STABILITY OF LARGE-AMPLITUDE VISCOUS SHOCK PROFILES OF HYPERBOLIC-PARABOLIC SYSTEMS

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Abstract. We establish nonlinear $L^1 \cap H^3 \to L^p$ orbital stability, $2 \le p \le \infty$, with sharp rates of decay, of large amplitude Lax-type shock profiles of a class of symmetric hyperbolic-parabolic systems, including compressible gas- and magnetohydrodynamics (MHD), under the necessary conditions of strong spectral stability, i.e., stable point spectrum of the linearized operator about the wave, transversality of the profile, and hyperbolic stability of the associated ideal shock. This yields in particular, together with the spectral stability results of [MN], the nonlinear stability of arbitrarily large-amplitude shock profiles of isentropic Navier–Stokes equations for a gamma-law gas with $\gamma \to 1$: the first complete large-amplitude stability result for a shock profile of a system with real (i.e., partial) viscosity. A corresponding small-amplitude result was established in [MZ.2–3] for general systems of Kawashima class by a combination of "Kawashima-type" energy estimates and pointwise Green function bounds, where the small amplitude assumption was used only to close the energy estimates. Here, under the mild additional assumption that hyperbolic characteristic speeds (relative to the shock) be not only nonzero but of a common sign, we close the estimates instead by use of a Goodman-type weighted norm [Go.1–2] designed to control estimates in the crucial hyperbolic modes. Spectral stability of general small-amplitude shock profiles has been shown for Kawashima class systems in [HuZ,PZ]; the determination of spectral stability for general large-amplitude profiles remains a fundamental open problem.

Section 1. Introduction

Consider the class of degenerate parabolic conservation laws of *dissipative, symmetric hyperbolic–parabolic type* in the sense of Kawashima [Kaw], i.e., systems

$$G(\tilde U)_t + F(\tilde U)_x = (B(\tilde U)\tilde U_x)_x, \tag{1.1}$$

satisfying

(A1) dF, dG, B symmetric, $dG > 0$, $B \ge 0$ (symmetric hyperbolic–parabolicity),

(A2) No eigenvector of $dF dG^{-1}$ lies in the kernel of BdG^{-1} (dissipativity),

and

(A3) $\tilde U = \begin{pmatrix} \tilde u \\ \tilde v \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \quad b > 0$ (block structure),

in some state space $\mathcal{U}$, where $\tilde U \in \mathbb{R}^n$, $\tilde u \in \mathbb{R}^{n-r}$, $\tilde v \in \mathbb{R}^r$, and $b \in \mathbb{R}^{r\times r}$. This class of equations includes many physical models arising in continuum mechanics, in

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particular compressible Navier–Stokes equations and the equations of compressible magnetohydrodynamics (MHD); see Section 2 or [Kaw] for further discussion.

Such systems are well-known to support *viscous shock profiles*, or (without loss of generality) standing wave solutions

$$\bar U(x) = (\bar u, \bar v)(x), \qquad \lim_{z\to\pm\infty} = \tilde U_\pm \tag{1.2}$$

connecting endstates $\tilde U_\pm$, corresponding to "ideal", or discontinuous shock waves of the associated hyperbolic system

$$G(\tilde U)_t + F(\tilde U)_x = 0. \tag{1.3}$$

See, for example, the treatment of existence of small-amplitude profiles for general systems in [P.2] or [Fre.2], or the treatments in [Gil] and [CS] of existence of large-amplitude profiles for the specific systems of gas dynamics and MHD, under standard[1] assumptions on the equation of state.

Small-amplitude profiles near a point of strict hyperbolicity of the associated hyperbolic equations (1.3), as are large-amplitude gas-dynamical profiles of the class constructed by Gilbarg [Gi], are necessarily of classic, "Lax" type, satisfying, besides the usual (stationary) *Rankine-Hugoniot conditions*

$$[F] = 0 \tag{RH}$$

(automatic, by the condition that $\tilde U_\pm$ be rest points of the standing-wave ODE), both the *Lax characteristic conditions*

$$a_{p-1}^- < 0 < a_p^-, \quad a_p^+ < 0 < a_{p+1}^+, \tag{1.4}$$

where $a_j^\pm$ denote the eigenvalues of $dFdG^{-1}(\tilde U_\pm)$ ordered with increasing size, i.e., the characteristic speeds of the associated symmetric hyperbolic system (1.3) at endstates $\tilde U_\pm$, and *transversality* as a connection of the standing wave ODE with boundary conditions $\tilde U_\pm$ (see [MZ.3] for a detailed discussion of the standing wave ODE). More precisely, (1.4) are the conditions for a Lax p-shock, where $1 \le p \le n$ is arbitrary; The distinguished characteristic speed a_p is known as the principal characteristic speed associated with the shock. Transversality refers to transversality of the intersection at $\bar U$ of the unstable manifold at $\tilde U_-$ and the stable manifold at $\tilde U_+$ of the standing wave ODE, in particular implying local uniqueness/persistence under perturbation of the connection $\bar U$. We shall restrict attention for simplicity

[1]Standard, but not the most general possible: in particular, excluding 'real' or van der Waals type equation of state modeling effects of phase transition. In this regime, the question of existence, or "structural stability" of the associated ideal shock, is open as far as we know. Likewise, hyperbolic, or "dynamical stability" may fail; see [Be,MeP].

to Lax profiles, also for large-amplitude profiles, though other types of connections are certainly possible; see, e.g., the treatment of MHD in [CS].

Remark 1.1. Extreme Lax-type shock profiles $p = 1$ or n (for example, gas-dynamical shock profiles) if they exist are always transverse; see Corollary 1.3, [MZ.3].

Existence and transversality of a viscous profile is sometimes referred to in the physical literature as "structural stability" of the associated ideal shock, and hyperbolic stability as "dynamical stability"; see, e.g., [BE]. In one dimension, the condition for hyperbolic stability (see [Ma.1–3]) is simply

$$\Delta := \det\,(r_1^-, \dots, r_{p-1}^-, [G], r_{p+1}^+, r_n^+) \neq 0, \tag{1.5}$$

where $r_j^\pm$ denote the eigenvectors of $dFdG^{-1}(\tilde U_\pm)$ associated with $a_j^\pm$ and $[G] := G_+ - G_-$ the jump across the shock: a refined version of the Lax characteristic condition (1.4).[2] These may be derived through formal, matched asymptotic expansion as approximate necessary conditions for validity of the hyperbolic shock wave theory as an approximation of the behavior of (1.1); for further discussion, see [Z.3], Section 1.3. However, rigorous validation requires a full, *viscous* stability analysis.

Such an analysis was carried out in the small in [MZ.2–3,HuZ], with the result that *all* small-amplitude profiles of Kawashima class systems are nonlinearly stable as solutions of the viscous system (1.1), in agreement with previous results of structural [P.1] and dynamical [Ma.1–3] stability. Our purpose in the present paper is to extend this analysis to the large-amplitude case, building on the large-amplitude *linearized* stability results of [MZ.3].

Specifically, defining *strong spectral stability* of $\bar U$, following [ZH,MZ.1–3], by

$$\sigma_p(L_G) \subset \{\lambda : \text{Re}\,\lambda \le 0\} \setminus \{0\}, \tag{1.6}$$

where $L_G G := (BG_x)_x - (AG)_x$ denotes the linearized operator about the wave (written with respect to the conservative G variable), with

$$B := B(\bar U) \qquad AW := dFdG^{-1}(\bar U)W - dBdG^{-1}(\bar U)W\bar G_x. \tag{1.7}$$

there was established in [MZ.3] the result that linearized $L^1 \cap L^p \to L^p$ stability is equivalent to the conditions of structural and dynamical stability, *augmented with the condition of strong spectral stability.* (Here and elsewhere, $\sigma(M)$ refers to the spectrum of an operator M, and $\sigma_p(M)$ to its point spectrum.) This has the important philosophical interpretation of rigorously validating the formal conclusions of the physical literature.

[2] Nonvanishing of determinant Δ corresponds to solubility of an algebraic system arising through Laplace transform, whereas the Lax condition, also derived from stability/well-posedness considerations [La]) corresponds to the weaker requirement that this system have the correct number of variables.

Here, we shall show that (at least for Lax-type profiles) these same three conditions in fact imply full nonlinear stability, thus completing the formal requirements of the physical literature by the addition of the single new condition of strong spectral stability. This may well have physical significance in the regime where transition to instability in fact occurs. For, as described in [BE], there is some uncertainty about where and how this transition occurs; in particular, it does not appear that this is determined precisely by structural and dynamical stability alone. The general consensus seems to be that viscous effects must be taken into consideration, yet different formal viscous analyses are as yet inconclusive, sometimes yielding opposite conclusions [BE]. Our rigorous identification of a third (spectral) stability condition incorporating all remaining viscous effects sets the stage for the serious investigation of this interesting physical problem. The resolution of this puzzle by determination of spectral stability of large-amplitude profiles by either analytical or numerical[3] means we regard as the outstanding open problem in the stability of shock waves. In particular, can spectral stability in some cases be more restrictive than either dynamical or structural stability, thus determining the transition to instability?

We now state our results more precisely. Suppose that we are given a profile (1.2) of system (1.1), satisfying in addition to (A1)–(A3), the standard hypotheses:

(H0) F, G, $B \in C^5$,

(H1) The eigenvalues of $dF_{11}dG_{11}^{-1}$ (necessarily real and semisimple) are: (i) of constant multiplicity; (ii) nonvanishing; and (iii) of common sign.

(H2) $\sigma(dF(\tilde U_\pm)) \neq 0$.

Here, condition (H0) gives the regularity needed for our later analysis.[4] Condition (H1)(ii) corresponds to the requirement that "hyperbolic," or unsmoothed modes in the solution be noncharacteristic relative to the shock speed, and is needed in order that the the traveling wave ODE be of nondegenerate type. Condition (H1)(i) is an additional technical assumption that was used in the detailed Green function analysis carried out in [MZ.2] for slightly more general systems; we suspect that it can be dropped in the present, symmetrizable case. Condition (H1)(iii) is the sole new assumption made in the large-amplitude analysis, a technical condition used in the energy estimates. Condition (H2) is the standard one that the ideal shock be noncharacteristic.

Remark 1.2. Conditions (H1)(i)–(iii) are always satisfied for compressible gas dynamics and MHD, for which $dF_{11}dG_{11}^{-1}$ is 1×1 and identically equal to particle velocity (u in Eulerian coordinates, 0 in Lagrangian coordinates, vs. acoustic characteristic speeds of $u \pm c$ and $\pm c$, respectively, where the sound speed c is

[3] The spectral stability condition may readily be tested numerically; see, e.g., [Br.1–3,BrZ].

[4] Specifically, we require C^4 for the energy estimates involved in our nonlinear analysis, and C^5 to apply the linearized bounds of [MZ.2]; we conjecture that the latter requirement could be reduced to C^3 with further effort.

nonvanishing under standard assumptions on the equation of state); see Section 2. Likewise, (H2) is generically satisfied for these equations, with failure typically marking the boundary of existence of the profile. Thus, all conclusions of this paper are applicable to these cases.

Definition 1.3. For a profile $\bar U$ that is (as in the Lax case) unique up to translation, we define *nonlinear orbital stability* as convergence of $\tilde U(\cdot,t)$ as $t\to\infty$ to a translate $\bar U(\cdot-\delta(t))$, where $\delta(\cdot)$ is an appropriately chosen function describing shock location, for any solution $\tilde U$ of (1.1) with initial data sufficiently close in some norm to the the original profile $\bar U$.

Then, the main result of this paper is:

Theorem 1.4. *Let there hold (A1)–(A3) and (H0)–(H2) for a general real viscosity model (1.1), and let $\bar U$ be a Lax-type profile (1.2) satisfying the conditions of structural stability (i.e., transversality), dynamical stability (i.e., hyperbolic stability, (1.5)), and strong spectral stability (necessary for stability [MZ.3]). Then, $\bar U$ is nonlinearly orbitally stable from $L^1\cap H^3$ to L^p, for all $p\geq 2$. More precisely, for initial perturbations $U_0:=\tilde U_0-\bar U$ such that $|U_0|_{L^1\cap H^3}\leq\zeta_0$, ζ_0 sufficiently small, the solution $\tilde U=(\tilde u,\tilde v)(x,t)$ of (1.1) with initial data $\tilde U_0$ satisfies*

$$|\tilde U(x,t)-\bar U(x-\delta(t))|_{L^p}\leq C\zeta_0(1+t)^{-\frac{1}{2}(1-1/p)} \tag{1.8}$$

and

$$|\left(\tilde v(x,t)-\bar v(x-\delta(t))\right)_x|_{L^p}\leq C\zeta_0(1+t)^{-\frac{1}{2}(1-1/p)}, \tag{1.9}$$

for all $2\leq p\leq\infty$, for some $\delta(t)$ satisfying

$$|\dot\delta(t)|\leq C\zeta_0(1+t)^{-\frac{1}{2}} \tag{1.10}$$

and

$$|\delta(t)|\leq C\zeta_0. \tag{1.11}$$

Remarks 1.5. 1. Useful geometric necessary conditions for viscous stability have been obtained in [Z.3] for a class of models including (1.1), (A1)–(A3), using the stability index of [GZ,BSZ]. Strengthened, *signed* versions $\Delta>0$ (under appropriate normalization) of the one-dimensional hyperbolic stability criterion (1.5), these readily yield examples of spectrally unstable large-amplitude profiles, similarly as in the strictly parabolic case (see, e.g., [GZ,FreZ,ZS,Z.3]). This shows that the stability conditions assumed in Theorem 1.4 are not vacuous in the large amplitude case. As discussed above, it is an interesting and physically important open question *which* of the three criteria is the more restrictive in different regimes; in particular, it would be very interesting to know whether spectral stability can be the determining factor: that is, is $\Delta > 0$ sufficient as well as necessary?

2. A corresponding multidimensional stability result has been obtained in [Z.4], for the same class of systems and profiles. However, large-amplitude multidimensional spectral stability has so far not been verified for a single physical example. (It is automatic in the scalar case, by the maximum principle; see [HoZ.2–3]. It has been verified in [PZ] for a class of artificial systems constructed for this purpose.)

3. In both one- and multi-dimensional analyses, the hypothesis B symmetric may be dropped in (A1) with no change in the results. It is an interesting question whether the stability index calculations of [Z.3] can also be generalized to this case.

Theorem 1.4 is established by essentially the same argument used to establish the corresponding small-amplitude result in [MZ.2], the single new ingredient being the extension to large amplitudes of the auxiliary energy estimate of Proposition 5.1, [MZ.2]. Following [MZ.2], define the nonlinear perturbation $U = (u, v)$ by

$$U(x,t) := \tilde U(x+\delta(t),t) - \bar U(x), \tag{1.12}$$

where the "shock location" δ is to be determined later. Evidently, decay of U is equivalent is equivalent to nonlinear orbital stability as described in (1.8). Then, the key energy estimate, and the main technical contribution of the paper, is as follows.

Proposition 1.6. *Under the hypotheses of Theorem 1.4, let $U_0 \in H^3$, and suppose that, for $0 \le t \le T$, both the supremum of $|\dot\delta|$ and the $H^2 \cap W^{2,\infty}$ norm of the solution $U = (u,v)^t$ of (1.1), (1.12) remain bounded by a sufficiently small constant $\zeta > 0$. Then,*

$$\begin{aligned} |U|_{H^3}^2(t) + &\int_0^t \big(|U_x|_{H^2}^2 + |v_x|_{H^3}^2\big)(s)ds \le \\ &C|U_0|_{H^3}^2 + C\int_0^t \Big(|U|_{L^\infty}(|U|_{L^\infty} + |U|_{L^2}^2) + \dot\delta^2\Big)(s)ds, \end{aligned} \tag{1.13}$$

for all $0 \le t \le T$.

Remark 1.7. Note, with the expected decay rates $|U|_{L^\infty}$, $|\dot\delta(t)| \sim C(1+t)^{-1/2}$, and $|U|_{L^2} \sim C(1+t^{-1/4})$, that the righthand side of (1.13) becomes order $\log(1+t)$, very nearly recovering the order one bound available in the constant-coefficient case studied in [Kaw,LZe] (used there, as here, to control higher derivatives in the nonlinear iteration yielding time-asymptotic decay).

The key new ingredient in the argument of Proposition 1.6 is the strengthened hypothesis (H1)(iii) requiring that hyperbolic modes be uniformly transverse to the shock. This allows us to control terms of form $\int \mathcal{O}(|\bar U_x|)|v|^2$ that were formerly controlled by the small-amplitude assumption $|\bar U_x| = \mathcal{O}(\varepsilon)$, $\varepsilon := |\tilde U_+ - \tilde U_-| << 1$, instead by a "Goodman-type" weighted norm estimate in the spirit of [Go.1–2], thus closing the argument. A similar device was used in the study of small-amplitude "real" viscous boundary layer profiles in [SZ], Appendix A, in the case, corresponding to full Dirichlet boundary conditions, that all hyperbolic modes are incoming with respect to the boundary. More general "dissipative" boundary conditions have been treated in [R] by a different technique.

We conclude this introduction with a simple example illustrating the various ideas discussed so far. Consider a traveling shock profile $(v,u)=(\bar v,\bar u)(x-st)$ of the equations of isentropic gas dynamics written in Lagrangian coordinates:

$$\begin{aligned} v_t - u_x &= 0,\\ u_t + p(v)_x &= ((\nu/v)u_x)_x, \end{aligned} \tag{1.14}$$

where $v>0$ denotes specific volume, u fluid velocity, $p>0$ pressure, and ν the coefficient of viscosity, under the minimal assumption $p_v<0$ corresponding to hyperbolicity of the associated first-order system. Under this assumption, it is readily verified that $(\bar v,\bar u)$, (1.14) satisfy (A1)–(A3), (H0)–(H2) when considered in terms of $U:=(-p,u)=d\eta$, where $\eta:=\int_v^\infty p(w)dw+u^2/2$ is a particular "convex entropy"– specifically, the mechanical energy– of the system. This observation is a special case of a general principle discussed in Section 2.

Reducing to the case of a standing profile by the change of coordinates $x\to x-st$, we find, following [Z.1], that the standing-wave ODE

$$\begin{aligned} -sv' - u' &= 0,\\ -su' + p(v)' &= ((\nu/v)u')', \end{aligned} \tag{1.15}$$

reduces to the first-order scalar equation

$$v' = (1/s\nu)v(v-v_-)(\sigma^2(v,v_-)-s^2), \tag{1.16}$$

where $\sigma^2(v_1,v_2):=-(p(v_1)-p(v_2))(v_1-v_2)$ denotes the shock speed for an ideal shock connecting endstates v_1 and v_2. Thus, connections if they exist are always transverse.

Moreover, taking without loss of generality $s>0$ ($s=0$ is prohibited by $p_v\neq 0$), we have evidently that a connection exists if and only if there holds the strict Liu–Oleinik entropy condition

$$\sigma(v_-,v_*)>\sigma(v_-,v_+)=s \text{ for all } v_-<v_*<v_+, \tag{E}$$

that is, whenever there exists an (hyperbolically) admissible ideal shock of the corresponding hyperbolic system: in the commonly occurring genuinely nonlinear case $p_{vv}>0$, precisely when $v_+>v_-$. Phrased in another way, *structural stability holds* *always* for isentropic gas dynamics, under the assumption $p_v < 0$ (more generally, for any equation of state and any shock speed $s \neq 0$)[5].

Likewise, taking for definiteness $u < 0$ (equivalently, $m \equiv \rho u < 0$, since $\rho > 0$), the Majda determinant Δ is readily computed [Z.1] as

$$\Delta = \det \begin{pmatrix} 1 & [v] \\ \sqrt{-p_v}(v_-) & [u] \end{pmatrix} = [v](-s - \sqrt{-p_v}(v_-)) \neq 0.$$

Thus, *dynamical stability holds also* for arbitrary profiles under the assumption $p_v < 0$.

According to Theorem 1.4, therefore, linearized and nonlinear stability is completely determined by *spectral stability* (or instability) of the profile. Here, the information is less complete. Small-amplitude profiles are known to be spectrally stable; see, e.g., [MN,Hu,HuZ,PZ]. However, spectral stability of large-amplitude profiles, in this generality, is an open problem. In the special case of gamma-law gas dynamics $p(v) = Cv^{-\gamma}$, on the other hand, spectral stability (precisely, the closely related issue of zero-mass stability) has been carefully investigated in [MN], with the result that, for a fixed lefthand state (v_-, u_-), the range of stability includes at least shocks of amplitude less than an explicit function $a(\gamma) \to \infty$ as $\gamma \to 1$. This yields a variety of examples of spectrally stable profiles for any given, fixed large amplitude. Moreover, in the distinguished case $\gamma = 1$, it implies that all shock profiles, of arbitrarily large amplitude, are stable. Thus, we have at least one example of large-amplitude nonlinear stability. The analytic determination of spectral stability for more general cases is an outstanding open problem in the theory.

Plan of the paper. In Section 2, we give an alternative characterization of (A1)–(A3), (H0)–(H2) for systems with a viscosity-compatible convex entropy, and use this to validate the hypotheses for gas dynamics and MHD (with or without electromagnetic transport). In Sections 3–4, we recall from [MZ.3] the basic linearized $L^q \to L^p$ bounds that we shall need in our nonlinear analysis, and carry out a linearized orbital stability analysis motivating our later treatment of the nonlinear case. In Section 5, we carry out the main work of the paper, establishing the key energy estimate Proposition 1.6. Together with the nonlinear stability argument of [MZ.2], this is sufficient to yield Theorem 1.4. We reproduce this argument, for completeness, in Section 6.

Section 2. Example systems.

We begin by giving an alternative and more easily verified characterization of systems (1.1), (A1)–(A2), and hypotheses (H0)–(H2). First, noting that $U \to G(U)$ is globally invertible, by symmetric positive definiteness of dG, we observe that

[5] For a van der Waals gas, p_v may change sign, and there do arise interesting stationary, phase-transitional ideal shocks to which the above discussion does not apply; see, e.g., [Sl.1–5,P.2,Z.5].

any system of form (1.1), (A1)–(A3) may be written in terms of the conservative variable $G := G(U)$ in the more familiar form

$$G_t + \tilde F(G)_x = (\tilde B(G)G_x)_x, \tag{2.1}$$

$$\tilde B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \quad \text{Re } \sigma(b_2) > 0, \tag{2.2}$$

(A2') No eigenvector of $d\tilde F$ lies in the kernel of $\tilde B$,

studied in [MZ.3], where $\tilde F(G) := F(U(G))$ and $\tilde B(G) := B(U(G))(\partial U/\partial G)$. In this section, we give a simple condition under which the converse is true, and use this observation to verify the conditions for a number of examples from gas dynamics and MHD.

Definition 2.1. Following [Kaw], we define functions $\eta(G)$, $q(G) : \mathbb{R}^n \to \mathbb{R}^1$ to be a convex entropy, entropy flux pair for (2.1), (2.2) if: (i) $d^2\eta > 0$ (η convex), (ii) $d\eta d\tilde F = dq$, and (iii) $d^2\eta\tilde B$ is symmetric (hence positive semidefinite, by (2.2)).

Proposition 2.2. *A system (2.1), (2.2), (A2') possessing a convex entropy, entropy flux pair η, q, considered as an equation in the "entropy variable" $U := d\eta(G)$, is of the symmetric form (1.1), (A1)–(A3).*

Proof. The change of coordinates $G \to U := d\eta(G)$ is globally invertible, since its Jacobian $d^2\eta$ is symmetric positive definite, by Definition 2.2 (i), and thus we can rewrite (2.1) as (1.1), as claimed, where $F(U) := \tilde F(G(U))$ and $B(U) := \tilde B(G(U))(\partial G/\partial U) = \tilde B d^2\eta^{-1}$. Then, symmetry of $dF = d\tilde F d^2\eta^{-1}$, $B = \tilde B d^2\eta^{-1}$ follow from symmetry of $d^2\eta d\tilde F$ and $d^2\eta\tilde B$, the first a consequence of Definition 2.2 (ii) that is well-known from the hyperbolic theory [G,M], the second just Definition 2.2(iii). That $\tilde B$ is similar to B implies that also Re $\sigma(B) \geq 0$, completing the verification of (A1). Condition (A2) is just (A2') in the new coordinates, so trivially satisfied. Finally, (A3) follows by symmetry of B and the fact that the u rows must vanish, by the corresponding property of left factor $\tilde B$. ∎

Conditions (H0),(H2) remain unchanged in the G coordinates; however, (H1) becomes (see [MZ.3]), for possibly nonzero shock speed s:

(H1') The eigenvalues of $A_* := d\tilde F_{11} - d\tilde F_{12} b_2^{-1} b_1$ (necessarily real and semisimple) are: (i) of constant multiplicity; (ii) different from the shock speed s; and (iii) of common sign relative to s.

We now use this observation to verify hypotheses (1.1), (A1)–(A3), (H0)–(H2) for various example systems.

Example 2.3. (isentropic gas dynamics) For equations (1.14), it is easily verified that $\eta(v,u) := \int_v^\infty p(w)dw + u^2/2$ is a (unique up to constant factor) convex entropy whenever $p_v < 0$, whence the system may be put into form (1.1), (A1), (A3) as claimed in the introduction. Condition (A2') is clearly satisfied, so that we need only check condition (H2'). But, $d\tilde{F}_{11}$, $b_1 = 0$, so that $A_* \equiv 0$, and (H2') is satisfied if $s \neq 0$, which is always the case when $p_v < 0$. Thus, all hypotheses are satisfied for $p_v < 0$.

Example 2.4. (full gas dynamics) The general Navier–Stokes equations of compressible gas dynamics, written in Lagrangian coordinates, appear as

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= ((\nu/v)u_x)_x, \\ (e+u^2/2)_t + (pu)_x &= ((\kappa/v)\theta_x + (\mu/v)uu_x)_x, \end{aligned} \tag{2.3}$$

where $v > 0$ denotes specific volume, u velocity, e internal energy, $\theta = \theta(v,e)$ temperature, $p = p(v,e) > 0$ pressure, and $\mu > 0$ and $\kappa > 0$ are coefficients of viscosity and heat conduction, respectively.

Referring to the calculations of Section 9 [LZe], we find that

$$\eta(v,u,e+u^2/2) := -s, \quad q(v,u,e+u^2/2) := 0 \tag{2.4}$$

is a convex entropy, entropy flux pair for (2.3), where s is the physical entropy, defined implicitly by the thermodynamic law

$$de = \theta ds - pdv, \tag{2.5}$$

provided

$$p_v < 0, p_e > 0, \theta_e > 0. \tag{2.6}$$

Straighforward calculation then gives $A_* = d\tilde{F}_{11} = d\tilde{F}_{12}b_2^{-1}b_1 \equiv 0$ as in the isentropic case, verifying (H2') and completing the verification of the hypothesese under assumptions (2.6) (for which, again, it is readily seen that $s \neq 0$ for any traveling shock profile).

Example 2.5. (MHD) Next, consider the equations of MHD:

$$
\begin{aligned}
&v_t - u_x = 0,\\
&u_{1t} + (p(v) + (1/2\mu_0)(B_2^2 + B_3^2))_x = ((\nu/v)u_{1x})_x,\\
&u_{2t} - ((1/\mu_0)B_1^* B_2)_x = ((\nu/v)u_{2x})_x,\\
&u_{3t} - ((1/\mu_0)B_1^* B_3)_x = ((\nu/v)u_{3x})_x,\\
&(vB_2)_t - (B_1^* u_2)_x = ((1/\sigma\mu_0 v)B_{2x})_x,\\
&(vB_3)_t - (B_1^* u_3)_x = ((1/\sigma\mu_0 v)B_{3x})_x,\\
&(e + (1/2)(u_1^2 + u_2^2 + u_3^2) + (1/2\mu_0)v(B_2^2 + B_3^2))_t\\
&\qquad + [(p + (1/2\mu_0)(B_2^2 + B_3^2))u_1 - (1/\mu_0)B_1^*(B_2u_2 + B_3u_3)]_x\\
&\qquad = [(\nu/v)u_1u_{1x} + (\mu/v)(u_2u_{2x} + u_3u_{3x}) +\\
&\qquad (\kappa/v)\theta_x + (1/\sigma\mu_0^2 v)(B_2B_{2x} + B_3B_{3x})]_x,
\end{aligned}
\tag{2.7}
$$

where v denotes specific volume, $u = (u_1, u_2, u_3)$ velocity, $p = P(v, e)$ pressure, $B = (B_1^*, B_2, B_3)$ magnetic induction, B_1^* constant, e internal energy, $\theta = \theta(v, e)$ temperature, and $\mu > 0$ and $\nu > 0$ the two coefficients of viscosity, $\kappa > 0$ the coefficient of heat conduction, $\mu_0 > 0$ the magnetic permeability, and $\sigma > 0$ the electrical resistivity.

Referring again to the calculations of Section 9 [LZe], we find as in (2.4) that, under assumptions (2.5)–(2.6), $\eta := -s$, $q := 0$ is an entropy, entropy flux pair for system (2.7), η, q now considered as functions of the conserved quantities

$$(v, u_1, u_2, u_3, vB_2, vB_3, E),$$

where $E := e + (1/2)(u_1^2 + u_2^2 + u_3^2) + (1/2\mu_0)v(B_2^2 + B_3^2)$ denotes total energy. Likewise, we obtain by straighforward calculation that $A_* = d\tilde{F}_{11} = d\tilde{F}_{12}b_2^{-1}b_1 \equiv 0$, verifying (H2') and completing the verification of the hypothesese under assumptions (2.6) (under which, as before, it is readily seen that $s \neq 0$ for any traveling shock profile).

Example 2.6. (MHD with infinite resistivity/permeability) An interesting variation that is of interest in certain astrophysical parameter regimes is the limit in which either electrical resistivity σ, magnetic permeability μ_0, or both, go to infinity, in which case the righthand sides of the fifth and sixth equations of (2.7) goes to zero and there is a three-dimensional set of hyperbolic modes (v, vB_2, vB_3) instead of the usual one. Nonetheless, $A_* = d\tilde{F}_{11} = d\tilde{F}_{12}b_2^{-1}b_1 \equiv 0$ in this case as well; that is, though now vectorial, hyperbolic modes still experience passive, scalar convection, and so (H2') remains valid under assumption (2.6). (Note: in Lagrangian coordinates, zero speed corresponds to "particle", or fluid velocity.) Reduction to symmetric form may be achieved by the same entropy, entropy flux pair as in the standard case of Example 2.5, under assumptions (2.5)–(2.6).

Example 2.7. (multi-species gas dynamics or MHD) Another simple example for which the hyperbolic modes are vectorial is the case of miscible, multi-species flow, neglecting species diffusion, in either gas- or magnetohydrodynamics. In this case, the hyperbolic modes consist of k copies of the hyperbolic modes for a single species, where k is the number of total species, with a single, scalar convection rate $A_* \equiv 0$, corresponding to passive convection by the common fluid velocity.

Summary. We have shown, under very general assumptions on the equation of state, that hypotheses (1.1), (A1)–(A3), (H0)–(H2) of Theorem 1.4 are satisfied for arbitrary traveling shock profiles of gas dynamics or MHD. As for the new technical hypothesis (H2)(iii), it is trivially satisfied due to the circumstance that, in each case considered, hyperbolic modes propagate by *passive, scalar convection.* It would be interesting to find a physical example for which the hyperbolic convection is not scalar: in particular, to investigate the circumstances under which (H2)(iii) is satisfied. For transverse, Lax-type shock profiles, Theorem 1.4 thus reduces the question of nonlinear stability to the investigation of hyperbolic, and spectral stability. For gas dynamics, the hyperbolic stability condition was completely evaluated in [Er.1] (see also, [Ma.1–3,Z.3, Appendix A]); extensions to MHD may be found in [B,BT.1–4,BTM.1–2]. Thus, existence and spectral stability are the remaining issues.

For gas dynamics, (2.3), existence of profiles has been established for arbitrary ideal shocks [Gi,W] under the additional assumptions ??? (TODO) on the equation of state. These may be recognized as the "strong condition" from [MeP], guaranteeing hyperbolic stability along with uniqueness of Riemann solutions (the main topic of study in [MeP]). For this type of gas, spectral stability is the only possible obstruction to nonlinear stability.

Section 3. Linearized estimates.

In the companion paper [MZ.3], we have investigated linearized stability and behavior of viscous profiles of systems of the general form (2.1), not necessarily symmetrizable, satisfying the standard set of conditions identified in [Z.3]. As remarked in [Z.3,MZ.2–3], these hold always in case (2.1) derives from a symmetric system (1.1) satisfying (A1)–(A3), (H0)–(H2), for an arbitrary transverse, Lax-type profile. (Indeed, they hold also for transverse profiles of nonclassical under- and overcompressive type; see [MZ.3].) More precisely, assuming the necessary conditions of structural, dynamical, and strong spectral stability (i.e., the hypotheses of Theorem 1.4), we obtained in [MZ.3] detailed pointwise bounds on the Green function $\mathcal{G}(x,t;y)$ (more properly speaking, distribution) of the linearized evolution equations

$$G_t = L_G G := -(A_G G)_x + (B_G G_x)_x, \tag{2.1}$$

$$B_G := \tilde B(\bar G(x)) = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \quad A_G V := d\tilde F V - d\tilde B(\bar G(x)) V \bar G'(x), \tag{2.2}$$

$\bar G := G(\bar U(x))$, associated with (2.1), as we shall now describe.

Let $a_j^\pm$, $j = 1, \dots (n)$ denote the eigenvalues of $A_G(\pm\infty)$, and $l_j^\pm$ and $r_j^\pm$ associated left and right eigenvectors, respectively, normalized so that $l_j^{\pm t} r_k^\pm = \delta_k^j$. In case $A_G^\pm$ is strictly hyperbolic, these are uniquely defined up to a scalar multiplier. In the general case, we require further that $l_j^\pm$, $r_j^\pm$ be left and right eigenvectors also of $P_j^\pm B_G^\pm P_j^\pm$, $P_j^\pm := R_j^\pm L_j^{\pm t}$, where $L_j^\pm$ and $R_j^\pm$ denote $m_j^\pm \times m_j^\pm$ left and right eigenblocks associated with the $m_j^\pm$-fold eigenvalue $a_j^\pm$, normalized so that $L_j^\pm R_j^\pm = I_{m_j^\pm}$. (Note: The matrix $P_j^\pm B_G^\pm P_j^\pm \sim L_j^{\pm t} B_G^\pm R_j^\pm$ is necessarily diagonalizable, by simultaneous symmetrizability of A_G, B_G).

Eigenvalues $a_j(x)$, and eigenvectors l_j, r_j correspond to large-time convection rates and modes of propagation of the degenerate model (2.1). Likewise, let $a_j^*(x)$, $j = 1, \dots, (n-r)$ denote the eigenvalues of

$$\begin{aligned} A_{G*} &:= A_{G,11} - A_{G,12} B_{G,22}^{-1} B_{G,21} \\ &= A_{11} (A_{11}^0)^{-1}, \end{aligned} \tag{2.3}$$

$A := dF(\bar U(x))$, $A^0 := dG(\bar U(x))$, and $l_j^*(x)$, $r_j^*(x) \in \mathbb{R}^{n-r}$ associated left and right eigenvectors, normalized so that $l_j^{*t} r_j \equiv \delta_k^j$. More generally, for an m_j^*-fold eigenvalue, we choose $(n-r) \times m_j^*$ blocks L_j^* and R_j^* of eigenvectors satisfying the *dynamical normalization*

$$L_j^{*t} \partial_x R_j^* \equiv 0,$$

along with the usual static normalization $L_j^{*t} R_j \equiv \delta_k^j I_{m_j^*}$; as shown in Lemma 4.9, [MZ.1], this may always be achieved with bounded L_j^*, R_j^*. Associated with L_j^*, R_j^*, define extended, $n \times m_j^*$ blocks

$$\mathcal{L}_j^* := \begin{pmatrix} L_j^* \\ 0 \end{pmatrix}, \quad \mathcal{R}_j^* := \begin{pmatrix} R_j^* \\ -B_{G,22}^{-1} B_{G,21} R_j^* \end{pmatrix}. \tag{2.4}$$

Eigenvalues a_j^* and eigenmodes $\mathcal{L}_j^*$, $\mathcal{R}_j^*$ correspond, respectively, to short-time hyperbolic characteristic speeds and modes of propagation for the reduced, hyperbolic part of degenerate system (2.1). Note that our discussion in the introduction of condition (H3) is validated by the second equality in (2.3), a nontrivial consequence of symmetry/block structure in the original system (1.1).

Define time-asymptotic, *scalar diffusion rates*

$$\beta_j^\pm := \left(l_j^t B_G r_j \right)_\pm, \quad j = 1, \dots, n, \tag{2.5}$$

and local, $m_j \times m_j$ *dissipation coefficients*

$$\eta_j^*(x) := -l_j^{*t} D_* r_j^*(x), \quad j = 1, \dots, J \le n - r, \tag{2.6}$$

where
(2.7)

$$D_*(x) := A_{G,12}B_{G,22}^{-1}\Big[A_{G,21} - A_{G,22}B_{G,22}^{-1}B_{G,21} + A_{G*}B_{G,22}^{-1}B_{G,21} + B_{G,22}\partial_x(B_{G,22}^{-1}B_{G,21})\Big]$$

is an effective dissipation analogous to the effective diffusion predicted by formal, Chapman–Enskog expansion in the (dual) relaxation case.

At $x = \pm\infty$, these reduce to the corresponding quantities identified by Zeng [Ze.1,LZe] in her study by Fourier transform techniques of decay to *constant solutions* $(\bar u, \bar v) \equiv (u_\pm, v_\pm)$ of hyperbolic–parabolic systems, i.e., of limiting equations

$$U_t = L_\pm U := -A_\pm U_x + B_\pm U_{xx}. \tag{2.8}$$

These arise naturally through Taylor expansion of the (frozen-coefficient) Fourier symbol $(-i\xi A - \xi^2 B)_\pm$, as described in Appendix B of [MZ.3]; in particular, as a consequence of dissipativity, (A2), we obtain (see, e.g., [Kaw,LZe,MZ.3])

$$\beta_j^\pm > 0, \quad \text{Re}\ \sigma(\eta_j^{*\pm}) > 0 \quad \text{for all } j. \tag{2.9}$$

However, note that the dynamical dissipation coefficient $D_*(x)$ *does not* agree with its static counterpart, possessing an additional term $B_{G,22}\partial_x(B_{G,22}^{-1}B_{G,21})$, and so we cannot conclude that (2.9) holds everywhere along the profile, but only at the endpoints. This is an important difference in the variable-coefficient case; see Remarks 1.11-1.12 of [MZ.3] for further discussion.

Proposition 2.1[MZ.3]. *Under assumptions (A1)–(A3), (H0)–(H3), for Lax-type shock profiles satisfying the conditions of transversality, hyperbolic stability, and strong spectral stabitiy, the Green function $\mathcal{G}(x,t;y)$ associated with the linearized evolution equations (2.1) may be decomposed as*

$$\mathcal{G}(x,t;y) = H + E + S + R, \tag{2.10}$$

where, for $y \le 0$:

$$\begin{aligned} H(x,t;y) &:= \sum_{j=1}^{J} a_j^{*-1}(x)a_j^*(y)\mathcal{R}_j^*(x)\zeta_j^*(y,t)\delta_{x-\bar a_j^* t}(-y)\mathcal{L}_j^{*t}(y) \\ &= \sum_{j=1}^{J} \mathcal{R}_j^*(x)\mathcal{O}(e^{-\eta_0 t})\delta_{x-\bar a_j^* t}(-y)\mathcal{L}_j^{*t}(y), \end{aligned} \tag{2.11}$$

(2.12)

$$E(x,t;y) := \sum_{a_k^- > 0} [c_{k,-}^0]\bar G'(x) l_k^{-t} \left(\text{errfn}\left(\frac{y + a_k^- t}{\sqrt{4\beta_k^- t}}\right) - \text{errfn}\left(\frac{y - a_k^- t}{\sqrt{4\beta_k^- t}}\right)\right),$$

and

(2.13)

$$S(x,t;y) := \chi_{\{t\ge 1\}} \sum_{a_k^- <0} r_k^- l_k^{-t} (4\pi\beta_k^- t)^{-1/2} e^{-(x-y-a_k^- t)^2/4\beta_k^- t}$$

$$+ \chi_{\{t\ge 1\}} \sum_{a_k^- >0} r_k^- l_k^{-t} (4\pi\beta_k^- t)^{-1/2} e^{-(x-y-a_k^- t)^2/4\beta_k^- t} \left(\frac{e^{-x}}{e^x+e^{-x}}\right)$$

$$+ \chi_{\{t\ge 1\}} \sum_{a_k^- >0,\, a_j^- <0} [c_{k,-}^{j,-}] r_j^- l_k^{-t} (4\pi\bar\beta_{jk}^- t)^{-1/2} e^{-(x-z_{jk}^-)^2/4\bar\beta_{jk}^- t} \left(\frac{e^{-x}}{e^x+e^{-x}}\right),$$

$$+ \chi_{\{t\ge 1\}} \sum_{a_k^- >0,\, a_j^+ >0} [c_{k,-}^{j,+}] r_j^+ l_k^{-t} (4\pi\bar\beta_{jk}^+ t)^{-1/2} e^{-(x-z_{jk}^+)^2/4\bar\beta_{jk}^+ t} \left(\frac{e^{x}}{e^x+e^{-x}}\right)$$

denote hyperbolic, excited, and scattering terms, respectively, $\eta_0 > 0$*, and* R *denotes a faster decaying residual.*[6] *Symmetric bounds hold for* $y \ge 0$*.*

Here, the averaged convection rates $\bar a_j^* = \bar a_j^*(x,t)$ *in (2.11) denote the time-averages over* $[0,t]$ *of* $a_j^*(x)$ *along backwards characteristic paths* $z_j^* = z_j^*(x,t)$ *defined by*

$$dz_j^*/dt = a_j^*(z_j^*), \quad z_j^*(t) = x, \tag{2.14}$$

and the dissipation matrix $\zeta_j^* = \zeta_j^*(x,t) \in \mathbb{R}^{m_j \times m_j}$ *is defined by the dissipative flow*

$$d\zeta_j^*/dt = -\eta_j^*(z_j^*)\zeta_j^*, \quad \zeta_j^*(0) = I_{m_j}. \tag{2.15}$$

Similarly, in (2.13),

$$z_{jk}^{\pm(y,t)} := a_j^\pm \left(t - \frac{|y|}{|a_k^-|}\right) \tag{2.16}$$

and

$$\bar\beta_{jk}^\pm(x,t;y) := \frac{|x^\pm|}{|a_j^\pm t|}\beta_j^\pm + \frac{|y|}{|a_k^- t|}\left(\frac{a_j^\pm}{a_k^-}\right)^2 \beta_k^-, \tag{2.17}$$

represent, respectively, approximate scattered characteristic paths and the time-averaged diffusion rates along those paths. In all equations, a_j*,* $a_j^{*\pm}$*,* l_j*,* $\mathcal{L}_j^{*\pm}$*,* r_j*,* $\mathcal{R}_j^{*\pm}$*,* $\beta_j^\pm$ *and* η_j^* *are as defined just above, and scattering coefficients* $[c_{k,-}^{j,i}]$*,* $i = -,0,+$*, are constants, uniquely determined by*

$$\sum_{a_j^- <0} [c_{k,-}^{j,-}] r_j^- + \sum_{a_j^+ >0} [c_{k,-}^{j,+}] r_j^+ + [c_{k,-}^0]\big(G(+\infty) - G(-\infty)\big) = r_k^- \tag{2.18}$$

[6] Bounds on R and its derivatives may be found in [MZ.2–3].

for each $k = 1, \dots n$, *and satisfying*

$$\sum_{a_k^- > 0} [c^0_{k,-}] l_k^- = \sum_{a_k^+ < 0} [c^0_{k,+}] l_k^+ = \pi, \tag{2.19}$$

where the constant vector π *is the left zero effective eigenfunction of* L_G *associated with the right eigenfunction* $\bar{G}'$.

Remark 2.2. Similar decompositions hold in the over- and undercompressive case; see [ZH,Z.2,MZ.1,MZ.3].

Proposition 2.1, the variable-coefficient generalization of the constant-coefficient results of [Ze.1,LZe], was established in [MZ.3] by Laplace transform (i.e., semigroup) techniques generalizing the Fourier transform approach of [Ze.1–2,LZe]; for discussion/geometric interpretation, see [Z.2,MZ.1–2]. In our stability analysis, we will use only a small part of the detailed information given in the proposition, namely $L^p \to L^q$ estimates on the time-decaying portion $H + S + R$ of the Green function $\mathcal{G}$ (see Lemma 4.1, below). However, the stationary portion E of the Green function must be estimated accurately for an efficient stability analysis.

Lemma 2.3. *Under the assumptions of Proposition 2.1, there hold, additionally,*

$$H(x, t; y)\Pi_2 \equiv 0 \tag{2.20}$$

and

$$\Pi_2 (A^0(x))^{-1} H(x, t; y) \equiv 0, \tag{2.21}$$

where $\Pi_2 := \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix}$ *denotes projection onto final* r *coordinates.*

Proof. Equivalently, we must show that $\mathcal{L}_j^{*tr}\Pi_2 \equiv 0$, $\Pi_2(A^0)^{-1}\mathcal{R}_j^* \equiv 0$, for all $1 \le j \le J$. This is most easily verified by the intrinsic property of $\mathcal{L}_j^*$ and $\mathcal{R}_j^*$ (readily seen from our formulae) that they lie, respectively, in the left and right kernel of B_G. This gives the first relation immediately, by the structure of B_G given in (2.2). Likewise, we find that

$$\begin{aligned} 0 \equiv B_G \mathcal{R}_j^* &:= \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} (A^0)^{-1} \mathcal{R}_j^* \\ &= \begin{pmatrix} 0 \\ b\Pi_2 (A^0)^{-1} \mathcal{R}_j^* \end{pmatrix}, \end{aligned} \tag{2.22}$$

yielding the second result by invertibility of b (see condition (A3)). ∎

Lemma 2.3 quantifies the observation that, in $U = (u, v)^t$ coordinates, data in the v coordinate is smoothed under the under the evolution of (2.1), whereas data in the v coordinate is not; likewise, the "parabolic" variable v experiences smoothing while the "hyperbolic" coordinate v does not. (Recall that $U = (A^0)^{-1}G$).

Section 4. Linearized stability.

As described in [MZ.2–3], linearized orbital stability follows immediately from the pointwise bounds of Proposition 2.1. We reproduce the argument here, both for completeness and to motivate the nonlinear argument to follow in Section 6. Similarly as in [Z.2,MZ.1–3], define the *linear instantaneous projection*:

$$\begin{aligned} \varphi(x,t) &:= \int_{-\infty}^{+\infty} E(x,t;y)G_0(y)\,dy \\ &=: -\delta(t)\bar G'(x), \end{aligned} \tag{4.1}$$

where G_0 denotes the initial data for (2.1), and $\bar G = G(\bar U(x))$ as usual. The amplitude δ may be expressed, alternatively, as

$$\delta(t) = -\int_{-\infty}^{+\infty} e(y,t)G_0(y)\,dy,$$

where

$$E(x,t;y) =: \bar G'(x)e(y,t), \tag{4.2}$$

i.e.,

$$e(y,t) := \sum_{a_k^-} \left(\text{errfn}\,\left(\frac{y+a_k^- t}{\sqrt{4\beta_k^- t}}\right) - \text{errfn}\,\left(\frac{y-a_k^- t}{\sqrt{4\beta_k^- t}}\right)\right) l_k^- \tag{4.3}$$

for $y \le 0$, and symmetrically for $y \ge 0$.

Then, the solution G of (2.1) satisfies

$$G(x,t) - \varphi(x,t) = \int_{-\infty}^{+\infty} (H + \tilde{\mathcal{G}})(x,t;0)G_0(y)\,dy, \tag{4.4}$$

where

$$\tilde{\mathcal{G}} := S + R \tag{4.5}$$

is the regular part and H the singular part of the time-decaying portion of the Green function $\mathcal{G}$.

Lemma 4.1 [MZ.2]. *For transverse, hyperbolically and strongly spectrally stable Lax-type shock profiles of (1.1), under assumptions (A1)–(A3), (H0)–(H3), there hold:*

$$|\int_{-\infty}^{+\infty} \tilde{\mathcal{G}}(\cdot,t;y)f(y)dy|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/r)}|f|_{L^q}, \tag{4.6}$$

$$|\int_{-\infty}^{+\infty} \tilde{\mathcal{G}}_y(\cdot,t;y)f(y)dy|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/r)-1/2}|f|_{L^q} + Ce^{-\eta t}|f|_{L^p}, \tag{4.7}$$

$$|\int_{-\infty}^{+\infty} \tilde{\mathcal{G}}_x(\cdot,t;y)f(y)dy|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/r)}|f|_{L^q} + Ce^{-\eta t}|f|_{L^p}, \tag{4.8}$$

$$|\int_{-\infty}^{+\infty} \tilde{\mathcal{G}}_{xy}(\cdot,t;y)f(y)dy|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/r)-1/2}|f|_{L^q} + Ce^{-\eta t}|f|_{W^{1,p}}, \tag{4.9}$$

and

$$|\int_{-\infty}^{+\infty} H(\cdot,t;y)f(y)dy|_{L^p} \le Ce^{-\eta t}|f|_{L^p}, \tag{4.10}$$

$$|\int_{-\infty}^{+\infty} H_x(\cdot,t;y)f(y)dy|_{L^p} \le Ce^{-\eta t}|f|_{W^{1,p}}, \tag{4.11}$$

for all $t \ge 0$, some C, $\eta > 0$, for any $1 \le q \le p$ (equivalently, $1 \le r \le p$) and $f \in L^q \cap W^{1,p}$, where $1/r + 1/q = 1 + 1/p$.

Proof. As described in [Z.2,MZ.1–3], bounds (4.6)–(4.9) follow by the Hausdorff-Young inequality together with bounds (2.13) and the more stringent bounds on R and its derivatives (omitted here; see [MZ.2–3]). Bound (4.10) follows by direct computation and the fact that particle paths $z_j(y,t)$ satisfy uniform bounds

$$1/C \le |(\partial/\partial y)z_j| < C,$$

for all y, t, by the fact that characteristic speeds $a_j(x)$ converge exponentially as $x \to \pm\infty$ to constant states. Finally, (4.11) follows from the relations

$$\delta_{x-\bar{a}_j^* t}(-y) = a_j^*(y)^{-1}\delta(t - \int_y^x (1/a_j^*)(z)dz) \tag{4.12}$$

and

$$(\partial/\partial x)\delta(t-\int_y^x (1/a_j^*)(z)dz) = a_j^*(y)a_j^*(x)^{-1}(\partial/\partial y)\delta(t-\int_y^x (1/a_j^*)(z)dz), \tag{4.13}$$

where $\tilde a_j^* = \tilde a_j^*(y,t)$ is defined as the average of $a_j^*(z)$ along the *forward* characteristic $\tilde z_j^*(y,t)$ defined by ${z_j^*}' = a_j^*(z_j)$, $z_j^*(y,0) = y$, which are special cases of the more general relations

$$f_y(x,y,t)\delta\big(f(x,y,t)\big) = \delta_{h(x,t)}(y) \tag{4.14}$$

and

$$(\partial/\partial x)\delta(f(x,y,t)) = (f_x/f_y)(\partial/\partial y)\delta(f(x,y,t)), \tag{4.15}$$

where $h(x,t)$ is defined by $f(x,h(x,t),t) \equiv 0$. ∎

Definition 4.2. For a profile $\bar U$ that is (as in the Lax case) unique up to translation, we define *linearized orbital stability* analogously as in Definition 1.3 as convergence as $t\to\infty$ of the solution U of the linearized system of (1.1) about $\bar U$ to a translate $-\delta(t)\bar U_x$, where $\delta(\cdot)$ is an appropriately chosen function describing linearized shock location, for any initial perturbation $U_0 := \tilde U_0 - \bar U$ sufficiently small in some norm.

Corollary 4.3 [MZ.2]. *Let $\bar U$ be a transverse, hyperbolically and strongly spectrally stable shock profile of (1.1), under assumptions (A1)–(A3), (H0)–(H3) (as in the hypotheses of Theorem 1.4). Then, $\bar U$ is $L^1\cap L^p \to L^p$ linearly orbitally stable, for any $p>1$. More precisely, for initial data $U_0 \in L^1\cap L^p$, the solution $U = (u,v)^t(x,t)$ of equations (1.1) linearized about $\bar U$ satisfies the linear decay bounds*

$$|U(\cdot,t)+\delta(t)\bar U'(\cdot)|_{L^p} \le C(1+t)^{-\frac12(1-1/p)}(|U_0|_{L^1}+|U_0|_{L^p}). \tag{4.16}$$

Moreover, provided $U_0\in W^{1,p}$, there hold also the derivative bounds

$$|\big(v(\cdot,t)+\delta(t)\bar v'(\cdot)\big)_x|_{L^p} \le C(1+t)^{-\frac12(1-1/p)}(|U_0|_{L^1}+|U_0|_{W^{1,p}}). \tag{4.17}$$

Proof. For (4.16), it is equivalent to show that, for initial data $G_0\in L^1\cap L^p$, the solution $G(x,t)$ of (2.1) satisfies

$$|G(\cdot,t)-\varphi(\cdot,t)|_{L^p} \le C(1+t)^{-\frac12(1-1/p)}(|G_0|_{L^1}+|G_0|_{L^p}). \tag{4.18}$$

But, this follows immediately from (4.4) and bounds (4.6) and (4.10), with $q=p$. Likewise, for (4.17), it is equivalent to show that, for initial data $G_0\in L^1\cap W^{1,p}$, the solution $G(x,t)$ of (2.1) satisfies

$$|\big(\Pi_2(A^0)^{-1}\big(G(\cdot,t)-\varphi(\cdot,t)\big)\big)_x|_{L^p} \le C(1+t)^{-\frac12(1-1/p)}(|G_0|_{L^1}+|G_0|_{W^{1,p}}). \tag{4.19}$$

This follows, similarly, from the derivative analog

(4.20)
$$\Big(\Pi_2(A^0)^{-1}\big(G(x,t)-\varphi(x,t)\big)\Big)_x=\int_{-\infty}^{+\infty}\Big(\Pi_2(A^0)^{-1}(H+\tilde{\mathcal{G}})\Big)_x(x,t;0)G_0(y)\,dy$$

of (4.4), together with (2.21) and bounds (4.6), (4.8), with $q=p$. ∎

Section 5. Energy estimates.

In this section, we do the main work of the paper, establishing the auxiliary energy estimate of Proposition 1.6. Define the nonlinear perturbation $U(x,t):=\tilde U(x+\delta(t),t)-\bar U(x)$ as in (1.12) where $\delta(t)$ (estimating shock location) is to be determined later; for definiteness, fix $\delta(0)=0$. Substituting (1.12) into (1.1), we obtain

$$G(\tilde U)_t+F(\tilde U)_x-(B(\tilde U)\tilde U_x)_x=\dot\delta G(\tilde U)_x, \tag{5.1}$$

and thereby the basic nonlinear perturbation equation

$$\Big(G(\tilde U)-G(\bar U)\Big)_t+\Big(F(\tilde U)-F(\bar U)\Big)_x-\Big(B(\tilde U)\tilde U_x-B(\bar U)\bar U_x\Big)_x=\dot\delta(t)G(\tilde U)_x, \tag{5.2}$$

where $\tilde U$ now denotes $\tilde U(x+\delta(t),t)$ and $\bar U$ denotes $\bar U(x)$.

Lemma 5.1 ([SK]). *Assuming (A1), condition (A2) is equivalent to either of:*

(K1) There exists a smooth skew-symmetric matrix $K(u)$ such that

$$\text{Re }\Big(K(A^0)^{-1}A+B\Big)(u)\ge\theta>0. \tag{5.3}$$

$A^0:=dG$, $A:=dF$, B as in (1.1).

(K2) For some $\theta>0$, there holds

$$\text{Re }\sigma(-i\xi(A^0)^{-1}A(u)-|\xi|^2(A^0)^{-1}B(u))\le-\theta|\xi|^2/(1+|\xi|^2), \tag{5.4}$$

for all $\xi\in\mathbb{R}$.

Proof. These and other useful equivalent formulations are established in [SK]. ∎

Lemma 5.2[MZ.3]. *Under assumptions (A1)–(A3), (H0)–(H2), standing wave solutions (1.2) satisfy*

$$|(d/dx)^k \bar U| \le Ce^{-\theta|x|}, \quad k = 1, \dots, 4, \tag{5.5}$$

as $x \to \pm\infty$, for some $\theta > 0$.

Proof. See [MZ.3], Lemma 1.3.∎

Proof of Proposition 1.6. Writing (5.2) in quasilinear form

$$\left(\tilde A^0 \tilde U_t - \bar A^0 \bar U_t\right) + \left(\tilde A \tilde U_x - \bar A \bar U_x\right) - \left(\tilde B \tilde U_x - \bar B \bar U_x\right)_x = \dot\delta(t) \tilde A^0 \tilde U_x, \tag{5.6}$$

where

$$\tilde A^0 := A^0(\tilde U), \bar A^0 := A^0(\bar U); \quad \tilde A := A(\tilde U), \bar A := A(\bar U); \quad \tilde B := B(\tilde U), \bar B := B(\bar U), \tag{5.7}$$

using the quadratic Leibnitz relation

$$A_2U_2 - A_1U_1 = A_2(U_2 - U_1) + (A_2 - A_1)U_1, \tag{5.8}$$

and recalling the block structure assumption (A3), we obtain the alternative perturbation equation:

$$\tilde A^0 U_t + \tilde A U_x - (\tilde B U_x)_x = M_1(U)\bar U_x + (M_2(U)\bar U_x)_x + \dot\delta(t)\tilde A^0 U_x + \dot\delta(t)\tilde A^0 \bar U_x, \tag{5.9}$$

where

$$M_1(U) := \tilde A - \bar A = \left(\int_0^1 dA(\bar U + \theta U)\, d\theta\right) U, \tag{5.10}$$

and

$$\begin{aligned} M_2(U) &:= \tilde B - \bar B \\ &= \begin{pmatrix} 0 & 0 \\ 0 & (\int_0^1 db(\bar U + \theta U)\, d\theta) U \end{pmatrix}, \end{aligned} \tag{5.11}$$

with

$$\tilde b := b(\tilde U), \bar b := b(\bar U). \tag{5.12}$$

We now carry out a series of successively higher order energy estimates of the the type formalized by Kawashima [Ka.1]. The origin of this approach goes back to [K,MN] in the context of gas dynamics; see, e.g., [HoZ.1] for further discussion/references. The novelty here is that the stationary background state $\bar U$ is not constant in x, nor even slowly varying as in [MZ.2]. We overcome this difficulty by introducing the additional ingredient of a Goodman-type weighted norm in order to control hyperbolic modes, assumed for this purpose (see hypothesis (H2)(iii)) to be uniformly transverse to the background shock profile.

Let $\tilde{K}$ denote the skew-symmetric matrix described in Lemma 5.1 associated with $\tilde{A}^0$, $\tilde{A}$, $\tilde{B}$. Then, regarding $\tilde{A}^0$, $\tilde{K}$, we have the bounds

$$\begin{aligned} &\tilde{A}^0_x = dA^0(\tilde{U})\tilde{U}_x, \quad \tilde{K}_x = dK(\tilde{U})\tilde{U}_x, \quad \tilde{A}_x = dA(\tilde{U})\tilde{U}_x, \quad \tilde{B}_x = dB(\tilde{U})\tilde{U}_x, \\ &\tilde{A}^0_t = dA^0(\tilde{U})\tilde{U}_t, \quad \tilde{K}_t = dK(\tilde{U})\tilde{U}_t, \quad \tilde{A}_t = dA(\tilde{U})\tilde{U}_t, \quad \tilde{B}_x = dB(\tilde{U})\tilde{U}_t, \end{aligned} \tag{5.13}$$

and (from defining equations (1.12)–(5.1)):

$$|\tilde{U}_x| = |U_x + \bar{U}_x| \le |U_x| + |\bar{U}_x| \tag{5.14}$$

and

$$\begin{aligned} |\tilde{U}_t| &\le C(|\tilde{U}_x| + |\tilde{v}_{xx}| + |\dot{\delta}||\tilde{U}_x|) \\ &\le C(|U_x| + |\bar{U}_x| + |v_{xx}| + |\bar{v}_{xx}| + |\dot{\delta}||U_x| + |\dot{\delta}||\bar{U}_x|) \\ &\le C(|U_x| + |\bar{U}_x| + |v_{xx}|). \end{aligned} \tag{5.15}$$

Thus, in particular

$$\begin{aligned} &|\dot{\delta}|,\, |\tilde{A}^0_x|,\, |\tilde{A}^0_{xx}|,\, |\tilde{K}_x|,\, |\tilde{K}_{xx}|,\, |\tilde{A}_x|,\, |\tilde{A}_{xx}|,\, |\tilde{B}_x|,\, |\tilde{B}_{xx}|,\, |\tilde{A}^0_t|, |\tilde{K}_t|,\, |\tilde{A}_t|,\, |\tilde{B}_t| \\ &\qquad\qquad\qquad\qquad\qquad\qquad \le C(\zeta + |\bar{U}_x|). \end{aligned} \tag{5.16}$$

Finally, we introduce the weighted norm and inner product

$$|f|_\alpha := |\alpha^{1/2} f|_{L^2}, \quad \langle f, g\rangle_\alpha := \langle \alpha f, g\rangle_{L^2}, \tag{5.17}$$

$\alpha(x)$ scalar, uniformly positive, and uniformly bounded. For the remainder of this section, we shall for notational convenience omit the subscript α, referring always to α-norms or -inner product unless otherwise specified. For later reference, we note the commutator relation

$$\langle f, g_x\rangle = -\langle f_x, g\rangle + \langle (\alpha_x/\alpha) f, g\rangle, \tag{5.18}$$

and the related identities

$$\langle f, S f_x\rangle = -(1/2)\langle f, \big(S_x + (\alpha_x/\alpha)S\big) f\rangle, \tag{5.19}$$

$$\langle f, (Sf)_x\rangle = (1/2)\langle f, \big(S_x - (\alpha_x/\alpha)S\big) f\rangle, \tag{5.20}$$

valid for symmetric operators S.

By (H2)(iii), we have that $\bar A_{11}(\bar A^0_{11})^{-1}$ has real spectrum of uniform sign, without loss of generality negative, so that the similar matrix

$$(\bar A^0_{11})^{-1/2}\bar A_{11}(\bar A^0_{11})^{-1/2} = (\bar A^0_{11})^{-1/2}\bar A_{11}(\bar A^0_{11})^{-1}(\bar A^0_{11})^{1/2}$$

has real, negative spectrum as well. (Recall, $\bar A^0_{11}$ is symmetric negative definite as a principal minor of the symmetric negative definite matrix $\bar A^0$.) It follows that $\bar A_{11}$ itself is uniformly symmetric negative definite, i.e.,

$$\bar A_{11} \le -\theta < 0. \tag{5.21}$$

Defining α, following Goodman [Go.2], by the ODE

$$\alpha_x = C_*|\bar U_x|\alpha, \quad \alpha(0) = 1, \tag{5.22}$$

where $C_* > 0$ is a large constant to be chosen later, we have by (5.21)

$$(\alpha_x/\alpha)\bar A_{11} \le -C_*\theta|\bar U_x|. \tag{5.23}$$

Note, because $|\bar U_x| \le Ce^{-\theta|x|}$ (see Lemma 5.2), that α is indeed positive and bounded from both zero and infinity, as the solution of the simple scalar exponential growth equation (5.22). In what follows, we shall need to keep careful track of the distinguished constant C_*.

Computing

$$-\langle U, \tilde A U_x\rangle = (1/2)\langle U, (\tilde A_x + (\alpha_x/\alpha)\tilde A)U\rangle \tag{5.24}$$

and expanding $\tilde A = \bar A + \mathcal{O}(\zeta)$, $\tilde A_x = \mathcal{O}(|\bar U_x| + \zeta)$, we obtain by (5.23) the key property

$$\begin{aligned} -\langle U, \tilde A U_x\rangle &= (1/2)\langle u, (\alpha_x/\alpha)\bar A_{11} u\rangle \\ &\quad + \mathcal{O}\Big(\langle |\bar U_x||U|, |U|\rangle + \langle (\alpha_x/\alpha)|U|, \zeta|U| + |v|\rangle\Big) \\ &\le -(C_*\theta/3)\langle |u|, |\bar U_x||u|\rangle + C\zeta|u|^2 + C(C_*)|v|^2, \end{aligned} \tag{5.25}$$

by which we shall control transverse modes, provided C_* is chosen sufficiently large, or, more generally,

$$-\langle \partial_x^k U, \tilde A \partial_x^k U_x\rangle \le -(C_*\theta/3)\langle |\partial_x^k u|, |\bar U_x||\partial_x^k u|\rangle + C\zeta|\partial_x^k u|^2 + C(C_*)|\partial_x^k v|^2. \tag{5.26}$$

Here and below, $C(C_*)$ denotes a suitably large constant depending on C_*, while C denotes a fixed constant independent of C_*: likewise, $\mathcal{O}(\cdot)$ indicates a bound independent of C_*.

"Friedrichs-type" estimates. We first perform standard, zeroth- and first-order "Friedrichs-type" estimates for symmetrizable hyperbolic systems. Taking the α-inner product of U against (5.9), we obtain after rearrangement, integration by parts using (5.18)–(5.19), and several applications of Young's inequality, the energy estimate

(5.27)
$$\begin{aligned}
\frac{1}{2}\langle U, \tilde A^0 U\rangle_t &= \langle U, \tilde A^0 U_t\rangle + \frac{1}{2}\langle U, \tilde A^0_t U\rangle\\
&= -\langle U, \tilde A U_x\rangle + \langle U, (\tilde B U_x)_x\rangle + \langle U, M_1(U)\bar U_x\rangle + \langle U, (M_2(U)\bar U_x)_x\rangle\\
&\qquad + \dot\delta(t)\langle U, \tilde A^0 U_x\rangle + \dot\delta(t)\langle U, \tilde A^0 \bar U_x\rangle + \frac{1}{2}\langle U, \tilde A^0_t U\rangle\\
&= \frac{1}{2}\langle U, (\tilde A_x + (\alpha_x/\alpha)\tilde A)U\rangle - \langle U_x - (\alpha_x/\alpha)U, \tilde B U_x\rangle + \langle U, M_1(U)\bar U_x\rangle\\
&\qquad - \langle U_x + (\alpha_x/\alpha)U, M_2(U)\bar U_x\rangle - \frac{1}{2}\dot\delta(t)\langle U, (\tilde A^0_x + (\alpha_x/\alpha)\tilde A^0)U\rangle\\
&\qquad + \dot\delta(t)\langle U, \tilde A^0 \bar U_x\rangle + \frac{1}{2}\langle U, \tilde A^0_t U\rangle\\
&\le -\langle U_x, \tilde B U_x\rangle + C(C_*)\int \alpha\Big((|U_x| + |\bar U_x|)|U|^2\\
&\qquad + |v_x||U|(|U_x| + |\bar U_x|) + |\dot\delta||U||\bar U_x|\Big)\\
&\le -\theta|v_x|^2 + C(C_*)\int \alpha\,\Big((|U_x|^2 + |U|^2)|U| + (|\dot\delta|^2 + |U|^2)|\bar U_x|\Big)\\
&\le -\theta|v_x|^2 + C(C_*)\Big(|U|_{L^\infty}(|U|_{L^\infty} + |U|^2_{L^2}) + |\dot\delta|^2\Big) + C(C_*)\zeta|U_x|^2,
\end{aligned}$$

where $\zeta > 0$ is as in the statement of the Proposition. Here, we have freely used the bounds (5.5), as well as (5.16). We have also used in a crucial way the block-diagonal form of M_2 in estimating $|\langle U_x, M_2(U)\bar U_x\rangle| \le C\int |v_x||U||\bar U_x|$ in the first inequality. In the final term of the first righthand side, we have bounded $|\tilde A^0_t| \le C|U_t|$, estimating U_t using $(\tilde A^0)^{-1}$ times (5.9), and integrating by parts where necessary to eliminate second derivatives. This estimate is essentially the same as the zero order estimate carried out in [MZ.2], with α-contributions treated as errors.

First and higher order. For higher derivative estimates, it is crucial to make use of the favorable terms (5.26) afforded by the introduction of α-weighted norms. Differentiating (5.9) with respect to x, taking the α-inner product of U_x against the resulting equation, and substituting the result into the first term on the righthand side of

$$\frac{1}{2}\langle U_x, \tilde A^0 U_x\rangle_t = \langle U_x, (\tilde A^0 U_t)_x\rangle - \langle U_x, \tilde A^0_x U_t\rangle + \frac{1}{2}\langle U_x, \tilde A^0_t U_x\rangle, \tag{5.28}$$

we obtain after various simplifications and integrations by parts:
(5.29)

$$\begin{aligned}
\frac{1}{2}\langle U_x, \tilde{A}^0 U_x\rangle_t &= -\langle U_x, (\tilde{A}U_x)_x\rangle + \langle U_x, (\tilde{B}U_x)_{xx}\rangle + \langle U_x, (M_1(U)\bar{U}_x)_x\rangle \\
&\quad + \langle U_x, (M_2(U)\bar{U}_x)_{xx}\rangle + \dot{\delta}(t)\langle U_x, (\tilde{A}^0 U_x)_x\rangle \\
&\quad + \dot{\delta}(t)\langle U_x, (\tilde{A}^0\bar{U}_x)_x\rangle - \langle U_x, \tilde{A}^0_x U_t\rangle + \frac{1}{2}\langle U_x, \tilde{A}^0_t U_x\rangle \\
&= -\langle U_x, \tilde{A}U_{xx}\rangle - \langle U_x, \tilde{A}_x U_x\rangle - \langle U_{xx} + (\alpha_x/\alpha)U_x, \tilde{B}U_{xx} + \tilde{B}_x U_x\rangle \\
&\qquad + \langle U_x, (M_1(U)\bar{U}_x)_x\rangle - \langle U_{xx} + (\alpha_x/\alpha)U_x, (M_2(U)\bar{U}_x)_x\rangle \\
&\qquad + \frac{1}{2}\dot{\delta}(t)\langle U_x, \big(\tilde{A}^0_x - (\alpha_x/\alpha)\tilde{A}^0\big)U_x\rangle + \dot{\delta}(t)\langle U_x, \tilde{A}^0_x\bar{U}_x\rangle \\
&\qquad + \dot{\delta}(t)\langle U_x, \tilde{A}^0\bar{U}_{xx}\rangle - \langle U_x, \tilde{A}^0_x U_t\rangle + \frac{1}{2}\langle U_x, \tilde{A}^0_t U_x\rangle.
\end{aligned}$$

Estimating the first term on the righthand side of (5.29) using (5.26), $k = 1$, and substituting $(\tilde{A}^0)^{-1}$ times (5.9) into the second to last term on the righthand side of (5.29), we obtain by (5.16) plus various applications of Young's inequality the next-order energy estimate:
(5.30)

$$\begin{aligned}
\frac{1}{2}\langle U_x, \tilde{A}^0 U_x\rangle_t &\le -\langle U_x, \tilde{A}U_{xx}\rangle - \langle U_{xx}, \tilde{B}U_{xx}\rangle \\
&\qquad + C(C_*)\int (|U|^2 + |\dot{\delta}|^2)(|\bar{U}_{xx}| + |\bar{U}_x|) \\
&\qquad + C(C_*)\langle |v_x| + \zeta|U_x|, (|U| + |U_x|)|\bar{U}_x| + |v_{xx}|\rangle \\
&\qquad + C\langle (|U_x| + |v_{xx}|), |\bar{U}_x|(|U| + |U_x|)\rangle \\
&\le -\theta|v_{xx}|^2 - (C_*\theta/3)\langle |\bar{U}_x||u|, |u|\rangle + C\zeta|u_x|^2 + C(C_*)|v_x|^2 \\
&\qquad + C(C_*)\Big((|U|^2_{L^\infty} + |\dot{\delta}|^2\Big) \\
&\qquad + C(C_*)\langle |v_x| + \zeta|U_x|, (|U| + |U_x|)|\bar{U}_x| + |v_{xx}|\rangle \\
&\qquad + C\langle (|U_x| + |v_{xx}|), |\bar{U}_x|(|U| + |U_x|)\rangle \\
&\le -(\theta/2)|v_{xx}|^2 - (C_*\theta/4)\langle |\bar{U}_x||u|, |u|\rangle + C(C_*)\zeta|u_x|^2 + C(C_*)|v_x|^2 \\
&\qquad + C(C_*)\Big((|U|^2_{L^\infty} + |\dot{\delta}|^2\Big),
\end{aligned}$$

provided C_* is sufficiently large and ζ sufficiently small.

"Kawashima-type" estimate. Next, we perform a nonstandard, "Kawashima-type" derivative estimate. Taking the α-inner product of U_x against $\tilde{K}(\tilde{A}^0)^{-1}$ times (5.9), and noting that (integrating by parts, and using skew-symmetry of $\tilde{K}$)

(STOPPED HERE)

$$\begin{aligned}\frac{1}{2}\langle U_x, \tilde KU\rangle_t &= \frac{1}{2}\langle U_x, \tilde KU_t\rangle + \frac{1}{2}\langle U_{xt}, \tilde KU\rangle + \frac{1}{2}\langle U_x, \tilde K_t U\rangle \\ &= \frac{1}{2}\langle U_x, \tilde KU_t\rangle - \frac{1}{2}\langle U_t, \tilde KU_x\rangle \\ &\quad - \frac{1}{2}\langle U_t, \tilde K_x U\rangle + \frac{1}{2}\langle U_x, \tilde K_t U\rangle \\ &= \langle U_x, \tilde KU_t\rangle + \frac{1}{2}\langle U, \tilde K_x U_t\rangle + \frac{1}{2}\langle U_x, \tilde K_t U\rangle,\end{aligned} \tag{5.31}$$

we obtain by calculations similar to the above the auxiliary energy estimate:

$$\begin{aligned}\frac{1}{2}\langle U_x, \tilde KU\rangle_t &\le -\langle U_x, \tilde K(\tilde A^0)^{-1}\tilde AU_x\rangle \\ &\quad + C(\bar\zeta+\varepsilon)|U_x|_{L^2}^2 + C\bar\zeta^{-1}|v_{xx}|_{L^2}^2 + C(|U|_{L^\infty}^2 + |\dot\delta(t)|^2),\end{aligned} \tag{5.32}$$

where $\bar\zeta > 0$ is an arbitrary constant arising through Young's inequality. Choosing $\varepsilon + \zeta << \bar\zeta << 1$, adding (5.27), (5.32), and (5.30) times a suitably large positive constant $M >> \bar\zeta^{-1}$, and recalling (5.3), we obtain, finally:

$$\begin{aligned}&\frac{1}{2}\Big(\langle U, \tilde A^0 U\rangle + \langle U_x, \tilde KU\rangle + M\langle U_x, \tilde A^0 U_x\rangle\Big)_t \\ &\qquad \le -\theta(|U_x|_{L^2}^2 + |v_{xx}|_{L^2}^2) + C\Big(|U|_{L^\infty}(|U|_{L^\infty} + |U|_{L^2}^2) + |\dot\delta|^2\Big),\end{aligned} \tag{5.33}$$

$\theta > 0$, for any ζ, ε sufficiently small, and M, $C > 0$ sufficiently large.

(OLD: to be merged with new notation) *Higher order estimates.* Performing the same procedure on the once- and twice-differentiated versions of equation (5.9), we obtain, likewise, the H^q estimates, $q = 2$, 3, of:

$$\begin{aligned}&\frac{1}{2}\Big(\langle \partial_x^{q-1}U, \tilde A^0\partial_x^{q-1}U\rangle + \langle \partial_x^{q-1}U_x, \tilde K\partial_x^{q-1}U\rangle + M\langle \partial_x^q U, \tilde A^0\partial_x^q U\rangle\Big)_t \\ &\qquad \le -\theta(|\partial_x^q U|_{L^2}^2 + |\partial_x^{q+1}v|_{L^2}^2) \\ &\qquad\quad + (\varepsilon+\zeta)|U_x|_{H^{q-2}}^2 + C\Big(|U|_{L^\infty}(|U|_{L^\infty} + |U|_{L^2}^2) + |\dot\delta|^2\Big).\end{aligned} \tag{5.34}$$

The calculations are entirely similar to those carried out already; see also the similar argument of Appendix A, [MZ.2].

Final estimate. Summing our H^q estimates from $q = 1$ to 3, and telescoping the sum of the righthand sides, we thus obtain, for ε, ζ sufficiently small:

$$\begin{aligned}&\sum_{q=1}^{3}\big(\langle \tilde A^0\partial_x^{q-1}U, \partial_x^{q-1}U\rangle + \frac{1}{2}\langle \partial_x^q U, \tilde K\partial_x^{q-1}U\rangle + M\langle \tilde A^0\partial_x^q U, \partial_x^q U\rangle\big)_t \\ &\qquad \le -\theta(|U_x|_{H^2}^2 + |v_x|_{H^3}^2) + C\Big(|U|_{L^\infty}(|U|_{L^\infty} + |U|_{L^2}^2) + |\dot\delta|^2\Big),\end{aligned} \tag{5.35}$$

or, integrating from 0 to t:

$$\begin{aligned}
\sum_{q=1}^{3} \big(\langle \tilde A^0 \partial_x^{q-1}U, \partial_x^{q-1}U\rangle + \frac{1}{2}\langle \partial_x^q U, \tilde K \partial_x^{q-1}U\rangle + M\langle \tilde A^0 \partial_x^q U, \partial_x^q U\rangle\big)\big|_0^t \\
\le -\int_0^t \theta(|U_x|_{H^2}^2 + |v_x|_{H^3}^2)(s)ds \\
+ C\int_0^t \Big(|U|_{L^\infty}(|U|_{L^\infty} + |U|_{L^2}^2) + |\dot\delta|^2\Big)(s)ds.
\end{aligned} \tag{5.36}$$

Noting that, for M sufficiently large, we have for each q, by Young's inequality, and positive definiteness of $\tilde A^0$:

$$\big(\langle \partial_x^{q-1}U, \tilde A^0 \partial_x^{q-1}U\rangle + \frac{1}{2}\langle \partial_x^q U, \tilde K \partial_x^{q-1}U\rangle + M\langle \partial_x^q U, \tilde A^0 \partial_x^q U\rangle\big)(t) \ge \theta(|\partial_x^{q-1}U|_{L^2}^2 + |\partial_x^q U|_{L^2}^2), \tag{5.37}$$

for some $\theta > 0$, we may rearrange (5.36) to obtain (1.13) as claimed. ∎

Remark 5.3. As pointed out in the introduction, the assumption (H2)(iii), and the Goodman-style weight α are used only to control terms of order $\mathcal{O}(\int |\bar U_x||v|^2)$ in the absence of a smallness assumption

$$|\partial_x^j \bar U| \le C\varepsilon e^{-\theta|x|}, \tag{5.38}$$

$\theta > 0$, $\varepsilon << 1$ as occurs in the small-amplitude case. Evidently, when (5.38) holds, we may obtain the same estimates without assumption (H2)(iii), and setting $\alpha \equiv 1$. Thus, *we completely recover the earlier small-amplitude results of [MZ.2].*

Section 6. Nonlinear stability.

Having established the energy estimate of Proposition 1.6, we may now obtain Theorem 1.4 by exactly the same argument as in [MZ.2], for completeness reproduced here. We follow the basic iteration scheme of [MZ.1-2,Z.2]; for precursors of this scheme, see [Go.2,K.1–2,LZ.1–2,ZH,HZ.1–2]. For this stage of the argument, it will be convenient to work again with the conservative variable

$$G := G(\tilde U) - G(\bar U), \tag{6.1}$$

writing (5.2) in the more standard form:

$$G_t - LG = N(G, G_x)_x + \dot\delta(t)(\bar G_x + G_x), \tag{6.2}$$

$\bar G := G(\bar U)$, where

$$\begin{aligned}
N(G, G_x) &= \mathcal{O}(|G|^2 + |G||v_x|), \\
N(G, G_x)_x &= \mathcal{O}(|G|^2 + |G||v_x| + |G_x||v_x| + |G||v_{xx}|),
\end{aligned} \tag{6.3}$$

so long as $|G|$, $|G_x|$ remain bounded. Here, v denotes the second coordinate of the alternative perturbation variable $U = (u, v)^t$ defined in (1.12).

By Duhamel's principle, and the fact that

$$\int_{-\infty}^{\infty} \mathcal{G}(x,t;y)\bar{G}_x(y)dy = e^{Lt}\bar{G}_x(x) = \bar{G}_x(x), \tag{6.4}$$

we have, formally,

$$\begin{aligned} G(x,t) = &\int_{-\infty}^{\infty} \mathcal{G}(x,t;y)G_0(y)dy \\ &+ \int_0^t \int_{-\infty}^{\infty} \mathcal{G}_y(x,t-s;y)(N(G,G_x) + \dot{\delta}G)(y,s)dyds \\ &+ \delta(t)\bar{G}_x. \end{aligned} \tag{6.5}$$

Defining, by analogy with the linear case, the *nonlinear instantaneous projection*:

$$\begin{aligned} \varphi(x,t) &:= -\delta(t)\bar{G}_x \\ &:= \int_{-\infty}^{\infty} E(x,t;y)G_0(y)dy \\ &\quad - \int_0^t \int_{-\infty}^{\infty} E_y(x,t-s;y)(N(G,G_x) + \dot{\delta}G)(y,s)dy, \end{aligned} \tag{6.6}$$

or equivalently, the *instantaneous shock location*:

$$\begin{aligned} \delta(t) = &-\int_{-\infty}^{\infty} e(y,t)G_0(y)dy \\ &+ \int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)(N(G,G_x) + \dot{\delta}G)(y,s)dyds, \end{aligned} \tag{6.7}$$

where E, e are defined as in (2.12), (4.3), and recalling (4.5), we thus obtain the *reduced equations*:

$$\begin{aligned} G(x,t) = &\int_{-\infty}^{\infty} (H + \tilde{\mathcal{G}})(x,t;y)G_0(y)dy \\ &+ \int_0^t \int_{-\infty}^{\infty} H(x,t-s;y)(N(G,G_x)_x + \dot{\delta}G_x)(y,s)dy\,ds \\ &- \int_0^t \int_{-\infty}^{\infty} \tilde{\mathcal{G}}_y(x,t-s;y)(N(G,G_x) + \dot{\delta}G)(y,s)dy\,ds, \end{aligned} \tag{6.8}$$

and, differentiating (6.7) with respect to t,

$$\begin{aligned} \dot{\delta}(t) = &-\int_{-\infty}^{\infty} e_t(y,t)G_0(y)dy \\ &+ \int_0^t \int_{-\infty}^{+\infty} e_{yt}(y,t-s)(N(G,G_x) + \dot{\delta}G)(y,s)dyds. \end{aligned} \tag{6.9}$$

Note: In deriving (6.9), we have used the fact that $e_y(y,s) \rightharpoondown 0$ as $s \to 0$, as the difference of approaching heat kernels, in evaluating the boundary term

$$\int_{-\infty}^{+\infty} e_y(y,0)(N(G,G_x)+\dot\delta G)(y,t)dy = 0. \tag{6.10}$$

(Indeed, $|e_y(\cdot,s)|_{L^1} \to 0$; see Remark 2.6, below).

The defining relation $\delta(t)\bar u_x := -\varphi$ in (6.6) can be motivated heuristically by

$$\begin{aligned}\tilde G(x,t) - \varphi(x,t) \sim G = \begin{pmatrix} u \\ v\end{pmatrix}(x+\delta(t),t) - \begin{pmatrix} \bar u \\ \bar v\end{pmatrix}(x) \\ \sim \tilde G(x,t) + \delta(t)\bar G_x(x),\end{aligned}$$

where $\tilde G$ denotes the solution of the linearized perturbation equations, and $\bar G$ the background profile. Alternatively, it can be thought of as the requirement that the instantaneous projection of the shifted (nonlinear) perturbation variable G be zero; see [HZ.1–2].

Lemma 6.1 [Z.2]. *The kernel e satisfies*

$$|e_y(\cdot,t)|_{L^p}, |e_t(\cdot,t)|_{L^p} \le Ct^{-\frac12(1-1/p)}, \tag{6.11}$$

$$|e_{ty}(\cdot,t)|_{L^p} \le Ct^{-\frac12(1-1/p)-1/2}, \tag{6.12}$$

for all $t>0$. Moreover, for $y \le 0$ we have the pointwise bounds

$$|e_y(y,t)|, |e_t(y,t)| \le Ct^{-\frac12}e^{-\frac{(y+a_-t)^2}{Mt}}, \tag{6.13}$$

$$|e_{ty}(y,t)| \le Ct^{-1}e^{-\frac{(y+a_-t)^2}{Mt}}, \tag{6.14}$$

for $M>0$ sufficiently large (i.e. $> 4b_\pm$), and symmetrically for $y \ge 0$.

Proof. For definiteness, take $y \le 0$. Then, (4.3) gives

$$e_y(y,t) = \left(\frac{1}{u_+ - u_-}\right)\left(K(y+a_-t,t) - K(y-a_-t,t)\right), \tag{6.15}$$

$$e_t(y,t) = \left(\frac{1}{u_+ - u_-}\right)\left((K+K_y)(y+a_-t,t) - (K+K_y)(y-a_-t,t)\right), \tag{6.16}$$

(6.17)
$$e_{ty}(y,t) = \left(\frac{1}{u_+ - u_-}\right)\left((K_y + K_{yy})(y + a_-t, t) - (K_y + K_{yy})(y - a_-t, t)\right),$$

where

$$K(y,t) := \frac{e^{-y^2/4b_-t}}{\sqrt{4\pi b_- t}} \tag{6.18}$$

denotes an appropriate heat kernel. The pointwise bounds (6.13)–(6.14) follow immediately for $t \geq 1$ by properties of the heat kernel, in turn yielding (6.11)–(6.12) in this case. The bounds for small time $t \leq 1$ follow from estimates

$$\begin{aligned} |K_y(y + a_-t, t) - K_y(y - a_-t, t)| &= |\int_{y+a_-t}^{y-a_-t} K_{yy}(z,t)dz| \\ &\leq Ct^{-3/2}\int_{y+a_-t}^{y-a_-t} e^{\frac{-z^2}{Mt}}dz \\ &\leq Ct^{-1/2}e^{-\frac{(y+a_-t)^2}{Mt}}, \end{aligned} \tag{6.19}$$

and, similarly,

$$\begin{aligned} |K_{yy}(y + a_-t, t) - K_{yy}(y - a_-, t)| &= |\int_{-a_-t}^{a_-t} K_{yyy}(z,t)dz| \\ &\leq Ct^{-2}\int_{y+a_-t}^{y-a_-t} e^{\frac{-z^2}{Mt}}dz, \\ &\leq Ct^{-1}e^{-\frac{(y+a_-t)^2}{Mt}}. \end{aligned} \tag{6.20}$$

The bounds for $|e_y|$ are again immediate. Note that we have taken crucial account of cancellation in the small time estimates of e_t, e_{ty}. $\square$

Remark 6.2: For $t \leq 1$, a calculation analogous to that of (6.19) yields $|e_y(y,t)| \leq Ce^{-\frac{(y+a_-t)^2}{Mt}}$, and thus $|e(\cdot, s)|_{L^1} \to 0$ as $s \to 0$.

With these preparations, we are ready to carry out our analysis:

Proof of Theorem 1.4. Define

$$\begin{aligned} \zeta(t) := \sup_{0\leq s\leq t,\, 2\leq p\leq\infty} \Big[&\big(|U(\cdot,s)|_{L^p} + |v_x(\cdot,s)|_{L^p}\big)(1+s)^{\frac{1}{2}(1-1/p)} \\ &+ |\dot\delta(s)|(1+s)^{1/2} + \big(|\delta(s)| + |U(\cdot,s)|_{H^2} + |U(\cdot,s)|_{W^{2,\infty}}\big)\Big]. \end{aligned} \tag{6.21}$$

We shall establish:

Claim. For all $t \geq 0$ for which a solution exists with ζ uniformly bounded by some fixed, sufficiently small constant, there holds

$$\zeta(t) \leq C_2(\zeta_0 + \zeta(t)^2). \tag{6.22}$$

From this result, it follows by continuous induction that, provided $\zeta_0 < 1/4C_2$, there holds

$$\zeta(t) \leq 2C_2\zeta_0 \tag{6.23}$$

for all $t \geq 0$ such that ζ remains small. By standard short-time theory/continuation, we find that the solution (unique, in this regularity class) in fact remains in H^2 for all $t \geq 0$, with bound (6.23), at once yielding existence and the claimed sharp L^p bounds, $2 \leq p \leq \infty$. Thus, it remains only to establish the claim above.

Proof of Claim. We must show that each of the quantities $|U|_{L^p}(1+s)^{\frac{1}{2}(1-1/p)}$, $|v_x|_{L^p}(1+s)^{\frac{1}{2}(1-1/p)}$, $|\dot\delta|(1+s)^{1/2}$, $|\delta|$, $|U|_{H^2}$, and $|U|_{W^{2,\infty}}$ are separately bounded by

$$C(\zeta_0 + \zeta(t)^2), \tag{6.24}$$

for some $C > 0$, all $0 \leq s \leq t$, so long as ζ remains sufficiently small.

Provided that we can establish the others, the final two bounds follow easily from the energy estimates of Proposition 1.6. For, by bounds (6.24) on $|\dot\delta|$ and $|U|_{L^\infty}$, and the assumption that $|U|_{H^2\cap W^{2,\infty}} \leq \zeta$ remains small, we obtain from (1.13) that

$$|U|_{H^3}(s) \leq C_3(\zeta_0 + \zeta(t)^2)\log(1+s), \tag{6.25}$$

for $0 \leq s \leq t$. Interpolating with assumed bound

$$|U|_{L^2}(s) \leq C_3(\zeta_0 + \zeta(t)^2)(1+s)^{-1/4},$$

we obtain

$$\begin{aligned} |U|_{H^2}(s) &\leq C_3(\zeta_0 + \zeta(t)^2)|U|_{H^3}^{2/6}|U|_{L^2}^{1/6} \\ &\leq C_3(\zeta_0 + \zeta(t)^2)(1+t)^{-1/25}, \end{aligned} \tag{6.26}$$

and, by Sobolev estimate,

$$|U|_{W^{2,\infty}} \leq |U|_{H^2}^{1/2}|U|_{H^3}^{1/2} \leq C_3(\zeta_0 + \zeta(t)^2)(1+t)^{-1/51}, \tag{6.27}$$

both better than claimed.

Thus, in order to establish the result, we have only to establish the remaining bounds, on $|U|_{L^p}$, $|v_x|_{L^p}$, $|\dot\delta|$, and $|\delta|$. These will be carried out using the Green function estimates of Lemma 4.1. Accordingly, we first convert the problem to conservative, G coordinates, via:

Observation 6.3. It is sufficient to establish corresponding bounds on $|G|_{L^p}$, $|v_{G,x}|_{L^p}$, $|\dot\delta|$, and $|\delta|$, where $v_G(x,t) := \Pi_2 A^0(x)^{-1}G(x,t)$, Π_2 defined as in Lemma 2.3 as projection onto final r coordinates.

Proof. We have

$$\begin{aligned} U = G^{-1}(\tilde G) - G^{-1}(\bar G) &= (A^0)^{-1}_{\text{ave}}(x,t)G \\ &:= \Big(\int_0^1 (A^0)^{-1}\big(\bar U(x) + \theta U(x,t)\big)d\theta\Big)\, G, \end{aligned} \tag{6.28}$$

where

$$|(A^0)^{-1}_{\text{ave}}(x,t) - A^0(x)^{-1}| \le C|U| \le C|G| \tag{6.29}$$

and

$$|\big((A^0)^{-1}_{\text{ave}}(x,t) - A^0(x)^{-1}\big)_x| \le C(|\bar U_x| + |U_x|) \le \zeta, \tag{6.30}$$

whence

$$|U|_{L^p} \le C|G|_{L^p}$$

and

$$|U_x - ((A^0(x))^{-1}G)_x|_{L^p} \le C(|G|_{L^p}|G_x|_{L^\infty} + \zeta|G|_{L^p}) \le C\zeta|G|_{L^p},$$

from which the result easily follows. ∎

By (6.8)–(6.9), we have

$$\begin{aligned} |G|_{L^p}(t) &\le |\int_{-\infty}^{\infty} (H + \tilde{\mathcal{G}})(x,t;y)G_0(y)dy|_{L^p} \\ &\quad + |\int_0^t\int_{-\infty}^{\infty} H(x,t-s;y)(N(G,G_x)_x + \dot\delta G_x)(y,s)dy\,ds|_{L^p} \\ &\quad + |\int_0^t\int_{-\infty}^{\infty} \tilde{\mathcal{G}}_y(x,t-s;y)(N(G,G_x) + \dot\delta G)(y,s)dy\,ds|_{L^p} \\ &=: I_a + I_b + I_c, \end{aligned} \tag{6.31}$$

$$\begin{aligned} |v_{G,x}|_{L^p}(t) := |\big(\Pi_2(A^0)^{-1}G\big)_x|_{L^p}(t) &\le |\int_{-\infty}^{\infty} \Big(\Pi_2(A^0)^{-1}\tilde{\mathcal{G}}\Big)_x(x,t;y)G_0(y)dy|_{L^p} \\ &\quad + |\int_0^t\int_{-\infty}^{\infty} \big(\Pi_2(A^0)^{-1}\tilde{\mathcal{G}}_y\big)_x(x,t-s;y)(N(G,G_x) + \dot\delta G)(y,s)dy\,ds|_{L^p} \\ &=: II_a + II_b, \end{aligned} \tag{6.32}$$

$$
\begin{aligned}
|\dot\delta|(t) &\le |\int_{-\infty}^{\infty} e_t(y,t)G_0(y)dy| \\
&\quad + |\int_0^t\int_{-\infty}^{+\infty} e_{yt}(y,t-s)(N(G,G_x)+\dot\delta G)(y,s)dyds| \\
&=: III_a + III_b,
\end{aligned}
\tag{6.33}
$$

and

$$
\begin{aligned}
|\delta|(t) &\le |\int_{-\infty}^{\infty} e(y,t)G_0(y)dy| \\
&\quad + |\int_0^t\int_{-\infty}^{+\infty} e_{y}(y,t-s)(N(G,G_x)+\dot\delta G)(y,s)dyds| \\
&=: IV_a + IV_b.
\end{aligned}
\tag{6.34}
$$

We estimate each term in turn, following the approach of [Z.2,MZ.1]. The linear term I_a satisfies bound

$$
I_a \le C\zeta_0(1+t)^{-\frac12(1-1/p)},
\tag{6.35}
$$

as already shown in the proof of Corollary 4.3. Likewise, applying the bounds of Lemma 4.1 together with (6.3) and definition (6.21), we have

$$
\begin{aligned}
I_b &= |\int_0^t\int_{-\infty}^{\infty} H(x,t-s;y)(N(G,G_x)_x+\dot\delta G_x)(y,s)dy\,ds|_{L^p} \\
&\le C\int_0^t e^{-\eta(t-s)}(|G|_{L^\infty}+|v_{G,x}|_{L^\infty}+|\dot\delta|)|G|_{W^{2,p}}(s)ds \\
&\le C\zeta(t)^2\int_0^t e^{-\eta(t-s)}(1+s)^{-1/2}ds \\
&\le C\zeta(t)^2(1+t)^{-1/2},
\end{aligned}
\tag{6.36}
$$

and (taking $q=2$ in (4.7))

$$
\begin{aligned}
I_c &= |\int_0^t\int_{-\infty}^{\infty} \tilde{\mathcal{G}}_y(x,t-s;y)(N(G,G_x)+\dot\delta G)(y,s)dy\,ds|_{L^p} \\
&\le C\int_0^t e^{-\eta(t-s)}(|G|_{L^\infty}+|v_{G,x}|_{L^\infty}+|\dot\delta|)|G|_{L^p}(s)ds \\
&\quad + C\int_0^t (t-s)^{-3/4+1/2p}(|G|_{L^\infty}+|v_{G,x}|_{L^\infty}+|\dot\delta|)|G|_{L^2}(s)ds \\
&\le C\zeta(t)^2\int_0^t e^{-\eta(t-s)}(1+s)^{-\frac12(1-1/p)-1/2}ds \\
&\quad + C\zeta(t)^2\int_0^t (t-s)^{-3/4+1/2p}(1+s)^{-3/4}ds \\
&\le C\zeta(t)^2(1+t)^{-\frac12(1-1/p)}.
\end{aligned}
\tag{6.37}
$$

Summing bounds (6.35)–(6.37), we obtain (6.24), as claimed, for $2 \le p \le \infty$. The desired bounds on II_a and II_b follow by an identical calculation, once we notice that, in Lemma 4.1, $\Big((A^0)^{-1}\tilde{\mathcal{G}}\Big)_x$ and $\Big((A^0)^{-1}\tilde{\mathcal{G}}_y\Big)_x$ satisfy essentially the same $L^q \to L^p$ (resp. $L^q \cap W^{1,p} \to L^p$) bounds as do $(H + \tilde{\mathcal{G}})$ and $(H + \tilde{\mathcal{G}})_y$, the only difference being a harmless (since integrable) additional factor of $t^{-1/2}$ for $t \le 1$.

Similarly, applying the bounds of Lemma 6.1 together with definition (6.21), we find that

$$\begin{aligned} III_a &= \Big|\int_{-\infty}^{\infty} e_t(y,t)G_0(y)dy\Big| \\ &\le |e_t(y,t)|_{L^\infty}(t)|G_0|_{L^1} \\ &\le C\zeta_0(1+t)^{-1/2} \end{aligned} \tag{6.38}$$

and

$$\begin{aligned} III_b &= \Big|\int_0^t \int_{-\infty}^{+\infty} e_{yt}(y,t-s)(N(G,G_x) + \dot\delta G)(y,s)dyds\Big| \\ &\le \int_0^t |e_{yt}|_{L^2}(t-s)(|G|_{L^\infty} + |v_{G,x}|_{L^\infty} + |\dot\delta|)|G|_{L^2}(s)ds \\ &\le C\zeta(t)^2 \int_0^t (t-s)^{-3/4}(1+s)^{-3/4}ds \\ &\le C\zeta(t)^2(1+t)^{-1/2}, \end{aligned} \tag{6.39}$$

while

$$\begin{aligned} IV_a &= \Big|\int_{-\infty}^{\infty} e(y,t)G_0(y)dy\Big| \\ &\le |e(y,t)|_{L^\infty}(t)|G_0|_{L^1} \\ &\le C\zeta_0 \end{aligned} \tag{6.40}$$

and

$$\begin{aligned} IV_b &= \Big|\int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)\dot\delta G(y,s)dyds\Big| \\ &\le \int_0^t |e_y|_{L^2}(t-s)(|G|_{L^\infty} + |v_{G,x}|_{L^\infty} + |\dot\delta|)|G|_{L^2}(s)ds \\ &\le C\zeta(t)^2 \int_0^t (t-s)^{-1/4}(1+s)^{-3/4}ds \\ &\le C\zeta(t)^2. \end{aligned} \tag{6.41}$$

Summing (6.38)–(6.39) and (6.40)–(6.41), we obtain (6.24) as claimed.

This completes the proof of the claim, and the result. ∎

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