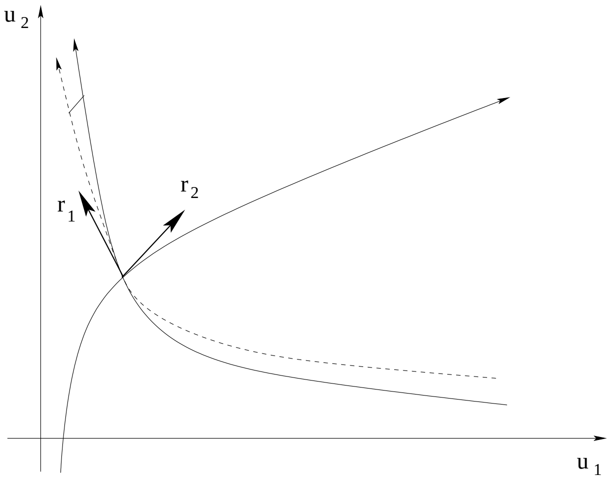

Figure 3.1

The main result of this chapter is the following (see [14]).

Theorem 3.0.2 *Consider a 2×2 system satisfying the assumption (H). Then there exist initial data $\phi : [a, b] \to \mathbb{R}^2$ having arbitrarily small total bounded variation for which the following holds. For every entropy weak solution u of (1.0.1), (1.0.5), with Tot.Var.$\{u(t, \cdot)\}$ remaining small for all t, the set of shocks in $u(t, \cdot)$ is dense on $[a, b]$, for each $t > 0$. In particular, $u(t, \cdot)$ cannot be constant.*

As a preliminary, in Section 3.1 we establish an Oleinik-type estimate on the decay of positive waves. This bound is of independent interest, and sharpens the results in [15], for systems satisfying the additional conditions (H).

As a consequence, this implies that positive waves are "weak", and cannot completely cancel a shock within finite time. The proof of Theorem 3.0.2 is then achieved by an induction argument. We show that, if the set of 1-shocks is dense on $[0, T] \times [a, b]$, then the set of points $P_j = (t_j, x_j)$ where two 1-shocks interact and create a new 2-shock is also dense on the same domain. Therefore, new shocks are constantly generated, and the solution can never be reduced to a constant. Details of the proof will be given in Section 3.2.

Throughout the following, we denote by $r_i(u)$, $l_i(u)$ the right and left i-eigenvectors of the Jacobian matrix $A(u) \doteq Df(u)$. As in [12], we write $\sigma \mapsto R_i(\sigma)(u_0)$ for the parametrized i-rarefaction curve through the state u_0, so that

$$\frac{d}{d\sigma} R_i(\sigma) = r_i\big(R_i(\sigma)\big), \qquad R_i(0) = u_0.$$

The i-shock curve through u_0 is denoted by $\sigma \mapsto S_i(\sigma)(u_0)$. It satisfies the Rankine-Hugoniot equations

$$f\big(S_i(\sigma)\big) - f(u_0) = \lambda_i(\sigma)\,\big(S_i(\sigma) - u_0\big),$$

for some shock speed λ_i. We recall (see [12, Chapter 5]) that the general Riemann problem is solved in terms of the composite curves

$$\Psi_i(u_0)(\sigma) = \begin{cases} R_i(u_0)(\sigma), & \text{if} \quad \sigma \geq 0, \\ S_i(u_0)(\sigma), & \text{if} \quad \sigma < 0. \end{cases} \tag{3.0.4}$$

3.1 Decay of Positive Waves

Throughout the following, we consider a 2×2 system of conservation laws

$$u_t + f(u)_x = 0\,, \tag{3.1.5}$$

satisfying the assumptions (H). Following [25], p. 128, we construct a set of Riemann coordinates (w_1, w_2). One can then choose the right eigenvectors of $Df(u)$ so that

$$r_i(u) = \frac{\partial u}{\partial w_i}\,, \qquad \frac{\partial \lambda_i}{\partial w_i} = D\lambda_i \cdot r_i > 0, \qquad i = 1, 2. \tag{3.1.6}$$

It will be convenient to perform most of the analysis on a special class of solutions: piecewise Lipschitz functions with finitely many shocks and no compression waves. Due to the geometric structure of the system, this set of functions turns out to be positively invariant for the flow generated by the hyperbolic system. We first derive several a priori estimate concerning these solutions, in particular on the strength and location of the shocks. We then observe that any BV solution can be obtained as limit of a sequence of piecewise Lipschitz solutions in our special class. Our estimates can thus be extended to general BV solutions.

Definition 3.1.1 *We call $\mathcal{U}$ the set of all piecewise Lipschitz functions $u : \mathbb{R} \to \mathbb{R}^2$ with finitely many jumps, such that:*

(i) at every jump, the corresponding Riemann problem is solved only in terms of shocks (no centered rarefactions);

(ii) no compression waves are present, i.e.: $w_{i,x}(x) \geq 0$ at almost every $x \in \mathbb{R}$, $i = 1, 2$.

The next lemma establishes the forward invariance of the set $\mathcal{U}$.

Lemma 3.1.1 *Consider the 2×2 system of conservation laws (3.1.5), satisfying the assumptions (H). Let $u = u(t, x)$ be the solution to a Cauchy problem, with small total variation, satisfying $u(0, \cdot) \in \mathcal{U}$. Then*

$$u(t, \cdot) \in \mathcal{U}, \tag{3.1.7}$$

for all $t \geq 0$.

PROOF. We have to show that, as time progresses, the total number of shocks does not increase and no compression wave is ever formed. This will be the case provided that

(i) the interaction of two shocks of the same family produces an outgoing shock of the other family;

(ii) the interaction of a shock with an infinitesimal rarefaction wave of the same family produces a rarefaction wave in the other family.

Both of the above conditions can be easily checked by analyzing the relative positions of shocks and rarefaction curves. We will do this for the first family, leaving the verification of the other case to the reader.

Call $\sigma \mapsto R_1(\sigma)$ the rarefaction curve through a state u_0, parametrized so that

$$\lambda_1\big(R_1(\sigma)\big) = \lambda_1(u_0) + \sigma\,.$$

It is well known that the shock curve through u_0 has a second order tangency with this rarefaction curve. Hence there exists a smooth function $c_1(\sigma)$ such that the point

$$S_1(\sigma) \doteq R_1(\sigma) + c_1(\sigma)\frac{\sigma^3}{6} r_2(u_0)$$

lies on this shock curve, for all σ in a neighborhood of zero. From the Rankine-Hugoniot equations it now follows

$$\chi(\sigma) \doteq \Big(f\big(R_1(\sigma)+c_1(\sigma)(\sigma^3/6)\,r_2(u_0)\big)-f(u_0)\Big)\wedge\Big(R_1(\sigma)+c_1(\sigma)(\sigma^3/6)\,r_2(u_0)-u_0\Big) = 0\,. \tag{3.1.8}$$

Differentiating the wedge product (3.1.8) four times at $\sigma = 0$ and denoting derivatives with upper dots, we obtain

$$\begin{aligned}
\frac{d^4\chi}{d\sigma^4}(0) \quad &= 4\big[\lambda_1(u_0)\dddot{R}_1(0) + 2\ddot{R}_1(0) + \lambda_2(u_0)\,c_1(0)r_2(u_0)\big]\wedge \dot{R}_1(0)\\
&\qquad +6\big[\lambda_1(u_0)\ddot{R}_1(0) + \dot{R}_1(0)\big]\wedge\ddot{R}_1(0) + 4\lambda_1(u_0)\dot{R}_1(0)\wedge\big[\dddot{R}_1(0) + c(0)r_2(u_0)\big]\\
&= 4\big(\lambda_2(u_0) - \lambda_1(u_0)\big)c_1(0)\,r_2(u_0)\wedge r_1(u_0) + 2(Dr_1\cdot r_1)(u_0)\wedge r_1(u_0)\\
&= 0
\end{aligned}$$

Hence, from (H),

$$c_1(0) = \frac{(Dr_1\cdot r_1)\wedge r_1}{2(\lambda_2-\lambda_1)(r_1\wedge r_2)} < 0\,. \tag{3.1.9}$$

By (3.1.9), the relative position of 1-shock and 1-rarefaction curves is as depicted in Figure 3.1. By the geometry of wave curves, the properties (i) and (ii) are now clear. Figure 3.2a illustrates the interaction of two 1-shocks, while Figure 3.2b shows the interaction between a 1-shock and a 1-rarefaction. By u_l, u_m, u_r we denote the left, middle and right states before the interaction, while u_m' is the middle state after the interaction. In the two cases, the solution of the Riemann problem contains a 2-shock and a 2-rarefaction, respectively. □

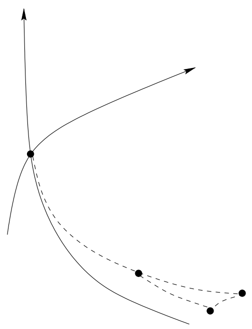

Figure 3.2a

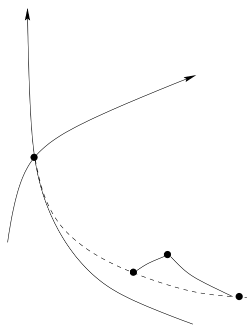

Figure 3.2b

The next lemma shows the decay of positive waves for solutions with small total variation, taking values inside $\mathcal{U}$.

Lemma 3.1.2 *Let $u = u(t,x)$ be a solution of the Cauchy problem for the 2×2 system (3.1.5) satisfying (H). Assume that*

$$u(t,\cdot) \in \mathcal{U}, \qquad t \geq 0. \tag{3.1.10}$$

Then there exist $\kappa, \delta > 0$ such that if Tot.Var.$(u(t,\cdot)) < \delta$ for all t, then its Riemann coordinates (w_1, w_2) satisfy

$$0 \leq w_{i,x}(t,x) \leq \frac{\kappa}{t}, \qquad t > 0,\ i = 1,2. \tag{3.1.11}$$

Proof. We consider the case $i = 1$. Fix any point $(\bar{t},\ \bar{x})$. Since centered rarefaction waves are not present, there exists a unique 1-characteristic through this point, which we denote as $t \mapsto x_1(t;\ \bar{t},\ \bar{x})$. It is the solution of the Cauchy problem

$$\dot{x}(t) = \lambda_1\big(u(t, x(t))\big), \qquad x(\bar{t}) = \bar{x}. \tag{3.1.12}$$

The evolution of $w_{1,x}$ along this characteristic is described by

$$\frac{d}{dt}w_{1,x}\big(t, x_1(t)\big) = w_{1,xt} + \lambda_1 w_{1,xx} = -(\lambda_1 w_{1,x})_x + \lambda_1 w_{1,xx} = -\frac{\partial\lambda_1}{\partial w_1}w_{1,x}^2 - \frac{\partial\lambda_1}{\partial\omega_2}w_{1,x}w_{2,x}.$$

Since the system is genuinely nonlinear there exists $k_1 > 0$ such that $\partial\lambda_1/\partial w_1 \geq k_1 > 0$, hence

$$\frac{d}{dt}w_{1,x}\big(t, x_1(t)\big) \leq -k_1 w_{1,x}^2 + \mathcal{O}(1)\cdot w_{1,x}w_{2,x}. \tag{3.1.13}$$

Moreover, at each time t_α where the characteristic crosses a 2-shock of strength $|\sigma_\alpha|$ we have the estimate

$$w_{1,x}(t_\alpha+) \leq \big(1 + \mathcal{O}(1)\cdot|\sigma_\alpha|\big)w_{1,x}(t_\alpha-). \tag{3.1.14}$$

Let $Q(t)$ be the total interaction potential at time t (see for example [12], p. 202) and let $V_2(t)$ be the total amount of 2-waves approaching our 1-wave located at $x_1(t)$. Repeating the arguments in [12], p.139, we can find a constant $C_0 > 0$ such that the quantity

$$\Upsilon(t) \doteq V_1(t) + C_0\, Q(t), \qquad t > 0,$$

is non-increasing. Moreover, for a.e. t one has

$$\dot\Upsilon(t) \leq -\big|\lambda_2-\lambda_1\big||w_{2,x}|\big(t,x_1(t)\big)\,,$$

while at times t_α where x_1 crosses a 2-shock of strength $|\sigma_\alpha|$ there holds

$$\Upsilon(t_\alpha-) \leq \Upsilon(t_\alpha+) - |\sigma_\alpha|\,.$$

Call $W(t) \doteq w_{1,x}\big(t,x_1(t)\big)$. By the previous estimates, from (3.1.13) and (3.1.14) it follows form

$$\dot W(t) \leq -k_1 W^2(t) - C\,\dot\Upsilon(t)W(t), \tag{3.1.15}$$

$$W(t_\alpha+) - W(t_\alpha-) \leq C\big[\Upsilon(t_\alpha+) - \Upsilon(t_\alpha-)\big]W(t_\alpha-), \tag{3.1.16}$$

for a suitable constant C. We now observe that

$$y(t) \doteq \frac{e^{-C\Upsilon(t)}}{\displaystyle\int_0^t k_1 e^{-C\Upsilon(s)}ds}$$

is a distributional solution of the equation

$$\dot y = -k_1 y^2 - C\,\dot\Upsilon(t)y\,,$$

with $y(t)\to\infty$ as $t\to 0+$. A comparison argument now yields $W(t)\leq y(t)$. Since Υ is positive and decreasing, we have

$$W(t) \leq \frac{1}{k_1}\frac{1}{\displaystyle\int_0^t e^{-C\Upsilon(s)}ds} \leq \frac{e^{C\Upsilon(0)}}{k_1 t},$$

for all $t>0$. This establishes (3.7) for $i=1$, with $\kappa \doteq e^{C\Upsilon(0)}/k_1$. The case $i=2$ is identical. □

We conclude this section by proving a decay estimate for positive waves, valid for general BV solutions of the system (3.1.5). For this purpose, we need to recall some definitions introduced in [15]. See also p. 201 in [12].

Let $u:\mathbb{R}\to\mathbb{R}^2$ have bounded variation. By possibly changing the values of u at countably many points, we can assume that u is right continuous. The distributional derivative $\mu\doteq D_x u$ is a vector measure, which can be decomposed into a continuous and an atomic part: $\mu=\mu_c+\mu_a$. For $i=1,2$, the scalar measures $\mu^i=\mu^i_c+\mu^i_a$ are defined as follows. The continuous part of μ^i is the Radon measure μ^i_c such that

$$\int \phi\; d\mu^i_c = \int \phi\, l_i(u)\cdot d\mu_c, \tag{3.1.17}$$

for every scalar continuous function ϕ with compact support. The atomic part of μ^i is the measure μ^i_a concentrated on the countable set $\{x_\alpha;\ \alpha=1,2,\ldots\}$ where u has a jump, such that

$$\mu^i_a\big(\{x_\alpha\}\big) = \sigma_{\alpha,i} \doteq E_i\big(u(x_\alpha-),\ u(x_\alpha+)\big) \tag{3.1.18}$$

is the size of the i-th wave in the solution of the corresponding Riemann problem with data $u(x_\alpha\pm)$. We regard μ^i as the *measure of i-waves* in the solution u. It can be decomposed in a positive and a negative part, so that

$$\mu^i = \mu^{i+} - \mu^{i-}, \qquad\qquad |\mu^i| = \mu^{i+} + \mu^{i-}. \tag{3.1.19}$$

The decay estimate in (3.1.11) can now be extended to general BV solutions. Indeed, we show that the density of positive i-waves decays as κ/t. By meas(J) we denote here the Lebesgue measure of a set J.

Lemma 3.1.3 *Let $u = u(t,x)$ be a solution of the Cauchy problem for the 2×2 system (3.1.5) satisfying (H). Then there exist $\kappa,\delta > 0$ such that if Tot.Var.$(u(t,\cdot)) < \delta$ for all t, then the measures μ_t^{1+}, μ_t^{2+} of positive waves in $u(t,\cdot)$ satisfy*

$$\mu_t^{i+}(J) \leq \frac{\kappa}{t}\, meas\,(J), \tag{3.1.20}$$

for every Borel set $J \subset \mathbb{R}$ and every $t > 0$, $i = 1,2$.

Proof. For every BV solution u of (3.1.5) we can construct a sequence of solutions u_ν with $u_\nu \to u$ as $\nu \to \infty$ and such that $u_\nu(t,\cdot) \in \mathcal{U}$ for all t. Calling $(w_1^\nu,\, w_2^\nu)$ the Riemann coordinates of u_ν, by Lemma 3.1.2, we have

$$0 \leq w_{i,x}^\nu(t,x) \leq \frac{\kappa}{t}, \qquad\qquad t > 0,\ i = 1,2, \qquad \nu \geq 1\,. \tag{3.1.21}$$

For a fixed $t > 0$, observe that the map $x \mapsto w_1^\nu(t,x)$ has upward jumps precisely at the points x_α where $u(t,\cdot)$ has a 2-shock. Define $\tilde\mu_\nu$ as the positive, purely atomic measure, concentrated on the finitely many points x_α where $u(t,\cdot)$ has a 2-shock, such that

$$\tilde\mu_\nu\big(\{x_\alpha\}\big) = w_1^\nu(t,\, x_\alpha+) - w_1^\nu(t,\, x_\alpha-) \leq C\,|\sigma_\alpha|^3, \tag{3.1.22}$$

for some constant C. By possibly taking a subsequence, we can assume the existence of a weak limit $\tilde\mu_\nu \rightharpoonup \tilde\mu$. Because of the estimate in (3.1.22), the measure $\tilde\mu$ is purely atomic, and is concentrated on the set of points x_β which are limits as $\nu \to \infty$ of a sequence of points x_α^ν where $u_\nu(t,\cdot)$ has a 2-shock of uniformly positive strength $|\sigma_\nu| \geq \delta > 0$. Therefore, $\tilde\mu$ is concentrated on the set of points where the limit solution $u(t,\cdot)$ has a 2-shock, and makes no contribution to the positive part of μ_t^{1+}. We thus conclude that the positive part of μ_t^{1+} is absolutely continuous w.r.t. Lebesgue measure, with density $\leq \kappa/t$. An analogous argument holds for μ_t^{2+}. □

Corollary 3.1.1 *Let $u = u(t,x)$ be a solution of the 2×2 system (1.0.1). Let the assumptions (H) hold. Fix $\varepsilon > 0$ and consider the subinterval $[a',b'] \doteq [a+\varepsilon,\ b-\varepsilon]$. Assume that, at time $t = 0$, the measures μ^{1+}, μ^{2+} of positive waves in $u(0,\cdot)$ on $[a,b]$ vanish identically. Then, for every $t > 0$ one has*

$$\mu_t^{i+}(J) \leq \frac{\kappa\lambda^*}{\varepsilon}\, meas\,(J), \tag{3.1.23}$$

for every Borel set $J \subset [a',b']$ and every $t > 0$, $i = 1,2$.

Indeed, recalling (3.0.1), the values of $u(t,\cdot)$ restricted to the interval $[a',b']$ can be obtained by solving a Cauchy problem, with initial data assigned on the whole interval $[a,b]$ at time $t-\varepsilon/\lambda^*$.

3.2 Proof of Theorem 3.0.2

Lemma 3.2.1 *In the same setting as Lemma 3.1.2, assume that there exists* $\kappa' > 0$ *such that*

$$0 \le w_{i,x}(t,x) \le \kappa', \qquad t \in [0,T], \quad i = 1,2. \tag{3.2.24}$$

Let $t \mapsto x(t)$ *be the location of a shock, with strength* $|\sigma(t)|$. *There exists a constant* $0 < c < 1$ *such that*

$$|\sigma(t)| \ge c|\sigma(s)|, \qquad 0 \le s < t \le T. \tag{3.2.25}$$

Proof. To fix the ideas, let $u(t,\cdot)$ have a 1-shock located at $x(t)$, with strength $|\sigma(t)|$. Outside points of interaction with other shocks, the strength satisfies an inequality of the form

$$\frac{d}{dt}|\sigma(t)| \ge -C\cdot\Big(w_{1,x}(t,x(t)+)+w_{1,x}(t,x(t)-)w_{2,x}(t,x(t)+)+w_{2,x}(t,x(t)-)\Big)|\sigma(t)|. \tag{3.2.26}$$

At times where our 1-shock interacts with other 1-shocks, its strength increases. Moreover, at each time t_α where our 1-shock interacts with a 2-shock, say of strength $|\sigma_\alpha|$, one has

$$|\sigma(t_\alpha+)| \ge |\sigma(t_\alpha-)|\,(1 - C'|\sigma_\alpha|). \tag{3.2.27}$$

for some constant C'. Assuming that the total variation remains small, the total amount of 2-shocks which cross any given 1-shock is uniformly small. Hence, (3.2.26)-(3.2.27) together imply (3.2.25). □

Lemma 3.2.2 *Let* $t \mapsto u(t,\cdot) \in \mathcal{U}$ *be a solution of the Cauchy problem for a genuinely nonlinear* 2×2 *system satisfying (3.0.3). Assume that there exists* $\kappa' > 0$ *such that*

$$w_{i,x}(t,x) \le \kappa', \qquad t \in [0,T], \quad i = 1,2. \tag{3.2.28}$$

Since no centered rarefactions are present, any two i*-characteristics, say* $x(t) < y(t)$*, can uniquely be traced backward up to time* $t = 0$. *There exists a constant* $L > 0$ *such that*

$$y(t) - x(t) \le L\,\big(y(s) - x(s)\big), \qquad 0 \le s < t \le T. \tag{3.2.29}$$

Proof. Consider the case $i = 2$. By definition, the characteristics are solutions of

$$\dot{x}(t) = \lambda_2\big(u(t,x(t))\big), \qquad \dot{y}(t) = \lambda_2\big(u(t,y(t))\big).$$

Since the characteristic speed λ_2 decreases across 2-shocks, we can write

$$\dot{y}(t) - \dot{x}(t) \le C\int_{x(t)}^{y(t)} |w_{1,x}(t,\xi)| + |w_{2,x}(t,\xi)|\,d\xi + C\sum_{\alpha\in\mathcal{S}_1[x,y]} |\sigma_\alpha(t)|, \tag{3.2.30}$$

where $\mathcal{S}_1[x,y]$ denotes the set of all 1-shocks located inside the interval $[x(t),\ y(t)]$. Introduce the function

$$\phi(t,x) \doteq \begin{cases} 0 & \text{if } x \le x(t), \\ \dfrac{x - x(t)}{y(t) - x(t)}, & \text{if } x(t) < x < y(t), \\ 1, & \text{if } x \ge y(t), \end{cases}$$

Moreover, define the functional

$$\Phi(t) \doteq \sum_{\alpha\in\mathcal{S}_1} \phi\big(t, x_\alpha(t)\big)\,\big|\sigma_\alpha(t)\big| + C_0\, Q(t)\,,$$

where the summation now refers to all 1-shocks in $u(t,\cdot)$ and Q is the usual interaction potential. Observe that the map $t\mapsto\Phi(t)$ is non-increasing. By (3.2.28) and (3.2.30), we can now write

$$\dot y(t) - \dot x(t) \le C'\,\big(1-\dot\Phi(t)\big)\,\big(y(t)-x(t)\big),$$

for some constant C'. This implies (3.2.29) with $L = \exp\big\{C'T + C'\Phi(0)\big\}$. □

The next result is the key ingredient toward the proof of Theorem 3.0.2. It provides the density of the set of interaction points where new shocks are generated.

Lemma 3.2.3 *Fix $\varepsilon>0$ and define $a'' = a+2\varepsilon$, $b'' = b-2\varepsilon$. Consider a 2×2 system of the form (1.0.1), satisfying (H). Let u be an entropy weak solution defined on $[0,\tau]\times[a,b]$, with $\tau\doteq\varepsilon/4\lambda^*$. Let (3.1.23) hold for all $t\in[0,\tau]$, and assume that $u(0,\cdot)$ has a dense set of 1-shocks on the interval $[a'', b'']$. Then, for $0\le t\le\tau$, the solution $u(t,\cdot)$ has a set of 1-shocks which is dense on $[a'',\ b'-\lambda^* t]$ and a set of 2-shocks which is dense on $[a'', b'']$.*

Proof. By the assumptions of the lemma, there exists a sequence of piecewise Lipschitz solutions $t\mapsto u_\nu(t)\in\mathcal{U}$ such that $u_\nu\to u$ in L^1,

$$0\le w^\nu_{i,x}(t,x)\le\frac{2\kappa\lambda^*}{\varepsilon}, \qquad\qquad i=1,2,\quad \nu\ge1\,,$$

and moreover the following holds. For every $\rho>0$, there exists $\delta>0$ such that each $u_\nu(0,\cdot)$ (with ν large enough) contains at least one 1-shock of strength $\big|\sigma_\nu(0)\big|\ge\delta$ on every subinterval $J\subset[a'', b'']$ having length $\ge\rho$.

To prove the first statement in Lemma 3.2.3, fix $t\in[0,\tau]$ and consider any non-trivial interval $[p,q]\subset[a'',\ b''-t\lambda^*]$. Call $s\mapsto p_\nu(s)$, $s\mapsto q_\nu(s)$ the backward characteristics through these points, relative to the solution u_ν. We thus have

$$\begin{cases} \dot p_\nu(s) &= \lambda_1\big(u_\nu(s,\,p_\nu(s))\big),\\ \dot q_\nu(s) &= \lambda_1\big(u_\nu(s,\,q_\nu(s))\big),\end{cases} \qquad\qquad \begin{cases} p_\nu(t) &= p,\\ q_\nu(t) &= q.\end{cases}$$

By Lemma 3.2.2, $q_\nu(0)-p_\nu(0)\ge\rho$ for some $\rho>0$ independent of ν. Hence, each solution u_ν contains a shock of strength $\big|\sigma_\nu(s)\big|\ge\delta$ located inside the interval $\big[p_\nu(0),\ q_\nu(0)\big]$. Lemma 3.1.3 now yields $\big|\sigma_\nu(t)\big|\ge c\delta$. By possibly taking a subsequence, we conclude that the limit solution $u(t,\cdot)$ contains a 1-shock of positive strength at the point $x(t)=\lim_\nu x_\nu(t)\in[p,q]$.

To prove the second statement, we will show that the set of points where two 1-shocks in u interact and produce a new 2-shock is dense on the triangle

$$\Delta\doteq\big\{(t,x)\,;\quad t\in[0,\tau],\quad a''<x<b''-\lambda^* t\big\}.$$

Indeed, let $t \in [0, \tau]$ and $p < q$ be as before. For each ν sufficiently large, let $t \mapsto x_\nu(t)$ be the location of a 1-shock in u_ν, with strength $\big|\sigma_\nu(t)\big| \geq \delta > 0$. Assume $x_\nu(\cdot) \to x(\cdot)$ as $\nu \to \infty$, and $x_\nu(t) \in [p, q]$, so that $x(t)$ is the location of a 1-shock of the limit solution u, say with strength $\big|\sigma(t)\big| > 0$.

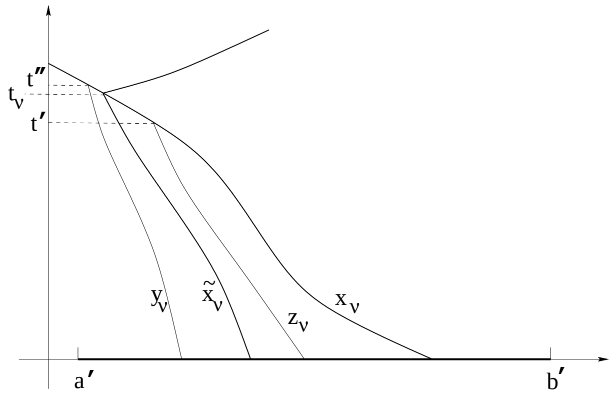

Figure 3.3

We claim that the set of times $\hat{t}$ where some other 1-shock σ' impinges on σ and generates a new 2-shock is dense on $[0, t]$. To see this, fix $0 < t' < t'' < t$. For each ν sufficiently large, consider the backward 1-characteristics $y_\nu,\ z_\nu$ impinging from the left on the shock x_ν at times t'', t' respectively (see Figure 3.3). These provide solutions to the Cauchy problems

$$\dot{y}_\nu(t) = \lambda_1\big(u_\nu(t, y_\nu(t))\big), \qquad\qquad y_\nu(t'') = x_\nu(t''),$$

$$\dot{z}_\nu(t) = \lambda_1\big(u_\nu(t, z_\nu(t))\big), \qquad\qquad z_\nu(t') = x_\nu(t'),$$

respectively. Observe that

$$z_\nu(0) - y_\nu(0) \geq \rho,$$

for some $\rho > 0$ independent of ν. Indeed, the genuine nonlinearity of the system implies

$$\lambda_1\big(u_\nu(t, x_\nu(t)-)\big) - \dot{x}_\nu(t) \geq \kappa \Big| u_\nu\big(t,\, x_\nu(t)+\big)\big) - u_\nu\big(t,\, x_\nu(t)-\big) \Big| \geq \kappa\delta.$$

Therefore,

$$x_\nu(t') - y_\nu(t') \geq \rho' > 0,$$

for some constant $\rho' > 0$ independent of ν. By Lemma 3.2.1, the interval $\big[y_\nu(0),\ z_\nu(0)\big]$ has uniformly positive length. Hence it contains a 1-shock of $u_\nu(0, \cdot)$ with uniformly positive strength $\big|\sigma_\nu(0)\big| \geq \delta > 0$. By Lemma 3.2.3, every u_ν has a 1-shock with strength $\big|\sigma_\nu(t)\big| \geq c\delta$ located along some curve $t \mapsto \tilde{x}_\nu(t)$ with

$$y_\nu(t) < \tilde{x}_\nu(t) < z_\nu(t), \qquad t \in [0, t'].$$

Clearly, this second 1-shock impinges on the shock x_ν at some time $t_\nu \in [t', t'']$, creating a new 2-shock with uniformly large strength. Letting $\nu \to \infty$ we obtain the result. $\square$

Proof of Theorem 3.0.2. Let $\delta_0 > 0$ be given. We can then construct an initial condition $u(0,\cdot) = \phi$, with $\text{Tot.Var.}\{\phi\} < \delta_0$, having a dense set of 1-shocks on the interval $[a,b]$, and no other waves. As a consequence, for any $\varepsilon > 0$ by Corollary 3.1.1 we have the estimate (3.1.23) on the density of positive waves away from the boundary.

Fix $\tau = \varepsilon/4\lambda^*$, and consider again the subinterval $[a'', b''] = [a + 2\varepsilon,\ b - 2\varepsilon]$. We can apply Lemma 3.2.3 first on the time interval $[0,\tau]$, obtaining the density of 2-shocks on the region $[0,\tau] \times [a'', b'']$. Then, by induction on m, the same argument is repeated on each time interval $t \in \big[m\tau,\ (m+1)\tau\big]$, proving the theorem. $\square$

Chapter 4

Asymptotic Stabilization

The result of this chapter is concerned with stabilization near a constant state. Assuming that for the system (1.0.1) all characteristic speeds are bounded away from zero, we show that the system with boundary conditions (1.0.6) and (1.0.9) can be asymptotically stabilized to any state $u^* \in U$, with quadratic rate of convergence (see [14, 3]).

The main statement of this chapter is the following.

Theorem 4.0.1 *Let K be a compact, connected subset of the open domain $U \subset \mathbb{R}^n$. Then there exist constants $C_0, \delta, \kappa > 0$ such that the following holds. For every constant state $u^* \in K$ and every initial data $u(0) = \phi : [a,b] \to K$ with Tot.Var.$\{\phi\} < \delta$, there exists an entropy weak solution $u = u(t,x)$ of (1.0.1) such that, for all $t > 0$,*

$$\textit{Tot.Var.}\{u(t,\cdot)\} \leq C_0\, e^{-2^{\kappa t}}, \tag{4.0.1}$$

$$\big\| u(t,\cdot) - u^* \big\|_{L^\infty} \leq C_0\, e^{-2^{\kappa t}}. \tag{4.0.2}$$

The proof will be given in Section 4.1.

Throughout the following, we denote by $r_i(u)$, $l_i(u)$ the right and left i-eigenvectors of the Jacobian matrix $A(u) \doteq Df(u)$. As in [12], we write $\sigma \mapsto R_i(\sigma)(u_0)$ for the parametrized i-rarefaction curve through the state u_0, so that

$$\frac{d}{d\sigma} R_i(\sigma) = r_i\big(R_i(\sigma)\big), \qquad R_i(0) = u_0.$$

The i-shock curve through u_0 is denoted by $\sigma \mapsto S_i(\sigma)(u_0)$. It satisfies the Rankine-Hugoniot equations

$$f\big(S_i(\sigma)\big) - f(u_0) = \lambda_i(\sigma)\,\big(S_i(\sigma) - u_0\big),$$

for some shock speed λ_i. We recall (see [12, Chapter 5]) that the general Riemann problem is solved in terms of the composite curves

$$\Psi_i(u_0)(\sigma) = \begin{cases} R_i(u_0)(\sigma), & \text{if} \quad \sigma \geq 0, \\ S_i(u_0)(\sigma), & \text{if} \quad \sigma < 0. \end{cases} \tag{4.0.3}$$

4.1 Proof of Theorem 4.0.1

The proof relies on the two following two lemmas.

Lemma 4.1.1 *In the setting of Theorem 4.0.1, there exists a time $T > 0$ such that the following holds. For every pair of states $\omega, \omega' \in K$ there exists an entropic solution $u = u(t, x)$ of (1.0.1) such that*

$$u(0, x) \equiv \omega, \qquad u(T, x) \equiv \omega', \tag{4.1.4}$$

for all $x \in [a, b]$.

PROOF. Consider the function

$$\Phi(\sigma_1, \ldots, \sigma_n\,;\ v,\, v') \doteq \Psi_n(\sigma_n) \circ \cdots \circ \Psi_{p+1}(\sigma_{p+1})(v') - \Psi_p(\sigma_p) \circ \cdots \circ \Psi_1(\sigma_1)(v). \tag{4.1.5}$$

Observe that, whenever $v = v'$, the $n \times n$ Jacobian matrix $\partial\Phi/\partial\sigma_1 \cdots \sigma_n$ computed at $\sigma_1 = \sigma_2 = \cdots = \sigma_n = 0$ has full rank. Indeed, the columns of this matrix are given by the linearly independent vectors $-r_1(v), \ldots, -r_p(v)$, $r_{p+1}(v), \ldots, r_n(v)$. By the Implicit Function Theorem and a compactness argument we can find $\delta > 0$ such that the following holds. For every $v, v' \in K$, with $|v - v'| \le \delta$, there exist unique values $\sigma_1, \ldots, \sigma_n$ such that

$$v'' \doteq \Psi_n(\sigma_n) \circ \cdots \circ \Psi_{p+1}(\sigma_{p+1})(v') = \Psi_p(\sigma_p) \circ \cdots \circ \Psi_1(\sigma_1)(v)\,. \tag{4.1.6}$$

Defining the time

$$\tau \doteq \max_{1 \le i \le n} \sup_{u \in U} \frac{b - a}{\big|\lambda_i(u)\big|}\,, \tag{4.1.7}$$

we claim that there exists an entropy weak solution $u : [0, 2\tau] \times [a, b] \to U$ such that

$$u(0, x) \equiv v, \qquad u(2\tau, x) \equiv v'. \tag{4.1.8}$$

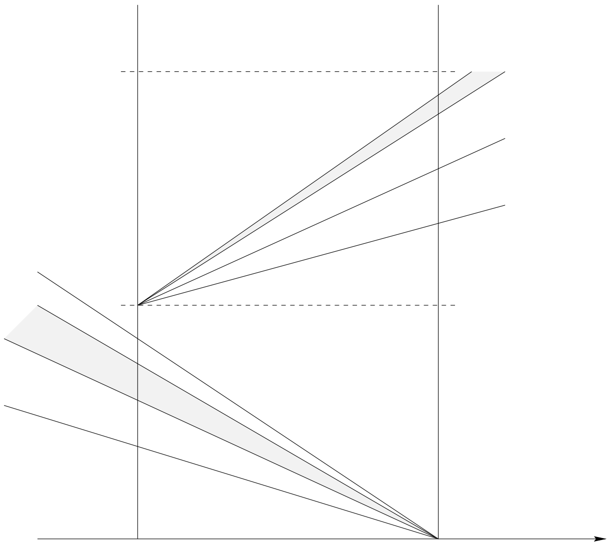

Figure 4.1

The function u is constructed as follows (see Figure 4.1). For $t \in [0,\tau]$, we let u be the solution of the Riemann problem

$$u(0,x) = \begin{cases} v, & \text{if} \quad x < b, \\ v'', & \text{if} \quad x > b. \end{cases} \tag{4.1.9}$$

Moreover, for $t \in [\tau,\ 2\tau]$, we define u as the solution of the Riemann problem

$$u(\tau,x) = \begin{cases} v', & \text{if} \quad x < a, \\ v'', & \text{if} \quad x > a. \end{cases} \tag{4.1.10}$$

It is now clear that the restriction of u to the domain $[0,\ 2\tau] \times [a,b]$ satisfies the conditions (4.1.8). Indeed, by (4.1.6), on $[0,\tau]$ the solution u contains only waves of families $\leq p$, originating at the point $(0,b)$. By (4.1.7) these waves cross the whole interval $[a,b]$ and exit from the boundary point a before time τ. Hence $u(\tau,x) \equiv v''$. Similarly, still by (4.1.6), for $t \in [\tau,\ 2\tau]$ the function u contains only waves of families $\geq p+1$, originating at the point (τ,a). By (4.1.7) these waves cross the whole interval $[a,b]$ and exit from the boundary point b before time 2τ. Hence $u(2\tau,x) \equiv v'$.

Next, given any two states $\omega,\omega' \in K$, by the connectedness assumption we can find a chain of points $\omega_0 = \omega, \omega_1, \ldots, \omega_N = \omega'$ in K such that $|\omega_i - \omega_{i-1}| < \delta$ for every $i = 1,\ldots,N$. Repeating the previous construction in connection with each pair of states (ω_{i-1},ω_i), we thus obtain an entropy weak solution $u : [0,\ 2N\tau] \times [a,b] \to U$ that satisfies the conclusion of the lemma, with $T = 2N\tau$. □

In the following, we shall construct the desired solution $u = u(t,x)$ as limit of a sequence of front tracking approximations. Roughly speaking, an ε-approximate front tracking solution is a piecewise constant function u^ε, having jumps along a finite set of straight lines in the t-x plane say $x = x_\alpha(t)$, which approximately satisfies the Rankine-Hugoniot equations:

$$\sum_\alpha \Big| f\big(u(t,x_\alpha+)\big) - f\big(u(t,x_\alpha-)\big) - \dot{x}_\alpha \left(u(t,x_\alpha+) - u(t,x_\alpha-)\right)\Big| < \varepsilon,$$

for all $t > 0$. For details, see [12], p.125.

Lemma 4.1.2 *In the setting of Theorem 4.0.1, for every state $u^* \in U$ there exist constants $C,\delta_0 > 0$ for which the following holds. For any $\varepsilon > 0$ and every piecewise constant function $\bar{u} : [a,b] \to U$ such that*

$$\rho \doteq \sup_{x\in[a,b]} \big|\bar{u}(x) - u^*\big| \leq \delta_0, \qquad\qquad \delta \doteq \mathit{Tot.Var.}\{\bar{u}\} \leq \delta_0, \tag{4.1.11}$$

there exists an ε-approximate front tracking solution $u = u(t,x)$ of (1.0.1), with $u(0,x) = \bar{u}(x)$, such that

$$\sup_{x\in[a,b]} \big|u(3\tau,\,x) - u^*\big| \leq C\delta^2, \qquad\qquad \mathit{Tot.Var.}\big\{u(3\tau)\big\} \leq C\delta^2\,. \tag{4.1.12}$$

Proof. On the domain $(t,x) \in [0,\tau] \times [a,b]$, we construct u as an ε-approximate front tracking solution in such a way that, whenever a front hits one of the boundaries $x = a$ or $x = b$, no reflected front is ever created (see Figure 4.2). Since all fronts emerging from the initial data $\bar u$ at time $t = 0$ exit from $[a, b]$ within time τ, it is clear that $u(\tau)$ can contain only fronts of second or higher generation order. In other words, the only fronts that can be present in $u(\tau, \cdot)$ are the new ones, generated by interactions at times $t > 0$ (the dotted lines in Figure 4.2). Therefore, using the interaction estimate (7.69) in [12] we obtain

$$\sup_{x\in[a,b]} \big|u(\tau, x) - u^*\big| = \mathcal{O}(1) \cdot (\rho + \delta), \qquad \text{Tot.Var.}\big\{u(\tau)\big\} = \mathcal{O}(1) \cdot \delta^2 . \tag{4.1.13}$$

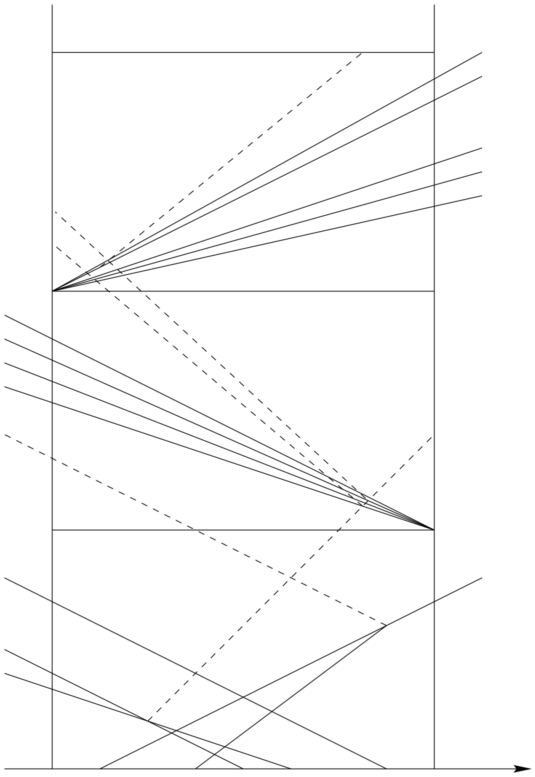

Figure 4.2

We now apply a similar procedure as in the proof of Lemma 4.1.1, and construct a solution on the interval $[\tau,\ 3\tau]$ in such a way that $u(3\tau) \approx u^*$. More precisely, to construct u on the domain $[\tau,\ 2\tau] \times [a, b]$, consider the state v'' implicitly defined by (4.1.5), with $v \doteq u(\tau, b-)$, $v' \doteq u^*$. On a forward neighborhood of the point (τ, b) we let u coincide with (a front-tracking approximation of) the solution to the Riemann problem

$$u(\tau, x) = \begin{cases} u(\tau, b-), & \text{if} \quad x < b, \\ v'', & \text{if} \quad x > b. \end{cases}$$

This procedure will introduce at the point (τ, b) a family of wave-fronts of families $i = 1, \ldots, p$, whose total strength is $\mathcal{O}(1)\cdot(\rho+\delta)$. Because of (4.1.7), all these fronts will exit from the boundary $x = a$ within time 2τ. Of course, they can interact with the other fronts present in $u(\tau, \cdot)$. In any case, the total strength of fronts in $u(2\tau,\cdot)$ is still estimated as

$$\text{Tot.Var.}\big\{u(2\tau)\big\} = \mathcal{O}(1)\cdot\delta^2\,. \tag{4.1.14}$$

Next, to define u for $t\in[2\tau,\,3\tau]$, consider the state v''' implicitly defined by

$$\begin{cases} u(2\tau,a+) & = \Psi_n(\sigma_n)\circ\cdots\circ\Psi_{p+1}(\sigma_{p+1})(v'''),\\ u^* & = \Psi_p(\sigma_p)\circ\cdots\circ\Psi_1(\sigma_1)(v'''). \end{cases} \tag{4.1.15}$$

On a forward neighborhood of the point $(2\tau,a)$ we let u coincide with (a front-tracking approximation of) the solution to the Riemann problem

$$u(2\tau,x) = \begin{cases} u(2\tau,\,a+), & \text{if} \quad x>a,\\ v''', & \text{if} \quad x<a. \end{cases}$$

This procedure introduces at the point $(2\tau,\,a)$ a family of wave-fronts of families $i=p+1,\ldots,n$, whose total strength is $\mathcal{O}(1)\cdot(\rho+\delta)$. Because of (4.1.7), all these fronts will exit from the boundary $x=b$ within time 3τ. Of course, they can interact with the other fronts present in $u(2\tau,\cdot)$. In any case, the total strength of fronts in $u(3\tau,\cdot)$ is still estimated as

$$\text{Tot.Var.}\big\{u(3\tau)\big\} = \mathcal{O}(1)\cdot\delta^2\,. \tag{4.1.16}$$

Moreover, the difference between the values $u(3\tau,x)$ and u^* will be of the same order of the total strength of waves in $u(\tau,\cdot)$, so that the first inequality in (4.1.12) will also hold. □

Proof of Theorem 4.0.1. Using the same arguments as in the proof of Lemma 4.1.1, for every $\varepsilon>0$ we can construct an ε-approximate front tracking solution $u=u(t,x)$ on $[0,\,2N\tau]\times[a,b]$ such that

$$\sup_{x\in[a,b]}\big|u(2N\tau,x)-u^*\big| = \mathcal{O}(1)\cdot\delta, \qquad \text{Tot.Var.}\big\{u(2N\tau)\big\} = \mathcal{O}(1)\cdot\delta\,. \tag{4.1.17}$$

Choosing $\delta>0$ sufficiently small, we can assume that, in (4.1.17), $\mathcal{O}(1)\cdot\delta<\delta_0<1/C$, the constant in Lemma 4.1.2. Calling $T\doteq 2N\tau$, we can now repeat the construction described in Lemma 4.1.2 on each interval $\big[T+3k\tau,\ T+3(k+1)\tau\big]$. This yields

$$\sup_{x\in[a,b]}\big|u(T+3k\tau,\,x)-u^*\big| \le \delta_k, \qquad \text{Tot.Var.}\big\{u(T+3k\tau)\big\} \le \delta_k\,, \tag{4.1.18}$$

where the constants δ_k satisfy the inductive relations

$$\delta_{k+1} \le C\delta_k^2. \tag{4.1.19}$$

Choosing a sequence of ε-approximate front tracking solutions u_ε satisfying (4.1.18)-(4.1.19) and taking the limit as $\varepsilon\to 0$, we obtain an entropy weak solution u which still satisfies the same estimates. The bounds (4.0.1)-(4.0.2) are now a consequence of (4.1.18)-(4.1.19), with a suitable choice of the constants C_0,κ. □

Chapter 5

Exact Controllability for Linear Systems

In this chapter we look for the finite time controllability of the following linear problem

$$\begin{cases} \omega_t + A \cdot \omega_x = 0, & a < x < b,\ t > 0, \\ \omega^+(t,a) = C_a \cdot \omega^-(t,a), & t > 0, \\ \omega^-(t,b) = C_b \cdot \omega^+(t,b) + \Gamma_b \cdot \alpha(t), & t > 0, \\ \omega(0,x) = \omega_0(x), & a < x < b, \end{cases} \tag{5.0.1}$$

where $A \in \mathbb{R}^{n\times n}$, $C_a \in \mathbb{R}^{(n-p)\times p}$, $C_b \in \mathbb{R}^{p\times(n-p)}$, $\Gamma_b \in \mathbb{R}^{p\times\nu}$ are constant matrices, $\alpha(\cdot) \in \mathbb{R}^\nu$ and we use the following notations

$$\omega^- \doteq \begin{pmatrix} \omega_1 \\ ... \\ \omega_p \end{pmatrix} \in \mathbb{R}^p, \qquad \omega^+ \doteq \begin{pmatrix} \omega_{p+1} \\ ... \\ \omega_n \end{pmatrix} \in \mathbb{R}^{n-p}, \qquad \omega \doteq \begin{pmatrix} \omega_1 \\ ... \\ \omega_n \end{pmatrix} \in \mathbb{R}^n,$$

$p \in \{1, ..., n\}$ and $\nu \in \mathbb{N}$. Moreover we assume that A is diagonal, namely

$$A \doteq \begin{pmatrix} \lambda_1 & ... & 0 \\ ... & ... & ... \\ 0 & ... & \lambda_n \end{pmatrix} \tag{5.0.2}$$

with eigenvalues

$$\lambda_1 < ... < \lambda_p < 0 < \lambda_{p+1} < ... < \lambda_n. \tag{5.0.3}$$

Fix a time $T > 0$ and an initial condition $\omega_0 \in L^1\big([a,b];\mathbb{R}^n\big)$, denote

$$\mathcal{A}_q(T,\omega_0) \doteq \big\{\omega(T,\cdot) \text{ solution of (5.0.1) for } \alpha \in L^q\big([0,T];\mathbb{R}^\nu\big)\big\},$$

we will care of the cases $q = 1, 2$.

Definition 5.0.1 *We say that (5.0.1) is exactly controllable at time $T > 0$ if and only if*

$$\mathcal{A}_1(T, \omega_0) = L^1\big([a,b]; \mathbb{R}^n\big),$$

for each initial condition $\omega_0 \in L^1\big([a,b]; \mathbb{R}^n\big)$.

Remark 5.0.1 Due to the linearity of the problem (5.0.1), we can split the effect of the initial condition and of the controls in the attainable set, namely the two following propositions are equivalent

i) (5.0.1) is exactly controllable at time T,

ii) $\mathcal{A}_1(T, 0) = L^1\big([a,b]; \mathbb{R}^n\big)$.

The main result of this chapter is the following (see [5]).

Theorem 5.0.1 *Assume that*

$$\operatorname{rank}\big(C_a\big) = n - p, \quad \operatorname{rank}\big(C_b|\Gamma_b\big) = p,$$

and

$$n - p = p, \quad T \geq l\left(\frac{b-a}{|\lambda_p|} + \frac{b-a}{\lambda_{p+1}}\right),$$
$$\operatorname{rank}\big((C_b \cdot C_a)^{l-1} \cdot \Gamma_b|...|\Gamma_b\big) = p,$$

or

$$n - p < p, \quad T \geq l\left(\frac{b-a}{|\lambda_p|} + \frac{b-a}{\lambda_{p+1}}\right) + \frac{b-a}{|\lambda_p|},$$
$$\operatorname{rank}\Big(C_a \cdot (C_b \cdot C_a)^{l-1} \cdot \Gamma_b|...|C_a \cdot \Gamma_b\big)\Big) = n - p,$$

then (5.0.1) is exactly controllable at time T, where $l \in \mathbb{N}$ such that

$$l - 1 < \frac{n-p}{\nu} \leq l.$$

The proof of this will be done in the next sections.

Fix $T > 0$ and consider the operator

$$C : L^1\big([0,T]; \mathbb{R}^\nu\big) \longrightarrow L^1\big([a,b]; \mathbb{R}^n\big), \tag{5.0.4}$$

where $C\alpha$ is the solution of (5.0.1) at time T with $\omega_0 \equiv 0$. Using the method of the characteristics (see [29]) one can prove the following.

Theorem 5.0.2 *The operator C is linear and bounded from $L^1\big([0,T]; \mathbb{R}^\nu\big)$ to $L^1\big([a,b]; \mathbb{R}^n\big)$ and from $L^2\big([0,T]; \mathbb{R}^\nu\big)$ to $L^2\big([a,b]; \mathbb{R}^n\big)$. Moreover*

$$\operatorname{Im}(C) = \mathcal{A}_1(T, 0), \qquad C\Big(L^2\big([0,T]; \mathbb{R}^\nu\big)\Big) = \mathcal{A}_2(T, 0), \tag{5.0.5}$$

and so (5.0.1) is exactly controllable at time T if and only if C is surjective.

5.1 The Adjoint Problem

Consider the initial boundary value problem

$$\begin{cases} v_t + A \cdot v_x = 0, & a < x < b,\ 0 < t < T, \\ v^-(t,a) = -(A^-)^{-1} \cdot C_a^T \cdot A^+ \cdot v^+(t,a), & 0 < t < T, \\ v^+(t,b) = -(A^+)^{-1} \cdot C_b^T \cdot A^- \cdot v^-(t,b), & 0 < t < T, \\ v(T,x) = v_T(x), & a < x < b, \end{cases} \tag{5.1.6}$$

where C_a^T and C_b^T are the transpose of C_a and C_b respectively,

$$A^- \doteq \begin{pmatrix} \lambda_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_p \end{pmatrix}, \qquad A^+ \doteq \begin{pmatrix} \lambda_{p+1} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_n \end{pmatrix}$$

and consider the operator

$$C^* : L^2\big([a,b];\mathbb{R}^n\big) \longrightarrow L^2\big([0,T];\mathbb{R}^\nu\big), \qquad C^* v_T \doteq -\Gamma_b^T \cdot A^- \cdot v^-(\cdot,b), \tag{5.1.7}$$

and Γ_b^T is the transpose of Γ_b. The mixed problem (5.1.6) is called *the adjoint problem* of (5.0.1)

Theorem 5.1.1 *The operator C^* is the adjoint one of the restriction of C to $L^2\big([0,T];\mathbb{R}^\nu\big)$.*

The proof of the previous theorem is based on the following.

Lemma 5.1.1 *Let $\omega = \omega(t,x)$, $v = v(t,x)$ be the solutions of (5.0.1) and (5.1.6) respectively, there results*

$$\big(v(T,\cdot),\omega(T,\cdot)\big)_{L^2} - \big(v(0,\cdot),\omega(0,\cdot)\big)_{L^2} = -\big(\Gamma_b^T \cdot A^- \cdot v^-(\cdot,b),\alpha\big)_{L^2}, \tag{5.1.8}$$

for each $T > 0$.

Proof. Let $\omega = \omega(t,x)$, $v = v(t,x)$ be the solutions of (5.0.1) and (5.1.6) respectively. Since A is symmetric we have

$$
\begin{aligned}
&\big(v(T,\cdot),\omega(T,\cdot)\big)_{L^2} - \big(v(0,\cdot),\omega(0,\cdot)\big)_{L^2} = \\
&= \int_a^b v(T,x)^T \cdot \omega(T,x)dx - \int_a^b v(0,x)^T \cdot \omega(0,x)dx = \\
&= \int_0^T \frac{d}{dt}\left(\int_a^b v(t,x)^T \cdot \omega(t,x)dx\right)dt = \\
&= \int_0^T \int_a^b \big(v_t(t,x)^T \cdot \omega(t,x) + v(t,x)^T \cdot \omega_t(t,x)\big)dxdt = \\
&= -\int_0^T \int_a^b \big(v_x(t,x)^T \cdot A \cdot \omega(t,x) + v(t,x)^T \cdot A \cdot \omega_x(t,x)\big)dxdt = \\
&= -\int_0^T \int_a^b \frac{\partial}{\partial x}\big(v(t,x)^T \cdot A \cdot \omega(t,x)\big)dxdt = \\
&= -\int_0^T \big(v(t,b)^T \cdot A \cdot \omega(t,b) - v(t,a)^T \cdot A \cdot \omega(t,a)\big)dt.
\end{aligned}
\tag{5.1.9}
$$

Observe that

$$
\begin{aligned}
&v(t,b)^T \cdot A \cdot \omega(t,b) = \\
&= \big(v^-(t,b)^T, v^+(t,b)^T\big) \cdot \begin{pmatrix} A^- & 0 \\ 0 & A^+ \end{pmatrix} \cdot \begin{pmatrix} \omega^-(t,b) \\ \omega^+(t,b) \end{pmatrix} = \\
&= \big(v^-(t,b)^T, v^+(t,b)^T\big) \cdot \begin{pmatrix} A^- \cdot \omega^-(t,b) \\ A^+ \cdot \omega^+(t,b) \end{pmatrix} = \\
&= v^-(t,b)^T \cdot A^- \cdot \omega^-(t,b) + v^+(t,b)^T \cdot A^+ \cdot \omega^+(t,b) = \\
&= v^-(t,b)^T \cdot A^- \cdot \big(C_b \cdot \omega^+(t,b) + \Gamma_b \cdot \alpha(t)\big) + v^+(t,b)^T \cdot A^+ \cdot \omega^+(t,b) = \\
&= v^-(t,b)^T \cdot A^- \cdot C_b \cdot \omega^+(t,b) + v^-(t,b)^T \cdot A^- \cdot \Gamma_b \cdot \alpha(t) - \\
&\quad - \big((A^+)^{-1} \cdot C_b^T \cdot A^- \cdot v^-(t,b)\big)^T A^+ \cdot \omega^+(t,b) = \\
&= v^-(t,b)^T \cdot A^- \cdot \Gamma_b \cdot \alpha(t).
\end{aligned}
\tag{5.1.10}
$$

and using the same arguments

$$
v(t,a)^T \cdot A \cdot \omega(t,a) = 0. \tag{5.1.11}
$$

By (5.1.9), (5.1.10) and (5.1.11), we obtain

$$
\begin{aligned}
&\big(v(T,\cdot),\omega(T,\cdot)\big)_{L^2} - \big(v(0,\cdot),\omega(0,\cdot)\big)_{L^2} = \\
&= -\int_a^b v^-(t,b)^T \cdot A^- \cdot \Gamma_b \cdot \alpha(t)dt = -\big(\Gamma_b^T \cdot A^- \cdot v^-(\cdot,b),\alpha\big)_{L^2},
\end{aligned}
\tag{5.1.12}
$$

so (5.1.8) is proved. ◻

PROOF OF THEOREM 5.1.1. If $\omega_0 \equiv 0$, we have

$$-\Gamma_b^T \cdot A^- \cdot v^-(\cdot, b) \equiv C^* v_T, \qquad \omega(T, \cdot) \equiv C\alpha$$

and by (5.1.8) we have

$$\begin{aligned}\big(C^* v_T, \alpha\big)_{L^2} &= -\big(\Gamma_b^T \cdot A^- \cdot v^-(\cdot, b), \alpha\big)_{L^2} = \\ &= -\big(v(T,\cdot), \omega(T,\cdot)\big)_{L^2} = \big(v_T, C\alpha\big)_{L^2},\end{aligned}$$

namely C^* is the adjoint operator of the restriction of C to $L^2\big([0,T]; \mathbb{R}^\nu\big)$. □

5.2 Necessary Conditions for Controllability

In this we show some necessary conditions for the exact controllability in finite time.

Theorem 5.2.1 *Let $T > 0$, if the problem (5.0.1) is exactly controllable at time T then*

$$T \geq \frac{b-a}{|\lambda_p|} + \frac{b-a}{|\lambda_{p+1}|}, \qquad \operatorname{rank}\big(C_a\big) = n - p, \qquad \operatorname{rank}\big(C_b | \Gamma_b\big) = p. \tag{5.2.13}$$

Moreover, there results

$$\begin{aligned}\operatorname{rank}\big(C_b|\Gamma_b\big) = p \quad &\Longrightarrow \quad p \leq n - p + \nu, \\ \operatorname{rank}\big(C_a\big) = n - p \quad &\Longrightarrow \quad n - p \leq p.\end{aligned} \tag{5.2.14}$$

PROOF. The proof of $T \geq \dfrac{b-a}{|\lambda_p|} + \dfrac{b-a}{|\lambda_{p+1}|}$ is direct consequence of [42, Theorem 3.6].

We prove the second identity of (5.2.13). Let $\bar{\omega}^+ \in \mathbb{R}^{n-p}$ be a constant state, since we are assuming that (5.0.1) is exactly controllable at time T, there exists a map $\alpha^+ \in L^1\big([0,T]; \mathbb{R}^\nu\big)$ such that

$$(C\bar{\alpha}^+)^+ \equiv \bar{\omega}^+.$$

Let $\omega = \omega(t,x)$ be the solution of (5.0.1) in correspondence of $\bar{\alpha}^+$ and $\omega_0 \equiv 0$, by definition,

$$\omega^+(T, \cdot) \equiv (C\bar{\alpha}^+)^+ \equiv \bar{\omega}^+.$$

Fix t such that

$$\max\left\{0, T - \frac{b-a}{|\lambda_n|}\right\} < t < T,$$

we have that

$$\omega^+(t, a) = \begin{pmatrix} \omega_{p+1}(T, \lambda_{p+1}(T-t) + a) \\ \dots \\ \omega_n(T, \lambda_n(T-t) + a) \end{pmatrix} = \bar{\omega}^+.$$

Moreover, by the boundary conditions in (5.0.1),

$$\omega^+(t, a) = C_a \cdot \omega^-(t, a) \in \operatorname{Im}(C_a),$$

so

$$\bar{\omega}^+ \in \mathrm{Im}(C_a).$$

Then

$$\mathrm{Im}(C_a) = \mathbb{R}^{n-p},$$

namely

$$\mathrm{rank}(C_a) = n - p.$$

We continue proving the third of (5.2.13). Let $\bar{\omega}^- \in \mathbb{R}^p$ be a constant state, since we are assuming that (5.0.1) is exactly controllable at time T, there exists a map $\alpha^- \in L^1\big([0,T];\mathbb{R}^\nu\big)$ such that

$$(C\bar{\alpha}^-)^- \equiv \bar{\omega}^-.$$

Let $\omega = \omega(t,x)$ be the solution of (5.0.1) in correspondence of $\bar{\alpha}^-$ and $\omega_0 \equiv 0$, by definition,

$$\omega^-(T,\cdot) \equiv (C\bar{\alpha}^-)^- \equiv \bar{\omega}^-.$$

Fix t such that

$$\max\left\{0, T - \frac{b-a}{|\lambda_1|}\right\} < t < T,$$

we have that

$$\omega^-(t,b) = \begin{pmatrix} \omega_1(T, \lambda_1(T-t)+b) \\ \dots \\ \omega_p(T, \lambda_p(T-t)+b) \end{pmatrix} = \bar{\omega}^-.$$

Moreover, by the boundary conditions in (5.0.1),

$$\omega^-(t,b) = C_b \cdot \omega^+(t,a) + \Gamma_b \alpha^-(t) = (C_b|\Gamma_b) \cdot \begin{pmatrix} \omega^+(t,a) \\ \alpha^-(t) \end{pmatrix} \in \mathrm{Im}(C_b|\Gamma_b),$$

so

$$\bar{\omega}^- \in \mathrm{Im}(C_b|\Gamma_b).$$

Then

$$\mathrm{Im}(C_b|\Gamma_b) = \mathbb{R}^p,$$

namely

$$\mathrm{rank}(C_b|\Gamma_b) = p.$$

This completes the proof of (5.2.13).

We prove the first of (5.2.14). Since

$$\dim\big(\mathrm{Im}\,(C_b|\Gamma_b)\big) = p, \qquad \dim\big(\mathrm{Dom}\,(C_b|\Gamma_b)\big) = n - p + \nu$$

and

$$\dim\big(\mathrm{Dom}\,(C_b|\Gamma_b)\big) = \dim\big(\mathrm{Ker}\,(C_b|\Gamma_b)\big) + \dim\big(\mathrm{Im}\,(C_b|\Gamma_b)\big),$$

there results

$$p = \dim\big(\mathrm{Im}\,(C_b|\Gamma_b)\big) \le \dim\big(\mathrm{Dom}\,(C_b|\Gamma_b)\big) = n - p + \nu.$$

Finally, to prove the second of (5.2.14) we have just to observe that

$$\dim\left(\mathrm{Im}\,(C_a)\right) = n - p, \qquad \dim\left(\mathrm{Dom}\,(C_a)\right) = p$$

and

$$n - p = \dim\left(\mathrm{Im}\,(C_a)\right) \le \dim\left(\mathrm{Dom}\,(C_a)\right) = \ p.$$

This concludes the proof. □

5.3 A Counterexample to the Exact Controllability

Unfortunately the necessary conditions stated in Theorem 5.2.1 are not sufficient. Look for example to the following.

Consider (5.0.1) in the case

$$n = 4, \qquad p = 2, \qquad \nu = 1,$$

with

$$C_a \doteq \begin{pmatrix} 0 & C_{a,12} \\ C_{a,21} & C_{a,22} \end{pmatrix}, \quad C_b \doteq \begin{pmatrix} C_{b,11} & 0 \\ C_{b,21} & 0 \end{pmatrix}, \quad \Gamma_b \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and assume

$$C_{a,21} \cdot C_{a,12} \cdot C_{b,21} \neq 0.$$

We want to prove that it is not exactly controllable in finite time for any $T > 0$. By Theorem 5.0.2, it is sufficient to prove that the kernel of the linear operator C^* defined in (5.1.7) does not coincide with $\{0\}$. Observe that in this case

$$C^* : v_T \longmapsto v^-(\cdot, b),$$

where $v = v(t,x)$ is the solution of (5.1.6) corresponding to v_T and the boundary conditions of (5.1.6) are the following

$$\begin{aligned}\begin{pmatrix} v_3(t,b) \\ v_4(t,b) \end{pmatrix} &= C_b^T \begin{pmatrix} v_1(t,b) \\ v_2(t,b) \end{pmatrix} = \\ &= \begin{pmatrix} C_{b,11} & C_{b,21} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1(t,b) \\ v_2(t,b) \end{pmatrix} = \begin{pmatrix} C_{b,11} v_1(t,b) + C_{b,21} v_2(t,b) \\ 0 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} v_1(t,a) \\ v_2(t,a) \end{pmatrix} &= C_a^T \begin{pmatrix} v_3(t,a) \\ v_4(t,a) \end{pmatrix} = \\ &= \begin{pmatrix} 0 & C_{a,21} \\ C_{a,12} & C_{a,22} \end{pmatrix} \begin{pmatrix} v_3(t,a) \\ v_4(t,a) \end{pmatrix} = \begin{pmatrix} C_{a,21} v_4(t,a) \\ C_{a,12} v_3(t,a) + C_{a,22} v_4(t,a) \end{pmatrix}.\end{aligned}$$

Clearly there results

$$v_4(t,b)=0, \qquad 0\le t\le T. \tag{5.3.15}$$

Let $v=v(t,x)$ be the solution of (5.1.6) corresponding to

$$v_T \equiv \begin{pmatrix} 0 \\ k_2 \\ k_3 \\ 0 \end{pmatrix},$$

where $k_2,\ k_3 \neq 0$ are constants. We claim that

$$v_1(t,b)=0, \qquad 0\le t\le T. \tag{5.3.16}$$

Distinguish three cases. If $T-\dfrac{b-a}{|\lambda_1|}\le t\le T$, there results

$$v_1(t,b)=v_1\big(T,b+\lambda_1(T-t)\big)=v_{T,1}(b+\lambda_1(T-t))=0.$$

If $T-\dfrac{b-a}{|\lambda_1|}-\dfrac{b-a}{|\lambda_4|}\le t<T-\dfrac{b-a}{|\lambda_1|}$, there results

$$\begin{aligned} v_1(t,b) \ &= v_1\left(t+\frac{b-a}{|\lambda_1|},a\right)=C_{a,21}v_4\left(t+\frac{b-a}{|\lambda_1|},a\right)= \\ &= C_{a,21}v_4\left(T,a+\lambda_4\left(T-t-\frac{b-a}{|\lambda_1|}\right)\right)= \\ &= C_{a,21}v_{T,4}\left(a+\lambda_4\left(T-t-\frac{b-a}{|\lambda_1|}\right)\right)=0. \end{aligned}$$

Finally, if $0\le t<T-\dfrac{b-a}{|\lambda_1|}-\dfrac{b-a}{|\lambda_4|}$, we have, by (5.3.15),

$$\begin{aligned} v_1(t,b) \ &= v_1\left(t+\frac{b-a}{|\lambda_1|},a\right)=C_{a,21}v_4\left(t+\frac{b-a}{|\lambda_1|},a\right)= \\ &= C_{a,21}v_{T,4}\left(t+\frac{b-a}{|\lambda_1|}+\frac{b-a}{|\lambda_4|},b\right)=0. \end{aligned}$$

This concludes the proof of (5.3.16), so we have

$$v_T\in \operatorname{Ker}(C^*).$$

On the other hand $v_T\not\equiv 0$, then

$$\operatorname{Ker}(C^*)\neq\{0\},$$

so

$$\operatorname{Im}(C)\neq L^1\big([a,b];\mathbb{R}^n\big),$$

and by Theorem 5.0.2, (5.0.1) is not exactly controllable in finite time.

Moreover since

$$(C_b|\Gamma_b)=\begin{pmatrix} C_{b,11} & 0 & 1 \\ C_{b,21} & 0 & 0 \end{pmatrix},$$

we have

$$\operatorname{rank}(C_b|\Gamma_b) = p = 2, \qquad \operatorname{rank}(C_a) = n - p = 2,$$

so the necessary conditions for the exact controllability in finite time of Theorem 5.2.1 are satisfied. Finally observe that

$$C_a \cdot \Gamma_b = \begin{pmatrix} 0 \\ C_{a,21} \end{pmatrix}, \qquad C_b \cdot C_a \cdot \Gamma_b = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

namely the assumptions of Theorem 5.0.1 are not satisfied.

5.4 Topological Properties of the Attainable Set

In this section we prove the closure of the attainable set.

Theorem 5.4.1 *Using the previous notations, $\mathcal{A}_1(T, \omega_0)$ is a closed subset of $L^1\big([a,b];\mathbb{R}^n\big)$ for each $\omega_0 \in L^1\big([a,b];\mathbb{R}^n\big)$ and $\mathcal{A}_2(T,\omega_0)$ is a closed subset of $L^2\big([a,b];\mathbb{R}^n\big)$, for each $\omega_0 \in L^1\big([a,b];\mathbb{R}^n\big)$ and*

$$T > \frac{b-a}{|\lambda_p|} + \frac{b-a}{|\lambda_{p+1}|}.$$

The fundamental step of the proof of the previous theorem is the following.

Theorem 5.4.2 *Using the previous notations, $\operatorname{Im} C$ is a closed subset of $L^1\big([a,b];\mathbb{R}^n\big)$ and $C\big(L^2\big([0,T];\mathbb{R}^\nu\big)\big)$ is a closed subset of $L^2\big([a,b];\mathbb{R}^n\big)$ for each*

$$T > \frac{b-a}{|\lambda_p|} + \frac{b-a}{|\lambda_{p+1}|}.$$

Remark 5.4.1 Using the method of the characteristics (see [29]) one can easily prove that the L^q–spaces are invariant for the operator C, namely

$$C\Big(L^q\big([0,T];\mathbb{R}^\nu\big)\Big) \subset L^q\big([a,b];\mathbb{R}^n\big),$$

for each $1 \le q \le +\infty$.

Lemma 5.4.1 *There exist two positive constants K_0, K_1 such that for each $T > \dfrac{b-a}{|\lambda_p|} + \dfrac{b-a}{|\lambda_{p+1}|}$, $\omega_0 \in L^1\big([a,b];\mathbb{R}^n\big)$ and $f \in L^1\big([0,T];\mathbb{R}^p\big)$, being $\omega = \omega(t,x)$ the solution of the mixed problem*

$$\begin{cases} \omega_t + A \cdot \omega_x = 0, & a < x < b,\ t > 0, \\ \omega^+(t,a) = C_a \cdot \omega^-(t,a), & 0 < t < T, \\ \omega^-(t,b) = C_b \cdot \omega^+(t,b) + f(t), & 0 < t < T, \\ \omega(0,x) = \omega_0(x), & a < x < b, \end{cases} \tag{5.4.17}$$

there results

$$\|f\|_{L^2} \leq K_0\|\omega_0\|_{L^2} + K_1\|\omega(T,\cdot)\|_{L^2}, \tag{5.4.18}$$

if $\omega_0 \in L^2\big([a,b];\mathbb{R}^n\big)$, $f \in L^2\big([0,T];\mathbb{R}^p\big)$ *and*

$$\|f\|_{L^1} \leq K_0\|\omega_0\|_{L^1} + K_1\|\omega(T,\cdot)\|_{L^1}, \tag{5.4.19}$$

if $\omega_0 \in L^1\big([a,b];\mathbb{R}^n\big)$, $f \in L^1\big([0,T];\mathbb{R}^p\big)$.

PROOF. (5.4.18) is a consequence of [42, (3.20)]. Since the proof of that formula is based on the method of the characteristics, using the same argument we can prove also (5.4.19). □

PROOF OF THEOREM 5.4.2. Let $q \in \{1,2\}$. Consider $\{\omega^\nu\} \subset C\big(L^q([0,T];\mathbb{R}^\nu)\big)$ and $\bar{\omega} \in L^q\big([a,b];\mathbb{R}^n\big)$ such that

$$\omega^\nu \longrightarrow \bar{\omega}, \qquad \text{in } L^q\big([a,b];\mathbb{R}^n\big), \tag{5.4.20}$$

we have to prove that

$$\bar{\omega} \in C\big(L^q([0,T];\mathbb{R}^\nu)\big). \tag{5.4.21}$$

By definition, there exists $\{\alpha^\nu\} \subset L^q\big([0,T];\mathbb{R}^\nu\big)$ such that

$$C\alpha^\nu \equiv \bar{\omega}^\nu, \qquad \nu \in \mathbb{N}.$$

Denote

$$f^\nu(t) = \Gamma_b\alpha^\nu(t), \qquad 0 \leq t \leq T, \quad \nu \in \mathbb{N}. \tag{5.4.22}$$

Consider the solution $\omega = \omega(t,x)$ of (5.4.17) corresponding to $\omega_0 \equiv 0$ and $f^\nu - f^\mu$, for some ν, $\mu \in \mathbb{N}$, by linearity, there results

$$\omega(T,\cdot) \equiv \omega^\nu - \omega^\mu,$$

so, by (5.4.18) or (5.4.19),

$$\|f^\nu - f^\mu\|_{L^q} \leq K_1\|\omega^\nu - \omega^\mu\|_{L^q}.$$

Then, by (5.4.20), $\{f^\nu\}$ is a Cauchy sequence in $L^q\big([0,T];\mathbb{R}^p\big)$, namely there exists a map $\bar{f} \in L^q\big([0,T];\mathbb{R}^p\big)$ such that

$$f^\nu \longrightarrow \bar{f}, \qquad \text{in } L^q\big([0,T];\mathbb{R}^p\big). \tag{5.4.23}$$

Since, by (5.4.22),

$$f^\nu(t) \in \mathrm{Im}(\Gamma_b), \qquad \text{a.e. } 0 \leq t \leq T, \quad \nu \in \mathbb{N},$$

and, by (5.4.23),

$$f^\nu(t) \longrightarrow \bar{f}(t), \qquad \text{a.e. } 0 \leq t \leq T,$$

we have that

$$\bar{f}(t) \in \mathrm{Im}(\Gamma_b), \qquad \text{a.e. } 0 \leq t \leq T.$$

Define the linear map

$$P : \mathrm{Im}(\Gamma_b) \longrightarrow \big(\mathrm{Ker}(\Gamma_b)\big)^\perp, \qquad P(\Gamma_b \cdot v) = v.$$

Since
$$C(Pf_\nu) = C(\alpha_\nu) = \omega_\nu, \qquad \nu \in \mathbb{N},$$
and P is continuous, we have
$$C(P\bar{f}) \equiv \bar{\omega}.$$
This concludes the proof. $\square$

Proof of Theorem 5.4.1. The proof is direct consequence of Theorems 5.0.2 and 5.4.2. $\square$

Corollary 5.4.1 *Using the previous notations and considering the restriction of C to $L^2\big([0,T];\mathbb{R}^\nu\big)$, there results*
$$C\big(L^2\big([0,T];\mathbb{R}^\nu\big)\big) = \big(\operatorname{Ker} C^*\big)^\perp. \tag{5.4.24}$$
Moreover, the following are equivalent

i) (5.0.1) is exactly controllable at time T,

ii) C is surjective,

iii) C^* is injective,

iv) $\mathrm{Im}(C)$ is dense in $L^1\big([a,b];\mathbb{R}^n\big)$,

v) $\mathcal{A}_2(T,0) = L^2\big([a,b];\mathbb{R}^n\big)$.

Proof . The claim is direct consequence of the Theorems 5.0.2, 5.4.1 5.4.2 and [18, Teorema II.18]. $\square$

5.5 Exact Controllability: The case $\nu \geq p$

In this section we begin the proof of Theorem 5.0.1. We start looking at the case
$$p \leq \nu,$$
the main result of this section is the following one.

Theorem 5.5.1 *Let $T > 0$, $A \in \mathbb{R}^{n\times n}$, $C_a \in \mathbb{R}^{(n-p)\times p}$, $C_b \in \mathbb{R}^{p\times(n-p)}$, $\Gamma_b \in \mathbb{R}^{p\times\nu}$ satisfying (5.0.2), (5.0.3) and the following ones*
$$\operatorname{rank}\big(C_b|\Gamma_b\big) = p, \quad \operatorname{rank}\big(C_a\big) = n-p, \quad \operatorname{rank}\big(\Gamma_b\big) = p, \qquad T \geq \frac{b-a}{|\lambda_p|} + \frac{b-a}{|\lambda_{p+1}|}, \tag{5.5.25}$$
then (5.0.1) is exactly controllable at time T.

Remark 5.5.1 Since

$$\operatorname{rank}\big(\Gamma_b\big) \leq \nu,$$

then, by (5.5.25), we have

$$p \leq \nu. \tag{5.5.26}$$

Lemma 5.5.1 (Null-Controllability) *With the assumption of Theorem 5.5.1, for each initial datum $\omega_0 \in L^1\big([a,b];\mathbb{R}^n\big)$ there exists a map $\alpha \in L^1\big([0,T]\big)$ such that the solution $\omega = \omega(t,x)$ of (5.0.1) satisfies the following*

$$\omega(T,x) = 0, \qquad a \leq x \leq b, \tag{5.5.27}$$

in this case we say that (5.0.1) is null-controllable at time T.

Proof. Let $\omega_0 \in L^1\big([a,b];\mathbb{R}^n\big)$ and consider the following control problem

$$\begin{cases} \omega_t + A \cdot \omega_x = 0, & a < x < b,\ t > 0, \\ \omega^+(t,a) = C_a \cdot \omega^-(t,a), & t > 0, \\ \omega^-(t,b) = C_b \cdot \omega^+(t,b) + f(t), & t > 0, \\ \omega(0,x) = \omega_0(x), & a < x < b, \end{cases} \tag{5.5.28}$$

where $f \in L^1\big([0,T];\mathbb{R}^p\big)$. By [42, Theorem 3.2], there exists $\bar{f} \in L^1\big([0,T];\mathbb{R}^p\big)$ such that the corresponding solution of (5.5.28) satisfies (5.5.27). Since, by (5.5.25), the following identity holds

$$L^1\big([0,T];\mathbb{R}^p\big) = \big\{\Gamma_b\alpha;\ \alpha \in L^1\big([0,T];\mathbb{R}^\nu\big)\big\},$$

there exists a map $\bar{\alpha} \in L^1\big([0,T];\mathbb{R}^\nu\big)$ such that

$$\bar{f} \equiv \Gamma_b\bar{\alpha},$$

then the solution of (5.0.1) corresponding to $\bar{\alpha}$ satisfies (5.5.27). □

Lemma 5.5.2 *With the assumption of Theorem 5.5.1 and the previous notations, let $\omega = \omega(t,x)$ be a solution of (5.1.6) such that*

$$v(T,\cdot) \in \operatorname{Ker}(C^*) \cap L^2\big([a,b];\mathbb{R}^n\big), \qquad v(0,\cdot) \equiv 0, \tag{5.5.29}$$

then

$$v(t,x) = 0, \tag{5.5.30}$$

for each $0 \leq t \leq T$ and $a \leq x \leq b$.

Proof. Denote

$$\widetilde{T} = \min\Big\{\frac{b-a}{|\lambda_1|}, \frac{b-a}{|\lambda_n|}\Big\}.$$

We begin proving that

$$v(t,x) = 0, \qquad a \leq x \leq b,\ 0 \leq t \leq \widetilde{T}. \tag{5.5.31}$$

Let be $0 \leq \widetilde{t} \leq \widetilde{T}$, by definition of $\widetilde{T}$, there results

$$v_i(\widetilde{t}, a) = v_i\left(0, a - \frac{\widetilde{t}}{|\lambda_i|}\right) = 0, \qquad i = 1, ..., p. \tag{5.5.32}$$

By (5.5.25),

$$\mathrm{Ker}\,(C_a^T) = \big(\mathrm{Im}\,(C_a)\big)^{\perp} = \{0\}, \tag{5.5.33}$$

since, by (5.1.6) and (5.5.32),

$$-(A^-)^{-1} \cdot C_a^T \cdot A^+ \cdot v^+(\widetilde{t}, a) = v^-(\widetilde{t}, a) = 0, \tag{5.5.34}$$

by (5.5.33), we have

$$v^+(\widetilde{t}, a) = 0. \tag{5.5.35}$$

On the other hand,

$$v_i(\widetilde{t}, b) = v_i\left(0, b - \frac{\widetilde{t}}{\lambda_i}\right) = 0, \qquad i = p+1, ..., n, \tag{5.5.36}$$

by (5.1.6) and (5.5.36),

$$-(A^+)^{-1} \cdot C_b^T \cdot A^- \cdot v^-(\widetilde{t}, b) = v^+(\widetilde{t}, b) = 0,$$

namely

$$C_b^T \cdot A^- \cdot v^-(\widetilde{t}, b) = 0. \tag{5.5.37}$$

Since,

$$\Gamma_b^T \cdot A^- \cdot v^-(\widetilde{t}, b) = 0, \tag{5.5.38}$$

by (5.5.37) and (5.5.38),

$$(C_b|\Gamma_b)^T \cdot A^- \cdot v^-(\widetilde{t}, b) = 0, \tag{5.5.39}$$

by (5.5.25)

$$v^-(\widetilde{t}, b) = 0. \tag{5.5.40}$$

By (5.5.32), (5.5.35), (5.5.36) and (5.5.40),

$$v(t, a) = v(t, b) = 0, \qquad 0 \leq t \leq \widetilde{T} \tag{5.5.41}$$

and this implies (5.5.31). Using again this argument in each rectangle of the type $\big[\,k\widetilde{T}, (k+1)\widetilde{T}\,\big]$ we obtain (5.5.30). □

Lemma 5.5.3 *With the assumption of Theorem 5.5.1 and the previous notations, if (5.0.1) is null controllable at time T, then it is exactly controllable at the same time.*

Proof. We begin proving that

$$C\big(L^2\big([0,T]; \mathbb{R}^{\nu}\big)\big) = L^2\big([a,b]; \mathbb{R}^n\big). \tag{5.5.42}$$

Let $v = v(t,x)$ be a solution of (5.1.6), such that

$$v(T,\cdot) \in \operatorname{Ker} C^* \cap L^2\big([a,b];\mathbb{R}^n\big). \tag{5.5.43}$$

We have to prove that

$$v(T,x) = 0, \qquad a \leq x \leq b. \tag{5.5.44}$$

By Lemma 5.5.1, there exists $\alpha \in L^2\big([0,T]\big)$ such that the solution $\omega = \omega(t,x)$ of (5.0.1) satisfies the following ones

$$\omega(T,\cdot) \equiv 0, \qquad \omega(0,\cdot) \equiv v(0,\cdot). \tag{5.5.45}$$

By (5.5.45), there results

$$\big(v(0,\cdot),\omega(0,\cdot)\big)_{L^2} = \|v(0,\cdot)\|^2_{L^2}, \tag{5.5.46}$$

by (5.5.43),

$$\big(\Gamma_b^T \cdot A^- \cdot v^-(\cdot,b),\alpha\big)_{L^2} = \big(C^* v(T,\cdot),\alpha\big)_{L^2} = 0, \tag{5.5.47}$$

so by (5.5.45),

$$\big(v(T,\cdot),\omega(T,\cdot)\big)_{L^2} = \big(v(T,\cdot),0\big)_{L^2} = 0. \tag{5.5.48}$$

Finally, by (5.1.8), (5.5.46), (5.5.47) and (5.5.48), we have

$$\begin{aligned} \|v(0,\cdot)\|^2_{L^2} &= \big(v(0,\cdot),\omega(0,\cdot)\big)_{L^2} = \\ &= \big(v(T,\cdot),\omega(T,\cdot)\big)_{L^2} + \big(\Gamma_b^T \cdot A^- \cdot v^-(\cdot,b),\alpha\big)_{L^2} = 0, \end{aligned} \tag{5.5.49}$$

then

$$v(0,x) = 0, \qquad a \leq x \leq b. \tag{5.5.50}$$

By Lemma 5.5.2, we have (5.5.44) and by Corollary 5.4.1, (5.5.42) is proved.

By the density of $L^2\big([a,b];\mathbb{R}^\nu\big)$ in $L^1\big([a,b];\mathbb{R}^\nu\big)$ and Corollary 5.4.1 the proof is done. □

Proof of Theorem 5.5.1. It is direct consequence of Lemmas 5.5.2 and 5.5.3. □

Corollary 5.5.1 *With the assumptions of Theorem 5.5.1, the problem (5.0.1) is asymptotically exponentially stabilizable.*

Proof. It is direct consequence of Theorem 5.5.1 and [43, Theorem 3.14]. □

5.6 Exact Controllability: The Case $n - p \leq \nu < p$

In this section we continue the proof of Theorem 5.0.1 considering the case

$$n - p \leq \nu \leq p,$$

the main result of this section is the following one.

Theorem 5.6.1 *Consider (5.0.1) and assume that*

$$\nu < p, \qquad T \geq 2\frac{b-a}{|\lambda_p|} + \frac{b-a}{|\lambda_{p+1}|}, \tag{5.6.51}$$

and

$$\operatorname{rank}\big(C_a\big) = n-p, \qquad \operatorname{rank}\big(C_a \cdot \Gamma_b\big) = n-p, \qquad \operatorname{rank}\big(C_b|\Gamma_b\big) = p. \tag{5.6.52}$$

Then (5.0.1) is exactly controllable at time T.

Remark 5.6.1 Since

$$\operatorname{rank}\big(C_a \cdot \Gamma_b\big) \leq \nu,$$

then, by (5.6.51) and (5.6.52), we have

$$n - p \leq \nu < p. \tag{5.6.53}$$

Consider now this new control problem

$$\begin{cases} u_t + A \cdot u_x = 0, & a < x < b,\ t > 0, \\ u^+(t,a) = C_a \cdot u^-(t,a) + f(t), & t > 0, \\ u^-(t,b) = C_b \cdot u^+(t,b), & t > 0, \\ u(0,x) = u_0(x), & a < x < b, \end{cases} \tag{5.6.54}$$

and define the linear operator

$$\widetilde{C} : L^2\big([0,T];\mathbb{R}^\nu\big) \longrightarrow L^2\big([a,b];\mathbb{R}^n\big),$$

where $\widetilde{C}f$ is the solution of (5.6.54) at time T with $u_0 \equiv 0$.

First of all remind that (5.6.54) is exactly controllable for each time $\geq T_p + T_{p+1}$ (see Theorem 5.5.1), where

$$T_i \doteq \frac{b-a}{|\lambda_i|}, \qquad i = 1, ..., n,$$

then there results

$$\operatorname{Im}(\widetilde{C}) = \widetilde{C}\Big(L^2\big([T_1,T];\mathbb{R}^\nu\big)\Big) = ... = \widetilde{C}\Big(L^2\big([T_p,T];\mathbb{R}^\nu\big)\Big), \tag{5.6.55}$$

where we use the following identifications

$$L^2\big([T_i,T];\mathbb{R}^\nu\big) \simeq \left\{g \in L^2\big([0,T];\mathbb{R}^\nu\big); g(t) = 0 \text{ if } 0 \leq t < T_i\right\}, \qquad i = 1, ..., n. \tag{5.6.56}$$

Lemma 5.6.1 *Assume (5.6.51) and (5.6.52). Let $\alpha \in L^2\big([0,T];\mathbb{R}^\nu\big)$ such that*

$$\alpha(t) = 0 \quad if \ T - T_p \leq t \leq T. \tag{5.6.57}$$

Define

$$f_\alpha(t) \doteq C_a g_\alpha(t) = C_a\big(g_{\alpha,1}(t), ..., g_{\alpha,\nu}(t)\big), \tag{5.6.58}$$

where

$$g_{\alpha,i}(t) = \begin{cases} 0, & \text{if } 0 \le t \le T_i, \\ \Gamma_{i1}\alpha_1(t-T_i) + \ldots + \Gamma_{i\nu}\alpha_\nu(t-T_i), & \text{if } T_i \le t \le T. \end{cases} \tag{5.6.59}$$

There results

$$C\alpha \equiv \widetilde{C} f_\alpha. \tag{5.6.60}$$

Proof. Let $\omega = \omega(t,x)$ be the solution of (5.0.1) corresponding to α with $u_0 \equiv 0$, by definition

$$\omega(T,x) = (C\alpha)(x), \qquad a \le x \le b,$$

and $u = u(t,x)$ be the one of (5.6.54) corresponding to f_α with $\omega_0 \equiv 0$, by definition

$$u(T,x) = \big(\widetilde{C} f_\alpha\big)(x), \qquad a \le x \le b.$$

Distinguish some cases. If $0 < t < T_1$ there results (see Figure 5.1)

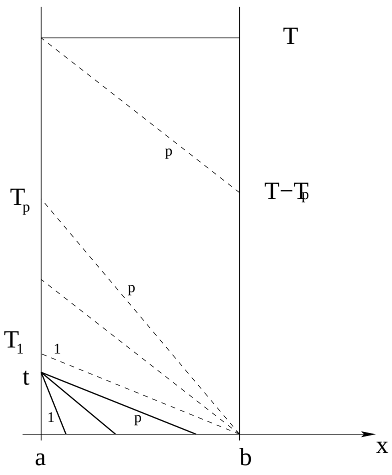

Figure 5.1

$$\omega^+(t,a) = C_a\omega^-(t,a) = C_a\big(\omega_1(0,\lambda_1 t + b), \ldots, \omega_p(0,\lambda_p t + b)\big) = 0$$

and

$$u^+(t,a) = C_a u^-(t,a) = C_a\big(u_1(0,\lambda_1 t + b), \ldots, u_p(0,\lambda_p t + b)\big) = 0.$$

If $T_1 \le t \le T_2$, there results (see Figure 5.2)

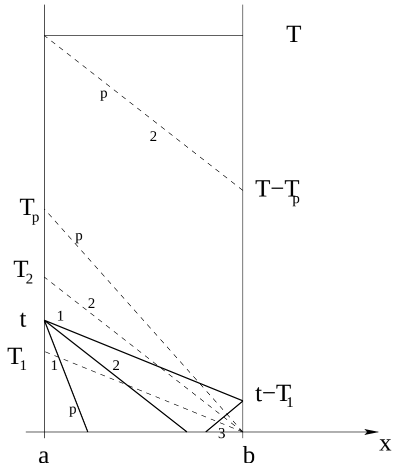

Figure 5.2

$$\begin{aligned}\omega^+(t,a) &= C_a\omega^-(t,a) = \\ &= C_a\big(\omega_1(t-T_1,a),\omega_2(0,\lambda_2 t+b),...,\omega_p(0,\lambda_p t+b)\big) = \\ &= C_a\big(\omega_1(t-T_1,a),0,...,0\big).\end{aligned}$$

Observe that,

$$\begin{aligned}\omega_1(t-T_1,b) &= \big[C_b\omega^+(t-T_1,b)\big]_1 + \Gamma_{11}\alpha_1(t-T_1)+...+\Gamma_{1\nu}\alpha_\nu(t-T_1) = \\ &= \big[C_b\omega^+(t-T_1,b)\big]_1 + g_1(t)\end{aligned}$$

and

$$\begin{aligned}\omega^+(t-T_1,b) &= \big(\omega_{p+1}(t-T_1,b),...,\omega_n(t-T_1,b)\big) = \\ &= \Big(\omega_{p+1}\big(0,\lambda_{p+1}(t-T_1)+a\big),...,\omega_n\big(0,\lambda_n(t-T_1)+a\big)\Big) = 0,\end{aligned}$$

then

$$\omega^+(t,a) = f_\alpha(t).$$

Moreover we have that

$$\begin{aligned}u^+(t,a) &= C_a\big(u_1(t-T_1,b),u_2(0,\lambda_2 t+b),...,u_p(0,\lambda_p t+b)\big) + f_\alpha(t) = \\ &= C_a\big(u_1(t-T_1,b),0,...,0\big) + f_\alpha(t).\end{aligned}$$

Observe that, for $k=1,...,j$,

$$u_1(t-T_1,b) = \big[C_b u^+(t-T_1,b)\big]_1$$

and

$$\begin{aligned}u^+(t-T_1,b) &= \big(u_{p+1}(t-T_1,b),...,u_n(t-T_1,b)\big) = \\ &= \Big(u_{p+1}\big(0,\lambda_{p+1}(t-T_1)+a\big),...,u_n\big(0,\lambda_n(t-T_1)+a\big)\Big) = 0,\end{aligned}$$

then

$$u^+(t,a) = f_\alpha(t).$$

If the positive characteristics have a reflection on the boundary $x = a$, we are reduced to the previous case. Arguing in this way one can prove that

$$\omega^+(t,a) = u^+(t,a), \qquad 0 \le t \le T. \tag{5.6.61}$$

Let $a \le x \le b$, observe that, by (5.6.61), for $k = p+1, ..., n$, (see Figure 5.3)

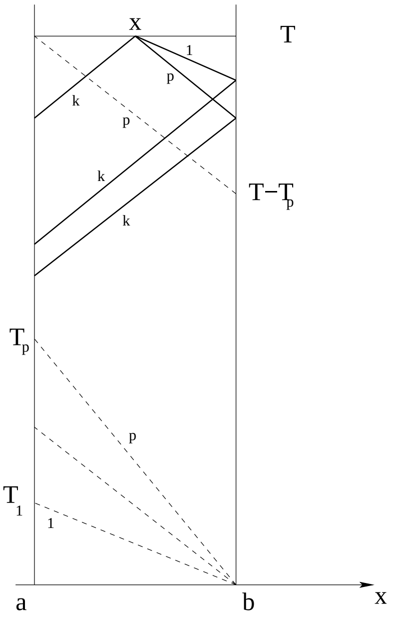

Figure 5.3

$$\begin{aligned}\Big[(C\alpha)(x)\Big]_k &= \omega_k(T,x) = \omega_k\left(T + \frac{x-a}{\lambda_k}, a\right) = \\ &= u_k\left(T + \frac{x-a}{\lambda_k}, a\right) = u_k(T,x) = \Big[(\widetilde{C} f_\alpha)(x)\Big]_k,\end{aligned}$$

moreover, by (5.6.57), for $k = 1, ..., p$, (see Figure 5.3)

$$\begin{aligned}&\Big[(C\alpha)(x)\Big]_k = \omega_k(T,x) = \omega_k\left(T + \frac{b-x}{\lambda_k}, b\right) = \\ &= \left[C_b\omega^+\left(T + \frac{b-x}{\lambda_k}, b\right)\right]_k + \Gamma_{k1}\alpha_1\left(T + \frac{b-x}{\lambda_k}\right) + ... + \Gamma_{k\nu}\alpha_\nu\left(T + \frac{b-x}{\lambda_k}\right) = \\ &= \left[C_b\omega^+\left(T + \frac{b-x}{\lambda_k}, b\right)\right]_k.\end{aligned}$$

Observe that, by (5.6.61),

$$\begin{aligned}
&\omega^+\left(T+\frac{b-x}{\lambda_k},b\right)=\\
&=\left(\omega_{p+1}\left(T+\frac{b-x}{\lambda_k},b\right),...,\omega_n\left(T+\frac{b-x}{\lambda_k},b\right)\right)=\\
&=\left(\omega_{p+1}\left(T+\frac{b-x}{\lambda_k}-T_{p+1},a\right),...,\omega_n\left(T+\frac{b-x}{\lambda_k}-T_n,a\right)\right)=\\
&=\left(u_{p+1}\left(T+\frac{b-x}{\lambda_k}-T_{p+1},a\right),...,u_n\left(T+\frac{b-x}{\lambda_k}-T_n,a\right)\right)=\\
&=\left(u_{p+1}\left(T+\frac{b-x}{\lambda_k},b\right),...,u_n\left(T+\frac{b-x}{\lambda_k},b\right)\right)=u^+\left(T+\frac{b-x}{\lambda_k},b\right),
\end{aligned}$$

then

$$\begin{aligned}
\Big[(C\alpha)(x)\Big]_k &=\omega_k(T,x)=\omega_k\left(T+\frac{b-x}{\lambda_k},b\right)=\\
&=\left[C_b\omega^+\left(T+\frac{b-x}{\lambda_k},b\right)\right]_k=\left[C_bu^+\left(T+\frac{b-x}{\lambda_k},b\right)\right]_k=\\
&=u_k\left(T+\frac{b-x}{\lambda_k},b\right)=u_k(T,x)=\Big[(\widetilde{C}f_\alpha)(x)\Big]_k,
\end{aligned}$$

and this proves (5.6.60). □

Define the linear operator

$$F:L^2\big([0,T];\mathbb{R}^\nu\big)\longrightarrow L^2\big([0,T];\mathbb{R}^\nu\big),$$

such that

$$(F\alpha)(t)\doteq C_a g_\alpha(t),\qquad 0\le t\le T,$$

where g_α is defined in (5.6.58) and (5.6.59) and denote

$$\widetilde{X}\doteq\big\{\alpha\in L^2\big([0,T];\mathbb{R}^\nu\big);\alpha(t)=0\quad\text{if }\ T-T_p\le t\le T\big\},\qquad X\doteq F(\widetilde{X}).$$

Clearly, by (5.6.60),

$$\widetilde{C}(X)\subset \mathrm{Im}(C). \tag{5.6.62}$$

Lemma 5.6.2 *Assume (5.6.51) and (5.6.52). There results*

$$\widetilde{C}(X)\ \textit{is dense in}\ L^2\big([a,b];\mathbb{R}^n\big), \tag{5.6.63}$$

where we use the previous notations.

Define the linear operators

$$F_1,\ ...,\ F_p:L^2\big([0,T];\mathbb{R}^\nu\big)\longrightarrow L^2\big([0,T];\mathbb{R}^\nu\big),$$

such that

$$\begin{aligned}
(F_i\alpha)(t)\ &\doteq\begin{cases}C_a g_\alpha(t), & \text{if } T_i\le t\le T_{i+1},\\ 0, & \text{otherwise},\end{cases}\qquad i=1,...,p-1,\\
(F_p\alpha)(t)\ &\doteq\begin{cases}C_a g_\alpha(t), & \text{if } T_p\le t\le T,\\ 0, & \text{otherwise},\end{cases}
\end{aligned}$$

there results

$$F\alpha = F_1\alpha + \ldots + F_p\alpha, \qquad \big(F_i\alpha, F_j\alpha\big)_{L^2} = 0, \qquad \alpha \in L^2\big([0,T];\mathbb{R}^\nu\big), \tag{5.6.64}$$

for each $i, j = 1, \ldots, p$ distinct.

Lemma 5.6.3 *Assume (5.6.51) and (5.6.52). Using the previous notations*

$$F_p(\widetilde{X}) \text{ is dense in } L^2\big([T_p,T];\mathbb{R}^\nu\big), \tag{5.6.65}$$

where we follow the identification of (5.6.56).

Proof. It is sufficient to prove that

$$\mathcal{P}_k \subset F_p(\widetilde{X}), \qquad k \in \mathbb{N}, \tag{5.6.66}$$

where $\mathcal{P}_k$ is the set of the polynomials defined on $[T_p, T]$ with values in $\mathbb{R}^\nu$ of degree $\leq k$. We argue by induction.

We begin proving

$$\mathcal{P}_0 \subset F_p(\widetilde{X}), \tag{5.6.67}$$

remind that $\mathcal{P}_0$ is the set of the constant map defined on $[T_p, T]$ with values in $\mathbb{R}^\nu$. Let be $h \in \mathbb{R}^\nu$, by (5.6.52), there exists a vector $\alpha \in \mathbb{R}^{n-p}$ such that

$$(C_a \cdot \Gamma_b)\alpha = h.$$

Since α is constant, we have

$$\alpha \in \widetilde{X}, \qquad F_p\alpha \equiv (C_a \cdot \Gamma_b)\alpha = h,$$

then

$$h \in F_p(\widetilde{X})$$

and this proves (5.6.67).

Let $k \in \mathbb{N}$, assume that

$$\mathcal{P}_k \subset F_p(\widetilde{X}), \tag{5.6.68}$$

we prove

$$\mathcal{P}_{k+1} \subset F_p(\widetilde{X}). \tag{5.6.69}$$

Let $f \in \mathcal{P}_{k+1}$, namely

$$f(t) = f_{k+1}t^{k+1} + \ldots + f_0,$$

where $f_0, \ldots, f_{k+1} \in \mathbb{R}^{n-p}$ are constants. By (5.6.52), there exists a constant $\bar{\alpha} \in \mathbb{R}^\nu$, such that

$$(C_a \cdot \Gamma_b)\bar{\alpha} = f_{k+1}.$$

Define

$$\hat{\alpha}(t) \doteq \bar{\alpha}t^{k+1},$$

by definition

$$\hat{\alpha} \in \widetilde{X}, \qquad (F_p\bar{\alpha})(t) = (C_a \cdot \Gamma_b)\bar{\alpha}t^{k+1} + p_k = f_{k+1}t^{k+1} + p_k,$$

where $p_k \in \mathcal{P}_k$. By (5.6.68), there exists $\tilde{\alpha} \in \widetilde{X}$ such that

$$(F_p\tilde{\alpha})(t) = f_k t^k + ... + f_0 - p_k(t),$$

then, by the linearity of F_p,

$$\big(F_p(\hat{\alpha} + \tilde{\alpha})\big)(t) = f_{k+1}t^{k+1} + ... + f_0 = f(t)$$

and this proves (5.6.69). By induction, (5.6.66) is done and by wellknown arguments the claim is proved. $\square$

Proof of Lemma 5.6.2. By (5.6.55) and (5.6.65),

$$\overline{\widetilde{C}(X)} = \overline{\widetilde{C}\big(F_p(\widetilde{X})\big)} = \overline{\widetilde{C}\Big(L^2\big([T_p, T]; \mathbb{R}^\nu\big)\Big)} = L^2\big([a,b]; \mathbb{R}^n\big),$$

this concludes the proof. $\square$

Proof of Theorem 5.6.1. Since, by Theorem 5.4.2, $\mathrm{Im}(C)$ is closed, by (5.6.62) and (5.6.63), we have

$$L^2\big([a,b]; \mathbb{R}^n\big) = \overline{\widetilde{C}(X)} \subset \mathrm{Im}(C) \subset L^2\big([a,b]; \mathbb{R}^n\big),$$

namely

$$\mathrm{Im}(C) = L^2\big([a,b]; \mathbb{R}^n\big).$$

Then, by Corollary 5.4.1, (5.0.1) is exactly controllable at time T. $\square$

5.7 Exact Controllability: The Case $\nu \le n - p = p$

In this section we continue the proof of Theorem 5.0.1 considering the case

$$\nu \le n - p = p,$$

the main result of this section is the following one.

Theorem 5.7.1 *Consider (5.0.1) and assume that*

$$\nu < n - p, \qquad T \ge l\left(\frac{b-a}{|\lambda_p|} + \frac{b-a}{|\lambda_{p+1}|}\right) = lT_0, \tag{5.7.70}$$

and

$$\mathrm{rank}\big(C_a\big) = n - p, \quad \mathrm{rank}\big((C_b \cdot C_a)^{l-1} \cdot \Gamma_b|...|\Gamma_b\big) = p, \quad \mathrm{rank}\big(C_b|\Gamma_b\big) = p, \tag{5.7.71}$$

where $l \in \mathbb{N}$ such that

$$l - 1 < \frac{n-p}{\nu} \le l.$$

Then (5.0.1) is exactly controllable at time T.

Consider now this new control problem

$$\begin{cases} u_t + A \cdot u_x = 0, & a < x < b,\ t > 0, \\ u^+(t,a) = C_a \cdot u^-(t,a) + f(t), & t > 0, \\ u^-(t,b) = C_b \cdot u^+(t,b), & t > 0, \\ u(0,x) = u_0(x), & a < x < b, \end{cases} \tag{5.7.72}$$

and define the linear operator

$$\widetilde{C} : L^1\big([0,T];\mathbb{R}^{n-p}\big) \longrightarrow L^1\big([a,b];\mathbb{R}^n\big),$$

where $\widetilde{C}f$ is the solution of (5.7.72) at time T with $u_0 \equiv 0$.

First of all remind that (5.7.72) is exactly controllable for each time $\geq T_p + T_{p+1}$ (see Theorem 5.5.1), where

$$T_i \doteq \frac{b-a}{|\lambda_i|}, \qquad i = 1, ..., n,$$

then there results

$$\operatorname{Im}(\widetilde{C}) = \widetilde{C}\Big(L^1\big([\tau,T];\mathbb{R}^\nu\big)\Big), \qquad 0 \leq \tau \leq (l-1)T_0, \tag{5.7.73}$$

where we use the following identification

$$L^1\big([\tau,T];\mathbb{R}^{n-p}\big) \simeq \big\{f \in L^1\big([0,T];\mathbb{R}^{n-p}\big); g(t) = 0 \text{ if } 0 \leq t \leq \tau\big\}, \qquad 0 \leq \tau \leq T. \tag{5.7.74}$$

Consider now this second control problem

$$\begin{cases} v_t + A \cdot v_x = 0, & a < x < b,\ t > 0, \\ v^+(t,a) = C_a \cdot v^-(t,a), & t > 0, \\ v^-(t,b) = C_b \cdot v^+(t,b) + g(t), & t > 0, \\ v(0,x) = v_0(x), & a < x < b, \end{cases} \tag{5.7.75}$$

and define the linear operator

$$\hat{C} : L^1\big([0,T];\mathbb{R}^p\big) \longrightarrow L^1\big([a,b];\mathbb{R}^n\big),$$

where $\hat{C}g$ is the solution of (5.7.75) at time T with $v_0 \equiv 0$. Clearly we have

$$C\alpha = \hat{C}(\Gamma_b\alpha), \qquad \alpha \in L^1\big([0,T];\mathbb{R}^\nu\big).$$

Remind that also (5.7.75) is exactly controllable for each time $\geq T_p + T_{p+1}$ (see Theorem 5.5.1), then there results

$$\operatorname{Im}(\hat{C}) = \hat{C}\Big(L^1\big([\tau,T];\mathbb{R}^\nu\big)\Big), \qquad 0 \leq \tau \leq (l-1)T_0, \tag{5.7.76}$$

where we use the identification in (5.7.74).

Define the linear operator

$$\widetilde{F} : L^1\big([0,(l-1)T_0 + T_p];\mathbb{R}^{n-p}\big) \longrightarrow L^1\big([0,lT_0];\mathbb{R}^{n-p}\big),$$

$$\big(\widetilde{F}(f)\big)_i(t)=\begin{cases} f_i(t-T_i), & \text{if } T_i\le t\le lT_0,\\ 0, & \text{otherwise,}\end{cases}\qquad i=p+1,...,n.$$

Observe that, if $f\in L^1\big([(j-1)T_0+T_p,jT_0+T_p];\mathbb{R}^{n-p}\big)$ for some $j=1,...,l-1$, then

$$\big[jT_0,(j+1)T_0\big]\subset \operatorname{supp}\big(\widetilde{F}(f)\big)\subset\big[(j-1)T_0+T_p+T_n,(j+1)T_0\big]. \tag{5.7.77}$$

Define this new linear operator

$$\hat{F}:L^1\big([0,(l-1)T_0];\mathbb{R}^p\big)\longrightarrow L^1\big([0,(l-1)T_0+T_p];\mathbb{R}^p\big),$$

$$\big(\hat{F}(g)\big)_i(t)=\begin{cases} g_i(t-T_i), & \text{if } T_i\le t\le (l-1)T_0+T_p,\\ 0, & \text{otherwise,}\end{cases}\qquad i=1,...,p.$$

Observe that, if $g\in L^1\big([(j-1)T_0,jT_0];\mathbb{R}^p\big)$ for some $j=1,...,l-1$, then

$$\big[(j-1)T_0+T_p,jT_0+T_p\big]\subset \operatorname{supp}\big(\widetilde{F}(g)\big)\subset\big[(j-1)T_0+T_1,jT_0+T_p\big]. \tag{5.7.78}$$

Arguing as in Lemma 5.6.1 we can prove the following.

Lemma 5.7.1 *Assume (5.7.70) and (5.7.71). Using the previous notations, there results*

$$\widetilde{C}(f)=\hat{C}\big(C_b\cdot\widetilde{F}(f)\big),\qquad \hat{C}(g)=\widetilde{C}\big(C_a\cdot\hat{F}(g)\big), \tag{5.7.79}$$

for each $f\in L^1\big([0,(l-1)T_0+T_p];\mathbb{R}^{n-p}\big)$ *and* $g\in L^1\big([0,(l-1)T_0];\mathbb{R}^p\big)$.

Finally, define the linear operator

$$F:L^1\big([0,(l-1)T_0];\mathbb{R}^p\big)\longrightarrow L^1\big([0,lT_0];\mathbb{R}^p\big),\qquad F(g)\doteq C_b\cdot\widetilde{F}\big(C_a\cdot\hat{F}(g)\big),$$

and observe that

$$\hat{C}(F(g))=\hat{C}(g),\qquad g\in L^1\big([0,(l-1)T_0];\mathbb{R}^p\big). \tag{5.7.80}$$

Indeed, by (5.7.79),

$$\hat{C}(F(g))=\hat{C}\big(C_b\cdot\widetilde{F}\big(C_a\cdot\hat{F}(g)\big)\big)=\widetilde{C}\big(C_a\cdot\hat{F}(g)\big)=\hat{C}(g), \tag{5.7.81}$$

for each $g\in L^1\big([0,(l-1)T_0];\mathbb{R}^p\big)$.

Remark 5.7.1 Let $k\in\mathbb{R}^p$ be a constant, $j\in\{1,...,l-1\}$ and consider the map

$$g(t)\doteq k\cdot\chi_{[(j-1)T_0,jT_0]}(t),\qquad 0\le t\le T,$$

then for $jT_0\le t\le (j+1)T_0$, there results

$$\big(F(g)\big)(t)=C_b\cdot C_a\cdot k.$$

Let $\alpha\in L^1\big([0,lT_0];\mathbb{R}^\nu\big)$, denote

$$\alpha_i\doteq\bar{\alpha}\cdot\chi_{[(i-1)T_0,iT_0]},\qquad i=1,...,l,$$

and observe

$$\big[(l-1)T_0,lT_0\big]\subset\operatorname{supp}\big(F^{l-i}(\Gamma_b\alpha_i)\big),\qquad i=1,...,l-1.$$

Remark 5.7.2 Observe that, if

$$\alpha_i \equiv k_i(\text{constant}) \qquad \text{on } \big[(i-1)T_0, iT_0\big],$$

then, by Remark 5.7.1,

$$\big(F^{l-i}(\Gamma_b \alpha_i)\big)(t) = (C_b \cdot C_a)^{l-1} \cdot \Gamma_g k_i \qquad \text{on } \big[(l-1)T_0, lT_0\big].$$

Define these linear operators

$$\mathcal{F} : L^1\big([0, lT_0]; \mathbb{R}^\nu\big) \longrightarrow L^1\big([0, lT_0]; \mathbb{R}^p\big), \qquad \mathcal{F}(\alpha) \doteq \sum_{i=1}^{l} F^{l-i}(\Gamma_b \alpha_i),$$

and

$$\widetilde{\mathcal{F}} : L^1\big([0, lT_0]; \mathbb{R}^\nu\big) \longrightarrow L^1\big([(l-1)T_0, lT_0]; \mathbb{R}^p\big), \qquad \widetilde{\mathcal{F}}(\alpha) \doteq \mathcal{F}(\alpha) \doteq \chi_{[(l-1)T_0, lT_0]}.$$

By definition and (5.7.81),

$$\hat{C}\big(\mathcal{F}(\alpha)\big) = C\alpha, \qquad \alpha \in L^1\big([0, lT_0]; \mathbb{R}^\nu\big), \tag{5.7.82}$$

so it is clear that

$$\operatorname{Im}(\widetilde{\mathcal{F}}) \subset \operatorname{Im}(\mathcal{F}) \subset L^1\big([(l-1)T_0, lT_0]; \mathbb{R}^p\big) \quad \text{and} \quad \operatorname{Im}\big(\hat{C}(\mathcal{F})\big) \subset \operatorname{Im}(C). \tag{5.7.83}$$

Lemma 5.7.2 *Assume (5.6.51) and (5.6.52). Using the previous notations*

$$\operatorname{Im}(\widetilde{\mathcal{F}}) \textit{ is dense in } L^1\big([(l-1)T_0, lT_0]; \mathbb{R}^p\big), \tag{5.7.84}$$

where we follow the identification of (5.7.74).

Proof. It is sufficient to prove that

$$\mathcal{P}_k \subset \operatorname{Im}(\widetilde{\mathcal{F}}), \qquad k \in \mathbb{N}, \tag{5.7.85}$$

where $\mathcal{P}_k$ is the set of the polynomials defined on $\big[(l-1)T_0, lT_0\big]$ with values in $\mathbb{R}^p$ of degree $\leq k$. We argue by induction.

We begin proving

$$\mathcal{P}_0 \subset \operatorname{Im}(\widetilde{\mathcal{F}}), \tag{5.7.86}$$

remind that $\mathcal{P}_0$ is the set of the constant map defined on $\big[(l-1)T_0, lT_0\big]$ with values in $\mathbb{R}^p$. Let $h \in \mathbb{R}^p$, by (5.7.71), there exists a vector $\alpha \in \mathbb{R}^\nu$ such that

$$\big((C_b \cdot C_a)^l \cdot \Gamma_b | ... | \Gamma_b\big)\alpha = h.$$

Since α is constant, by Remark 5.7.2, we have

$$\widetilde{\mathcal{F}}\alpha \equiv \big((C_b \cdot C_a)^l \cdot \Gamma_b | ... | \Gamma_b\big)\alpha = h,$$

then

$$h \in \mathrm{Im}(\widetilde{\mathcal{F}})$$

and this proves (5.7.86).

Let $k \in \mathbb{N}$, assume that

$$\mathcal{P}_k \subset \mathrm{Im}(\widetilde{\mathcal{F}}), \tag{5.7.87}$$

we prove

$$\mathcal{P}_{k+1} \subset \mathrm{Im}(\widetilde{\mathcal{F}}). \tag{5.7.88}$$

Let $f \in \mathcal{P}_{k+1}$, namely

$$f(t) = f_{k+1}t^{k+1} + ... + f_0,$$

where $f_0, ..., f_{k+1} \in \mathbb{R}^p$ are constants. By (5.7.71), there exists a constant $\bar{\alpha} \in \mathbb{R}^\nu$, such that

$$\big((C_b \cdot C_a)^l \cdot \Gamma_b|...|\Gamma_b\big)\bar{\alpha} = f_{k+1}.$$

Define

$$\hat{\alpha}(t) \doteq \bar{\alpha}t^{k+1},$$

by definition,

$$(\widetilde{\mathcal{F}}\bar{\alpha})(t) = \big((C_b \cdot C_a)^l \cdot \Gamma_b|...|\Gamma_b\big)\bar{\alpha}t^{k+1} + p_k = f_{k+1}t^{k+1} + p_k,$$

for some $p_k \in \mathcal{P}_k$. By (5.7.87), there exists $\tilde{\alpha} \in L^1\big([0, lT_0]; \mathbb{R}^\nu\big)$ such that

$$(\widetilde{\mathcal{F}}\tilde{\alpha})(t) = f_k t^k + ... + f_0 - p_k(t),$$

then, by the linearity of $\widetilde{\mathcal{F}}$,

$$\big(\widetilde{\mathcal{F}}(\hat{\alpha} + \tilde{\alpha})\big)(t) = f_{k+1}t^{k+1} + ... + f_0 = f(t)$$

and this proves (5.7.88). By induction, (5.7.85) is done and by wellknown arguments the claim is proved. $\square$

Proof of Theorem 5.7.1. Since, by Theorem 5.4.2, $\mathrm{Im}(C)$ is closed, by (5.7.76) and (5.7.83), we have

$$L^1\big([a,b]; \mathbb{R}^n\big) = \overline{\hat{C}(\mathrm{Im}\mathcal{F})} \subset \mathrm{Im}(C) \subset L^1\big([a,b]; \mathbb{R}^n\big),$$

namely,

$$\mathrm{Im}(C) = L^1\big([a,b]; \mathbb{R}^n\big).$$

Then, by Corollary 5.4.1, (5.0.1) is exactly controllable at time T. $\square$

5.8 Exact Controllability: The Case $\nu \le n-p < p$

In this section we conclude the proof of Theorem 5.0.1 considering the case

$$\nu \le n-p < p,$$

the main result of this section is the following one.

Theorem 5.8.1 *Consider (5.0.1) and assume that*

$$\nu < n-p, \qquad T \ge l\left(\frac{b-a}{|\lambda_p|} + \frac{b-a}{|\lambda_{p+1}|}\right) + \frac{b-a}{|\lambda_p|}, \tag{5.8.89}$$

and

$$\operatorname{rank}\big(C_a\big) = n-p, \quad \operatorname{rank}\big(C_a \cdot (C_b \cdot C_a)^{l-1} \cdot \Gamma_b|...|C_a \cdot \Gamma_b\big) = n-p, \quad \operatorname{rank}\big(C_b|\Gamma_b\big) = p, \tag{5.8.90}$$

where $l \in \mathbb{N}$ such that

$$l - 1 < \frac{n-p}{\nu} \le l.$$

Then (5.0.1) is exactly controllable at time T.

Consider now the linear operators

$$\widetilde{C} : L^1\big([0,T];\mathbb{R}^{n-p}\big) \longrightarrow L^1\big([a,b];\mathbb{R}^n\big), \qquad \hat{C} : L^1\big([0,T];\mathbb{R}^p\big) \longrightarrow L^1\big([a,b];\mathbb{R}^n\big),$$

defined in Section 5.7 and denote

$$T_0 \doteq \frac{b-a}{|\lambda_p|} + \frac{b-a}{|\lambda_{p+1}|}, \qquad T_i \doteq \frac{b-a}{|\lambda_i|}, \qquad i = 1,...,n.$$

Define the linear operator

$$\widetilde{F} : L^1\big([0,(l-1)T_0 + 2T_p];\mathbb{R}^{n-p}\big) \longrightarrow L^1\big([0, lT_0 + T_p];\mathbb{R}^{n-p}\big),$$

$$\big(\widetilde{F}(f)\big)_i(t) = \begin{cases} f_i(t-T_i), & \text{if } T_i \le t \le lT_0 + T_p, \\ 0, & \text{otherwise,} \end{cases} \qquad i = p+1,...,n,$$

where we use the identification in (5.7.74). Observe that, if $f \in L^1\big([(j-1)T_0 + T_p, jT_0 + T_p];\mathbb{R}^{n-p}\big)$ for some $j = 1,...,l-1$, then

$$\big[jT_0,(j+1)T_0\big] \subset \operatorname{supp}\big(\widetilde{F}(f)\big) \subset \big[(j-1)T_0 + T_p + T_n,(j+1)T_0\big]. \tag{5.8.91}$$

Define this new linear operator

$$\hat{F} : L^1\big([0,lT_0];\mathbb{R}^p\big) \longrightarrow L^1\big([0,lT_0 + T_p];\mathbb{R}^p\big),$$

$$\big(\hat{F}(g)\big)_i(t) = \begin{cases} g_i(t-T_i), & \text{if } T_i \le t \le lT_0 + T_p, \\ 0, & \text{otherwise,} \end{cases} \qquad i = 1,...,p.$$

Observe that, if $g \in L^1\big([(j-1)T_0, jT_0];\mathbb{R}^p\big)$ for some $j = 1,...,l-1$, then

$$\big[(j-1)T_0 + T_p, jT_0 + T_p\big] \subset \operatorname{supp}\big(\widetilde{F}(g)\big) \subset \big[(j-1)T_0 + T_1, jT_0 + T_p\big]. \tag{5.8.92}$$

Arguing as in Lemma 5.6.1 we can prove the following.

Lemma 5.8.1 *Assume (5.7.70) and (5.7.71). Using the previous notations, there results*

$$\widetilde{C}(f) = \hat{C}\big(C_b \cdot \widetilde{F}(f)\big), \qquad \hat{C}(g) = \widetilde{C}\big(C_a \cdot \hat{F}(g)\big), \tag{5.8.93}$$

for each $f \in L^1\big([0,(l-1)T_0 + 2T_p];\mathbb{R}^{n-p}\big)$ *and* $g \in L^1\big([0,lT_0];\mathbb{R}^p\big)$.

Let $\alpha \in L^1\big([0,lT_0];\mathbb{R}^\nu\big)$, denote

$$\alpha_i \doteq \bar{\alpha} \cdot \chi_{[(i-1)T_0,iT_0]}, \qquad i = 1,...,l,$$

and observe

$$\big[(l-1)T_0 + T_p, lT_0 + T_p\big] \subset \operatorname{supp}\Big(C_a\hat{F}\Big(\big(F^{l-i}(\Gamma_b\alpha_i)\big)\Big)\Big), \qquad i = 1,...,l-1.$$

Remark 5.8.1 Observe that, if

$$\alpha_i \equiv k_i(\text{constant}) \qquad \text{on } \big[(i-1)T_0, iT_0\big],$$

then, by Remark 5.7.1,

$$C_a\hat{F}\Big(\big(F^{l-i}(\Gamma_b\alpha_i)\big)\Big)(t) = C_a \cdot (C_b \cdot C_a)^{l-1} \cdot \Gamma_g k_i \qquad \text{on } \big[(l-1)T_0 + T_p, lT_0 + T_p\big].$$

Define these linear operators

$$\mathcal{G} : L^1\big([0,lT_0];\mathbb{R}^\nu\big) \longrightarrow L^1\big([0,lT_0+T_p];\mathbb{R}^{n-p}\big), \qquad \mathcal{G}(\alpha) \doteq \sum_{i=1}^{l} C_a\hat{F}\Big(F^{l-i}(\Gamma_b\alpha_i)\Big),$$

and

$$\widetilde{\mathcal{G}} : L^1\big([0,lT_0];\mathbb{R}^\nu\big) \longrightarrow L^1\big([(l-1)T_0+T_p, lT_0+T_p];\mathbb{R}^{n-p}\big), \qquad \widetilde{\mathcal{G}}(\alpha) \doteq \mathcal{F}(\alpha) \doteq \chi_{[(l-1)T_0FF, lT_0FF]}.$$

By Lemma 5.7.1 and (5.7.82), we have

$$\widetilde{C}\big(\mathcal{G}(\alpha)\big) = C\alpha, \qquad \alpha \in L^1\big([0,lT_0];\mathbb{R}^\nu\big),$$

so it is clear that

$$\operatorname{Im}(\widetilde{\mathcal{G}}) \subset \operatorname{Im}(\mathcal{G}) \subset L^1\big([(l-1)T_0 + T_p, lT_0 + T_p];\mathbb{R}^{n-p}\big) \quad \text{and} \quad \operatorname{Im}\big(\widetilde{C}(\mathcal{G})\big) \subset \operatorname{Im}(C). \tag{5.8.94}$$

Lemma 5.8.2 *Assume (5.8.89) and (5.8.90). Using the previous notations,*

$$\operatorname{Im}(\widetilde{\mathcal{G}}) \text{ is dense in } L^1\big([(l-1)T_0 + T_p, lT_0 + T_p];\mathbb{R}^{n-p}\big), \tag{5.8.95}$$

where we follow the identification of (5.7.74).

Proof. It is sufficient to prove that

$$\mathcal{P}_k \subset \operatorname{Im}(\widetilde{\mathcal{G}}), \qquad k \in \mathbb{N}, \tag{5.8.96}$$

where $\mathcal{P}_k$ is the set of the polynomials defined on $\big[(l-1)T_0+T_p, lT_0+T_p\big]$ with values in $\mathbb{R}^{n-p}$ of degree $\leq k$. We argue by induction.

We begin proving

$$\mathcal{P}_0 \subset \mathrm{Im}(\widetilde{\mathcal{G}}), \tag{5.8.97}$$

remind that $\mathcal{P}_0$ is the set of the constant maps defined on $\big[(l-1)T_0+T_p, lT_0+T_p\big]$ with values in $\mathbb{R}^{n-p}$. Let be $h \in \mathbb{R}^{n-p}$, by (5.8.90), there exists a vector $\alpha \in \mathbb{R}^{\nu}$ such that

$$\big(C_a \cdot (C_b \cdot C_a)^{l-1} \cdot \Gamma_b | ... | C_a \cdot \Gamma_b\big)\alpha = h.$$

Since α is constant, by Remark 5.8.1, we have

$$\widetilde{\mathcal{G}}\alpha \equiv \big(C_a \cdot (C_b \cdot C_a)^{l-1} \cdot \Gamma_b | ... | C_a \cdot \Gamma_b\big)\alpha = h,$$

then

$$h \in \mathrm{Im}(\widetilde{\mathcal{G}})$$

and this proves (5.8.97).

Let $k \in \mathbb{N}$, assume that

$$\mathcal{P}_k \subset \mathrm{Im}(\widetilde{\mathcal{G}}), \tag{5.8.98}$$

we prove

$$\mathcal{P}_{k+1} \subset \mathrm{Im}(\widetilde{\mathcal{G}}). \tag{5.8.99}$$

Let $f \in \mathcal{P}_{k+1}$, namely

$$f(t) = f_{k+1}t^{k+1} + ... + f_0,$$

where $f_0, ..., f_{k+1} \in \mathbb{R}^{n-p}$ are constants. By (5.8.90), there exists a constant $\bar{\alpha} \in \mathbb{R}^{\nu}$, such that

$$\big(C_a \cdot (C_b \cdot C_a)^{l-1} \cdot \Gamma_b | ... | C_a \cdot \Gamma_b\big)\bar{\alpha} = f_{k+1}.$$

Define

$$\hat{\alpha}(t) \doteq \bar{\alpha}t^{k+1},$$

by definition,

$$(\widetilde{\mathcal{G}}\bar{\alpha})(t) = \big(C_a \cdot (C_b \cdot C_a)^{l-1} \cdot \Gamma_b | ... | C_a \cdot \Gamma_b\big)\bar{\alpha}t^{k+1} + p_k = f_{k+1}t^{k+1} + p_k,$$

for some $p_k \in \mathcal{P}_k$. By (5.8.98), there exists $\tilde{\alpha} \in L^1\big([0, lT_0]; \mathbb{R}^{\nu}\big)$ such that

$$(\widetilde{\mathcal{G}}\tilde{\alpha})(t) = f_k t^k + ... + f_0 - p_k(t),$$

then, by the linearity of $\widetilde{\mathcal{G}}$,

$$\big(\widetilde{\mathcal{F}}(\hat{\alpha} + \tilde{\alpha})\big)(t) = f_{k+1}t^{k+1} + ... + f_0 = f(t)$$

and this proves (5.8.99). By induction, (5.8.96) is done and by wellknown arguments the claim is proved. $\square$

Proof of Theorem 5.8.1. Since, by Theorem 5.4.2, $\mathrm{Im}(C)$ is closed, by (5.7.76) and (5.8.94), we have

$$L^1\big([a,b];\mathbb{R}^n\big) = \overline{\widetilde{C}(\mathrm{Im}\mathcal{G})} \subset \mathrm{Im}(C) \subset L^1\big([a,b];\mathbb{R}^n\big),$$

namely,

$$\mathrm{Im}(C) = L^1\big([a,b];\mathbb{R}^n\big).$$

Then, by Corollary 5.4.1, (5.0.1) is exactly controllable at time T. □

Proof of Theorem 5.0.1. It is direct consequence of Theorems 5.5.1, 5.6.1, 5.7.1 and 5.8.1. □

Chapter 6

Asymptotic Stabilization and Exact Controllability for Linear Systems

In this section we look for the asymptotic stabilization of

$$\begin{cases} \omega_t + A \cdot \omega_x = 0, & a < x < b,\ t > 0, \\ \omega^+(t,a) = C_a \cdot \omega^-(t,a), & t > 0, \\ \omega^-(t,b) = C_b \cdot \omega^+(t,b) + \Gamma_b \cdot \alpha(t), & t > 0, \\ \omega(0,x) = \omega_0(x), & a < x < b. \end{cases} \tag{6.0.1}$$

Definition 6.0.1 *Let $1 \leq q \leq +\infty$. The problem (6.0.1) is L^q-asymptotically stabilizable if and only if there exists $K \in \mathbb{R}^{\nu \times (n-p)}$ such that the solution $\omega = \omega(t,x)$ of*

$$\begin{cases} \omega_t + A \cdot \omega_x = 0, & a < x < b,\ t > 0, \\ \omega^+(t,a) = C_a \cdot \omega^-(t,a), & t > 0, \\ \omega^-(t,b) = \big(C_b + \Gamma_b \cdot K\big)\omega^+(t,b), & t > 0, \\ \omega(0,x) = \omega_0(x), & a < x < b, \end{cases} \tag{6.0.2}$$

satisfies the following condition

$$\|\omega(t,\cdot)\|_{L^q} \longrightarrow 0, \qquad \text{as } t \longrightarrow +\infty,$$

for each $\omega_0 \in L^q\big([a,b];\mathbb{R}^n\big)$. Moreover (6.0.1) is L^q-asymptotically exponentially stabilizable if and only if there exist $K \in \mathbb{R}^{\nu \times (n-p)}$ and $k_0,\ k_1 > 0$ such that the solution $\omega = \omega(t,x)$ of (6.0.2) satisfies the following condition

$$\|\omega(t,\cdot)\|_{L^q} \leq k_0 e^{-k_1 t}\|\omega_0\|_{L^q}, \qquad t \geq 0,$$

for each $\omega_0 \in L^q\big([a,b];\mathbb{R}^n\big)$.

The chapter is organized as follows. In Section 6.1 we prove some sufficient condition for the L^1-asymptotic exponential stabilizability (see [5]), in Section 6.2 we prove some sufficient condition for the L^2-asymptotic exponential stabilizability (see [42]) and finally in Section 6.3 we show with some examples that the assumptions of the other Sections are independent.

6.1 Asymptotic Stabilization in L^1

The main result of this section is the following (see [5]).

Theorem 6.1.1 *Using the previous notations, if*

$$\max\left\{\|C_a\|\cdot\|C_b\|^{1/2},\ \|C_a\|^{1/2}\cdot\|C_b\|\right\}\leq\log\left(\frac{3}{2}\right), \tag{6.1.3}$$

and let $\omega=\omega(t,x)$ be the solution of

$$\begin{cases}\omega_t+A\cdot\omega_x=0, & a<x<b,\ t>0,\\ \omega^+(t,a)=C_a\cdot\omega^-(t,a), & t>0,\\ \omega^-(t,b)=C_b\cdot\omega^+(t,b), & t>0,\\ \omega(0,x)=\omega_0(x), & a<x<b,\end{cases} \tag{6.1.4}$$

then there exist two positive constants k_0, k_1, depending only on C_a, C_b, A, such that

$$\|\omega(t,\cdot)\|_{L^1}\leq k_0e^{-k_1t}\|\omega_0\|_{L^1},\qquad t\geq 0, \tag{6.1.5}$$

for each $\omega_0\in L^1\big([a,b];\mathbb{R}^n\big)$.

In this section we shall use the following notations

$$\hat{T}\doteq\frac{b-a}{|\lambda_p|}+\frac{b-a}{\lambda_{p+1}},\qquad \tau\doteq\frac{b-a}{|\lambda_1|}+\frac{b-a}{\lambda_n}, \tag{6.1.6}$$

since $\tau\leq\hat{T}$ there exists a unique integer k such that

$$k\tau<\hat{T}\leq(k+1)\tau. \tag{6.1.7}$$

We begin proving some lemmas.

Lemma 6.1.1 *Using the previous notations, and let $\omega=\omega(t,x)$ be the solution of (6.1.4), there results*

$$\|\omega^-(\hat{T},\cdot)\|_{L^1}\leq\|C_a\|\cdot\|C_b\|\left(\int_0^{\hat{T}-\tau}|\omega^-(t,b)|dt+\|\omega^-(0,\cdot)\|_{L^1}\right) \tag{6.1.8}$$

and

$$\|\omega^+(\hat{T},\cdot)\|_{L^1}\leq\|C_a\|\cdot\|C_b\|\left(\int_0^{\hat{T}-\tau}|\omega^+(t,b)|dt+\|\omega^+(0,\cdot)\|_{L^1}\right), \tag{6.1.9}$$

for each $\omega_0\in L^1\big([a,b];\mathbb{R}^n\big)$.

Proof. We begin proving (6.1.8). By (5.0.3), (6.1.4) and the definitions of $\hat{T}$ and τ, we have

$$\begin{aligned}
&\int_a^b |\omega^-(\hat{T},x)|dx \leq \int_{\hat{T}-\frac{b-a}{|\lambda_p|}}^{\hat{T}} |\omega^-(t,b)|dt \leq \|C_b\| \int_{\hat{T}-\frac{b-a}{|\lambda_p|}}^{\hat{T}} |\omega^+(t,b)|dt \leq \\
&\leq \|C_b\| \int_{\hat{T}-\frac{b-a}{|\lambda_p|}-\frac{b-a}{\lambda_{p+1}}}^{\hat{T}-\frac{b-a}{\lambda_n}} |\omega^+(t,a)|dt = \|C_b\| \int_0^{\hat{T}-\frac{b-a}{\lambda_n}} |\omega^+(t,a)|dt \leq \\
&\leq \|C_a\|\|C_b\| \int_0^{\hat{T}-\frac{b-a}{\lambda_n}} |\omega^-(t,a)|dt \leq \qquad (6.1.10) \\
&\leq \|C_a\|\|C_b\| \Big(\int_0^{\hat{T}-\frac{b-a}{\lambda_n}-\frac{b-a}{|\lambda_1|}} |\omega^-(t,b)|dt + \int_a^b |\omega^-(0,x)|dx \Big) = \\
&= \|C_a\|\|C_b\| \Big(\int_0^{\hat{T}-\tau} |\omega^-(t,b)|dt + \int_a^b |\omega^-(0,x)|dx \Big).
\end{aligned}$$

Then, by (6.1.10),

$$\begin{aligned}
&\|\omega^-(\hat{T},\cdot)\|_{L^1} = \int_a^b |\omega^-(\hat{T},x)|dx \leq \\
&\leq \|C_a\| \cdot \|C_b\| \int_0^{\hat{T}-\tau} |\omega^-(t,b)|dt + \int_a^b |\omega^-(0,x)|dx \leq \qquad (6.1.11) \\
&\leq \|C_a\| \cdot \|C_b\| \Big(\int_0^{\hat{T}-\tau} |\omega^-(t,b)|dt + \|\omega^-(0,\cdot)\|_{L^1} \Big).
\end{aligned}$$

So (6.1.8) is proved. Using the same argument we can prove also (6.1.9). □

Lemma 6.1.2 *Using the previous notations and let $\omega = \omega(t,x)$ be the solution of (6.1.4), there results*

$$\begin{aligned}
&\int_0^{\hat{T}-\tau} |\omega^-(t,b)|dt \leq \big(\|C_a\| \cdot \|C_b\|\big)^{h-1} \int_0^{\hat{T}-h\tau} |\omega^-(t,b)|dt + \\
&+ \Big(\sum_{j=1}^{h-1} \big(\|C_a\| \cdot \|C_b\|\big)^j \Big) \|\omega^-(0,\cdot)\|_{L^1} + \Big(\sum_{j=1}^{h-1} \|C_a\|^{j-1}\|C_b\|^j \Big) \|\omega^+(0,\cdot)\|_{L^1} \qquad (6.1.12)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\hat{T}-\tau} |\omega^+(t,b)|dt \leq \big(\|C_a\| \cdot \|C_b\|\big)^{h-1} \int_0^{\hat{T}-h\tau} |\omega^+(t,b)|dt + \\
&+ \Big(\sum_{j=1}^{h-1} \big(\|C_a\| \cdot \|C_b\|\big)^j \Big) \|\omega^+(0,\cdot)\|_{L^1} + \Big(\sum_{j=1}^{h-1} \|C_a\|^j\|C_b\|^{j-1} \Big) \|\omega^-(0,\cdot)\|_{L^1}, \qquad (6.1.13)
\end{aligned}$$

for each $h \in \{1, ..., k\}$ and $\omega_0 \in L^1\big([a,b];\mathbb{R}^n\big)$.

Proof. We begin proving (6.1.12). We proceed by induction on h.

If $h = 1$, (6.1.12) becomes

$$\int_0^{\hat{T}-\tau} |\omega^+(t,b)|dt \leq \int_0^{\hat{T}-\tau} |\omega^+(t,b)|dt$$

that is trivial. Assume that (6.1.12) holds for one $h \in \{1, ..., k-1\}$, we prove that it holds also for $h+1$. Observe that, by (6.1.4),

$$
\begin{aligned}
&\int_0^{\hat{T}-h\tau} |\omega^-(t,b)|dt \leq \|C_b\| \int_0^{\hat{T}-h\tau} |\omega^+(t,b)|dt \leq \\
&\leq \|C_b\| \int_0^{\hat{T}-h\tau-\frac{b-a}{\lambda_n}} |\omega^+(t,a)|dt + \|C_b\| \int_a^b |\omega^+(0,x)|dx \leq \\
&\leq \|C_a\|\|C_b\| \int_0^{\hat{T}-h\tau-\frac{b-a}{\lambda_n}} |\omega^-(t,a)|dt + \|C_b\| \int_a^b |\omega^+(0,x)|dx \leq \\
&\leq \|C_a\|\|C_b\| \int_0^{\hat{T}-h\tau-\frac{b-a}{\lambda_n}-\frac{b-a}{|\lambda_1|}} |\omega^-(t,b)|dt+ \\
&+\|C_a\|\|C_b\| \int_a^b |\omega^-(0,x)|dx + \|C_b\| \int_a^b |\omega^+(0,x)|dx = \\
&= \|C_a\|\|C_b\| \int_0^{\hat{T}-(h+1)\tau} |\omega^-(t,b)|dt+ \\
&+\|C_a\|\|C_b\| \int_a^b |\omega^-(0,x)|dx + \|C_b\| \int_a^b |\omega^+(0,x)|dx,
\end{aligned}
\tag{6.1.14}
$$

then

$$
\begin{aligned}
&\int_0^{\hat{T}-h\tau} |\omega^-(t,b)|dt \leq \\
&\leq \|C_a\| \cdot \|C_b\| \int_0^{\hat{T}-(h+1)\tau} |\omega^-(t,b)|dt+ \\
&+\|C_a\| \cdot \|C_b\| \cdot \|\omega^-(0,\cdot)\|_{L^1} + \|C_b\| \cdot \|\omega^+(0,\cdot)\|_{L^1}.
\end{aligned}
\tag{6.1.15}
$$

Since (6.1.12) holds for h, by (6.1.15), there results

$$
\begin{aligned}
&\int_0^{\hat{T}-\tau} |\omega^-(t,b)|dt \leq \big(\|C_a\| \cdot \|C_b\|\big)^{h-1} \int_0^{\hat{T}-h\tau} |\omega^-(t,b)|dt+ \\
&+\Big(\sum_{j=1}^{h-1} \big(\|C_a\| \cdot \|C_b\|\big)^j\Big) \|\omega^-(0,\cdot)\|_{L^1} + \Big(\sum_{j=1}^{h-1} \|C_a\|^{j-1}\|C_b\|^j\Big) \|\omega^+(0,\cdot)\|_{L^1} \leq \\
&\leq \big(\|C_a\| \cdot \|C_b\|\big)^{h-1} \Bigg[\|C_a\| \cdot \|C_b\| \int_0^{\hat{T}-(h+1)\tau} |\omega^-(t,b)|dt+ \\
&+\|C_a\| \cdot \|C_b\| \cdot \|\omega^-(0,\cdot)\|_{L^1} + \|C_b\| \cdot \|\omega^+(0,\cdot)\|_{L^1}\Bigg]+
\end{aligned}
\tag{6.1.16}
$$

$$
\begin{aligned}
&+\Big(\sum_{j=1}^{h-1}\big(\|C_a\|\cdot\|C_b\|\big)^j\Big)\|\omega^-(0,\cdot)\|_{L^1}+\Big(\sum_{j=1}^{h-1}\|C_a\|^{j-1}\|C_b\|^j\Big)\|\omega^+(0,\cdot)\|_{L^1}=\\
&=\big(\|C_a\|\cdot\|C_b\|\big)^h\int_0^{\hat T-(h+1)\tau}|\omega^-(t,b)|dt+\\
&+\big(\|C_a\|\cdot\|C_b\|\big)^h\|\omega^-(0,\cdot)\|_{L^1}+\|C_a\|^{h-1}\|C_b\|^h\|\omega^+(0,\cdot)\|_{L^1}+\\
&+\Big(\sum_{j=1}^{h-1}\big(\|C_a\|\cdot\|C_b\|\big)^j\Big)\|\omega^-(0,\cdot)\|_{L^1}+\Big(\sum_{j=1}^{h-1}\|C_a\|^{j-1}\|C_b\|^j\Big)\|\omega^+(0,\cdot)\|_{L^1}=\\
&=\big(\|C_a\|\cdot\|C_b\|\big)^h\int_0^{\hat T-(h+1)\tau}|\omega^-(t,b)|dt+\\
&+\Big(\sum_{j=1}^{h}\big(\|C_a\|\cdot\|C_b\|\big)^j\Big)\|\omega^-(0,\cdot)\|_{L^1}+\Big(\sum_{j=1}^{h}\|C_a\|^{j-1}\|C_b\|^j\Big)\|\omega^+(0,\cdot)\|_{L^1},
\end{aligned}
$$

so (6.1.12) holds for $h+1$. By the Induction Principle, (6.1.12) holds for each $h\in\{1,...,k\}$. Using the same argument you can prove also (6.1.13). $\square$

Lemma 6.1.3 *Using the previous notations and let $\omega=\omega(t,x)$ be the solution of (6.1.4), there results*

$$
\int_0^{\hat T-k\tau}|\omega^-(t,b)|dt\le\|C_a\|\cdot\|C_b\|\cdot\|\omega^-(0,\cdot)\|_{L^1}+\|C_b\|\cdot\|\omega^+(0,\cdot)\|_{L^1} \tag{6.1.17}
$$

and

$$
\int_0^{\hat T-k\tau}|\omega^+(t,b)|dt\le\|C_a\|\cdot\|C_b\|\cdot\|\omega^+(0,\cdot)\|_{L^1}+\|C_a\|\cdot\|\omega^-(0,\cdot)\|_{L^1}, \tag{6.1.18}
$$

for each $\omega_0\in L^1\big([a,b];\mathbb{R}^n\big)$.

Proof. We begin proving (6.1.17). By (5.0.3), (6.1.4) and the definitions of $\hat T$ and τ, we have

$$
\begin{aligned}
&\int_0^{\hat T-k\tau}|\omega^-(t,b)|dt\le\|C_b\|\int_0^{\hat T-k\tau}|\omega^+(t,b)|dt\le\\
&\le\|C_b\|\int_0^{\max\{\hat T-k\tau-\frac{b-a}{\lambda_n},0\}}|\omega^+(t,a)|dt+\|C_b\|\int_a^b|\omega^+(0,x)|dx\le\\
&\le\|C_a\|\|C_b\|\int_0^{\max\{\hat T-k\tau-\frac{b-a}{\lambda_n},0\}}|\omega^-(t,a)|dt+\|C_b\|\int_a^b|\omega^+(0,x)|dx\le\\
&\le\|C_a\|\|C_b\|\int_0^{\max\{\hat T-k\tau-\frac{b-a}{\lambda_n}-\frac{b-a}{|\lambda_1|},0\}}|\omega^-(t,b)|dt+\\
&+\|C_a\|\|C_b\|\int_a^b|\omega^-(0,x)|dx+\|C_b\|\int_a^b|\omega^+(0,x)|dx=\\
&=\|C_a\|\|C_b\|\int_0^{\max\{\hat T-(k+1)\tau,0\}}|\omega^-(t,b)|dt+\\
&+\|C_a\|\|C_b\|\int_a^b|\omega^-(0,x)|dx+\|C_b\|\int_a^b|\omega^+(0,x)|dx.
\end{aligned} \tag{6.1.19}
$$

By (6.1.7),

$$\max\{\hat{T}-(k+1)\tau,0\}=0,$$

so, by (6.1.19),

$$\int_0^{\hat{T}-k\tau}|\omega^-(t,b)|dt \le \|C_a\|\|C_b\|\int_a^b|\omega^-(0,x)|dx+\|C_b\|\int_a^b|\omega^+(0,x)|dx \tag{6.1.20}$$

and then

$$\begin{aligned}&\int_0^{\hat{T}-k\tau}|\omega^-(t,b)|dt\le\\ &\le\|C_a\|\|C_b\|\int_a^b|\omega^-(0,x)|dx+\|C_b\|\int_a^b|\omega^+(0,x)|dx\le\\ &\le\|C_a\|\cdot\|C_b\|\cdot\|\omega^-(0,\cdot)\|_{L^1}+\|C_b\|\cdot\|\omega^+(0,\cdot)\|_{L^1}.\end{aligned}\tag{6.1.21}$$

So (6.1.17) is proved. Using the same argument we can prove also (6.1.18). □

Lemma 6.1.4 *Using the previous notations, if*

$$K\doteq\sum_{j=1}^{k+1}\left(\|C_a\|\cdot\|C_b\|\right)^j+\max\left\{\Big(\sum_{j=2}^{k+1}\|C_a\|^{j-1}\|C_b\|^j\Big),\ \Big(\sum_{j=2}^{k+1}\|C_a\|^j\|C_b\|^{j-1}\Big)\right\}<1, \tag{6.1.22}$$

and let $\omega=\omega(t,x)$ be the solution of (6.1.4), then there exist two positive constants k_0, k_1, depending only on C_a, C_b, A, such that

$$\|\omega(t,\cdot)\|_{L^1}\le k_0e^{-k_1t}\|\omega_0\|_{L^1},\qquad t\ge0, \tag{6.1.23}$$

for each $\omega_0\in L^1\big([a,b];\mathbb{R}^n\big)$.

Proof. By Lemmas 6.1.1, 6.1.2 and 6.1.3, we have

$$\begin{aligned}&\|\omega^-(\hat{T},\cdot)\|_{L^1}\le\|C_a\|\cdot\|C_b\|\textstyle\int_0^{\hat{T}-\tau}|\omega^-(t,b)|dt+\|C_a\|\cdot\|C_b\|\cdot\|\omega^-(0,\cdot)\|_{L^1}\le\\ &\le\|C_a\|\cdot\|C_b\|\Bigg[\left(\|C_a\|\cdot\|C_b\|\right)^{k-1}\int_0^{\hat{T}-k\tau}|\omega^-(t,b)|dt+\\ &+\Big(\sum_{j=1}^{k-1}\left(\|C_a\|\cdot\|C_b\|\right)^j\Big)\|\omega^-(0,\cdot)\|_{L^1}+\\ &+\Big(\sum_{j=1}^{k-1}\|C_a\|^{j-1}\|C_b\|^j\Big)\|\omega^+(0,\cdot)\|_{L^1}\Bigg]+\|C_a\|\cdot\|C_b\|\cdot\|\omega^-(0,\cdot)\|_{L^1}=\\ &=\left(\|C_a\|\cdot\|C_b\|\right)^k\int_0^{\hat{T}-k\tau}|\omega^-(t,b)|dt+\Big(\sum_{j=2}^{k}\left(\|C_a\|\cdot\|C_b\|\right)^j\Big)\|\omega^-(0,\cdot)\|_{L^1}+\\ &+\Big(\sum_{j=2}^{k}\|C_a\|^{j-1}\|C_b\|^j\Big)\|\omega^+(0,\cdot)\|_{L^1}+\|C_a\|\cdot\|C_b\|\cdot\|\omega^-(0,\cdot)\|_{L^1}=\end{aligned}\tag{6.1.24}$$

$$
\begin{aligned}
&= \big(\|C_a\|\cdot\|C_b\|\big)^k \int_0^{\hat{T}-k\tau} |\omega^-(t,b)|dt+ \\
&+\Big(\sum_{j=1}^{k}\big(\|C_a\|\cdot\|C_b\|\big)^j\Big)\|\omega^-(0,\cdot)\|_{L^1}+\Big(\sum_{j=2}^{k}\|C_a\|^{j-1}\|C_b\|^j\Big)\|\omega^+(0,\cdot)\|_{L^1}\le \\
&\le \big(\|C_a\|\cdot\|C_b\|\big)^k\Big[\|C_a\|\cdot\|C_b\|\cdot\|\omega^-(0,\cdot)\|_{L^1}+\|C_b\|\cdot\|\omega^+(0,\cdot)\|_{L^1}\Big]+ \\
&+\Big(\sum_{j=1}^{k}\big(\|C_a\|\cdot\|C_b\|\big)^j\Big)\|\omega^-(0,\cdot)\|_{L^1}+\Big(\sum_{j=2}^{k}\|C_a\|^{j-1}\|C_b\|^j\Big)\|\omega^+(0,\cdot)\|_{L^1}= \\
&= \big(\|C_a\|\cdot\|C_b\|\big)^{k+1}\|\omega^-(0,\cdot)\|_{L^1}+\|C_a\|^k\|C_b\|^{k+1}\|\omega^+(0,\cdot)\|_{L^1}+ \\
&+\Big(\sum_{j=1}^{k}\big(\|C_a\|\cdot\|C_b\|\big)^j\Big)\|\omega^-(0,\cdot)\|_{L^1}+\Big(\sum_{j=2}^{k}\|C_a\|^{j-1}\|C_b\|^j\Big)\|\omega^+(0,\cdot)\|_{L^1}= \\
&= \Big(\sum_{j=1}^{k+1}\big(\|C_a\|\cdot\|C_b\|\big)^j\Big)\|\omega^-(0,\cdot)\|_{L^1}+\Big(\sum_{j=2}^{k+1}\|C_a\|^{j-1}\|C_b\|^j\Big)\|\omega^+(0,\cdot)\|_{L^1}
\end{aligned}
$$

and analogously

$$
\|\omega^+(\hat{T},\cdot)\|_{L^1}\le\Big(\sum_{j=1}^{k+1}\big(\|C_a\|\cdot\|C_b\|\big)^j\Big)\|\omega^+(0,\cdot)\|_{L^1}+\Big(\sum_{j=2}^{k+1}\|C_a\|^j\|C_b\|^{j-1}\Big)\|\omega^-(0,\cdot)\|_{L^1}. \tag{6.1.25}
$$

Hence, by (6.1.22), (6.1.24) and (6.1.25),

$$
\begin{aligned}
&\|\omega(\hat{T},\cdot)\|_{L^1}=\|\omega^-(\hat{T},\cdot)\|_{L^1}+\|\omega^+(\hat{T},\cdot)\|_{L^1}\le \\
&\le\Big(\sum_{j=1}^{k+1}\big(\|C_a\|\cdot\|C_b\|\big)^j\Big)\|\omega(0,\cdot)\|_{L^1}+ \\
&+\Big(\sum_{j=2}^{k+1}\|C_a\|^{j-1}\|C_b\|^j\Big)\|\omega^+(0,\cdot)\|_{L^1}+\Big(\sum_{j=2}^{k+1}\|C_a\|^j\|C_b\|^{j-1}\Big)\|\omega^-(0,\cdot)\|_{L^1}\le \\
&\le K\cdot\|\omega(0,\cdot)\|_{L^1}.
\end{aligned} \tag{6.1.26}
$$

Moreover, using this argument, we are able to deduce

$$
\|\omega(\hat{T}+t,\cdot)\|_{L^1}\le K\cdot\|\omega(t,\cdot)\|_{L^1},\qquad t\ge 0. \tag{6.1.27}
$$

Fixed $t\ge 0$, there exist an integer μ and $0\le s<\hat{T}$ such that

$$
t=\mu\hat{T}+s,\qquad \mu\hat{T}\le t<(\mu+1)\hat{T}, \tag{6.1.28}
$$

namely

$$
\frac{t}{\hat{T}}-1<\mu\le\frac{t}{\hat{T}}. \tag{6.1.29}
$$

By (6.1.27) and (6.1.28),

$$
\|\omega(t,\cdot)\|_{L^1}\le K^\mu\|\omega(s,\cdot)\|_{L^1}. \tag{6.1.30}
$$

Moreover, there exists $K_0>0$ such that

$$
\|\omega(\theta,\cdot)\|_{L^1}\le K_0\|\omega(0,\cdot)\|_{L^1},\qquad 0\le\theta\le\hat{T}. \tag{6.1.31}
$$

By (6.1.22), (6.1.29), (6.1.30) and (6.1.31)

$$\|\omega(t,\cdot)\|_{L^1} \le \frac{K_0}{K}\cdot K^{t/\hat{T}}\|\omega(0,\cdot)\|_{L^1} = k_0\cdot e^{-k_1 t}\|\omega(0,\cdot)\|_{L^1}, \tag{6.1.32}$$

where

$$k_0 \doteq \frac{K_0}{K}, \qquad k_1 \doteq \frac{\big|\log(K)\big|}{\hat{T}}.$$

So the proof is done. □

Proof of Theorem 6.1.1. In the following we shall assume that

$$\|C_a\| \le \|C_b\|. \tag{6.1.33}$$

By (6.1.3), we have

$$\|C_a\| < 1, \tag{6.1.34}$$

and then

$$K = \sum_{j=1}^{k+1}\left(\|C_a\|\cdot\|C_b\|\right)^j + \sum_{j=2}^{k+1}\|C_a\|^{j-1}\|C_b\|^j, \tag{6.1.35}$$

where K is the constant defined in Lemma 6.1.4. Moreover, by (6.1.3) and (6.1.34),

$$\|C_a\|\cdot\|C_b\| \le \|C_a\|^{1/2}\cdot\|C_b\| \le \log\left(\frac{3}{2}\right) \tag{6.1.36}$$

and since

$$\frac{j}{2} \le j-1, \qquad j\in\mathbb{N}\setminus\{0,\,1\},$$

by (6.1.34), we have

$$\|C_a\|^{j-1}\|C_b\|^j \le \|C_a\|^{j/2}\|C_b\|^j. \tag{6.1.37}$$

By (6.1.3), (6.1.35), (6.1.36) and (6.1.37),

$$\begin{aligned} K \quad &\le \sum_{j=1}^{k+1}\left(\|C_a\|\cdot\|C_b\|\right)^j + \sum_{j=2}^{k+1}\left(\|C_a\|^{1/2}\|C_b\|\right)^j < \\ &< e^{\|C_a\|\cdot\|C_b\|} + e^{\|C_a\|^{1/2}\|C_b\|} - 2 \le \frac{3}{2}+\frac{3}{2}-2 = 1, \end{aligned}$$

then, by Lemma 6.1.4, the claim is done. □

Theorem 6.1.2 *If*

$$\inf_{v\in\mathrm{Im}(\Gamma_b)}\|C_b+v\| \le \max\left\{\frac{1}{\|C_a\|^2}\log^2\left(\frac{3}{2}\right),\ \frac{1}{\|C_a\|^{\frac{1}{2}}}\log\left(\frac{3}{2}\right)\right\} \tag{6.1.38}$$

then there exist two positive constants k_2, k_3, depending only on C_a, C_b, A, Γ_b, and $\alpha\in L^1(\mathbb{R}^+)$ such that

$$\|\omega(t,\cdot)\|_{L^1} \le k_2 e^{-k_3 t}\|\omega_0\|_{L^1}, \qquad t\ge 0, \tag{6.1.39}$$

for each $\omega_0\in L^1\big([a,b];\mathbb{R}^n\big)$, namely (6.0.1) is L^1 asymptotically exponentially stable, where $\omega=\omega(t,x)$ is the solution of (6.0.1).

Proof. By (6.1.38), there exists $u \in \mathbb{R}^{\nu \times n-p}$ such that

$$\|C_b + \Gamma_b u\|^{1/2} \le \frac{1}{\|C_a\|} \log\left(\frac{3}{2}\right), \qquad \|C_b + \Gamma_b u\| \le \frac{1}{\|C_a\|^{1/2}} \log\left(\frac{3}{2}\right),$$

so, we have only to set

$$\alpha(t) \doteq u \cdot \omega^+(t,b), \qquad t \ge 0,$$

and use Theorem 6.1.1. □

Remark 6.1.1 Since $0 \in \mathrm{Im}(\Gamma_b)$, Theorem 6.1.1 is direct consequence of Theorem 6.1.2. Indeed, if (6.1.3) holds, then also (6.1.38) holds and so in this case we do not need of a feedback control to stabilize the system.

6.2 Asymptotic Stabilization in L^2

The main result of this section is the following (see [42]).

Theorem 6.2.1 *If there exists $\gamma > 0$ such that*

$$\big(\omega^+, (A^+ + C_b^T \cdot A^- \cdot C_b)\omega^+\big) \ge \gamma|\omega^+|^2, \qquad \omega^+ \in \mathbb{R}^{n-p}, \tag{6.2.40}$$

$$\big(\omega^-, (A^- + C_a^T \cdot A^+ \cdot C_a)\omega^-\big) \le 0, \qquad \omega^+ \in \mathbb{R}^p, \tag{6.2.41}$$

and let $\omega = \omega(t,x)$ be the solution of

$$\begin{cases} \omega_t + A \cdot \omega_x = 0, & a < x < b,\ t > 0, \\ \omega^+(t,a) = C_a \cdot \omega^-(t,a), & t > 0, \\ \omega^-(t,b) = C_b \cdot \omega^+(t,b), & t > 0, \\ \omega(0,x) = \omega_0(x), & a < x < b, \end{cases} \tag{6.2.42}$$

then there exist two positive constants μ, μ_1 such that

$$\|\omega(t,\cdot)\|_{L^2} \le \mu_1 e^{-\mu t}\|\omega_0\|_{L^2}, \qquad t > 0, \tag{6.2.43}$$

for each $\omega_0 \in L^2\big([a,b];\mathbb{R}^n\big)$.

We begin with some lemmas.

Lemma 6.2.1 *For each $T > \dfrac{b-a}{|\lambda_p|} + \dfrac{b-a}{|\lambda_{p+1}|}$ and v_0, $v_T \in L^2\big([a,b];\mathbb{R}^n\big)$ there exists a map $f \in L^2\big([0,T]\big)$ such that the solution $v = v(t,x)$ of the problem*

$$\begin{cases} v_t + A \cdot v_x = 0, & a < x < b,\ t > 0, \\ v^-(t,a) = -(A^-)^{-1} \cdot C_a^T \cdot A^+ \cdot v^+(t,a), & 0 < t < T, \\ v^+(t,b) = -(A^+)^{-1} \cdot C_b^T \cdot A^- \cdot v^-(t,b) + f(t), & 0 < t < T, \\ v(T,x) = v_T(x), & a < x < b, \end{cases} \tag{6.2.44}$$

satisfies the following

$$v(0,\cdot) \equiv v_0.$$

Moreover, there exist two positive constants K_0, K_1 such that

$$\|f\|_{L^2} \leq K_0\|v_0\|_{L^2} + K_1\|v_T\|_{L^2}. \tag{6.2.45}$$

PROOF. It is direct consequence of [42, (3.20)]. □

Lemma 6.2.2 *Let $\omega = \omega(t,x)$ be the solution of (6.2.42). The map*

$$E : t \geq 0 \longmapsto \|\omega(t,\cdot)\|_{L^2}$$

is decreasing if and only if (6.2.41) and

$$\big(\omega^+, (A^+ + C_b^T \cdot A^- \cdot C_b)\omega^+\big) \geq 0, \qquad \omega^+ \in \mathbb{R}^{n-p}, \tag{6.2.46}$$

hold. Moreover, if (6.2.40) and (6.2.41) hold there results

$$\frac{d}{dt}E^2(t) \leq -\gamma|\omega^+(t,b)|, \tag{6.2.47}$$

for each $t \geq 0$.

PROOF. Assume that (6.2.41) and (6.2.46) holds, we have to prove that E is decreasing. Since A is symmetric there results

$$\begin{aligned}
&\frac{d}{dt}\int_a^b |\omega(t,x)|^2 dx = 2\int_a^b \big(\omega_t(t,x), \omega(t,x)\big)dx = \\
&= -\int_a^b \big(\omega_x(t,x), A\cdot\omega(t,x)\big)dx - \int_a^b \big(\omega(t,x), A\cdot\omega_x(t,x)\big)dx = \\
&= -\int_a^b \frac{\partial}{\partial x}\big(\omega(t,x), A\cdot\omega(t,x)\big)dx = \\
&= \big(\omega(t,a), A\cdot\omega(t,a)\big) - \big(\omega(t,b), A\cdot\omega(t,b)\big).
\end{aligned} \tag{6.2.48}$$

By (6.2.41), we have

$$\begin{aligned}
&\big(\omega(t,a), A\cdot\omega(t,a)\big) = \left(\begin{pmatrix}\omega^-(t,a)\\ \omega^+(t,a)\end{pmatrix}, \begin{pmatrix}A^-\cdot\omega^-(t,a)\\ A^+\cdot\omega^+(t,a)\end{pmatrix}\right) = \\
&= \left(\begin{pmatrix}\omega^-(t,a)\\ C_a\cdot\omega^-(t,a)\end{pmatrix}, \begin{pmatrix}A^-\cdot\omega^-(t,a)\\ A^+\cdot C_a\cdot\omega^-(t,a)\end{pmatrix}\right) = \\
&= \big(\omega^-(t,a), A^-\cdot\omega(t,a)\big) + \big(C_a\cdot\omega^-(t,a), A^+\cdot C_a\cdot\omega^-(t,a)\big) = \\
&= \big(\omega^-(t,a), (A^- + C_a^T\cdot A^+\cdot C_a)\omega^-(t,a)\big) \leq 0,
\end{aligned} \tag{6.2.49}$$

by (6.2.46), we have

$$
\begin{aligned}
&\big(\omega(t,b), A\cdot\omega(t,b)\big) = \left(\begin{pmatrix}\omega^-(t,b)\\ \omega^+(t,b)\end{pmatrix}, \begin{pmatrix}A^-\cdot\omega^-(t,b)\\ A^+\cdot\omega^+(t,b)\end{pmatrix}\right) = \\
&= \left(\begin{pmatrix}C_b\cdot\omega^+(t,b)\\ \omega^+(t,b)\end{pmatrix}, \begin{pmatrix}A^-\cdot C_b\cdot\omega^+(t,b)\\ A^+\cdot\omega^+(t,b)\end{pmatrix}\right) = \\
&= \big(C_b\cdot\omega^+(t,b), A^-\cdot C_b\cdot\omega^+(t,b)\big) + \big(\omega^+(t,b), A^+\cdot\omega(t,b)\big) = \\
&= \big(\omega^+(t,b), (A^+ + C_b^T\cdot A^-\cdot C_b)\omega^+(t,b)\big) \geq 0,
\end{aligned}
\tag{6.2.50}
$$

and analogously by (6.2.40), we have

$$\big(\omega(t,b), A\cdot\omega(t,b)\big) \geq \gamma|\omega^+(t,b)|^2 \tag{6.2.51}$$

Then if (6.2.41) and (6.2.46) hold, substituting (6.2.49) and (6.2.50) in (6.2.48), there results

$$\frac{d}{dt}\int_a^b |\omega(t,x)|^2dx \leq 0, \tag{6.2.52}$$

instead if (6.2.41) and (6.2.40) hold, substituting (6.2.49) and (6.2.51) in (6.2.48), there results

$$\frac{d}{dt}\int_a^b |\omega(t,x)|^2dx \leq -\gamma|\omega^+(t,b)|^2, \tag{6.2.53}$$

and then (6.2.47) holds.

Now assume that E is decreasing, we begin proving (6.2.41). Let $\bar{\omega}^- \in \mathbb{R}^p$ and $\bar{\omega} = \bar{\omega}(t,x)$ be the solution of (6.2.42) such that

$$\bar{\omega}_i(0,\cdot) \equiv \begin{cases}\bar{\omega}_i, & \text{if } i = 1,...,p,\\ 0, & \text{otherwise.}\end{cases}$$

Fix $0 < t < \dfrac{b-a}{|\lambda_1|}$, there results

$$\bar{\omega}^+(t,b) = 0, \qquad \bar{\omega}(t,a) = \bar{\omega}^-,$$

since, by assumption, E is decreasing, we have, by (6.2.48), (6.2.49) and (6.2.50),

$$
\begin{aligned}
0 \leq\ & \frac{d}{dx}\int_a^b |\bar{\omega}(t,x)|^2dx = \\
&= \big(\bar{\omega}^-(t,a), (A^- + C_a^T\cdot A^+\cdot C_a)\bar{\omega}^-(t,a)\big) - \big(\bar{\omega}^+(t,b), (A^+ + C_b^T\cdot A^-\cdot C_b)\bar{\omega}^+(t,b)\big) = \\
&= \big(\bar{\omega}^-, (A^- + C_a^T\cdot A^+\cdot C_a)\bar{\omega}^-\big),
\end{aligned}
$$

then (6.2.41) holds. Using the same argument, we can prove also (6.2.46). This concludes the proof. □

PROOF OF THEOREM 6.2.1. Using the same argument of the previous proof we can deduce (6.2.53). Let $T > \dfrac{b-a}{|\lambda_p|} + \dfrac{b-a}{\lambda_{p+1}}$, by (6.2.47), we have

$$\|\omega(T,\cdot)\|_{L^2} - \|\omega(0,\cdot)\|_{L^2} \le -\gamma \int_0^T |\omega^+(t,b)|dt = -\gamma\|\omega^+(\cdot,b)\|_{L^2}. \tag{6.2.54}$$

Let be $v = v(t,x)$ the solution of (6.2.44) such that

$$v(T,\cdot) \equiv \omega(T,\cdot),$$

by Lemma 6.2.1 there exists $f \in L^2\big([0,T]\big)$ such that

$$v(0,\cdot) \equiv 0$$

and there results

$$\|f\|_{L^2} \le K_0\|v(T,\cdot)\|_{L^2} + K_0\|v(0,\cdot)\|_{L^2} = K_0\|\omega(T,\cdot)\|_{L^2}.$$

Moreover

$$\|\omega(T,\cdot)\|^2_{L^2} = \big(\omega(T,\cdot),v(T,\cdot)\big)_{L^2} - \big(\omega(0,\cdot),v(0,\cdot)\big)_{L^2} = \int_0^T \frac{d}{dt}\big(\omega(t,\cdot),v(t,\cdot)\big)_{L^2}dt \tag{6.2.55}$$

and

$$\begin{aligned}
&\frac{d}{dt}\big(\omega(t,\cdot),v(t,\cdot)\big)_{L^2} = \frac{d}{dt}\int_a^b \big(\omega(t,x),v(t,x)\big)dx = \\
&= \int_a^b \Big(\big(\omega_t(t,x),v(t,x)\big) + \big(\omega(t,x),v_t(t,x)\big)\Big)dx = \\
&= -\int_a^b \Big(\big(A\cdot\omega_x(t,x),v(t,x)\big) + \big(\omega(t,x),A\cdot v_x(t,x)\big)\Big)dx = \\
&= -\int_a^b \frac{\partial}{\partial x}\big(A\cdot\omega(t,x),v(t,x)\big)dx = \\
&= -\big(A\cdot\omega(t,b),v(t,b)\big) + \big(A\cdot\omega(t,a),v(t,a)\big).
\end{aligned} \tag{6.2.56}$$

Since

$$\begin{aligned}
&\big(A\cdot\omega(t,b),v(t,b)\big) = \left(\begin{pmatrix} A^-\cdot\omega^-(t,b) \\ A^+\cdot\omega^+(t,b)\end{pmatrix}, \begin{pmatrix} v^-(t,b) \\ v^+(t,b)\end{pmatrix}\right) = \\
&= \big(A^-\cdot\omega^-(t,b),v^-(t,b)\big) + \big(A^+\cdot\omega^+(t,b),v^+(t,b)\big) = \\
&= \big(A^-\cdot C_b\cdot\omega^+(t,b),v^-(t,b)\big) + \\
&\quad +\big(A^+\cdot\omega^+(t,b), -(A^+)^{-1}\cdot C_b^T\cdot A^-\cdot v^-(t,b) + f(t)\big) = \\
&= \big(A^-\cdot C_b\cdot\omega^+(t,b),v^-(t,b)\big) - \\
&\quad -\big(A^-\cdot C_b\cdot\omega^+(t,b),v^-(t,b)\big) + \big(A^+\cdot\omega^+(t,b),f(t)\big) = \\
&= \big(A^+\cdot\omega^+(t,b),f(t)\big),
\end{aligned} \tag{6.2.57}$$

and

$$\begin{aligned}
&\big(A\cdot\omega(t,a),v(t,a)\big) = \left(\begin{pmatrix} A^-\cdot\omega^-(t,a) \\ A^+\cdot\omega^+(t,a)\end{pmatrix}, \begin{pmatrix} v^-(t,a) \\ v^+(t,a)\end{pmatrix}\right) = \\
&= \big(A^-\cdot\omega^-(t,a),v^-(t,a)\big) + \big(A^+\cdot\omega^+(t,a),v^+(t,a)\big) = \\
&= \big(A^-\omega^-(t,a), -(A^-)^{-1}\cdot C_a^T\cdot A^+\cdot v^+(t,a)\big) + \\
&\quad +\big(A^+\cdot C_a\cdot\omega^-(t,a),v^+(t,a)\big) = 0,
\end{aligned} \tag{6.2.58}$$

substituting (6.2.57) and (6.2.58) in (6.2.55) we obtain

$$\begin{aligned}\|\omega(T,\cdot)\|_{L^2}^2 &= \int_0^T \big(A^+\cdot\omega^+(t,b), f(t)\big)dt \le \\ &\le \alpha\|\omega^+(\cdot,b)\|_{L^2}\|f\|_{L^2} \le \alpha K_0\|\omega^+(\cdot,b)\|_{L^2}\|\omega(T,\cdot)\|_{L^2},\end{aligned} \tag{6.2.59}$$

namely

$$\frac{\|\omega(T,\cdot)\|_{L^2}}{\alpha K_0} \le \|\omega^+(\cdot,b)\|_{L^2}. \tag{6.2.60}$$

By (6.2.54), we obtain

$$\frac{\|\omega(T,\cdot)\|_{L^2}}{\alpha K_0} \le \|\omega(0,\cdot)\|_{L^2} - \|\omega(T,\cdot)\|_{L^2} \tag{6.2.61}$$

and so

$$\|\omega(T,\cdot)\|_{L^2} \le M\|\omega(0,\cdot)\|_{L^2}, \tag{6.2.62}$$

where

$$M \doteq \frac{1}{1+\dfrac{\gamma}{\alpha K_0}}(<1).$$

Let $t \ge 0$ there exists $k \in \mathbb{N}$ such that

$$kT \le t \le (k+1)T,$$

by Lemma 6.2.2 and since $M < 1$, there results

$$\begin{aligned}\|\omega(t,\cdot)\|_{L^2} &\le \|\omega(kT,\cdot)\|_{L^2} \le M^k\|\omega(0,\cdot)\|_{L^2} \le \\ &\le M^{\frac{t}{T}-1}\|\omega(0,\cdot)\|_{L^2} = \mu_1 e^{-\mu t}\|\omega(0,\cdot)\|_{L^2},\end{aligned} \tag{6.2.63}$$

where

$$\mu_1 \doteq \frac{1}{M}, \qquad \mu \doteq \frac{\big|\log(M)\big|}{T}.$$

So the proof is concluded. □

Theorem 6.2.2 *If (6.2.41) holds and there exist $K \in \mathbb{R}^{\nu\times(n-p)}$ and $\gamma > 0$ such that*

$$\big(\omega^+, \big(A^+ + (C_b + \Gamma_b\cdot K)^T\cdot A^-\cdot(C_b+\Gamma_b\cdot K)\big)\omega^+\big) \ge \gamma|\omega^+|^2, \quad \omega^+ \in \mathbb{R}^{n-p} \tag{6.2.64}$$

for each $\omega^+ \in \mathbb{R}^{n-p}$ then for each $\omega_0 \in L^2\big([a,b];\mathbb{R}^n\big)$ there exists $\alpha \in L^2(\mathbb{R}_+)$ such that the solution $\omega = \omega(t,x)$ of (6.0.1) satisfies the following

$$\|\omega(t,\cdot)\|_{L^2} \le \mu_1 e^{-\mu t}\|\omega_0\|_{L^2}, \qquad t \ge 0, \tag{6.2.65}$$

where μ, μ_1 are two positive constants depending only on C_a, C_b, A, Γ_b, K, namely (6.0.1) is L^2 asymptotically exponentially stable.

Proof. We have only to set

$$\alpha(t) \doteq K\cdot\omega^+(t,b), \qquad t \ge 0,$$

and use Theorem 6.2.1. □

6.3 Examples

In this section we show with some examples that the previous conditions are sufficient but not necessary and that they are independent.

Example 6.3.1 Consider the case $n = 2$, $p = 1$, with

$$\lambda_1 = -\lambda_2 = 1, \qquad C_a = \alpha, \qquad C_b = \beta, \tag{6.3.66}$$

such that

$$\alpha > 1, \quad 0 < \beta < 1, \quad \alpha^2\beta < \log^2\left(\frac{3}{2}\right), \tag{6.3.67}$$

and denote

$$T \doteq \frac{b-a}{\lambda_1} = \frac{b-a}{|\lambda_2|} = b - a. \tag{6.3.68}$$

There results

$$A^- + C_a^T \cdot A^+ \cdot C_a = -1 + \alpha^2 > 0, \qquad A^+ + C_b^T \cdot A^- \cdot C_b = 1 - \beta^2 > 0, \tag{6.3.69}$$

namely (6.2.46) holds, but (6.2.41) does not. However, the system is L^2 exponentially stable. Indeed,

$$\|C_a \cdot C_b\| = \alpha\beta < \alpha^2\beta < \log^2\left(\frac{3}{2}\right) < 1, \tag{6.3.70}$$

and for each $t \geq 0$, we have

$$\begin{aligned}\int_a^b |\omega(t+2T,x)|^2 dx &= \int_a^b \left(|\omega_1(t+2T,x)|^2 + |\omega_2(t+2T,x)|^2\right) dx = \\ &= \int_a^b \left(\beta^2|\omega_2(t+T,x)|^2 + \alpha^2|\omega_1(t+T,x)|^2\right) dx = \\ &= \int_a^b \left(\alpha^2\beta^2|\omega_1(t,x)|^2 + \alpha^2\beta^2|\omega_2(t,x)|^2\right) dx = \alpha^2\beta^2 \int_a^b |\omega(t,x)|^2 dx,\end{aligned} \tag{6.3.71}$$

and then for each $n \in \mathbb{N}$,

$$\int_a^b |\omega(t+2nT,x)|^2 dx = (\alpha\beta)^{2n} \int_a^b |\omega(t,x)|^2 dx. \tag{6.3.72}$$

Finally observe that

$$\|C_a\|\|C_b\|^{\frac{1}{2}} = \alpha\sqrt{\beta} < \log\left(\frac{3}{2}\right), \quad \|C_a\|^{\frac{1}{2}}\|C_b\| = \sqrt{\alpha}\beta < \alpha\sqrt{\beta} < \log\left(\frac{3}{2}\right),$$

namely (6.1.3) holds.

Example 6.3.2 Consider the case $n = 4$, $p = 2$, with

$$A^- \doteq \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^+ \doteq \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}, \quad C_a \doteq \begin{pmatrix} 1\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad C_b \doteq \begin{pmatrix} 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}. \tag{6.3.73}$$

There results

$$\begin{aligned} C_a \cdot C_b &= \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3/2} \\ 0 & 0 \end{pmatrix}, \\ C_b \cdot C_a &= \begin{pmatrix} 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} = 0. \end{aligned} \tag{6.3.74}$$

and then

$$\|C_a \cdot C_b\| = \sqrt{\frac{3}{2}} > 1, \qquad \|C_b \cdot C_a\| = 0. \tag{6.3.75}$$

Moreover

$$\begin{aligned} A^+ + C_b^T \cdot A^- \cdot C_b &= \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \sqrt{3} & 0 \end{pmatrix} \cdot \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \sqrt{3} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -2\sqrt{3} \\ 0 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} > 0, \end{aligned} \tag{6.3.76}$$

and

$$\begin{aligned} A^- + C_a^T \cdot A^+ \cdot C_a &= \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -3/2 & 0 \\ 0 & -1 \end{pmatrix} < 0. \end{aligned} \tag{6.3.77}$$

So (6.2.40) and (6.2.41) hold, but

$$\|C_b\| = \sqrt{3} > 1, \qquad \|C_a \cdot C_b\| = \sqrt{\frac{3}{2}} > 1. \tag{6.3.78}$$

Example 6.3.3 Consider the system (6.2.42) in the case

$$n = 4, \quad \lambda_1 = -2, \quad \lambda_2 = -1, \quad \lambda_3 = 1, \quad \lambda_4 = 2$$

and

$$C_a = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}, \qquad C_b = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

Observe that

$$C_a \cdot C_b = C_b \cdot C_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned}
A^+ + C_b^T \cdot A^- \cdot C_b &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} = 0, \\
A^- + C_a^T \cdot A^+ \cdot C_a &= \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} = \\
&= \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 0.
\end{aligned}$$

Arguing as in Lemma 6.2.2, we deduce

$$\|\omega(t,\cdot)\|_{L^2} = \|\omega_0\|_{L^2}, \qquad t \geq 0,$$

then (6.2.43) does not hold.

Example 6.3.4 Consider the case $\nu = 1$ and assume that (6.2.46) holds and there exists two matrices $M_1 \in \mathbb{R}^{p\times 1}$ and $M_2 \in \mathbb{R}^{1\times(n-p)}$ such that

$$C_b = M_1 \cdot M_2, \qquad M_1^T \cdot A^- \cdot \Gamma_b \neq 0. \tag{6.3.79}$$

Denote

$$K \doteq \rho M_2, \qquad \gamma_1 \doteq M_1^T \cdot A^- \cdot \Gamma_b, \tag{6.3.80}$$

for some constant $\rho \in \mathbb{R}$. There results

$$\begin{aligned}
\big(\omega^+, \big(A^+ + (C_b + \Gamma_b \cdot K)^T &\cdot A^- \cdot (C_b + \Gamma_b \cdot K)\big)\omega^+\big) = \\
&= \big(\omega^+, \big(A^+ + C_b^T \cdot A^- \cdot C_b\big)\omega^+\big) + \\
&+ 2\big(\omega^+, \big(C_b^T \cdot A^- \cdot \Gamma_b \cdot K)\big)\omega^+\big) + \\
&+ \big(\omega^+, \big((\Gamma_b \cdot K)^T \cdot A^- \cdot (\Gamma_b \cdot K)\big)\omega^+\big).
\end{aligned} \tag{6.3.81}$$

Observe that, by (6.3.79) and (6.3.80),

$$\begin{aligned}
C_b^T \cdot A^- \cdot \Gamma_b \cdot K &= \rho (M_1 \cdot M_2)^T \cdot A^- \cdot \Gamma_b \cdot M_2 = \\
&= \rho M_2^T \cdot \big(M_1^T \cdot A^- \cdot \Gamma_b\big) \cdot M_2 = \rho \gamma_1 M_2^T \cdot M_2,
\end{aligned} \tag{6.3.82}$$

let ρ with the same sign of γ_3, so $C_b^T \cdot A^- \cdot \Gamma_b \cdot K$ is positively definited. Let $\gamma_2 > 0$ such that

$$\big(\omega^+, M_2^T \cdot M_2 \omega^+\big) \geq \gamma_2 |\omega^+|^2, \qquad \omega^+ \in \mathbb{R}^{n-p}, \tag{6.3.83}$$

so there results

$$\big(\omega^+, C_b^T \cdot A^- \cdot \Gamma_b \cdot K \omega^+\big) \geq \rho \gamma_1 \gamma_2 |\omega^+|^2, \qquad \omega^+ \in \mathbb{R}^{n-p}. \tag{6.3.84}$$

Moreover, we have

$$(\Gamma_b \cdot K)^T \cdot A^- \cdot (\Gamma_b \cdot K) = K^T \cdot (\Gamma_b^T \cdot A^- \cdot \Gamma_b) \cdot K, \tag{6.3.85}$$

by (5.0.3),

$$-\gamma_3 \doteq \Gamma_b^T \cdot A^- \cdot \Gamma_b = \sum_{i=1}^{p} \left[\Gamma_b\right]_i^2 \lambda_i < 0, \tag{6.3.86}$$

and, by (6.3.83),

$$\left(\omega^+, K^T \cdot K\omega^+\right) \geq \rho^2 \gamma_2 |\omega^+|^2, \qquad \omega^+ \in \mathbb{R}^{n-p}. \tag{6.3.87}$$

Finally, fix ρ such that

$$2\rho\gamma_1 - \rho^2\gamma_3 \geq 0, \tag{6.3.88}$$

since the eigenvalues of $C_b^T \cdot A^- \cdot \Gamma_b \cdot K$ are bigger than $2\rho\gamma_1\gamma_2$ and the ones of $(\Gamma \cdot K)^T \cdot A^- \cdot \Gamma_b \cdot K$ are less than $-\rho^2\gamma_2\gamma_3$, by (6.3.88), there results

$$\left(\omega^+, \left(A^+ + (C_b + \Gamma_b \cdot K)^T \cdot A^- \cdot (C_b + \Gamma_b \cdot K)\right)\omega^+\right) \geq (2\rho\gamma_1 - \rho^2\gamma_3)\gamma_2 |\omega^+|^2, \tag{6.3.89}$$

and so (6.2.64) holds.

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