STABILITY OF LARGE-AMPLITUDE SHOCK PROFILES FOR GENERAL RELAXATION SYSTEMS

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Abstract. Building on previous analyses carried out in [MZ.1, MZ.4], we establish nonlinear $L^1 \cap H^2 \to L^p$ orbital stability, $2 \le p \le \infty$, with sharp rates of decay, of large-amplitude Lax-type shock profiles for a general class of relaxation systems that includes most models in common use, under the necessary conditions of strong spectral stability, i.e., stable point spectrum of the linearized operator about the wave, transversality of the profile, and hyperbolic stability of the associated ideal shock. In particular, our results apply to standard moment closure systems, answering a question left open in [MZ.1]. The argument combines the basic nonlinear stability argument introduced [MZ.1] with an improved "Goodman-style" weighted energy estimate similar to that used in [MZ.4] to treat large-amplitude profiles of systems with real viscosity.

Section 1. Introduction

In [MZ.1], there was carried out a detailed study of linearized and nonlinear stability of traveling front solutions, or shock profiles

$$(u,v)(x,t) = (\bar u, \bar v)(x-st), \qquad \lim_{z\to\pm\infty}(\bar u,\bar v) = (u_\pm, v_\pm) = (u_\pm, v^*(u_\pm)), \tag{1.1}$$

of relaxation systems

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}_x = \begin{pmatrix} 0 \\ q(u,v) \end{pmatrix}, \tag{1.2}$$

$u,\ f \in \mathbb{R}^n$, $v,\ g,\ q \in \mathbb{R}^r$, where

$$\text{Re } \sigma\big(q_v(u, v^*(u))\big) < 0 \tag{1.3}$$

along a smooth equilibrium manifold $v = v^*(u)$ defined by

$$q(u, v^*(u)) \equiv 0. \tag{1.4}$$

The linearized results obtained in [MZ.1] are extremely general, and appear to be optimal. However, the nonlinear results are restricted to arbitrary-amplitude profiles of special, discrete kinetic models, defined as systems (1.2) for which f and g are linear, and small-amplitude profiles of general, simultaneously symmetrizable models, defined as systems (1.2) for which there exists a smooth, symmetric positive

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definite matrix function $A^0 = A^0(u,v)$ for which A^0A and A^0Q are symmetric, where $A := (df^t, dg^t)^t$ and $Q := (0, dq^t)^t$.

The first result is quite satisfactory, and has important applications to the physically interesting class of systems obtained by discretization of kinetic models such as Boltzmann or Vlasov–Poisson equations. On the other hand, the equally important class of systems obtained by moment closure approximation of kinetic models is not contained in the class of discrete kinetic models, and so the question of large-amplitude stability for these models was left open in [MZ.1]; indeed, as discussed in Remark 1.16 of [MZ.1], it was not at all clear from the analysis of [MZ.1] whether this was a technical artifact or represented a true qualitative difference in behavior between these two types of approximation. Moreover, even the requirement of simultaneous symmetrizability appears to be overly restrictive in the large-amplitude case. As described in [Yo], many systems of physical interest are simultaneously symmetrizable along the equilibrium manifold $v = v^*(u)$, including (standard versions of) both discrete and moment-closure approximations of kinetic models; however, so far as we can see, simultaneous symmetrizability does not typically hold away from equilibrium, in particular along a shock profile as assumed in [MZ.1].[1] This distinction is unimportant in the small-amplitude case, for which the argument of [MZ.1] goes through by continuity assuming only simultaneous symmetrizability at the endstates $(u_-, v^*(u_-))$;[2] however, it becomes significant in the large-amplitude case, for which profiles may feature arbitrarily large excursions from equilibrium.

The difficulty in the analysis of the general case was control of higher-derivative source terms arising in the nonlinear iteration through Taylor expansion of the variable coefficient matrix $A = (df^t, dg^t)^t$. Such terms do not arise in the case of discrete kinetic models, for which A is constant, and this made it possible to carry out the entire nonlinear stability analysis using linearized (i.e., Green function) estimates alone. In the general case, we found it necessary to augment these bounds with coupled energy estimates in order to close the iteration, and these estimates, as implemented in [MZ.1] used both global symmetrizability and the small-amplitude assumption in important ways.

Similar difficulties arose in the closely related study [MZ.2] of stability of shock profiles for real viscosity models, initially limiting this analysis also to the small-amplitude, globally symmetrizable case. Recently, however, these obstacles were overcome and the corresponding restrictions removed in [MZ.4], by the introduction of a modified energy estimate incorporating "Goodman-type" weighted norms in the style of [Go]. The purpose of this brief note is to show that a similar approach can be applied in the relaxation case to yield a satisfactory nonlinear stability theory applying to large-amplitude profiles of the physically correct class of systems that

[1] This corrects a misstatement in [MZ.1], Remark 1.15, where the calculations of [Yo] were misquoted as asserting global simultaneous symmetrizability.

[2] Specifically, in the notation of Lemma 7.4, [MZ.1], the key property Re $(kA - A^0Q) > 0$ persists under perturbation.

are simultaneously symmetrizable at equilibrium: indeed, to the considerably more general class of equations that was considered in the linearized analysis of [MZ.1].

This brings the analysis of [MZ.1] to a satisfying conclusion, putting under a common framework most of the relaxation models commonly studied, and in particular resolving in the negative the question posed in [MZ.1] whether there might be a true qualitative difference in behavior between discrete and moment-closure approximations of kinetic models. We point out that our results yield new information even in the case of (non-globally-symmetrizable) discrete kinetic models, since we require significantly less regularity on the data ($L^1 \cap H^2$ vs. $W^{3,1} \cap W^{3,\infty}$).

We now describe our results in more detail. Of system (1.2), we assume the following structural properties:

(A1) $A(u,v) = (df^t, dg^t)^t$ is *symmetrizable* in the sense that there exists a C^3 positive definite matrix-valued function $A^0(u,v)$ such that A^0A is symmetric.

(A2) The equilibrium states $(u_\pm, v^*(u_\pm))$ are *strictly dissipative* (strictly spectrally stable) in the sense that $\operatorname{Re}\sigma(i\xi A_\pm + Q_\pm) \le \dfrac{-\theta|\xi|^2}{1+|\xi|^2}$, $\theta > 0$, for all $\xi \in \mathbb{R}$, where $A_\pm := A(u_\pm, v^*(u_\pm))$, $Q_\pm := Q(u_\pm, v^*(u_\pm)$, and $Q(u,v) := (0, dq(u,v))^t$.

Condition (A1) is connected with well-posedness of (1.2), while (A2) is connected with time-asymptotic stability of equilibrium states $(u_\pm, v^*(u_\pm))$ [Kaw]. As discussed variously in [Kaw,Ze,Yo,BN], sufficient conditions for (A2) are either simultaneous symmetrizability, A^0Q symmetric, or else weak dissipativity, $Q(A^0)^{-1} \le 0$ and $(Q(A^0)^{-1})_{22} < 0$,[3] together with *genuine coupling*:

(GC) No eigenvector of $A_\pm$ lies in the kernel of $Q_\pm$.

Regarding the profile (1.1), we assume:

(H0) f, g, $q \in C^3$.

(H1) The eigenvalues of $A(x) := (df^t, dg^t)^t(\bar u, \bar v)(x)$ are (i) different from s and (ii) of constant multiplicity.

(H2) (i) The eigenvalues $a_j^{*\pm}$ of $A_\pm^* := df^*(u_\pm)$, $f^*(u) := f(u, v^*(u))$, are real and different from s;[4] moreover, (ii) when ordered with increasing size, they satisfy the strict Lax characteristic conditions [La]

$$a_{p-1}^{*-} < 0 < a_p^{*-}, \quad a_p^{*+} < 0 < a_{p+1}^{*+} \tag{1.5}$$

[3] That is, $\tilde Q := Q(A^0)^{-1} =$ block-diag $\{0, \tilde q\}$ with $\tilde q < 0$, whence we may replace $\tilde Q$ with $\operatorname{Re}\tilde Q$ and proceed as in the simultaneously symmetrizable case, noting that $\ker A^0 \operatorname{Re}\tilde Q = \ker A^0 \tilde Q = (I_{n-r}, 0)$, whence (GC) is preserved. The example $\tilde q = \begin{pmatrix} 0 & a \\ -a & 1 \end{pmatrix}$, $a \neq 0$, shows that $\tilde q > 0$ is necessary, since $\operatorname{Re}\sigma\tilde q < 0$ but $\operatorname{Re}\tilde q =$ block-diag $\{0, \tilde q\}$ is only semidefinite.

[4] This differs from hypothesis (H2) of [MZ.1], in which the eigenvalues were also required to be distinct. However, it was noted in Remark 1.12 of [MZ.1] that this requirement may be dropped when $A_\pm^*$ and $Q_\pm$ are simultaneously symmetrizable, with essentially no change in either results or notation; the same argument shows that this requirement may likewise be dropped in the general case, at the expense of further complications (specifically, the matrix-valued diffusion waves

for some $1 \leq p \leq n$ (the principal characteristic field of the shock).

(H3) The Liu-Majda determinant condition [Ma.1–3, L.1]

$$\Delta := \det\,(r_1^{*-}, \dots, r_{p-1}^{*-}, [u], r_{p+1}^{*+}, r_n^{*+}) \neq 0 \tag{1.6}$$

is satisfied, where $r_j^{*\pm}$ denote the eigenvectors of $A_\pm^*$ associated with $a_j^{*\pm}$, and $[u] := u_+ - u_-$ the jump in u across the shock (dynamical stability).

(H4) The profile $(\bar u, \bar v)(\cdot)$ is a transverse connection of the associated traveling-wave ODE (structural stability), in particular, therefore, locally unique up to translation.

(H5) The point spectrum of the linearized operator $L := -\partial_x(A - sI) + Q$ about the wave is contained in $\{\lambda : \text{Re } \lambda < 0\} \cup \{0\}$ (strong spectral stability).

Condition (H0) gives the regularity needed both for our analysis here and in order to apply the linearized bounds of [MZ.1]. Condition (H1)(i) is a standard assumption [YoZ, Z.3, MZ.1] ensuring that the traveling-wave ordinary differential equation (ODE) be of nondegenerate type [MZ.1]. It is not clearly necessary, however, and at least for discrete kinetic models it can be relaxed, as we discuss in Section 4; indeed, in that setting it is rather unnatural. Condition (H1)(ii) is a technical assumption that was used in the pointwise Green function analysis of [MZ.1]; at the expense of some detail in the pointwise description of linearized behavior, it may be removed altogether [Z.4]. Constant multiplicity holds automatically for discrete kinetic models, but for moment closure models may be difficult to verify. Together, (A1) and (H1)(ii) are equivalent to semisimplicity plus constant multiplicity of $\sigma(A)$. Condition (H2)(i) expresses hyperbolicity and noncharacteristicity of the associated "equilibrium system"

$$u_t + f^*(u)_x = 0 \tag{1.7}$$

obtained from (1.2) by formal Chapman–Enskog expansion. at the endstates u_-, u_+ with respect to the corresponding ideal shock (u_-, u_+) of (1.7). Note that hyperbolicity of the equilibrium system is not required *along the profile*, thus allowing applications to interesting nonhyperbolic situations as discussed in [JX,BGDV,BV]. Condition (H2)(ii) restricts attention for simplicity to the standard case of a classical, Lax-type shock (u_-, u_+) of (1.7); the treatment of nonclassical over- and under-compressive shocks we leave for the future.[5]

$e^{(x-a_j^{*\pm})^2(4\pi\beta_j^{*\pm}t)^{-1}}$ of the remark must be replaced by the fundamental solution of $v_t + a_j^{*\pm}v_x = \beta_j^{*\pm}v_{xx}$, where $\beta_j^{*\pm} := l_j^{*\pm t}B^{*\pm}r_j^{*\pm}$ are no longer necessarily diagonal, and the matrix-valued error-functions appearing in excited term E by its spatial integral; finally, though the precise form of scattering term S in Proposition 1.10 [MZ.1] is no longer clear, it is easily verified that it satisfies pointwise bounds yielding the same rates of $L^q \to L^p$ decay as needed for subsequent stability arguments).

[5] See [MZ.1] for a linearized analysis and [ZH,Z.3,HZ] for related nonlinear analyses in the viscous or viscous–dispersive case.

Conditions (H3)–(H5) are together equivalent to the Evans function condition

($\mathcal{D}$) The Evans function $D(\cdot)$ associated with L has precisely one zero on $\{\lambda : \text{Re } \lambda \geq 0\}$ (necessarily at $\lambda = 0$).[6]

The generalized spectral stability condition ($\mathcal{D}$) was shown in [MZ.1] to be necessary and sufficient for linearized stability under assumptions (A1)–(A2) and (H0)–(H2); the main point of this paper is to show that these conditions are sufficient also for nonlinear stability. The conditions (H3) and (H4) correspond to the classical physical notions of dynamical and structural stability (see, e.g. [BE]), whereas (H5) encodes heretofor neglected relaxation effects; for further discussion in the closely related viscous case, see [Z.3] and especially [MZ.4]. All three of conditions (H3)–(H5) hold automatically in the small-amplitude case; see [Ma.1–3], [YZ, MZ.1], and [PZ], respectively.

Definition 1.1. For a profile $\bar{U} = (\bar{u}, \bar{v})$ that is (as in the Lax case) unique up to translation, we define *nonlinear orbital stability* as convergence of $U = (u, v)(\cdot, t)$ as $t \to \infty$ to a translate $\bar{U}(\cdot - \delta(t))$, where $\delta(\cdot)$ is an appropriately chosen function describing shock location, for any solution U of (1.2) with initial data sufficiently close in some norm to the the original profile $\bar{U}$.

Then, the main result of this paper is:

Theorem 1.2. *Let $\bar{U} = (\bar{u}, \bar{v})$ be a profile (1.1) of a relaxation system (1.2), under assumptions (A1)–(A2) and (H0)–(H5). Then, $\bar{U}$ is nonlinearly orbitally stable from $L^1 \cap H^2$ to L^p, for all $p \geq 2$. More precisely, for initial perturbations $U_0 := \tilde{U}_0 - \bar{U}$ such that $|U_0|_{L^1 \cap H^2} \leq \zeta_0$, ζ_0 sufficiently small, the solution $\tilde{U} = (\tilde{u}, \tilde{v})(x, t)$ of (1.2) with initial data $\tilde{U}_0$ satisfies*

$$|\tilde{U}(x,t) - \bar{U}(x - \delta(t))|_{L^p} \leq C\zeta_0 (1+t)^{-\frac{1}{2}(1-1/p)} \tag{1.8}$$

for all $2 \leq p \leq \infty$, for some $\delta(t)$ satisfying

$$|\dot{\delta}(t)| \leq C\zeta_0 (1+t)^{-\frac{1}{2}} \tag{1.9}$$

and

$$|\delta(t)| \leq C\zeta_0. \tag{1.10}$$

Remark 1.3. Useful geometric necessary conditions for viscous stability have been obtained in [Z.1, Z.3, G] in the simultaneously symmetrizable case (A3)(ii) using the stability index of [GZ,BSZ]. Strengthened, *signed* versions $\Delta > 0$ (under appropriate normalization) of the dynamical stability condition (1.6), these readily yield examples of spectrally unstable large-amplitude profiles, similarly as in the

[6] For a precise definition of the Evans function, and a proof of the equivalence of ($\mathcal{D}$) and (H3)–(H5), see [MZ.1], or Appendix A2, [Z.3].

strictly parabolic case (see, e.g., [GZ,FreZ,ZS,Z.3]). This shows that the stability conditions assumed in Theorem 1.2 are not vacuous in the large amplitude case. Moreover, as discussed in [Z.3], Section 6.2, the signed version of the Majda condition can serve as a physical selection principle in situations for neither of the classical criteria of structural or dynamical stability suffice. As discussed further in [Z.3, MZ.4], it is an extremely interesting open problem which of the stability conditions (H3) and (H5) is in practice most restrictive.

Theorem 1.2 is a consequence of the following auxiliary energy estimate, the extension to the large-amplitude case of an identical estimate used (implicitly) in the small-amplitude argument of [MZ.1]. Following [MZ.1], define the nonlinear perturbation $U = (u, v)$ by

$$U(x,t) := \tilde{U}(x + \delta(t), t) - \bar{U}(x), \tag{1.11}$$

where the "shock location" δ is to be determined later. Evidently, decay of U is equivalent to nonlinear orbital stability as described in (1.8). Then, the key energy estimate, and the main technical contribution of the paper, is as follows.

Proposition 1.4. *Under the hypotheses of Theorem 1.2, let $U_0 \in H^2$, and suppose that, for $0 \le t \le T$, both the supremum of $|\dot\delta|$ and the $H^1 \cap W^{1,\infty}$ norm of the solution $U = (u, v)^t$ of (1.11) remain bounded by a sufficiently small constant $\zeta > 0$. Then, for all $0 \le t \le T$,*

$$|U|^2_{H^2}(t) + \int_0^t |U_x|^2_{H^1}(s)ds \le C(\zeta^2 + |U_0|^2_{H^2}) + C\int_0^t \big(|U|^2_{H^1} + \dot\delta^2\big)(s)ds. \tag{1.12}$$

With Proposition 1.4, Theorem 1.2 follows by exactly the same argument used in [MZ.1] to treat the small-amplitude case; for, the proof of (1.12) was the single place in [MZ.1] where the small-amplitude assumption was actually used. For the sake of brevity, we will not repeat the argument here, referring the reader instead to Section 7 of [MZ.1], or to the closely related analysis of Section 4, [MZ.4] in the real viscosity case.

Plan of the paper. In Section 2, we the preliminary lemmas needed in the analysis. In Section 3, we carry out the proofs of Proposition 1.4 and Theorem 1.2, completing the paper.

Section 2. Preliminaries.

As in [MZ.4], our starting point consists in the following two lemmas.

Lemma 2.1[MZ.3]. *Under assumptions (A1)–(A2), (H0)–(H2), standing wave solutions (1.1) satisfy*

$$|(d/dx)^k \bar{U} - U_\pm| \le C|\bar{U}_x| \le Ce^{-\theta|x|}, \quad k = 0, \dots, 4, \tag{2.1}$$

as $x \to \pm\infty$, *for some* $\theta > 0$, $U_\pm = (u_\pm, v_\pm) = (u_\pm, v^*(u_\pm))$.

Proof. Equivalently, the standing-wave equations may be expressed as a nondegenerate ODE with hyperbolic rest points; see the proof of Lemma 1.2, [MZ.1].∎

Lemma 2.2 ([SK]). *Let* A *and* Q *denote simultaneously symmetrizable matrices and* A^0 *their symmetrizer, with* $A^0Q \le 0$. *Then, genuine coupling (GC) is equivalent to strict dissipativity (A2), and also to either of:*

(K1) There exists a smooth skew-symmetric matrix-valued function $K(A, Q, A^0)$ *such that*

$$\text{Re } \big(A^0Q - KA\big) < 0. \tag{2.2}$$

(K2) block-diag $LQR_\pm < 0$, *where* $L := \tilde{O}^t(\tilde{A}^0)^{1/2}$ *and* $R := (\tilde{A}^0)^{-1/2}\tilde{O}$ *are matrices of left and right eigenvectors of* A *block-diagonalizing* LAR, *with* O *orthonormal Here,* block-diag M *denotes the matrix formed from the diagonal blocks of* M, *with each blocks of dimension equal to the multiplicity of corresponding eigenvalues of* LAR.

Proof. These and other useful equivalent formulations are established in [SK]. The main implication for our purposes, (K2) $\Rightarrow$ (K1), follows readily from Lemma 2.3, below, by first converting to the case of symmetric A, Q by the transformations $A \to (A^0)^{1/2}AA^0)^{-1/2}$, $Q \to (A^0)^{1/2}QA^0)^{-1/2}$, from which the orginal result follows by the fact that $M > 0 \Leftrightarrow (A^0)^{1/2}M(A^0)^{1/2} > 0$, then converting by an orthonormal change of coordinates to the case that A is diagonal and Q symmetric. Variable multiplicity eigenvalues may be handled by partition of unity/interpolation, noting that Re $(Q - KA) < 0$ persists under perturbation.

Under the assumed symmetry of LQR, (A2) $\Rightarrow$ (K2) follows by Taylor expansion at infinity of the spectrum of the symbol $i\xi A_\pm + Q_\pm$, from which we may deduce Re $\sigma($ block-diag $LQR_\pm) < 0$; see, e.g., Appendix B of [MZ.1]. That (K1) $\Rightarrow$ (A2) follows upon rearrangement of energy estimate

$$\langle (A^0 + |\xi|^2 A^0 - i\xi K)w, (\lambda + i\xi A + |\xi|^2 B)w\rangle = 0.$$

Finally, (K2) $\Leftrightarrow$ (GC) is clear. ∎

For our purposes, we shall require the following slight extension of Lemma 2.2, whose proof gives at the same time an explicit description of K of which we shall later make important use. We note that an equivalent version of this result was obtained independently and previously to ours by Humpherys [Hu].

Lemma 2.3. *Let A be diagonal, with real entries appearing with prescribed multiplicity in order of increasing size, and let Q be arbitrary. Then, there exists a smooth skew-symmetric matrix-valued function $K(A,Q)$ such that*

$$\text{Re } (Q - KA) = \text{Re block-diag}\, Q, \tag{2.3}$$

where block-diag Q *denotes the block-diagonal part of Q, with blocks of dimension equal to the multiplicity of the corresponding eigenvalues of A.*

Proof. It is straightforward to check that the symmetric matrix Re $KA = (1/2)(KA - A^tK)$ may be prescribed arbitrarily on off-diagonal blocks, by setting $K_{ij} := (a_i - a_j)^{-1}M_{ij}$, where M_{ij} is the desired block, $i \neq j$. Choosing $M = \text{Re } Q$, we obtain Re $(Q - KA) = \text{Re block-diag}\,(Q)$ as claimed. ∎

We shall also need the following two elementary results.

Lemma 2.4. *Given (A2), there exist block-diagonalizing matrices $L_\pm$, $R_\pm$, $LAR_\pm$ blockdiagonal, $LR_\pm = I$, such that*

$$\text{block-diag}\, LQR_\pm < 0. \tag{2.4}$$

Proof. Again, Re $\sigma(\text{ block-diag}\, \tilde{L}Q\tilde{R}_\pm) < 0$ follows from (A2) by Taylor expansion at infinity of the spectrum of the symbol $i\xi A_\pm + Q_\pm$, for any blockdiagonalizing transformations $\tilde{L}_\pm$, $\tilde{R}_\pm$; see Appendix B, [MZ.1]. By a standard linear-algebraic lemma (see, e.g., Proposition A.9, p. 361 of [St]), block-diag $S^{-1}\tilde{L}Q\tilde{R}S_\pm < 0$, $S_\pm :=$ block-diag $\{S_1, \dots, S_k\}_\pm$ for some choice of nonsingular $S_j^\pm$. Taking $L_\pm := S^{-1}\tilde{L}_\pm$, $R_\pm := \tilde{R}S_\pm$, we are done. ∎

Lemma 2.5. *There is a correspondence between symmetric positive definite symmetrizers A^0, A^0A symmetric, and diagonalizing transformations L, R, LAR diagonal, given by*

$$A^0 = L^*L, \tag{2.5}$$

or equivalently $L = O^*(A^0)^{1/2}$, where O is an orthonormal matrix diagonalizing the symmetric matrix $(A^0)^{1/2}A(A^0)^{-1/2}$. Here, O (or equivalently L) may be chosen with the same degree of smoothness as A^0, on any simply connected domain.

Proof. The first assertion follows by direct calculation. The second is clear in the strictly hyperbolic case, for which the correspondence is also one-to-one; in the general (constant-multiplicity) case, it follows by a standard lemma of Kato [Kat]. ∎

Remark 2.6. Lemma 2.5 hints at the strategy we shall follow in carrying out energy estimates, which is to "effectively diagonalize" by the use of a symmetrizer.

That is, rather than working with LAR as we should like, we work with $A^0A = L^*LA = L^*(LAR)L$, thereby avoiding the problem that there may exist a nonlinear change of coordinates with Jacobian L. Conjugation by L of course does not affect the energy estimates.

Finally, for convenience of the reader, we recall the standard relations

$$\langle U, SU_x\rangle = -\frac{1}{2}\langle U, S_xU\rangle \tag{2.6}$$

and

$$\frac{1}{2}\langle U, KU_x\rangle_t = \langle U, KU_{xt}\rangle + \frac{1}{2}\langle U, K_tU_x\rangle + \frac{1}{2}\langle U, K_xU_x\rangle, \tag{2.7}$$

valid, respectively, for symmetric S and skew-symmetric K.

Section 3. Proofs of the main results.

Proof of Proposition 1.4. Define the nonlinear perturbation $U(x,t) := \tilde{U}(x+\delta(t),t)-\bar{U}(x)$ as in (1.11) where $\delta(t)$ (estimating shock location) is to be determined later; for definiteness, fix $\delta(0) = 0$. Substituting (1.11) into (1.2), we obtain

$$\tilde{U}_t + \tilde{A}\tilde{U}_x - \begin{pmatrix} 0 \\ q \end{pmatrix}(\tilde{U}) = \dot{\delta}\tilde{U}_x, \tag{3.1}$$

where $\tilde{A} := (df, dg)^t(\tilde{u}, \tilde{v})$, and thereby

$$(\tilde{U}-\bar{U})_t + (\tilde{A}\tilde{U}_x - \bar{A}\bar{U}_x) - \Big(\begin{pmatrix} 0 \\ q \end{pmatrix}(\tilde{U}) - \begin{pmatrix} 0 \\ q \end{pmatrix}(\bar{U})\Big) = \dot{\delta}(t)\tilde{U}_x, \tag{3.2}$$

where $\tilde{U}$ now denotes $\tilde{U}(x+\delta(t),t)$ and $\bar{U}$ denotes $\bar{U}(x)$.

Expanding $(\tilde{A}\tilde{U}_x - \bar{A}\bar{U}_x)$ using the quadratic Leibnitz relation

$$\tilde{A}\tilde{U}_x - \bar{A}\bar{U}_x = \tilde{A}(\tilde{U}_x - \bar{U}_x) + (\tilde{A}-\bar{A})\bar{U}_x, \tag{3.3}$$

and Taylor expanding $\Big(\begin{pmatrix} 0 \\ q \end{pmatrix}(\tilde{U}) - \begin{pmatrix} 0 \\ q \end{pmatrix}(\bar{U})\Big)$ about $\tilde{U}$, we obtain the basic nonlinear perturbation equation

$$U_t - \tilde{A}U_x - \tilde{Q}U = M_1(U)\bar{U}_x + (0, I_r)^t M_2(U) + \dot{\delta}(t)(\bar{U}_x + U_x), \tag{3.4}$$

where $\tilde{Q} := (0, dq)^t(\tilde{u}, \tilde{v})$ and

$$\begin{aligned} M_1(U) &= \mathcal{O}(|U|) := \tilde{A}(x,t) - \bar{A}(x), \\ M_2(U) &= \mathcal{O}(|U|^2) := \begin{pmatrix} 0 \\ q \end{pmatrix}(\tilde{U}) - \begin{pmatrix} 0 \\ q \end{pmatrix}(\bar{U}) - \tilde{Q}(\tilde{U}-\bar{U}). \end{aligned} \tag{3.5}$$

Differentiating (3.4) twice with respect to x, we obtain, finally,

(3.6)
$$U_{xxt} - (\tilde{A}U_x)_{xx} - (\tilde{Q}U)_{xx} = (M(U)\bar{U}_x)_{xx} + (0, I_r)^t M_2(U)_{xx} + \dot{\delta}(t)(\bar{U}_{xxx} + U_{xxx}).$$

Let $\tilde{A}^0 := A^0(\tilde{U})$ denote the symmetrizer of $\tilde{A}$ guaranteed by (A1), and factor

$$\tilde{A}^0\tilde{A} = (\tilde{A}^0)^{1/2}\tilde{O}\tilde{D}\tilde{O}^t(\tilde{A}^0)^{1/2}, \tag{3.7}$$

or

$$\tilde{A} = (\tilde{A}^0)^{-1/2}\tilde{O}\tilde{D}\tilde{O}^t(\tilde{A}^0)^{1/2}, \tag{3.8}$$

where $\tilde{O}$ is orthogonal, $\tilde{O}^t = \tilde{O}^{-1}$, and C^3 as a function of (u, v) (see Lemma 2.5) and $\tilde{D} =$ block-diag $\{\tilde{a}_1, \dots, \tilde{a}_l\}$, where $\tilde{a}_j$ denote the eigenvalues of $\tilde{A}$, indexed in increasing order

$$a_1 \le \cdots \le a_k < 0 < a_{k+1} \le \cdots \le a_l.$$

Define the "Goodman-type" [Go] weighting matrix

$$\alpha(x) := \text{ block-diag}\,\{\alpha_1, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_l\}, \tag{3.9}$$

where $\alpha_j > 0$ are defined by ODE

$$\alpha_\pm' = -C_* \text{sgn } a_j |\bar{U}_x| \qquad \alpha_\pm(0) := 1, \tag{3.10}$$

$C_* > 0$ a sufficiently large constant to be determined later. This definition, together with $|a_j| > 0$, (H2)(i), gives the key inequality

$$\alpha_x \bar{D} \le -\theta_1 C_* |\bar{U}_x| \alpha, \tag{3.11}$$

where $\bar{A} =: (\bar{A}^0)^{-1/2}\bar{O}\bar{D}\bar{O}^t(\bar{A}^0)^{1/2}$.

Note, by factorization (3.8), that

$$[(\tilde{A}^0)^{1/2}\tilde{O}\alpha\tilde{O}^t(\tilde{A}^0)^{1/2}]\tilde{A} = (\tilde{A}^0)^{1/2}\tilde{O}(\alpha\tilde{D})\tilde{O}^t(\tilde{A}^0)^{1/2} \tag{3.12}$$

is symmetric, i.e., the symmetric positive definite matrix

$$(\tilde{A}^0)^{1/2}\tilde{O}\alpha\tilde{O}^t(\tilde{A}^0)^{1/2} \tag{3.13}$$

is also a viable symmetrizer for $\tilde{A}$. Moreover, setting $L := \tilde{O}^t(\tilde{A}^0)^{1/2}$ and $R := (\tilde{A}^0)^{-1/2}\tilde{O}$, and noting that, by Lemma 2.5 and constant multiplicity of eigenvalues of A, we have the freedom to smoothly (C^3) redefine L and R so that they take on prescribed values at $x \to \pm\infty$. Thus, by Lemma 2.4, we may assume without loss of generality that

$$\text{Re block-diag}\left(\alpha\tilde{O}^t(\tilde{A}^0)^{1/2}\tilde{Q}(\tilde{A}^0)^{-1/2}\tilde{O}\right)_\pm \le -\theta(C_*) < 0, \tag{3.14}$$

and thereby, appealing to (2.1), $k=0$,

$$\text{Re block-diag}\big(\alpha\tilde O^t(\tilde A^0)^{1/2}\tilde Q(\tilde A^0)^{-1/2}\tilde O\big)\le -\theta(C_*)+C(C_*)\zeta+C|\bar U_x|\alpha, \tag{3.15}$$

where

$$\big[(\tilde A^0)^{1/2}\tilde O\alpha\tilde O^t(\tilde A^0)^{1/2}\big]\tilde Q=(\tilde A^0)^{1/2}\tilde O\big[\alpha\tilde O^t(\tilde A^0)^{1/2}\tilde Q(\tilde A^0)^{-1/2}\tilde O\big]\tilde O^t(\tilde A^0)^{1/2}, \tag{3.16}$$

and the exponent θ now refers to the minimum of the constants used elsewhere in the argument and that appearing in (2.1). The constants $\theta(C_*)$ and $C(C_*)$ measure the conditioning of matrices $\alpha_\pm$ and in fact decay, resp. grow, exponentially with respect to C_*; however, this is unimportant for our argument.

Defining

$$K_1:=K\big(\tilde D,\alpha\tilde O^t(\tilde A^0)^{1/2}\tilde Q(\tilde A^0)^{-1/2}\tilde O+N\big),$$

where $K(\cdot)$ is as defined in Lemma 2.3, and N is *an arbitrary matrix with* $|N|_{C^1(x,t}\le C(C_*)$ *and vanishing on diagonal blocks*, to be determined later, we have, therefore,

$$\begin{aligned}
&\text{Re}\,\Big(-\tilde K_\alpha\tilde A+(\tilde A^0)^{1/2}\tilde O N\tilde O^t(\tilde A^0)^{1/2}+\big[(\tilde A^0)^{1/2}\tilde O\alpha\tilde O^t(\tilde A^0)^{1/2}\big]\tilde Q\Big)=\\
&\quad\text{Re}\,(\tilde A^0)^{1/2}\tilde O\big(-K_1D+\alpha\tilde O^t(\tilde A^0)^{1/2}\tilde Q(\tilde A^0)^{-1/2}\tilde O+N\big)\tilde O^t(\tilde A^0)^{1/2}\\
&\quad=\text{Re}\,(\tilde A^0)^{1/2}\tilde O\,\text{block-diag}\big(\alpha\tilde O^t(\tilde A^0)^{1/2}\tilde Q(\tilde A^0)^{-1/2}\tilde O\big)\tilde O^t(\tilde A^0)^{1/2}\\
&\quad\le -\theta(C_*)+C(C_*)\zeta+C|\bar U_x|(\tilde A^0)^{1/2}\tilde O\alpha\tilde O^t(\tilde A^0)^{1/2}
\end{aligned} \tag{3.17}$$

for the skew-symmetric matrix

$$\tilde K_\alpha:=(\tilde A^0)^{1/2}\tilde O K_1\tilde O^t(\tilde A^0)^{1/2}, \tag{3.18}$$

by (3.15) together with Lemma 2.3. By (3.11), we have also

$$\begin{aligned}
(\tilde A^0)^{1/2}\tilde O(\alpha_x\tilde D)\tilde O^t(\tilde A^0)^{1/2}&=(\tilde A^0)^{1/2}\tilde O(\alpha_x\bar D)\tilde O^t(\tilde A^0)^{1/2}+\mathcal O(C_*\zeta)\\
&\le -\theta C_*|\bar U_x|(\tilde A^0)^{1/2}\tilde O\alpha\tilde O^t(\tilde A^0)^{1/2}+C(C_*)\zeta,
\end{aligned} \tag{3.19}$$

for possibly still smaller $\theta>0$ (ζ as defined in the statement of the proposition).

Finally, through smooth dependence on $\tilde U=\bar U+U$, we have

$$|\tilde A_x|,\,|\tilde Q_x|,\,|\tilde A^0_x|,\le C(|\bar U_x|+\zeta) \tag{3.20}$$

and

$$|\tilde A_t|,\,|\tilde Q_t|,\,|\tilde A^0_t|,\le C\zeta, \tag{3.21}$$

and, through smooth dependence on $\tilde U=\bar U+U$ and N,

$$|\tilde K_{\alpha,x}|,|\tilde K_{\alpha,t}|\le C(C_*). \tag{3.22}$$

Friedrichs-type estimate. As in [MZ.1], we first perform a standard "Friedrichs-type" estimate for symmetric hyperbolic systems, now incorporating the weight α. Taking the L^2 inner product of

$$\tilde{A}^0_\alpha U_{xx} := (\tilde{A}^0)^{1/2}\tilde{O}\alpha\tilde{O}^t(\tilde{A}^0)^{1/2}U_{xx} \tag{3.23}$$

against (3.6), we obtain in the usual way, by an application of (2.6) together with rearrangement/integration by parts and several applications of Young's inequality, the energy estimate

$$\begin{aligned}
&\frac{1}{2}\langle \tilde{A}^0_\alpha U_{xx}, U_{xx}\rangle_t\\
&\quad = \langle \tilde{A}^0_\alpha U_{xx}, U_{xxt}\rangle + \frac{1}{2}\langle (\tilde{A}^0_\alpha)_t U_{xx}, U_{xx}\rangle\\
&\quad = \langle U_{xx}, (\tilde{A}^0)^{1/2}\tilde{O}(\alpha_x \tilde{D})\tilde{O}^t(\tilde{A}^0)^{1/2}U_{xx}\rangle\\
&\qquad + \langle U_{xx}, (\tilde{A}^0)^{1/2}\tilde{O}\alpha\tilde{O}^t(\tilde{A}^0)^{1/2}\tilde{Q}U_{xx}\rangle + \langle U_{xx}, (\tilde{A}^0)^{1/2}\tilde{O}N\tilde{O}^t(\tilde{A}^0)^{1/2}U_{xx}\rangle\\
&\qquad + \langle U_{xx}, \mathcal{O}(|\bar{U}_x|)(\tilde{A}^0)^{1/2}\tilde{O}\alpha\tilde{O}^t(\tilde{A}^0)^{1/2}U_{xx}\rangle\\
&\qquad + \mathcal{O}(C(C_*)(\zeta+\bar{\zeta}))|U_{xx}|^2_{L^2} + \mathcal{O}(C(C_*,\bar{\zeta})(|U|^2_{H^1} + |\dot{\delta}(t)|^2),
\end{aligned} \tag{3.24}$$

where $\bar{\zeta} > 0$ is an arbitrarily small constant arising through Young's inequality, with $\zeta << \bar{\zeta} << 1$, the term involving N is the sum of the "off-block-diagonal" parts (meaning off-block-diagonal after conjugation by $(\tilde{A}^0)^{1/2}\tilde{O}$) of the $\int \mathcal{O}(|\bar{U}_x||\alpha||U_{xx}|^2)$ error terms arising through x-differentiation of factors other than α, with the corresponding $\int \mathcal{O}(|\bar{U}_x||\alpha||U_{xx}|^2)$ terms separated out and accounted for in the $\mathcal{O}(C(C_*)\zeta)$ error term of the final line, and the $\int \mathcal{O}(|\bar{U}_x||\alpha||U_{xx}|^2)$ part of term

$$\begin{aligned}
&\langle U_{xx}, (\tilde{A}^0)^{1/2}\tilde{O}\alpha\tilde{O}^t(\tilde{A}^0)^{1/2}(M_1(U)\bar{U}_x)_{xx}\rangle\\
&\qquad = \langle U_{xx}, (\tilde{A}^0)^{1/2}\tilde{O}\big(\alpha\tilde{O}^t(\tilde{A}^0)^{1/2}(M_1(U)\bar{U}_x)_{xx}(\tilde{A}^0)^{-1/2}\tilde{O}\big)\tilde{O}^t(\tilde{A}^0)^{1/2}\rangle
\end{aligned}$$

and $\mathcal{O}(\cdot)$ denotes a uniform constant independent of C_* and $\bar{\zeta}$. Note that $|N|_{C^1(x,t} \le C(C_*)$ indeed holds, since N by definition is of form $J(\tilde{U}, \alpha, \bar{U}_x)$, with $J(\cdot)$ smooth, and $|\tilde{U}|_{C^1(x,t)} \le C\zeta$, $|\alpha|_{C^1(x,t)} \le C_* \sup|\alpha| \le C(C_*)$. Likewise, the "block-diagonal" parts (meaning block-diagonal after conjugation by $(\tilde{A}^0)^{1/2}\tilde{O}$) of these error terms may be estimated as

$$\langle U_{xx}, \mathcal{O}(|\bar{U}_x|(\tilde{A}^0)^{1/2}\tilde{O}\alpha\tilde{O}^t(\tilde{A}^0)^{1/2}U_{xx}\rangle$$

as claimed, since all such terms are either of form

$$\langle U_{xx}, (\tilde{A}^0)^{1/2}\tilde{O}\big(\alpha P\big)\tilde{O}^t(\tilde{A}^0)^{1/2}U_{xx}\rangle$$

or of form

$$\langle U_{xx}, (\tilde{A}^0)^{1/2}\tilde{O}\big(P\alpha\big)\tilde{O}^t(\tilde{A}^0)^{1/2}U_{xx}\rangle,$$

where P (arising through x-differentiation) is $\mathcal{O}(|\bar{U}_x|)$, so that the jth diagonal block of αP or $P\alpha$ involves the correct factor $\alpha_j\mathcal{O}(|\bar{U}_x|)$.

Remark 3.1. The treatment of error term N, above, we regard as the most delicate and novel aspect of our argument. Without complete cancellation of off-diagonal terms, we see no way that such a "Goodman-type" estimate can close, due to exponential growth in C_* of $\sup|\alpha|$.

Kawashima-type estimate. Next, we perform a "Kawashima-type" estimate of the type formalized in [Kaw]. Taking the L^2 inner product of U_{xx} against $\tilde{K}_\alpha$ times the singly differentiated equation

$$U_{xt} + (\tilde{A}U_x)_x - (\tilde{Q}U)_x = (M(U)\bar{U}_x)_x + (0, I_r)^t M_2(U)_x + \dot{\delta}(t)(\bar{U}_{xx} + U_{xx}), \tag{3.25}$$

where $\tilde{K}_\alpha$ is as in (3.18), with N as implicitly defined in (3.24) substituting into (2.7), and bounding all lower-derivative terms by Young's inequality, we obtain the auxiliary energy estimate:

(3.26)

$$\begin{aligned}\frac{1}{2}\langle \tilde{U}_{xx}, \tilde{K}_\alpha U_x\rangle_t &= -\langle U_{xx}, \tilde{K}_\alpha\tilde{A}U_{xx}\rangle + \dot{\delta}(t)\langle \tilde{K}_\alpha U_{xx}, U_{xx}\rangle \\ &\quad + \mathcal{O}(\bar{\zeta})|U_{xx}|_{L^2}^2 + \mathcal{O}(\bar{\zeta}^{-1}C(C_*))(|U|_{H^1}^2 + |\dot{\delta}(t)|^2) \\ &\le -\langle U_{xx}, \tilde{K}_\alpha\tilde{A}U_{xx}\rangle + C(\bar{\zeta}+\zeta)|U_{xx}|_{L^2}^2 + C(\bar{\zeta}, C_*)(|U|_{H^1}^2 + |\dot{\delta}(t)|^2).\end{aligned}$$

Adding (3.24) and (3.26), and recalling (3.17) and (3.19), we obtain, finally,

(3.27)

$$\begin{aligned}\frac{1}{2}(\langle \tilde{A}^0U_{xx}, U_{xx}\rangle + \langle U_{xx}, \tilde{K}_\alpha U_x\rangle)_t &\le (-\theta C_* + C)\langle U_{xx}, (\tilde{A}^0)^{1/2}\tilde{O}|\bar{U}_x|\alpha\tilde{O}^t(\tilde{A}^0)^{1/2}U_{xx}\rangle \\ &\quad + \big(-\theta(C_*) + C(C_*)(\zeta+\bar{\zeta})\big)|U_{xx}|_{l^2}^2 \\ &\quad + C(\bar{\zeta}, C_*)(|U|_{H^1}^2 + |\dot{\delta}(t)|^2), \\ &\le -\frac{1}{2}\theta(C_*)|U_{xx}|_{L^2}^2 + C(\bar{\zeta}, C_*)(|U|_{H^1}^2 + |\dot{\delta}(t)|^2),\end{aligned}$$

provided that C_* is taken sufficiently large and $\bar{\zeta}$, ζ sufficiently small that $C \le \theta C_*$ and $-\theta(C_*) + C(C_*)(\zeta+\bar{\zeta}) \le -\theta(C_*)/2$.

Integrating (3.27) from 0 to t thus yields

(3.28)

$$\begin{aligned}(\langle \tilde{A}^0_\alpha U_{xx}, U_{xx}\rangle + \langle U_{xx}, \tilde{K}_\alpha U_x\rangle)|_0^t + \theta(C_*)\int_0^t |U_{xx}|_{L^2}^2(s)\,ds \le \\ C(C_*, \bar{\zeta})\int_0^t (|U|_{H^1}^2 + |\dot{\delta}|^2)(s)\,ds.\end{aligned}$$

Rearranging, using positive definiteness of $\tilde{A}^0_\alpha$, and using Young's inequality to bound

$$\langle U_{xx}, K_\alpha U_x\rangle(t) \le \bar{\zeta}|U_{xx}|^2_{L^2}(t) + C\bar{\zeta}^{-1}|U_x|^2_{L^2}(t) \le \bar{\zeta}|U_{xx}|^2_{L^2}(t) + C\bar{\zeta}^{-1}\zeta^2, \tag{3.29}$$

and recalling, by assumption, that $(\langle \tilde{A}^0_\alpha U_x, U_x\rangle + \langle U_x, \tilde{K}_\alpha U\rangle)(0) \le C\zeta^2$, we obtain (1.12) as claimed. ∎

Proof of Theorem 1.2. Theorem 1.2 now follows by the same argument used in [MZ.1], pages 865–872, to treat the small-amplitude case, with energy estimate (1.12) replacing the one (analogous to ((7.48–7.49) of the reference) implicitly obtained from the Friedrichs-type estimate just above (7.70) together with the Kawashima-type estimate (7.72) of the reference. ∎

Section 4. The characteristic case.

Finally, we briefly discuss the uniformly characteristic case in which (H1)(i) is violated for a shock profile of a discrete kinetic model. This cannot occur for the simplest examples of the Broadwell or Jin–Xin models, for the reason that it would violate the subcharacteristic condition

$$a_j < a_j^{*\pm} < a_{j+r}, \tag{4.1}$$

which is in turn necessary for strict dissipativity, (A2); see, e.g., [Yo, MZ.1, MZ.3]. Indeed, this holds in general for models with the property that r consecutive characteristics take on only two values ($r = 2$ for Broadwell; $r = n/2$ for Jin–Xin, but the total number of characteristic values is 2); for, the subcharacteristic condition (4.1) then implies that $a_p^{*\pm}$ lie between the neighboring characteristic values a_p and a_{p+r}, whence, by the Lax condition $a_p^{*-} > s > a_p^{*+}$, the speed s does as well.

This is clearly an accident of low dimension, however, and for general models there is no physical reason that (H1)(i) should be satisfied. Indeed, though it is evidently satisfied generically, there is ample reason to discard this hypothesis. For, discretizing the Boltzmann equations

$$f_t + \xi f_x = Q(\xi, f), \quad \xi \in \mathbb{R}^1, \tag{4.2}$$

by velocity ξ, where ξ denotes velocity, $f(\xi, x)$ the probability distribution of speeds at spatial location x, and $Q(\xi, f)$ a collision term (local in x but nonlocal in ξ), we find as the velocity mesh goes to zero that (H1)(i) is more and more poorly satisfied, so that uniformity of our estimates (or even the ball for which small-amplitude profiles are guaranteed to exist) is lost.

This is hardly the main difficulty in proceeding to the Boltzmann limit, which is rather the reverse problem of unboundedness of the multiplication operator $f \to \xi f$

(in our notation, blowup of the spectrum of A), and the associated lack of spectral gap between zero and the essential spectrum of the operator $A^{-1}Q$ appearing in the travelling-wave and eigenvalue ODE; see, e.g., [CN]. Nonetheless, it is an issue that arises and should be addressed.

Fortunately, there is a simple fix, at least for discrete kinetic models. Namely, in case characteristics $a_j, \dots, a_k$ coincide with shock speed s, we may substitute for (H1)(i) the more general hypotheses

$$\text{Re } (L_j \quad \cdots \quad L_k)\, dQ \begin{pmatrix} R_j \\ \vdots \\ R_k \end{pmatrix} \le -\theta < 0 \tag{4.3}$$

for some fixed left and right zero eigenbases L_i and R_i, and

$$\ker \tilde{A} \cap \ker dq = \emptyset, \tag{4.4}$$

where without loss of generality A is taken to be diagonal, $\tilde{A} := A - s = dF - s$. This holds necessarily at $x = \pm\infty$ for some choice of diagonalizing transformation, by strict dissipativity, (A2), (recall that Re $\sigma(\text{ block-diag } \tilde{L}Q\tilde{R}_\pm) < 0$ follows from (A2) by Taylor expansion at infinity of the spectrum of the symbol $i\xi A_\pm + Q_\pm$; likewise, (A2) implies genuine coupling, (GC), of which (4.4) is a weakened form), hence is always satisfied in the small-amplitude case. Whether or not it holds globally for physically interesting examples we do not know.

Review of the argument of Section 3 shows that auxiliary energy estimate (1.12) goes through under this hypothesis with constant weights $\alpha_j = \alpha_{j+1} = \cdots = \alpha_k \equiv 1$ in the zero-speed modes, since there are no error terms in these modes to be overcome and there is a uniformly good contribution by (4.3). Likewise, review of the arguments of [MZ.1] shows that the results obtained there carry though as well, with appropriate modification of the proofs. Namely, (4.3) and (4.4) together imply that

$$\begin{pmatrix} df \\ dQ_j \\ \vdots \\ dQ_k \\ dg_2 \end{pmatrix}, \qquad g =: \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad g_1 \in \mathbb{R}^{k-j+1},$$

is full rank under some choice of coordinate system, whence we can again rewrite both traveling-wave and eigenvalue/resolvent equations as nondegenerate first-order systems and proceed as before. We omit the details as aside from our main purpose.

Note, in particular, that we obtain small-amplitude existence for fixed speed s by a minor adjustment of the argument of [MZ.1] (namely, fixing the speed and letting endstates vary), without any assumption on the base state other than simplicity of the principal eigenvalue and strong dissipativity; this generalizes earlier results of [YoZ, MZ.1].

For $a_j, \dots, a_k$ close to s, we may recover our previous results with uniform estimates by a singular perturbation version of the same argument. It would be interesting to extend this approach to more general relaxation models for which the speeds a_i are not constant, in particular the case for which they are sometimes but not always characteristic.

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