A Specific Hyperbolic System of Conservation Laws Admitting Delta-shock Wave Type Solutions

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Abstract. We construct δ-shock type solutions of the Cauchy problem for the system of conservation laws

$$u_t + \big(f(u) - v\big)_x = 0, \quad v_t + \big(g(u)\big)_x = 0,$$

where $f(u)$ and $g(u)$ are polynomials of degree n and $n+1$, respectively, n is even. A well known particular case of this system was studied in [**17**], [**16**] by B. L. Keyfitz and H. C. Kranzer. In this paper a techniques of the *weak asymptotics method* and the definition of a δ-shock type solution introduced by V. G. Danilov and V. M. Shelkovich [**6**]– [**8**], are used.

Geometric and physics sense of the Rankine–Hugoniot conditions for δ-shocks is given for the above system, for the system

$$u_t + \big(f(u)\big)_x = 0, \quad v_t + \big(g(u)v\big)_x = 0,$$

and for the well-known zero-pressure gas dynamics system. The geometric aspect of δ-shock formation from sufficiently smooth compactly supported initial data is considered. Namely, the construction for the position of δ-shock in a breaking wave is given.

1. Introduction and basic results

1. Consider the system of equations

$$(1.1) \qquad \begin{aligned} L_1[u,v] = u_t + \Big(F(u,v)\Big)_x &= 0, \\ L_2[u,v] = v_t + \Big(G(u,v)\Big)_x &= 0, \end{aligned}$$

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where $F(u,v)$ and $G(u,v)$ are smooth functions, such that $F(u,v)$, $G(u,v)$ are *linear* with respect to v, $u = u(x,t)$, $v = v(x,t) \in \mathbb{R}$, and $x \in \mathbb{R}$. As is well known, such a system, even in the case of smooth (and, moreover, in the case of discontinuous) initial data $(u^0(x), v^0(x))$, can have a discontinuous *shock wave* type solution. In this case, it is said that the pair of functions $(u(x,t), v(x,t)) \in L^\infty\big(\mathbb{R} \times (0,\infty); \mathbb{R}^2\big)$ is a generalized solution of the Cauchy problem (1.1) with the initial data $(u^0(x), v^0(x))$ if the integral identities

$$
\begin{aligned}
\int_0^\infty \int \Big(u\varphi_t + F(u,v)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty \int \Big(v\varphi_t + G(u,v)\varphi_x\Big)\,dx\,dt + \int v^0(x)\varphi(x,0)\,dx &= 0
\end{aligned}
\tag{1.2}
$$

hold for all compactly supported test functions $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\,\infty))$, where $\int \,\cdot\, dx$ denotes an improper integral $\int_{-\infty}^{\infty} \,\cdot\, dx$.

Let us consider the Cauchy problem for system (1.1) with the initial data

$$
u^0(x) = u_0 + u_1 H(-x), \qquad v^0(x) = v_0 + v_1 H(-x),
\tag{1.3}
$$

where u_0, u_1, v_0, v_1 are constants and $H(\xi)$ is the Heaviside function. It is well known [**1**], [**6**]– [**18**], [**29**], that in order to solve this problem for some "nonclassical cases", it is necessary to introduce new elementary singularities called δ*-shock waves (singular shock waves).* These are generalized solutions of the Cauchy problem of the form

$$
\begin{aligned}
u(x,t) &= u_0 + u_1 H(-x+ct),\\
v(x,t) &= v_0 + v_1 H(-x+ct) + e(t)\delta(-x+ct),
\end{aligned}
\tag{1.4}
$$

where $e(0) = 0$ and $\delta(\xi)$ is the Dirac delta function.

There is *no standard definition* of δ-shocks. This reflects the fact that to define a δ-shock wave type solution, we need to define the *product of the Heaviside function and the delta function*. We also need to define *in which sense* the distributional solution (1.4) satisfies a nonlinear system.

In what follows, we present a short review of well-known methods used to solve problems close to those studied in this paper.

In order to construct a δ-shock wave type solution of the system

$$
\begin{aligned}
u_t + (u^2)_x &= 0,\\
v_t + (uv)_x &= 0,
\end{aligned}
\tag{1.5}
$$

in [**15**] the parabolic regularization

$$
u_t + (u^2)_x = \varepsilon u_{xx}, \qquad v_t + (uv)_x = \varepsilon v_{xx}.
$$

is used.

In [**13**], in order to construct a δ-shock wave type solution of the system

$$
\begin{aligned}
u_t + \big(f(u)\big)_x &= 0, \\
v_t + \big(g(u)v\big)_x &= 0,
\end{aligned}
\tag{1.6}
$$

this system is reduced to a system of Hamilton–Jacobi equations, and then the Lax formula is used. In [**18**], for the case $g(u) = f'(u)$, to construct a δ-shocks wave type solution the problem of multiplication of distributions is solved by using the definition of Volpert's averaged superposition [**30**].

In [**29**] for system (1.5) and in [**1**] for the system of "zero-pressure gas dynamics"

$$
\begin{aligned}
v_t + \big(vu\big)_x &= 0, \\
(vu)_t + \big(vu^2\big)_x &= 0,
\end{aligned}
\tag{1.7}
$$

(here $v \geq 0$ is the density, u is the velocity) with the initial data (1.3), the δ-shock wave type solution is defined as a measure-valued solution. In [**10**], the global δ-shock wave type solution was obtained for system (1.7). In [**14**], the uniqueness of the weak solution is proved for the case when the initial data is a Radon measure. System (1.7) describes the motion of free particles which stick under collision. In multidimensional case this system was used to describe the formation of large-scale structures in the universe [**32**].

In [**12**] for system (1.7) and in [**24**] for some classes of systems, *approximate solutions* of the Cauchy problem are constructed, by using the Colombeau theory approach.

The system

$$
\begin{aligned}
L_{01}[u,v] &= u_t + (u^2 - v)_x = 0, \\
L_{02}[u,v] &= v_t + (\tfrac{1}{3}u^3 - u)_x = 0
\end{aligned}
\tag{1.8}
$$

with the initial data (1.3) is studied in [**16**], [**17**]. In [**17**] in order to construct *approximate solutions* the Colombeau theory approach, as well as the Dafermos–DiPerna regularization, and the box approximations are used. But the notion of a *singular solution* of system (1.8) has *not* been defined. Some problems for system (1.8) are considered in [**26**].

In the papers of V. G. Danilov and V. M. Shelkovich [**4**]– [**9**], [**28**] (see also [**2**], [**27**]) a new analytic method for studying the *dynamics of propagation and interaction* of different singularities of nonlinear equations and hyperbolic systems of conservation laws was developed (infinitely narrow δ-solitons, shocks, δ-shocks). It is the so-called *weak asymptotics method*. The summary of this method see in [**3**]. One of the main ideas of this method is based on V. P. Maslov's approach that permits deriving the Rankine–Hugoniot conditions directly from the differential equations *considered in the weak sense* [**20**], [**23**] [**2**] (see also [**31**, 2.7]). Maslov's *algebras of singularities* [**21**], [**22**], [**2**] are essentially used in the *weak asymptotics method*.

In the framework of the *weak asymptotics method*, in [**8**], for systems (1.6), (1.7), (1.8) the propagation of δ-shock waves was described. In [**6**], [**7**], for system (1.6) formulas describing the propagation and interaction of δ-shock waves are constructed. In these papers for some classes of hyperbolic systems of conservation laws a *new definition of a δ-shock wave type solution* was introduced. This definition is *close* to the standard definition of a shock wave type solution (1.2) and *relevant* to the notion of δ-shocks.

In [**28**], in the framework of the *weak asymptotics method* the Cauchy problem to the system

$$
\begin{aligned}
L_{11}[u,v] &= u_t + \big(f(u)-v\big)_x = 0,\\
L_{12}[u,v] &= v_t + \big(g(u)\big)_x = 0,
\end{aligned} \tag{1.9}
$$

with piecewise constant initial data was solved. Here

$$
f(u) = \sum_{k=0}^{n} A_k u^k, \quad A_n \neq 0, \qquad g(u) = \sum_{k=0}^{n+1} B_k u^k, \quad B_{n+1} \neq 0,
$$

are polynomials, n is an even number, $u = u(x,t), v = v(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$. System (1.8) is a well known particular case of system (1.9).

2. In this paper, generalizing results obtained in [**28**], in the framework of the *weak asymptotics method*, we solve the Cauchy problem to system (1.9) with the initial data of the form

$$
\begin{aligned}
u^0(x) &= u_0^0(x) + u_1^0(x)H(-x),\\
v^0(x) &= v_0^0(x) + v_1^0(x)H(-x) + e^0\delta(-x),
\end{aligned} \tag{1.10}
$$

where $u_k^0(\mathrm{x})$, $v_k^0(x)$, $k = 0,1$ are given smooth functions, e^0 is a given constant. This means that we study the problem of the propagation of δ-shocks. We use the *definition of a δ-shock wave type solution* introduced by V. G. Danilov and V. M. Shelkovich [**7**], [**8**]. The initial data (1.10) *can contain* δ-function, but as a rule, in the well-known papers on δ-shocks, the initial data without δ-function is considered. This situation is related to the fact that the technical base of these papers is connected with self-similar solutions.

REMARK 1.1. The systems (1.9), (1.8) differ from above systems (1.5), (1.6), (1.7) and have a *specific* property. Namely, in systems (1.9) and (1.8) *there is no balance* of singularities. Let (u,v) be a δ-shock type solution of (1.8). Hence, u contains the Heaviside function H, and v contains the Heaviside function H and δ-function. Thus, $u^2 - v$ contains the distributions H, δ, and $\frac{1}{3}u^3 - u$ contains the distribution H. It is easily seen that, the term $(u^2 - v)_x$ contains the distributions H, δ, δ', but the term u_t contains *only* the distributions H and δ. Analogously, the term v_t contains the distributions H, δ, δ', but the term $(\frac{1}{3}u^3 - u)_x$ contains *only* the distributions H, δ. Nevertheless, we prove that the last systems have *exact* δ-shock type solutions.

The eigenvalues of the characteristic matrix of system (1.9) are

$$\lambda_{\pm}(u) = \frac{1}{2}\Big(f'(u) \pm \sqrt{\big(f'(u)\big)^2 - 4g'(u)}\Big), \quad \big(f'(u)\big)^2 \geq 4g'(u).$$

As in [**11**], [**17**], [**24**], [**29**], we assume that the "overcompression" condition is satisfied:

$$\lambda_-(u_+) \leq \lambda_+(u_+) \leq \sigma_\delta \leq \lambda_-(u_-) \leq \lambda_+(u_-), \tag{1.11}$$

where σ_δ is the speed of propagation of δ-shock waves, and u_- and u_+ are respective left- and right-hand values of u on the discontinuity curve. Condition (1.11) serves as the admissibility condition for the δ-shocks and means that all characteristics on both sides of the discontinuity are in-coming.

In Section 2 we solve the Cauchy problem (1.9), (1.10) using a *Definition* 1.2 *of a δ-shock wave type solution* given below.

Let us introduce a definition of a *δ-shock type solution* of system (1.1). Suppose that $\Gamma = \{\gamma_i : i \in I\}$ is a connected graph in the upper half-plane $\{(x,t) : x \in \mathbb{R},\ t \in [0,\infty)\} \in \mathbb{R}^2$ containing smooth arcs γ_i, $i \in I$, and I is a finite set. By I_0 we denote a subset of I such that an arc γ_k for $k \in I_0$ starts from the points of the x-axis; $\Gamma_0 = \{x_k^0 : k \in I_0\}$ is the set of initial points of arcs γ_k, $k \in I_0$.

Consider the initial data of the form $(u^0(x), v^0(x))$, where

$$v^0(x) = V^0(x) + e^0\delta(\Gamma_0),$$

$e^0\delta(\Gamma_0) = \sum_{k\in I_0} e_k^0\delta(x - x_k^0)$, $u^0, V^0 \in L^\infty\big(\mathbb{R};\mathbb{R}\big)$, e_k^0 are constants, $k \in I_0$.

Definition 1.2. ([**7**], [**8**]) A pair of distributions $(u(x,t), v(x,t))$ and graph Γ, where $v(x,t)$ is represented in the form of the sum

$$v(x,t) = V(x,t) + e(x,t)\delta(\Gamma),$$

$u, V \in L^\infty\big(\mathbb{R}\times(0,\,\infty);\mathbb{R}\big)$, $e(x,t)\delta(\Gamma)=\sum_{i\in I} e_i(x,t)\delta(\gamma_i)$, $e_i(x,t)\in C^1(\Gamma)$, $i\in I$, is called a *generalized δ-shock wave type solution* of system (1.1) with the initial data $(u^0(x), v^0(x))$ if the integral identities

$$
\begin{aligned}
\int_0^\infty\int\Big(u\varphi_t+F(u,V)\varphi_x\Big)\,dx\,dt+\int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty\int\Big(V\varphi_t+G(u,V)\varphi_x\Big)\,dx\,dt \qquad\qquad & \\
+\sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl \qquad & \\
+\int V^0(x)\varphi(x,0)\,dx+\sum_{k\in I_0} e_k^0\varphi(x_k^0,0) &= 0,
\end{aligned}
\tag{1.12}
$$

hold for all test functions $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\,\infty))$, where $\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}$ is the tangential derivative on the graph Γ, $\int_{\gamma_i}\,\cdot\,dl$ is a line integral over the arc γ_i.

REMARK 1.3. The system of integral identities (1.12) generalizes the *usual system of integral identities* (1.2) which is the definition of a shock wave type solution. The integral identities (1.12) for δ-*shocks* differ from integral identities (1.2) for *shocks* by an additional term

$$
\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl=\sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl
$$

in the second identity. This term appears due to the so-called *Rankine–Hugoniot deficit* and reflects the fact that for δ-shocks the Rankine–Hugoniot conditions are defined by the fifth and sixth equations of systems (2.4)

$$
\begin{aligned}
\dot\phi(t) &= \tfrac{[f(u)]-[v]}{[u]},\\
\dot e(t) &= [g(u)]-[v]\tfrac{[f(u)]-[v]}{[u]},
\end{aligned}
$$

where the fifth equation is the *standard* Rankine–Hugoniot condition, $\dot{} = \frac{d}{dt}$.

According to Definition 1.2 a *generalized δ-shock wave type solution* is a pair of *distributions* $(u(x,t), v(x,t))$ unlike the Definition of measure-solutions given in [**1**], [**29**], where $v(dx,t)$ is a *measure* and $u(x,t)$ is understood as a *measurable function which is defined* $v(dx,t)$ *a.e.*.

Next, we introduce a definition of a *weak asymptotic solution*, which is one of the most important notions in the *weak asymptotics method*.

Denote by $O_{\mathcal{D}'}(\varepsilon^\alpha)$ a distribution $f(x,t,\varepsilon)\in\mathcal{D}'(\mathbb{R}_x)$ such that

$$
\langle f(x,t,\varepsilon),\psi(x)\rangle=O(\varepsilon^\alpha),
$$

for any test function $\psi(x) \in \mathcal{D}(\mathbb{R}_x)$. Moreover, $\langle f(x,t,\varepsilon), \psi(x)\rangle$ is a continuous function in t, where the estimate $O(\varepsilon^\alpha)$ is understood in the standard sense and is uniform with respect to t.

Definition 1.4. ([**7**], [**8**]) A pair of functions $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ smooth as $\varepsilon > 0$ is called a *weak asymptotic solution* of system (1.1) with the initial data $(u^0(x), v^0(x))$ if

$$\begin{aligned}
\int L_1[u(x,t,\varepsilon), v(x,t,\varepsilon)]\psi(x)\,dx &= o(1),\\
\int L_2[u(x,t,\varepsilon), v(x,t,\varepsilon)]\psi(x)\,dx &= o(1),\\
\int \Big(u(x,0,\varepsilon) - u^0(x)\Big)\psi(x)\,dx &= o(1),\\
\int \Big(v(x,0,\varepsilon) - v^0(x)\Big)\psi(x)\,dx &= o(1), \quad \varepsilon \to +0,
\end{aligned}$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R})$.

The last relations can be rewritten as

$$\begin{aligned}
L_1[u(x,t,\varepsilon), v(x,t,\varepsilon)] &= o_{\mathcal{D}'}(1),\\
L_2[u(x,t,\varepsilon), v(x,t,\varepsilon)] &= o_{\mathcal{D}'}(1),\\
u(x,0,\varepsilon) &= u^0(x) + o_{\mathcal{D}'}(1),\\
v(x,0,\varepsilon) &= v^0(x) + o_{\mathcal{D}'}(1), \quad \varepsilon \to +0,
\end{aligned} \tag{1.13}$$

where the first two estimates are uniform in t.

Within the framework of the *weak asymptotics method*, we find the *generalized δ-shock wave type solution* $(u(x,t), v(x,t))$ of the Cauchy problem as the limit

$$\begin{aligned}
u(x,t) &= \lim\nolimits_{\varepsilon\to+0} u(x,t,\varepsilon),\\
v(x,t) &= \lim\nolimits_{\varepsilon\to+0} v(x,t,\varepsilon),
\end{aligned} \tag{1.14}$$

of the *weak asymptotic solution* $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ of this problem, where limits are understood in the weak sense (in the sense of the space of distributions $\mathcal{D}'(\mathbb{R} \times [0,\infty))$). Constructing the *weak asymptotic solution* and multiplying the first two relations (1.13) by a test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0,\infty))$, integrating these relations by parts and then passing to the limit as $\varepsilon \to +0$, we obtain that the pair of distributions (1.14) satisfy integral identities (1.12). Thus, we will prove that the left-hand sides of the following relations

$$\begin{aligned}
\lim_{\varepsilon\to+0} \int_0^\infty \int L_1[u(x,t,\varepsilon), v(x,t,\varepsilon)]\varphi(x,t)\,dx\,dt &= 0,\\
\lim_{\varepsilon\to+0} \int_0^\infty \int L_2[u(x,t,\varepsilon), v(x,t,\varepsilon)]\varphi(x,t)\,dx\,dt &= 0,
\end{aligned}$$

coincide with the left-hand side of (1.12).

In this paper we only consider the problem of propagation of δ-shock waves and, consequently, the graph Γ contains only one arc. Suppose this arc has the form $\Gamma = \{(x,t) : x = \phi(t)\}$, and hence $e(x,t)\Big|_{\Gamma} = e(t)$.

Now we will describe the scheme of the our technique.

a. According to the *weak asymptotics method*, we must seek a δ-*shock wave type solution* in the form of the *singular ansatz*

(1.15)
$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t)H(-x+\phi(t)), \\ v(x,t) &= v_0(x,t) + v_1(x,t)H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{aligned}$$

which *corresponds* to the structure of initial data (1.10). Here $u_k(x,t)$, $v_k(x,t)$, $k=0,1$, $e(t)$, $\phi(t)$ are the desired functions.

b. In the framework of our approach, we construct a *weak asymptotic solution* in the form of the *smooth ansatz*:

$$\begin{aligned} u(x,t,\varepsilon) &= \tilde{u}(x,t,\varepsilon) + R_u(x,t,\varepsilon), \\ v(x,t,\varepsilon) &= \tilde{v}(x,t,\varepsilon) + R_v(x,t,\varepsilon), \end{aligned}$$

where a pair of functions $(\tilde{u}(x,t,\varepsilon), \tilde{v}(x,t,\varepsilon))$ is a *regularization* of the singular ansatz (1.15) *with respect to singularities* $H(-x+\phi(t))$, $\delta(-x+\phi(t))$, and the so-called *corrections* $R_u(x,t,\varepsilon)$, $R_v(x,t,\varepsilon)$ are functions which must admit the estimates:

$$R_j(x,t,\varepsilon) = o_{\mathcal{D}'}(1), \quad \frac{\partial R_j(x,t,\varepsilon)}{\partial t} = o_{\mathcal{D}'}(1), \qquad \varepsilon \to +0. \tag{1.16}$$

$j = u, v$. Thus, we must seek a *weak asymptotic solution* in the following form:

$$\begin{aligned} u(x,t,\varepsilon) &= u_0(x,t) + u_1(x,t)H_u(-x+\phi(t),\varepsilon) \\ &\qquad + R_u(x,t,\varepsilon), \\ v(x,t,\varepsilon) &= v_0(x,t) + v_1(x,t)H_v\big(-x+\phi(t),\varepsilon\big) \\ &\qquad + e(t)\delta_v\big(-x+\phi(t),\varepsilon\big) + R_v(x,t,\varepsilon), \end{aligned} \tag{1.17}$$

where $u_k(x,t)$, $v_k(x,t)$, $k=0,1$, $e(t)$, $\phi(t)$, $R_u(x,t,\varepsilon)$, $R_v(x,t,\varepsilon)$ are the desired functions,

$$\delta_v(x,\varepsilon) = \varepsilon^{-1}\omega_\delta\big(x/\varepsilon\big) \tag{1.18}$$

is a regularization of the δ-function,

$$H_j(x,\varepsilon) = \omega_{0j}\Big(\frac{x}{\varepsilon}\Big) = \int_{-\infty}^{x/\varepsilon} \omega_j(\eta)\,d\eta, \quad j = u, v, \tag{1.19}$$

are regularizations of the Heaviside function $H(x)$. The mollifiers $\omega_u(\eta)$, $\omega_v(\eta)$, $\omega_\delta(\eta)$ have the following properties: (a) $\omega(\eta) \in C^\infty(\mathbb{R})$, (b) $\omega(\eta)$ has a compact support or decreases sufficiently rapidly as $|\eta| \to \infty$, (c) $\int \omega(\eta)\,d\eta = 1$, (d) $\omega(\eta) \ge 0$, (e) $\omega(-\eta) = \omega(\eta)$. It is clear that $\omega_{0j}(\eta) \in C^\infty(\mathbb{R})$, $\lim_{\eta\to+\infty} \omega_{0j}(\eta) = 1$, $\lim_{\eta\to-\infty} \omega_{0j}(\eta) = 0$, $j = u, v$.

In order to construct a regularization $f(x, \varepsilon)$ of the distribution $f(x) \in \mathcal{D}'(\mathbb{R})$ we use the representation

$$f(x, \varepsilon) = f(x) * \frac{1}{\varepsilon}\omega\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0,$$

where $*$ is a convolution, and $\omega(\eta)$ is a mollifier.

Since the *generalized δ-shock wave type solution* (1.14) is defined as a weak limit of (1.17), in view of the estimates (1.16), the corrections *do not make a contribution* to the generalized solution of the problem. Otherwise, setting *corrections* equal to zero, i.e., without introducing these terms, we cannot solve the Cauchy problem with an arbitrary initial data (see Remark 2.6 below). It is clear that we can construct the *weak asymptotic solution*, using the *correction* of a different structure. Note, that *choosing the corrections* is an *essential* part of the “right” construction of the *weak asymptotic solution*.

A *weak asymptotic solution* of the Cauchy problem (1.9), (1.10) is constructed in Theorem 2.1. If $e^0 = 0$, and the initial data is piecewise constant, our results about a *weak asymptotic solution* of system (1.8) coincide with the main statements of [**17**] (see Corollary 2.5 and Remark 2.6). In particular, the Rankine–Hugoniot deficit $\dot{e}(t) = \frac{[u^3]}{3} - [u] - [v]\frac{[u^2]-[v]}{[u]}$ is positive. Note that in [**17**] a particular case of the approximate solution (1.17), (2.1) of the Cauchy problem (1.8), (1.10) with piecewise constant initial data was constructed.

c. Using the *weak asymptotic solution*, in Theorem 2.2 we construct a *generalized δ-shock wave type solution* (1.15) of the Cauchy problem (1.9), (1.10) as the weak limit of (1.17). The system (2.4) describes the dynamics of singularity and defines the smooth functions $u_k(x,t)$, $v_k(x,t)$, $k = 0, 1$, $e(t)$, $\phi(t)$. Theorem 2.3 gives a *generalized δ-shock wave type solution* (1.15) of the Cauchy problem (1.8), (1.10).

Remark 1.5. Using a *weak asymptotic solution* (1.17), constructed in Theorem 2.1, and (2.12), (2.13), (2.5), we obtain the following relations

$$\begin{aligned} f\big(u(x,t,\varepsilon)\big) - v(x,t,\varepsilon) \\ = f(u_0) - v_0 + \Big[f(u) - v\Big]H(-x+\phi(t)) + o_{\mathcal{D}'}(1), \qquad (1.20) \\ g\big(u(x,t,\varepsilon)\big) = g(u_0) + \Big[g(u)\Big]H(-x+\phi(t)) \end{aligned}$$

$$+e(t)\frac{\big[f(u)\big]}{[u]}\delta(-x+\phi(t))+o_{\mathcal{D}'}(1). \quad \varepsilon\to+0. \tag{1.21}$$

In the framework of the *weak asymptotics method* by (1.20), (1.21), in fact, we define the *superposition of the Heaviside function and the delta function*. In the background of formulas (1.20), (1.21) there is the *construction of multiplication of distributions*. We can introduce the *"right" singular superpositions* by the following definition:

$$\begin{aligned} f\big(u(x,t)\big)-v(x,t) &\stackrel{def}{=} \lim_{\varepsilon\to+0}\Big(f\big(u(x,t,\varepsilon)\big)-v(x,t,\varepsilon)\Big) \\ &= f(u_0)-v_0+\Big[f(u)-v\Big]H(-x+\phi(t)), \\ g\big(u(x,t)\big) &\stackrel{def}{=} \lim_{\varepsilon\to+0}\Big(g\big(u(x,t,\varepsilon)\big)\Big) \\ &= g(u_0)+\Big[g(u)\Big]H(-x+\phi(t))+e(t)\frac{\big[f(u)\big]}{[u]}\delta(-x+\phi(t)), \end{aligned}$$

where distributions $u(x,t)$, $v(x,t)$ are defined in (1.15) and the limits are understood in the weak sense. It is clear that, in general, the weak limits of $f\big(u(x,t,\varepsilon)\big)-v(x,t,\varepsilon)$ and $g\big(u(x,t,\varepsilon)\big)$ *depend* on the regularization of the Heaviside function and delta function. But the above *unique "right" singular superpositions* can be obtained *only by the construction of a weak asymptotic solution*. In this paper we omit the algebraic aspects of our technique which is given in detail in [**2**], [**3**], [**27**].

By substituting *"right" singular superpositions* of $f\big(u(x,t)\big)-v(x,t)$ and $g\big(u(x,t)\big)$ into system (1.9), Theorem 2.2 can be proved directly.

In Section 3.1,2. the geometric and physics sense of the Rankine–Hugoniot conditions for δ-shocks for systems (1.9), (1.6) and (1.7) is considered. Suppose that the flux functions of (1.1) are normalized so that $F(0,0)=0$, $G(0,0)=0$. It is well known that if a pair of compactly supported functions $(u(x,t),v(x,t))\in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}^2\big)$ with respect to x is a generalized solution of system (1.1) then integrals of the solution on the whole space

$$\begin{aligned} \int_{-\infty}^{+\infty} u(x,t)\,dx &= \int_{-\infty}^{+\infty} u^0(x)\,dx, \\ \int_{-\infty}^{+\infty} v(x,t)\,dx &= \int_{-\infty}^{+\infty} v^0(x)\,dx, \quad t\ge 0, \end{aligned} \tag{1.22}$$

(that is, the total area, mass, momentum, energy, etc.) are independent of time, where $(u^0(x),v^0(x))$ is initial data.

For δ-shock wave type solution this fact does not hold. However, there is a "generalized" analog of conservation laws (1.22). According to Theorems 3.1, 3.5, if a pair of distributions (u, v) is compactly supported generalized δ-shock wave type solution of systems (1.9) or (1.6) then the integral

$$\int_{-\infty}^{+\infty} u(x,t)\,dx = \int_{-\infty}^{+\infty} u^0(x)\,dx,$$

and the sum

$$\int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx + e(t)$$

$$= \int_{-\infty}^{0} v^0(x)\,dx + \int_{0}^{+\infty} v^0(x)\,dx + e^0 \tag{1.23}$$

are independent of time, where $\Gamma = \{(x,t) : x = \phi(t)\}$ is the discontinuity line. Here

$$\begin{array}{rcl} S_1(t) &=& \int_{-\infty}^{+\infty} u(x,t)\,dx, \\ S_2(t) &=& \int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx \end{array}$$

are the *areas* under the graphs $y = u(x,t)$, $y = v(x,t)$, respectively. From formula (1.23), we can see that the sense of amplitude $e(t)$ of δ function is the *"area"* of the discontinuity line. Moreover, the *"total area"* $S_2(t) + e(t)$ is independent of time. Thus, for the *Rankine–Hugoniot deficit* we have

$$\dot{e}(t) = -\dot{S}_2(t).$$

According to Theorem 3.6, if (u, v) is compactly supported generalized δ-shock wave type solution of system "zero-pressure gas dynamics" (1.7), we have

$$\begin{array}{rcl} m(t) + e(t) &=& \text{const}, \\ p(t) + e(t)\dot{\phi}(t) &=& \text{const}. \end{array} \tag{1.24}$$

Since v is the density, u is the velocity,

$$m(t) = \int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx$$

is the *mass* and

$$p(t) = \int_{-\infty}^{\phi(t)} u(x,t)v(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)v(x,t)\,dx$$

is the *momentum*. From formula (1.24), we can see that the sense of amplitude $e(t)$ of δ function is the *"mass"* of discontinuity line and the sense of the term $e(t)\dot{\phi}(t)$ is the *"momentum"* of discontinuity line. Moreover, the *"total mass"* $m(t) + e(t)$ and the *"total momentum"*

$p(t)+e(t)\dot{\phi}(t)$ are independent of time. Thus, for the left-hand sides of the Rankine–Hugoniot conditions for δ-shocks, i.e., the fifth and sixth equations of system (3.9), we have

$$\begin{aligned} \dot{e}(t) &= -\dot{m}(t), \\ \frac{d\big(e(t)\dot{\phi}(t)\big)}{dt} &= -\dot{p}(t). \end{aligned}$$

According to (3.14), for a special form of the initial data, the discontinuity line $x=\phi(t)$ moves at the velocity

$$\dot{\phi}(t)=\frac{p(t)}{m(t)}$$

i.e., in such a way as if the total mass were concentrated at the point $x=\phi(t)$. Thus the point $x=\phi(t)$ can be in a sense considered as the system barycenter.

The model of "zero-pressure gas dynamics" cab be described at a discrete level by a finite collection of particles. In view of (3.8) and (3.9) the *Rankine–Hugoniot deficit* $\dot{e}(t)$ is located between $[u]v_+$ and $[u]v_-$, where $[u]=u_- - u_+$ is a jump in function $u(x,t)$ across the discontinuity curve $x=\phi(t)$. That is, $\dot{e}(t)>0$. It means that the particles stick more and more as the time increases, i.e., there is a *concentration process* on the discontinuity curve $x=\phi(t)$. Thus, at collision the colliding particles get stuck together and form a new massive particle at the point of the system barycenter $x=\phi(t)$.

In Section 3.3 the geometric aspect of the process of δ-shock formation from sufficiently smooth compactly supported initial data is considered. Namely, the construction for the position of a δ-shock in a breaking wave is given.

2. Construction of δ-shock wave type solutions

1. Let us consider the propagation of a single δ-shock wave of system (1.9), i.e., consider the Cauchy problem (1.9), (1.10). The first step is to find a *weak asymptotic solution* of the problem.

Here we choose the *corrections* in the special form

$$\begin{aligned} R_u(x,t,\varepsilon) &= P(t)\tfrac{1}{\varepsilon^{1/n}}\Omega_P\Big(\tfrac{-x+\phi(t)}{\varepsilon}\Big) \\ &\qquad +Q(t)\tfrac{1}{\varepsilon^{1/(n+1)}}\Omega_Q\Big(\tfrac{-x+\phi(t)}{\varepsilon}\Big), \\ R_v(x,t,\varepsilon) &= 0, \end{aligned} \tag{2.1}$$

where $P(t)$, $Q(t)$ are continuously differentiable functions for all $t>0$, $\frac{1}{\varepsilon}\Omega_P^n\big(x/\varepsilon\big)$, $\frac{1}{\varepsilon}\Omega_Q^{n+1}\big(x/\varepsilon\big)$ are regularizations (1.18) of the delta function, mollifiers $\Omega_P(\eta)$, $\Omega_Q(\eta)$ have properties (a)–(c).

It is clear that estimates (1.16) hold.

Moreover, we can choose mollifiers $\Omega_P(\eta)$, $\Omega_Q(\eta)$ such that

$$\begin{aligned} \int \Omega_P^k(\eta)\Omega_Q^{n+1-k}(\eta)\,d\eta &= 0, \quad k=1,2,\dots n+1, \\ \int \Omega_Q^{n+1}(\eta)\,d\eta &\neq 0, \\ \int \Omega_P^{n}(\eta)\,d\eta &\neq 0. \end{aligned} \tag{2.2}$$

If $f(u)=u^2$, $g(u)=\frac{1}{3}u^3-u$ relation (2.2) has the form

$$\int \Omega_P^3(\eta)\,d\eta = 0, \quad \int \Omega_P^2(\eta)\Omega_Q(\eta)\,d\eta = 0, \quad \int \Omega_P(\eta)\Omega_Q^2(\eta)\,d\eta = 0.$$

In this case, for example, we can choose $\Omega_P(\eta) = \eta e^{-\eta^2}$, $\Omega_Q(\eta) = \big(1-2\eta^2\big)e^{-\eta^2}$.

THEOREM 2.1. *Let*

$$\lambda_+(u_0^0(0)) \le \frac{[f(u^0)]-[v^0]}{[u^0]}\bigg|_{x=0} \le \lambda_-(u_0^0(0)+u_1^0(0)), \tag{2.3}$$

then there exists $T>0$ such that, for $t\in[0,\,T)$, the Cauchy problem (1.9), (1.10) *has a weak asymptotic solution* (1.17), (2.1), (2.2) *if and only if*

$$\begin{aligned} L_{11}[u_+,v_+] &= 0, \quad x>\phi(t), \\ L_{11}[u_-,v_-] &= 0, \quad x<\phi(t), \\ L_{12}[u_+,v_+] &= 0, \quad x>\phi(t), \\ L_{12}[u_-,v_-] &= 0, \quad x<\phi(t), \\ \dot\phi(t) &= \tfrac{[f(u)]-[v]}{[u]}, \\ \dot e(t) &= [g(u)]-[v]\tfrac{[f(u)]-[v]}{[u]}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} P(t) &= \left(\tfrac{e(t)}{aA_n}\right)^{1/n}, \\ Q(t) &= \bigg\{\tfrac{e(t)}{cB_{n+1}}\bigg(\tfrac{[f(u)]-[v]}{[u]} - \tfrac{1}{A_n}\Big(B_n + \\ &\qquad (n+1)B_{n+1}\big(u_0+\tfrac{b}{a}u_1\big)\Big|_{x=\phi(t)}\Big)\bigg)\bigg\}^{1/(n+1)}, \end{aligned} \tag{2.5}$$

where $u_+=u_0$, $v_+=v_0$, $u_-=u_0+u_1$, $v_-=v_0+v_1$,

$$\big[h(u(x,t),v(x,t))\big] = \Big(h\big(u_-(x,t),v_-(x,t)\big) - h\big(u_+(x,t),v_+(x,t)\big)\Big)\Big|_{x=\phi(t)}$$

is a jump in function $h(u(x,t),v(x,t))$ *across the discontinuity curve* $x=\phi(t)$*,*

$$
\begin{array}{rcl}
a &=& \int \Omega_P^n(\eta)\,d\eta > 0, \\
b &=& \int \omega_{0u}(\eta)\Omega_P^n(\eta)\,d\eta, \\
c &=& \int \Omega_Q^{n+1}(\eta)\,d\eta \neq 0.
\end{array}
\tag{2.6}
$$

The initial data for system (2.4), (2.5) *are defined from* (1.10), *and*

$$
\begin{array}{rcl}
e(0) &=& e^0, \\
P(0) &=& \Big(\frac{e^0}{aA_n}\Big)^{1/n}, \\
Q(0) &=& \Big\{\frac{e^0}{cB_{n+1}}\Big(\frac{[f(u)]-[v]}{[u]} - \frac{1}{A_n}\Big(B_n \\
&& +(n+1)\big(u_0+\frac{b}{a}u_1\big)B_{n+1}\Big)\Big)\Big\}^{1/(n+1)}\Big|_{x=0}.
\end{array}
$$

PROOF. In order to find a *weak asymptotic solution* of the Cauchy problem (1.9), (1.10) we need to construct the weak asymptotics of some products of regularizations of distributions.

Obviously,

$$
\big(H(x,\varepsilon)\big)^r = H(x) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0, \qquad r=1,2,\dots. \tag{2.7}
$$

Let $\delta_k(x,\varepsilon) = \frac{1}{\varepsilon}\omega_k\Big(\frac{x}{\varepsilon}\Big)$, $k=1,2$ be regularizations (1.18) of the delta function. Since $\omega_1(\eta)\omega_2^r(\eta)$ decreases sufficiently rapidly as $|\eta| \to \infty$, making the change of variables $x=\varepsilon\eta$, we obtain

$$
\begin{aligned}
J(\varepsilon) &= \Big\langle \frac{1}{\varepsilon}\omega_1\Big(\frac{x}{\varepsilon}\Big)\Big(\omega_2\Big(\frac{x}{\varepsilon}\Big)\Big)^r, \psi(x)\Big\rangle \\
&= \int \omega_1(\eta)\omega_2^r(\eta)\psi(\varepsilon\eta)\,d\eta = A_r\psi(0) + O(\varepsilon), \quad \varepsilon \to +0,
\end{aligned}
$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R})$, i.e.,

$$
\delta_1(x,\varepsilon)\Big(\omega_2\Big(\frac{x}{\varepsilon}\Big)\Big)^r = A_r\delta(x) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0, \tag{2.8}
$$

where $A_r = \int \omega_1(\eta)\omega_2^r(\eta)\,d\eta$, $r=1,2,\dots$.

Let $H(x,\varepsilon) = \omega_0\big(\frac{x}{\varepsilon}\big) = \int_{-\infty}^{\frac{x}{\varepsilon}} \tilde{\omega}(\eta)\,d\eta$ be regularization (1.19) of the Heaviside function $H(x)$ and $\delta(x,\varepsilon) = \frac{1}{\varepsilon}\omega\big(\frac{x}{\varepsilon}\big)$ be regularization (1.18) of the delta function. Making the change of variables $x=\varepsilon\eta$, we obtain

$$
\begin{aligned}
J(\varepsilon) &= \Big\langle \frac{1}{\varepsilon}\omega\big(\frac{x}{\varepsilon}\big)\Big(\omega_0\Big(\frac{x}{\varepsilon}\Big)\Big)^r, \psi(x)\Big\rangle \\
&= \int \omega_0^r(\eta)\omega(\eta)\psi(\varepsilon\eta)\,d\eta = B_r\psi(0) + O(\varepsilon), \quad \varepsilon \to +0,
\end{aligned}
$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R})$, i.e.,

$$\delta(x,\varepsilon)\Big(H(x,\varepsilon)\Big)^r = B_r\delta(x) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0, \tag{2.9}$$

where $B_r = \int \omega_0^r(\eta)\omega(\eta)\,d\eta$, $r = 1, 2, \dots$.

Using (2.2), (2.6), (2.8), (2.9), we find the weak asymptotics

$$\begin{aligned} R^k(x,t,\varepsilon) &= o_{\mathcal{D}'}(1), \quad k \le n-1, \\ R^n(x,t,\varepsilon) &= aP^n(t)\delta(-x+\phi(t)) \\ &\qquad + o_{\mathcal{D}'}(1), \\ R^{n+1}(x,t,\varepsilon) &= cQ^{n+1}(t)\delta(-x+\phi(t)) \\ &\qquad + o_{\mathcal{D}'}(1), \\ H(-x+\phi(t),\varepsilon)R^n(x,t,\varepsilon) &= bP^n(t)\delta(-x+\phi(t)) \\ &\qquad + o_{\mathcal{D}'}(1), \end{aligned} \tag{2.10}$$

where a, b, c are defined by (2.6).

Using (2.7)–(2.9), one can calculate

$$\begin{aligned} \big(u(x,t,\varepsilon)\big)^k &= u_0^k + \big((u_0+u_1)^k - u_0^k\big)H(-x+\phi(t)) \\ &\qquad + o_{\mathcal{D}'}(1), \quad k \le n-1, \\ \big(u(x,t,\varepsilon)\big)^n &= u_0^n + \big((u_0+u_1)^n - u_0^n\big)H(-x+\phi(t)) \\ &\qquad + R^n(x,t,\varepsilon) + o_{\mathcal{D}'}(1), \\ \big(u(x,t,\varepsilon)\big)^{n+1} &= u_0^{n+1} \\ &\quad + \big((u_0+u_1)^{n+1} - u_0^{n+1}\big)H(-x+\phi(t)) \\ &\qquad + (n+1)\big(u_0 + u_1H(-x+\phi(t),\varepsilon)\big) \\ &\quad \times R^n(x,t,\varepsilon) + R^{n+1}(x,t,\varepsilon) + o_{\mathcal{D}'}(1). \end{aligned} \tag{2.11}$$

In particular, we have

$$\begin{aligned} \big(u(x,t,\varepsilon)\big)^2 &= u_0^2 + \Big((u_0+u_1)^2 - u_0^2\Big)H(-x+\phi(t)) \\ &\qquad + aP^2(t)\delta(-x+\phi(t)) + o_{\mathcal{D}'}(1), \\ \big(u(x,t,\varepsilon)\big)^3 &= u_0^3 + \Big((u_0+u_1)^3 - u_0^3\Big)H(-x+\phi(t)) \\ &\quad + \Big((3au_0 + 3bu_1)P^2(t) + cQ^3(t)\Big)\delta(-x+\phi(t)) \\ &\qquad + o_{\mathcal{D}'}(1), \quad \varepsilon \to +0. \end{aligned}$$

Taking into account relations (2.10), (2.11), we obtain the following weak asymptotics

$$\begin{aligned} f\big(u(x,t,\varepsilon)\big) &= f(u_0) + \Big(f(u_0+u_1) - f(u_0)\Big)H(-x+\phi(t)) \\ &\qquad + aA_nP^n(t)\delta(-x+\phi(t)) + o_{\mathcal{D}'}(1), \\ g\big(u(x,t,\varepsilon)\big) &= g(u_0) + \Big(g(u_0+u_1) - g(u_0)\Big)H(-x+\phi(t)) \\ &\quad + \Big\{aB_nP^n(t) + (n+1)\big(au_0 + bu_1\big)B_{n+1}P^n(t) \end{aligned} \tag{2.12}$$

$$+cB_{n+1}Q^{n+1}(t)\Big\}\delta(-x+\phi(t))+o_{\mathcal{D}'}(1),\quad \varepsilon\to+0. \tag{2.13}$$

Substituting the smooth ansatz (1.17) and (2.12), (2.13) into the left-hand side of system (1.9), we obtain, up to $o_{\mathcal{D}'}(1)$, the following relations

$$\begin{aligned}
L_{11}[u(x,t,\varepsilon),v(x,t,\varepsilon)] &= L_{11}[u_0,v_0]\\
&+\Big\{\frac{\partial u_1}{\partial t}+\frac{\partial}{\partial x}\big[f(u)-v\big]\Big\}H(-x+\phi(t))\\
&+\Big\{[u]\dot\phi(t)-\big[f(u)-v\big]\Big\}\delta(-x+\phi(t))\\
&+\Big\{e(t)-aA_nP^n(t)\Big\}\delta'(-x+\phi(t))+o_{\mathcal{D}'}(1),
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
L_{12}[u(x,t,\varepsilon),v(x,t,\varepsilon)] &= L_{22}[u_0,v_0]\\
&+\Big\{\frac{\partial v_1}{\partial t}+\frac{\partial}{\partial x}\Big[g(u)\Big]\Big\}H(-x+\phi(t))\\
&=\Big\{[v]\dot\phi(t)+\dot e(t)-\Big[g(u)\Big]\Big\}\delta(-x+\phi(t))\\
+\Big\{e(t)\dot\phi(t)-aB_nP^n(t)&-(n+1)\big(au_0+bu_1\big)B_{n+1}P^n(t)\\
-cB_{n+1}Q^{n+1}(t)\Big\}&\delta'(-x+\phi(t))+o_{\mathcal{D}'}(1),\quad \varepsilon\to+0.
\end{aligned} \tag{2.15}$$

Here we take into account estimates (1.16).

Setting the left-hand side of (2.14), (2.15) equal to zero, we obtain the necessary and sufficient conditions for the first two equalities (1.13), i.e., systems (2.4), (2.5).

Consider the Cauchy problem

$$\begin{array}{rclll}
L_{11}[u,V] &=& 0, & u(x,0)=u^0(x),\\
L_{12}[u,V] &=& 0, & V(x,0)=V^0(x)=v_0^0(x)+v_1^0(x)H(-x),
\end{array} \tag{2.16}$$

assuming that condition (2.3) holds. The last condition means that $(u^0(x),V^0(x))$ is entropy initial data.

According to [**19**, Ch.4.2.], we extend a pair of functions $(u_+^0(x)=u_0^0(x),V_+^0(x)=v_0^0(x))$ $((u_-^0(x)=u_0^0(x)+u_1^0(x),V_-^0(x)=v_0^0(x)+v_1^0(x)))$ to $x\le 0$ $(x\ge 0)$ in a bounded C^1 fashion and continue to denote the extended functions by $(u_\pm^0(x),V_\pm^0(x))$. By $(u_\pm(x,t),V_\pm(x,t))$ we denote the C^1 solutions of the problems

$$\begin{array}{rclcrcl}
L_{11}[u,V] &=& 0, & & u_\pm(x,0) &=& u_\pm^0(x),\\
L_{12}[u,V] &=& 0, & & V_\pm(x,0) &=& V_\pm^0(x),
\end{array}$$

which, according to [**19**, Ch.2.1.], [**25**, Ch.I,§8.], exist for small enough time interval $[0,\ T_1]$. The pair $(u_\pm(x,t), V_\pm(x,t))$ determine a two-sheeted covering of the plane (x,t). Next, we define the function $x = \phi(t)$ as a solution of the problem

$$\dot{\phi}(t) = \frac{f(u_-(x,t)) - f(u_+(x,t)) - V_-(x,t)) + V_+(x,t)}{u_-(x,t) - u_+(x,t)}\Big|_{x=\phi(t)},$$

$\phi(0) = 0$. It is clear that there exists a unique function $\phi(t)$ for sufficiently short times $[0,\ T_2]$. To this end, for $T = \min(T_1, T_2)$ we define the shock solution by

$$(u(x,t), V(x,t)) = \begin{cases} (u_+(x,t), V_+(x,t)), & x > \phi(t), \\ (u_-(x,t), V_-(x,t)), & x < \phi(t). \end{cases}$$

Thus the first five equations of system (2.4) define a unique solution of the Cauchy problem (2.16) for $t \in [0,\ T)$. Solving this problem, we obtain $u(x,t)$, $V(x,t)$, $\phi(t)$.

Then, substituting these functions into (2.4), (2.5), we obtain $e(t)$, $v(x,t) = V(x,t) + e(t)\delta(-x + \phi(t))$, and $P(t)$, $Q(t)$. It is clear that mollifiers $\Omega_P(\eta)$, $\Omega_Q(\eta)$ can be chosen such that relations (2.2) hold. □

2. Using the *weak asymptotic solution* constructed by Theorem 2.1 we obtain a generalized solution of the Cauchy problem (1.9), (1.10).

THEOREM 2.2. *There exists $T > 0$ given by Theorem 2.1 such that the Cauchy problem (1.9), (1.10), (2.3) for $t \in [0, T)$ has a unique generalized solution*

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t)H(-x + \phi(t)), \\ v(x,t) &= v_0(x,t) + v_1(x,t)H(-x + \phi(t)) + e(t)\delta(-x + \phi(t)), \end{aligned}$$

which satisfies the integral identities (1.12):

$$\begin{aligned} \int_0^T \int \Big(u\varphi_t + \big(f(u) - V\big)\varphi_x\Big)\, dx\, dt \qquad & \\ + \int u^0(x)\varphi(x,0)\, dx &= 0, \\ \int_0^T \int \Big(V\varphi_t + g(u)\varphi_x\Big)\, dx\, dt + \int V^0(x)\varphi(x,0)\, dx \qquad & \\ + \int_\Gamma e(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\, dl + e^0\varphi(0,0) &= 0, \end{aligned} \tag{2.17}$$

where $\Gamma = \{(x,t) : x = \phi(t), \quad t \in [0,\ T)\}$, and

$$\int_\Gamma e(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\, dl = \int_0^T e(t)\Big(\varphi_t(\phi(t),t) + \dot{\phi}(t)\varphi_x(\phi(t),t)\Big)\, dt,$$

$V(x,t) = v_0(x,t)+v_1(x,t)H(-x+\phi(t))$ and functions $u_k(x,t)$, $v_k(x,t)$, $\phi(t)$, $e(t)$ are defined by system (2.4).

PROOF. By Theorem 2.1 we have the following estimates:

$$L_{11}[u(x,t,\varepsilon)] = o_{\mathcal{D}'}(\varepsilon), \quad L_{12}[u(x,t,\varepsilon), v(x,t,\varepsilon)] = o_{\mathcal{D}'}(\varepsilon).$$

Let us apply the left-hand and right-hand sides of these relations to an arbitrary test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,T))$. Since for $\varepsilon > 0$ the functions $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ are smooth, integrating by parts, we obtain

$$\int_0^T \int \Big(u(x,t,\varepsilon)\varphi_t(x,t) + \Big(f(u(x,t,\varepsilon)) - v(x,t,\varepsilon)\Big)\varphi_x(x,t)\Big)\,dxdt + \int u(x,0,\varepsilon)\varphi(x,0)\,dx = o(1),$$

$$\int_0^T \int \Big(v(x,t,\varepsilon)\varphi_t(x,t) + g(u(x,t,\varepsilon))\varphi_x(x,t)\Big)\,dxdt + \int v(x,0,\varepsilon)\varphi(x,0)\,dx = o(1), \quad \varepsilon\to+0.$$

Passing to the limit as $\varepsilon \to +0$ and taking into account (1.17), (1.16), (1.20), (1.21), and the fact that

$$\lim_{\varepsilon\to+0}\int_0^T\int_{-\infty}^{\infty} e(t)\delta_v\big(-x+\phi(t),\varepsilon\big)\varphi(x,t)\,dxdt = \int_0^T e(t)\varphi(\phi(t),t)\,dt,$$

$$\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} e(0)\delta_v\big(-x,\varepsilon\big)\varphi(x,0)\,dx = e(0)\varphi(0,0),$$

we obtain the integral identities (2.17).

In view of the above remark system (2.4) has a unique solution. □

The fifth and sixth equations of systems (2.4) are the Rankine–Hugoniot conditions of δ-shocks. Here the right-hand side of the fifth equation is the so-called *Rankine–Hugoniot deficit*:

$$\dot{e}(t) = [g(u)] - [v]\frac{[f(u)]-[v]}{[u]}.$$

If $A_n > 0$, $e^0 \ge 0$, according to (2.5), the amplitude $e(t)$ of δ-function is positive.

In particular, for system (1.8) we have the following result.

THEOREM 2.3. *There exists $T > 0$ given by Theorem 2.1 such that the Cauchy problem (1.8), (1.10),*

$$u_0^0(0) + 1 \le \frac{[(u^0)^2] - [v^0]}{[u^0]}\bigg|_{x=0} \le u_0^0(0) + u_1^0(0) - 1, \tag{2.18}$$

for $t \in [0, T)$ has a unique generalized solution

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t)H(-x+\phi(t)), \\ v(x,t) &= v_0(x,t) + v_1(x,t)H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{aligned}$$

which satisfies the integral identities (2.17), where $f(u) = u^2$, $g(u) = \frac{1}{3}u^3 - u$, and functions $u_k(x,t)$, $v_k(x,t)$, $\phi(t)$, $e(t)$ are defined by system (2.4).

Let the initial data (1.10) be piecewise constant, i.e $u_0^0 = u_0$, $u_1^0 = u_1$, $v_0^0 = v_0$, $v_1^0 = v_1$. Then from Theorems 2.2, 2.3 we have the following corollaries.

COROLLARY 2.4. *For $t \in [0, \infty)$, the Cauchy problem (1.9), (1.10), (2.3), with piecewise constant initial data has a unique generalized solution*

$$\begin{aligned} u(x,t) &= u_0 + u_1 H(-x+\phi(t)), \\ v(x,t) &= v_0 + v_1 H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{aligned}$$

where

$$\begin{aligned} \phi(t) &= \tfrac{[f(u)]-[v]}{[u]}t, \\ e(t) &= e^0 + \Big([g(u)] - [u] - [v]\tfrac{[u^2]-[v]}{[u]}\Big)t. \end{aligned}$$

COROLLARY 2.5. *For $t \in [0, \infty)$, the Cauchy problem (1.8), (1.10), (2.18), with piecewise constants initial data has a unique generalized solution*

$$\begin{aligned} u(x,t) &= u_0 + u_1 H(-x+\phi(t)), \\ v(x,t) &= v_0 + v_1 H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{aligned}$$

where

$$\begin{aligned} \phi(t) &= \tfrac{[u^2]-[v]}{[u]}t, \\ e(t) &= e^0 + \Big(\tfrac{[u^3]}{3} - [u] - [v]\tfrac{[u^2]-[v]}{[u]}\Big)t. \end{aligned}$$

Moreover, if $e^0 = 0$, the Rankine–Hugoniot deficit is positive:

$$\dot{e}(t) = \frac{[u^3]}{3} - [u] - [v]\frac{[u^2]-[v]}{[u]} > 0$$

(as in [**17**]).

Here $\dot{e}(t) > 0$ according to the seventh equation (2.5).

REMARK 2.6. To find a generalized solution of the Cauchy problem (1.9), (1.10) we construct a weak asymptotic solution of problem (1.17), where the functions $\phi(t)$, $e(t)$, u_k, v_k, $k = 0, 1$ are determined by Theorem 2.2 and the functions $\omega_{0u}(\eta)$, $\Omega_P(\eta)$, $\Omega_Q(\eta)$, $P(t)$, $Q(t)$ are determined by relations (2.2), (2.5), (2.6). In view of estimate (1.16) (see also (1.20), (1.21)), the generalized solution (1.15) of the Cauchy problem *does not depend* on relations (2.2), (2.5).

Without introducing the terms

$$P(t)\varepsilon^{-1/n}\Omega_P\Big(\frac{-x+\phi(t)}{\varepsilon}\Big), \quad Q(t)\varepsilon^{-1/(n+1)}\Omega_Q\Big(\frac{-x+\phi(t)}{\varepsilon}\Big),$$

according to (2.5), we cannot solve the Cauchy problem, which admits δ-shocks. If we introduce only the first term, we cannot solve the Cauchy problem with an *arbitrary initial value* (1.10), but *only* for initial values determined by the relation

$$\frac{[f(u)]-[v]}{[u]} = \frac{1}{A_n}\bigg(B_n + (n+1)\Big(u_0 + \frac{b}{a}u_1\Big)B_{n+1}\bigg), \tag{2.19}$$

where the constants a, b are defined by (2.6).

In [**17**], in the framework of the Colombeau theory, in order to construct an approximate δ-shock solution for system (1.8) only a term of the type

$$P(t)\varepsilon^{-1/2}\Omega_P\Big(\frac{-x+\phi(t)}{\varepsilon}\Big)$$

is introduced. In this case relation (2.19) has the following form

$$\frac{u_0+u_1-\frac{v_1}{u_1}}{u_1} = \frac{b}{a},$$

where $a = \int \Omega_P^2(\eta)\,d\eta$, $b = \int \omega_{0u}(\eta)\Omega_P^2(\eta)\,d\eta$. This relation can be rewritten as

$$\frac{u_0-\frac{v_1}{u_1}}{u_1} = \frac{\dot\phi(t)-u_-}{u_1} = \frac{b-a}{a}, \tag{2.20}$$

where $u_- = u_0 + u_1$. In [**17**] the parameter $a = \int \Omega_P^2(\eta)\,d\eta$ was set to be 1. Hence (see (1.19))

$$\frac{b-a}{a} = \int \Big(\omega_{0u}(\eta)-1\Big)\Omega_P^2(\eta)\,d\eta < 1.$$

Here relation (2.20) coincides with the second relation [**17**, Proposition 2] and the last inequality coincides with the statement of [**17**, Lemma 1]. However in this case relation (2.20) still leaves one degree of freedom, to connect $u_- = u_0 + u_1$ and $u_+ = u_0$ (see [**17**, Proposition 2]).

3. Geometric and physics sense of the Rankine–Hugoniot conditions

1. Suppose that the flux functions of system (1.8) are normalized so that

$$f(0) = 0, \qquad g(0) = 0. \tag{3.1}$$

Let a pair of distributions $(u(x,t), v(x,t))$ be the generalized δ-shock wave type solution of system (1.8), where $v(x,t) = V(x,t) + e(t)\delta(\Gamma)$, $\Gamma = \{(x,t) : x = \phi(t)\}$ is the discontinuity line, $u(x,t)$, $V(x,t)$ are compactly supported functions with respect to x. Denote by

$$\begin{array}{rcl} S_1(t) & = & \int_{-\infty}^{\phi(t)} u(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)\,dx, \\ S_2(t) & = & \int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx, \\ S_3(t) & = & \int_{-\infty}^{\phi(t)} u(x,t)v(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)v(x,t)\,dx, \\ S_1(0) & = & \int_{-\infty}^{0} u^0(x)\,dx + \int_{0}^{+\infty} u^0(x)\,dx, \\ S_2(0) & = & \int_{-\infty}^{0} V^0(x)\,dx + \int_{0}^{+\infty} V^0(x)\,dx, \\ S_3(0) & = & \int_{-\infty}^{0} u^0(x)V^0(x)\,dx + \int_{0}^{+\infty} u^0(x)V^0(x)\,dx, \end{array}$$

the areas under the graphs $y = u(x,t)$, $y = v(x,t)$, $y = u(x,t)v(x,t)$, and $y = u^0(x)$, $y = v^0(x)$, $y = u^0(x)v^0(x)$, respectively.

THEOREM 3.1. *Let the pair of distributions $(u(x,t), v(x,t))$ be a generalized δ-shock wave type solution of the Cauchy problem* (1.9), (1.10), *where $u(x,t)$, $V(x,t)$ are compactly supported functions with respect to x. Assume that condition* (3.1) *is satisfied. Then*

$$\begin{array}{rcl} \dot{S}_1(t) & = & 0, \\ \dot{S}_2(t) & = & -\dot{e}(t), \end{array} \tag{3.2}$$

where $\dot{e}(t) = [g(u)] - [v]\frac{[f(u)]-[v]}{[u]}$ is the Rankine–Hugoniot deficit, $t \in [0,\, T)$. Thus,

$$\begin{aligned} &\int_{-\infty}^{\phi(t)} u(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)\,dx \\ &\qquad = \int_{-\infty}^{0} u^0(x)\,dx + \int_{0}^{+\infty} u^0(x)\,dx, \\ &\int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx + e(t) \\ &\qquad = \int_{-\infty}^{0} V^0(x)\,dx + \int_{0}^{+\infty} V^0(x)\,dx + e^0. \end{aligned} \tag{3.3}$$

PROOF. Let us prove the second relation (3.2). We denote $v_{\pm} = \lim_{x\to\phi(t)\pm 0} v(x,t)$. Using the second equation of system (1.9), we obtain

$$\dot{S}_2(t) = v_-\dot{\phi}(t) - v_+\dot{\phi}(t) + \int_{-\infty}^{\phi(t)} v_t(x,t)\,dx + \int_{\phi(t)}^{+\infty} v_t(x,t)\,dx$$

$$= [v]\dot{\phi}(t) - \int_{-\infty}^{\phi(t)} \big(g(u(x,t))\big)_x\,dx - \int_{\phi(t)}^{+\infty} \big(g(u(x,t))\big)_x\,dx$$

$$= [v]\dot{\phi}(t) + g(u(-\infty,t)) - g(u(+\infty,t)) - [g(u)].$$

Taking into account that $g(u(-\infty,t)) = g(u(+\infty,t)) = g(0) = 0$ and using the expression for $\dot{\phi}(t)$, we have

$$\dot{S}_2(t) = [v]\frac{[f(u)] - [v]}{[u]} - [g(u)].$$

The first relation (3.2) is the well-known relation for scalar conservation law. The proof of this relation is carried out in the same way. Integrating expressions (3.2), we obtain (3.3). □

2. In the paper [**8**] of V. G. Danilov and V. M. Shelkovich, in the framework of Definition 1.2 a δ-shock wave type solution of the Cauchy problem (1.6), (1.10) was constructed. The eigenvalues of the characteristic matrix of system (1.6) are $\lambda_1(u) = f'(u)$, $\lambda_2(u) = g(u)$. We assume that

$$f''(u) > 0, \quad g'(u) > 0, \quad f'(u) \le g(u) \tag{3.4}$$

and the "overcompression" conditions

$$\lambda_1(u_+) \le \lambda_2(u_+) \le \dot{\phi}(t) \le \lambda_1(u_-) \le \lambda_2(u_-).$$

are satisfied.

In [**7**], [**8**] the following theorem was proved.

THEOREM 3.2. ([**7**], [**8**]) *Suppose that* $u_1^0(0) > 0$ *and conditions* (3.4) *hold. Then there exists* $T > 0$ *such that, for* $t \in [0, T)$, *the Cauchy problem* (1.6), (1.10), *has a unique generalized solution*

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t)H(-x+\phi(t)),\\ v(x,t) &= v_0(x,t) + v_1(x,t)H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{aligned}$$

which satisfies the integral identities (1.12), *where* $F(u,v) = f(u)$, $G(u,v) = vg(u)$, *and functions* $u_0 = u_+$, $v_0 = v_+$, $u_0 + u_1 = u_-$,

$v_0 + v_1 = v_-$, $\phi(t)$, $e(t)$ *are defined by the system:*

$$
\begin{array}{rcl}
L_{11}[u_+] &=& 0, \quad x > \phi(t), \\
L_{11}[u_-] &=& 0, \quad x < \phi(t), \\
L_{12}[u_+, v_+] &=& 0, \quad x > \phi(t), \\
L_{12}[u_-, v_-] &=& 0, \quad x < \phi(t), \\
\dot{\phi}(t) &=& \frac{[f(u)]}{[u]}, \\
\dot{e}(t) &=& [vg(u)] - [v]\frac{[f(u)]}{[u]}.
\end{array} \tag{3.5}
$$

The initial data for system (3.5) *are defined from* (1.10).

In [**8**] the Cauchy problem for the system of zero-pressure gas dynamics (1.7) was also solved. The initial data for system (1.7) is the following (see [**8**, Remark 1.1.])

$$
\begin{array}{rcl}
u^0(x) &=& u_0^0(x) + u_1^0(x)H(-x), \\
v^0(x) &=& v_0^0(x) + v_1^0(x)H(-x) + e^0\delta(-x). \\
\dot{\phi}(t)\big|_{t=0} &=& \phi^1,
\end{array} \tag{3.6}
$$

where ϕ^1 is given constant and $u_1^0(0) > 0$. Thus, in addition to the initial data (1.10) we add the *initial velocity* $\dot{\phi}(0)$ to the initial data for system (1.7).

Now we introduce the definition of a *δ-shock wave type solution* for systems (1.7) from [**8**]. Suppose that arcs of the graph $\Gamma = \{\gamma_i : i \in I\}$ have the form $\gamma_i = \{(x,t) : x = \phi_i(t)\}$, $i \in I$.

Definition 3.3. ([**8**]) A pair of distributions $(u(x,t), v(x,t))$ and graph Γ from Definition 1.2 is called a *generalized δ-shock wave type solution* of system (1.7) with the initial data $(u^0(x), v^0(x); \dot{\phi}_i(0), i \in I_0)$ if the integral identities

$$
\begin{array}{rcl}
\displaystyle\int_0^\infty \int \Big(V\varphi_t + uV\varphi_x\Big)\,dx\,dt + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl + \int V^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} e_k^0\varphi(x_k^0,0) &=& 0, \\
\displaystyle\int_0^\infty \int \Big(uV\varphi_t + u^2V\varphi_x\Big)\,dx\,dt + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\dot{\phi}_i(t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl + \int u^0(x)V^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} e_k^0\dot{\phi}_k(0)\varphi(x_k^0,0) &=& 0,
\end{array} \tag{3.7}
$$

hold for all $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0,\,\infty))$.

System (1.7) has a double eigenvalue $\lambda_1(u) = \lambda_2(u) = u$. In this case the entropy condition is

$$u_+ \le \dot\phi(t) \le u_-. \tag{3.8}$$

In [**8**] the following theorem was proved.

THEOREM 3.4. ([**8**]) *There exists $T > 0$ such that the Cauchy problem* (1.7), (3.6) *for $t \in [0, T)$ has a unique generalized solution*

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t)H(-x+\phi(t)), \\ v(x,t) &= v_0(x,t) + v_1(x,t)H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{aligned}$$

which satisfies the integral identities (3.7), *where functions $u_0 = u_+$, $v_0 = v_+$, $u_0 + u_1 = u_-$, $v_0 + v_1 = v_-$, $\phi(t)$, $e(t)$ are defined by the system*

$$\begin{aligned} L_{31}[u_+, v_+] &= 0, \quad x > \phi(t), \\ L_{31}[u_-, v_-] &= 0, \quad x < \phi(t), \\ L_{32}[u_+, v_+] &= 0, \quad x > \phi(t), \\ L_{32}[u_-, v_-] &= 0, \quad x < \phi(t), \\ \dot e(t) &= [uv] - [v]\dot\phi(t), \\ \frac{d\big(e(t)\dot\phi(t)\big)}{dt} &= [u^2 v] - [uv]\dot\phi(t), \end{aligned} \tag{3.9}$$

and initial data are defined from (3.6).

The fifth and sixth equations of system (3.5), (3.9) are the *Rankine–Hugoniot conditions of δ-shocks.*

Using Theorems 3.2, 3.4 from [**8**], we prove the following analogs of Theorem 3.1.

THEOREM 3.5. *Let the pair of distributions $(u(x,t), v(x,t))$ be a generalized δ-shock wave type solution of the Cauchy problem* (1.6), (1.10), *where $u(x,t)$, $V(x,t)$ are compactly supported functions with respect to x. Assume that condition* (3.1) *is satisfied. Then*

$$\begin{aligned} \dot S_1(t) &= 0, \\ \dot S_2(t) &= -\dot e(t), \end{aligned} \tag{3.10}$$

where $\dot{e}(t) = [vg(u)] - [v]\frac{[f(u)]}{[u]}$ *is the Rankine–Hugoniot deficit,* $t \in [0, T)$*. Thus,*

$$
\begin{aligned}
&\int_{-\infty}^{\phi(t)} u(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)\,dx \\
&\qquad = \int_{-\infty}^{0} u^0(x)\,dx + \int_{0}^{+\infty} u^0(x)\,dx, \\
&\int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx + e(t) \\
&\qquad = \int_{-\infty}^{0} V^0(x)\,dx + \int_{0}^{+\infty} V^0(x)\,dx + e^0.
\end{aligned}
\tag{3.11}
$$

In order to prove this theorem, we use system (3.5) and the same calculations as those carried out above. We omit them here.

We remind that for the system of "zero-pressure gas dynamics" $v(x,t)$ is density and $u(x,t)$ is velocity. Hence, the area $S_2(t) = m(t)$ is mass and the area $S_3(t) = p(t)$ is momentum.

THEOREM 3.6. *Let the pair of distributions* $(u(x,t), v(x,t))$ *be a generalized* δ*-shock wave type solution of the Cauchy problem* (1.7)*,* (3.6)*,* $u(x,t)$*,* $V(x,t)$ *are compactly supported functions with respect to* x*. Then*

$$
\begin{aligned}
\dot{m}(t) &= -\dot{e}(t), \\
\dot{p}(t) &= -\frac{d\big(e(t)\dot{\phi}(t)\big)}{dt},
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
\dot{e}(t) &= [uv] - [v]\dot{\phi}(t), \\
\frac{d\big(e(t)\dot{\phi}(t)\big)}{dt} &= [u^2 v] - [uv]\dot{\phi}(t),
\end{aligned}
$$

$\dot{\phi}(t)$ *is the phase velocity,* $t \in [0, T)$*. Thus,*

$$
\begin{aligned}
m(t) + e(t) &= m(0) + e^0, \\
p(t) + e(t)\dot{\phi}(t) &= p(0) + e^0\phi^1,
\end{aligned}
\tag{3.13}
$$

where $m(0) = S_2(0)$*,* $p(0) = S_3(0)$ *are initial mass and momentum respectively. Moreover, if we choose the initial data such that*

$$
m_0 = -e_0, \qquad p_0 = -e^0\phi^1,
$$

we have

$$
\dot{\phi}(t) = \frac{p(t)}{m(t)}. \tag{3.14}
$$

PROOF. The proof of the first relation (3.12) is based on the same calculations as the proof of the second relation (3.2). Let us prove the second relation (3.12).

Using the second equation of system (1.7), we obtain

$$\dot{p}(t) = \big[uv\big]\dot{\phi}(t) + \int_{-\infty}^{\phi(t)} (uv)_t\,dx + \int_{\phi(t)}^{+\infty} (uv)_t\,dx$$

$$= [uv]\dot{\phi}(t) - \int_{-\infty}^{\phi(t)} \big(vu^2\big)_x\,dx - \int_{\phi(t)}^{+\infty} \big(vu^2\big)_x\,dx$$

$$= [v]\dot{\phi}(t) - [vu^2] + \Big(vu^2\Big)(-\infty,t) - \Big(vu^2\Big)(+\infty,t).$$

In view of the sixth equation of (3.9), we have

$$\dot{p}(t) = -\frac{d\big(e(t)\dot{\phi}(t)\big)}{dt}.$$

Integrating (3.12), we obtain (3.13) and (3.14). □

3. Consider the geometric aspect of δ-shock formation from sufficiently smooth compactly supported initial data $(u^0(x), v^0(x))$ (here $u_1^0(x) = v_1^0(x) = e^0 = 0$) for systems (1.9) and (1.6). In a similar way, the geometric aspect of δ-shock wave formation for system (1.7) can be considered.

It is well known that the solution u and v must become *multivalued* at finite time. Any multivalued part of the wave profile must be replaced by an appropriate discontinuity. Construction for the position of δ-shock in a breaking wave will be given below.

Let $t = t^*$ be the time of δ-shock formation. Then, according to (1.22), (1.23), (for $t = t^*$) the *correct* initial positions for δ-shock discontinuities in u and v are such that these discontinuities must cut off lobes of equal area, as on Fig. 1..

If $t > t^*$, the *correct* initial positions for δ-shock discontinuities in u and v are such that the discontinuity in u must cut off lobes of equal area $B_u(t) = A_u(t)$ (see Fig. 1.), while the discontinuity in v must cut off lobes whose areas satisfy the following relation $B_v(t) = A_v(t) + e(t)$ (see Fig. 2.), where $A_u(t)$, $A_v(t)$ are the areas of the lobes to the left of discontinuity, $B_u(t)$, $B_v(t)$ are the areas of the lobes to the right of discontinuity. Note, that at the time $t = t^*$ of δ-shock wave formation the area, mass, momentum are *continuous* functions with respect to t but their derivatives have the *jumps*.

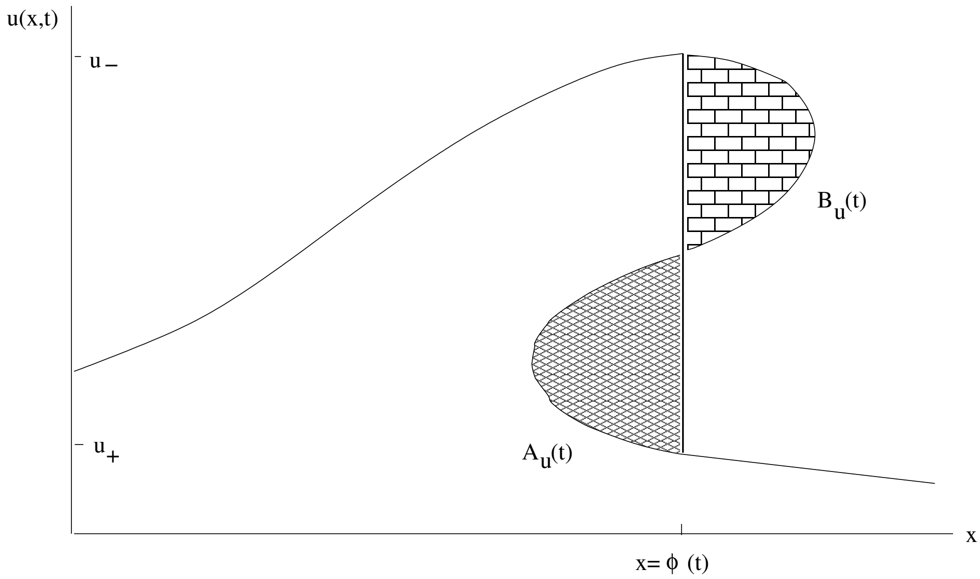

Fig. 1. Equal area construction for the position of the delta−shock in a breaking wave u(x,t).

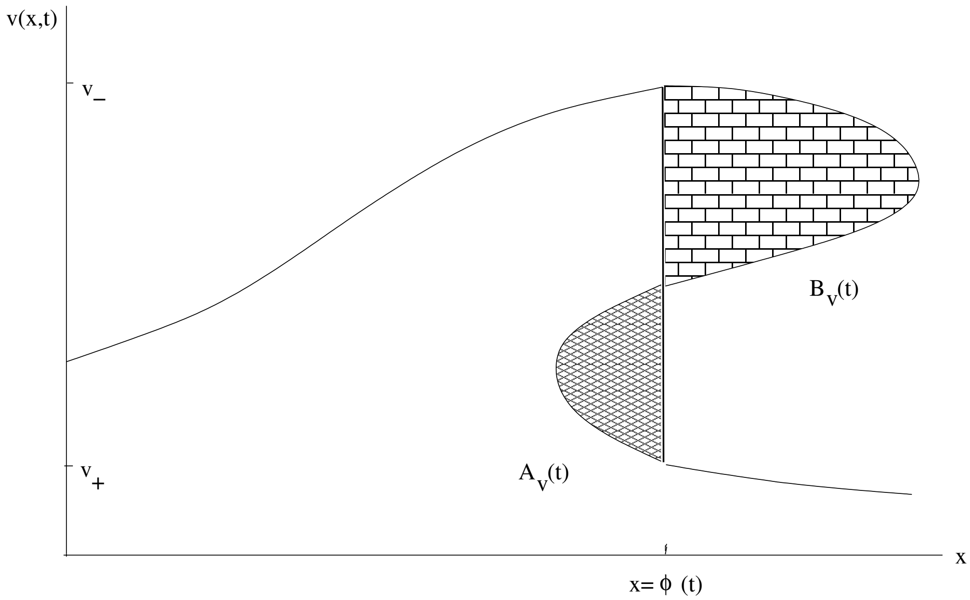

Fig. 2. Nonequal area construction for the position of the delta−shock in a breaking wave v(x,t).

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