On the Connection between some Riemann-Solver Free Approaches to the Approximation of Multi-Dimensional Systems of Hyperbolic Conservation Laws

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Abstract

In this paper, we present some interesting connections between a number of Riemann-solver free approaches to the numerical solution of multi-dimensional systems of conservation laws. As a main part, we present a new and elementary derivation of Fey's Method of Transport (MoT) (respectively the second author's ICE version of the scheme) and the state decompositions which form the basis of it. The only tools that we use are quadrature rules applied to the moment integral used in the gas kinetic derivation of the Euler equations from the Boltzmann equation, to the integration in time along characteristics and to space integrals occuring in the finite volume formulation. Thus, we establish a connection between the *MoT approach* and the *kinetic approach*. Further more, Ostkamp's equivalence result between her Evolution Galerkin scheme and the Method of Transport is lifted up from the level of discretizations to the level of exact evolution operators, introducing a new connection between the MoT and the *Evolution Galerkin approach*. At the same time, we clarify some important differences between these two approaches.

Key words: systems of conservation laws, Fey's Method of Transport, Euler equations, Boltzmann equation, kinetic schemes, bicharacteristic theory, state decompositions, flux decompositions, exact and approximate evolution operators, quadrature rules
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Contents

1 Introduction

Many physical problems, for example the behaviour of a compressible fluid, can be modelled as systems of hyperbolic conservation laws,

$$\partial_t \boldsymbol{U} + \nabla_{\underline{x}} \cdot \underline{\boldsymbol{F}}(\boldsymbol{U}) = 0, \tag{1.1}$$

if certain effects (for instance viscosity) are neglected. Here, $\underline{x} \in \mathbb{R}^d$, $\underline{\boldsymbol{F}} : \Omega \to \mathbb{R}^{m \times d}$, and $\boldsymbol{U} : \mathbb{R}^d \times [0, \infty) \to \Omega \subset \mathbb{R}^m$. Most classical numerical methods for systems of hyperbolic conservation laws are based on solving one-dimensional Riemann problems. Schemes for multi-dimensional systems are then obtained by either performing a dimensional splitting or by a finite volume approach. See for example the textbooks of LeVeque [18], Kröner [17], Godlewski and Raviart [12], and Toro [32].

Since about 15 years, however, there is an ongoing discussion (see for example Roe et al [29,7]) whether one-dimensional Riemann solvers do justice to the multi-dimensional effects arising in such systems. As discussed e. g. in the introduction of [23], there were a number of approaches which therefore purposely dispensed with Riemann solvers. The following three of them form the main subject of the current paper:

- the Method of Transport (MoT), originally developed by Fey [8–10] and later modified by Noelle [23],
- the Evolution Galerkin (EG) approach of Butler [3], Morton et al [19] (exploiting the transport collapse operator of Brenier [2]), Ostkamp [24,25], Lukáčová, Morton, Warnecke [20] as well as (based on this) the Finite Volume Evolution Galerkin (FVEG) approach of Lukáčová, Morton, Saibertová and Warnecke [22,21], and
- the kinetic approach of Deshpande [6] and Perthame [26,27].

In this paper, we present some (new, as we hope) connections between these approaches. As a main part of this paper, this includes a new derivation of the second author's [23] version of the MoT, called MoT-ICE, and the state decompositions (called wave models in [23]) which form the basis of it. The only tools used in this derivation are quadrature rules (some of them classical, some of them new), applied to:

- the moment integral used in the gas kinetic derivation of the Euler equations from the Boltzmann equation,
- the integration in time along characteristics, and
- space integrals occuring in the finite volume formulation.

We would like to spend some more words on the first item, because the other two quadratures are standard in most numerical schemes.

It is a well-known fact that the Euler equations of gas dynamics (which we will basically consider) are just a first order approximation of Boltzmann's equation, see for example Cercignani [4]. The idea to construct a numerical scheme based on the kinetic formulation of gas dynamics is not new, see e. g. Deshpande [6] and Perthame [26,27].

In the current paper, however, the kinetic theory is not utilized directly as the basis for a scheme. Rather, we will see that there is a certain class of quadratures of the moment integral all of which lead to decompositions (2.1) and (2.2) of the vector of conservative variables $\boldsymbol{U}$ and the flux matrix $\underline{\boldsymbol{F}}(\boldsymbol{U})$, respectively. These decompositions, which we call flux-consistent state decompositions (briefly flux decompositions), form the basis of the MoT. If would be interesting to see if flux decompositions could be constructed for general systems of conservation laws via the BGK models constructed recently by Bouchut [1].

The already mentioned EG and FVEG schemes by Ostkamp [24,25] and Lukáčová, Morton, Saibertová, Warnecke [20,22,21] are also derived from an exact evolution operator using quadrature rules. This evolution operator (which we will call the EG operator) is based on the classical characteristic theory, see for example Courant and Hilbert [5]. Ostkamp already showed that there is a close connection between her characteristic Galerkin scheme and Fey's [8] original version of the Method of Transport for the linearized, constant coefficient Euler equations. We are now able to lift this connection up from the level of schemes to the level of exact evolution operators. More precisely, we can find a canonical continuous state decomposition (derived from the classical characteristic theory) for which our exact evolution operator becomes identical to the EG operator for linear, constant coefficient equations.

Having established this connection, we would also like to point out an important difference between the EG approach and the MoT: The state decomposition leading to the EG formulation is not flux-consistent, i. e. it is not a flux decomposition. This shows that the two schemes which Ostkamp [24,25] was able to identify are derived from rather different sources. In fact, Ostkamp's proof relies on a number of rather restrictive assumptions which seem to be tailored to the equations of gas dynamics.

All important connections within and between the mentioned approaches are schematically shown in the diagram in figure 1.1. As a side remark, we would like to mention that our general evolution operator, applied to a trivial state decomposition, leads to the standard integral form of conservation laws.

The unifying themes of the paper are state decompositions, flux decompositions and the corresponding exact and approximate evolution operators: In section 2, we define state and flux decompositions and give some simple examples. As a more involved example, we derive flux-consistent state decompositions from gas-kinetic theory in section 3. Especially we will see that Fey's [10] discrete flux decomposition can be obtained this way. In section 4, we derive exact and approximate evolution operators from state and flux decompositions, respectively. In section 5, we derive another more involved state decomposition (which is not flux-consistent) from bicharacteristic theory and use it to identify the EG operator with our general evolution operator (but not with the MoT-ICE).

Extensive numerical tests [16] show that the MoT-ICE is a very diffusive scheme and thus shares also experimental properties with the kinetic schemes. Therefore, the current version of the MoT-ICE seems to be more of theoretical than of practical interest.

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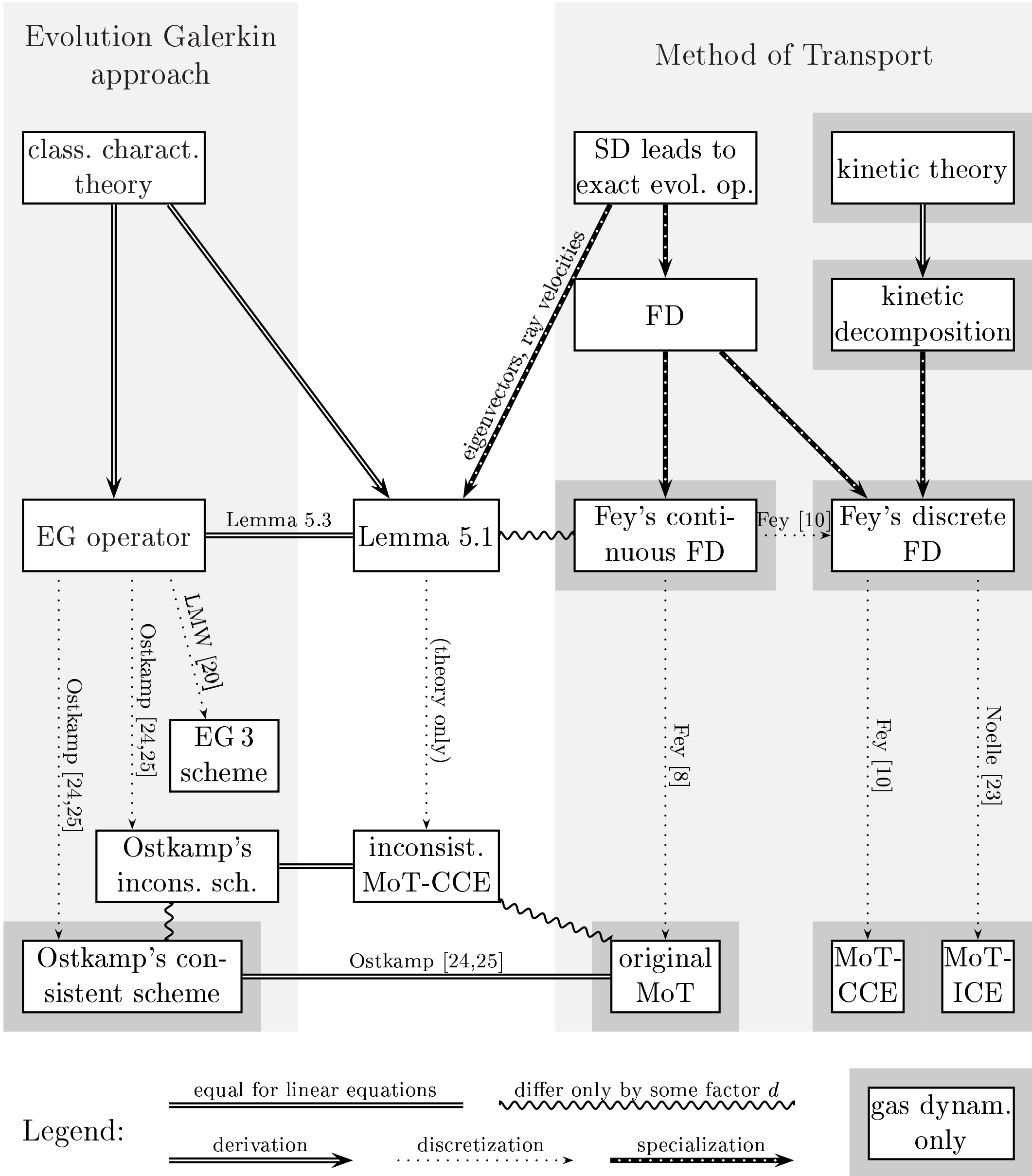

Fig. 1.1. Overview of state decompositions (SDs), flux decompositions (FDs), the classical bicharacteristic and the kinetic theories, the Evolution Galerkin (EG) schemes and Fey's Method of Transport (MoT).

2 Decompositions of hyperbolic systems

In this section, we will define state decompositions and flux-consistent state decomposition for arbitrary systems of hyperbolic conservation laws (1.1), and give some simple examples.

Definition 2.1 *Let $L \in \mathbb{N}$, and for every $l = 1, \ldots, L$ let $\boldsymbol{S}_l$ be some $\mathbb{R}^m$-valued and $\underline{a}_l$ some $\mathbb{R}^d$-valued continuously differentiable functions on the set*

Ω of physical states $\boldsymbol{U}$. The family $(\boldsymbol{S}_l, \underline{a}_l)_{l=1,\dots,L}$ is called a state decomposition for (1.1) *if*

$$\sum_{l=1}^{L} \boldsymbol{S}_l(\boldsymbol{U}) = \boldsymbol{U} \tag{2.1}$$

holds.

Definition 2.2 *A state decomposition is called flux-consistent if additionally*

$$\sum_{l=1}^{L} \boldsymbol{S}_l(\boldsymbol{U}) \cdot \underline{a}_l(\boldsymbol{U}) = \underline{\boldsymbol{F}}(\boldsymbol{U}) \tag{2.2}$$

holds.

A flux-consistent state decompositions will sometimes simply be called a flux decomposition. In section 5, we will also work with a *continuous state decomposition* where the sum over l is replaced with an integral.

We would like to emphasize that this is different from the Fluctuation Splitting approach of Deconinck, Roe and Struijs [7] where the divergence $\boldsymbol{\nabla} \cdot \underline{\boldsymbol{F}}(\boldsymbol{U})$ is decomposed rather than the flux matrix $\underline{\boldsymbol{F}}$ itself.

Example 2.3 *A very simple state decomposition for any system is given by $L = 1$, $\boldsymbol{S}_1 = \boldsymbol{U}$, $\underline{a}_1 = \underline{0}$. It is not flux-consistent except in the trivial case $\underline{\boldsymbol{F}} \equiv \underline{\boldsymbol{0}}$.*

In remark 4.3, we will recover the classical integral form of conservation laws from this example.

Example 2.4 *If we consider a one-dimensional system of conservation laws, i. e. $d = 1$, and further assume that $\boldsymbol{F}$ is homogeneous, i. e. $\boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{F}'(\boldsymbol{U})\cdot\boldsymbol{U}$, then we can set $L = m$, $\boldsymbol{S}_l = \boldsymbol{r}^l$ and $a_l = \lambda^l$ where $\boldsymbol{r}^l$ are the eigenvectors of $\boldsymbol{F}'(\boldsymbol{U})$ (normalized such that they sum up to $\boldsymbol{U}$) and λ^l are the corresponding eigenvalues. The resulting state decomposition is flux-consistent.*

This decomposition has been used by Steger and Warming in their well-known flux-vector splitting scheme [30].

Example 2.5 *The previous example can be generalized to the multi-dimensional case if one assumes the Jacobian matrices $\boldsymbol{F}_s$ to commute (i. e. to be simultaneously diagonalizable): If $\boldsymbol{r}^l$ are the common eigenvectors and λ_s^l is the corresponding l-th eigenvalue of the s-th Jacobian matrix $\boldsymbol{F}'_s(\boldsymbol{U})$, set $\boldsymbol{S}_l = r^l$ as before and $\underline{a}_l = (\lambda_1^l, \dots, \lambda_d^l)$. This again results in a flux-consistent state decomposition.*

For the general case in which the Jacobian matrices are not simultaneously diagonalizable, there does not seem to be such a simple mechanism to construct a flux-consistent state decomposition. In the next section, we will show that the kinetic theory quite naturally leads to a flux decomposition for Euler's equations of gas dynamics, and in section 5, we will present a continuous state decomposition—based on bicharacteristic theory—which is not flux-consistent.

3 Gaskinetic derivation of flux-consistent state decompositions

In this section, we will briefly recall well-known facts about gaskinetic theory, its connection to Euler's equations of gas dynamics and the main idea of the kinetic schemes based on it. Then, we will describe how quadrature rules quite naturally lead to flux decompositions for Euler's equations.

3.1 Euler's equations

For the Euler equations, the vector of conservative variables reads

$$\boldsymbol{U} = \begin{pmatrix} \rho \\ \underline{m}^{\mathrm{T}} \\ E \end{pmatrix}.$$

Here, ρ is the density, $\underline{m} = \rho \underline{u}$ is the momentum (denoted as a line vector so that $\underline{m}^{\mathrm{T}}$ is a column vector) and E is the total energy.

The flux matrix for Euler's equations then reads

$$\underline{\boldsymbol{F}}(\boldsymbol{U}) = \begin{pmatrix} \underline{m} \\ \underline{m}^{\mathrm{T}} \underline{u} + p\underline{\mathbf{1}} \\ (E+p)\underline{u} \end{pmatrix}$$

where p is the presure. We consider a polytropic gas with the following equation of state:

$$E = \frac{1}{2}\rho \underline{u}^2 + \frac{K+d}{4\lambda}\rho = \frac{1}{2}\rho \underline{u}^2 + \frac{p}{\gamma - 1}.$$

Here, K is the number of rotational degrees of freedom of a molecule (2 for oxygen for example), d is the space dimension (usually 3, even in the case that we consider solutions that only depend on one or two space dimensions), and λ is proportional to the inverse temperature and satisfies $1/2\lambda = p/\rho$. Finally, γ is the adiabatic coefficient and equals $1 + 2/(K + d)$.

Euler's equations now read

$$\partial_t \boldsymbol{U} + \nabla_{\underline{x}} \cdot \underline{\boldsymbol{F}}(\boldsymbol{U}) = 0. \tag{3.1}$$

3.2 Boltzmann's equation and kinetic schemes

Let $f = f(\underline{x}, t, \underline{v}, \xi)$ be the density of particles at position $\underline{x}$ and time t with speed $\underline{v}$ and rotational momentum ξ. Boltzmann's equation for the evolution of f is

$$\partial_t f + \underline{v} \cdot \nabla_{\underline{x}} f = Q(f, f), \tag{3.2}$$

where Q is a function which models the interactions of particles.

For any macroscopic state $\boldsymbol{U}$, there is a unique microscopic particle distribution $g = g(\boldsymbol{U}, \underline{v}, \xi)$ which models the given equilibrium state,

$$g(\boldsymbol{U}, \underline{v}, \xi) = \rho M_K^\lambda(\xi) M_d^\lambda(\underline{v} - \underline{u}), \tag{3.3}$$

where

$$M_K^\lambda(\xi) = \left(\frac{\lambda}{\pi}\right)^{K/2} \mathrm{e}^{-\lambda\xi^2} \qquad \text{and} \qquad M_d^\lambda(\underline{v} - \underline{u}) = \left(\frac{\lambda}{\pi}\right)^{d/2} \mathrm{e}^{-\lambda(\underline{v} - \underline{u})^2}$$

are Maxwell's equilibrium functions. If Q is given by elastic binary collisions, then $Q(g, g) = 0$, see Cercignani [4].

Lemma 3.1 *Let*

$$\boldsymbol{\psi}(\underline{v}, \xi) = \begin{pmatrix} 1 \\ \underline{v}^{\mathrm{T}} \\ \frac{1}{2}(\underline{v}^2 + \xi^2) \end{pmatrix}.$$

Then,

$$\boldsymbol{U} = \iint \boldsymbol{\psi} g \, \mathrm{d}\underline{v} \, \mathrm{d}\xi \qquad \textit{and} \qquad \underline{\boldsymbol{F}}(\boldsymbol{U}) = \iint \boldsymbol{\psi} \underline{v} g \, \mathrm{d}\underline{v} \, \mathrm{d}\xi. \tag{3.4}$$

The integrals in (3.4) are called moment integrals.

We omit the proof of the lemma, which is just a straight-forward computation.

Lemma 3.1 relates the moments of the particle distribution function f to the conservative variables $\boldsymbol{U}$ and the fluxes $\underline{\boldsymbol{F}}(\boldsymbol{U})$ of the Euler equations. Indeed, a classical Hilbert or Chapman–Enskog expansion (see for example Cercignani [4]) shows that the Euler equations are a first order approximation of Boltzmann's equation.

Deshpande [6] and Perthame [26,27] constructed kinetic schemes by considering (at the beginning of the time step) an initial particle distribution in equilibrium, carrying out a collision free transport for some finite time step, and finally projecting back the resulting particle distribution (which is not in equilibrium anymore) onto the corresponding equilibrium distribution (at the end of the time step). This may be interpreted as realizing all collisions at once. The resulting schemes have the advantage of preserving positivity of density and pressure, but are rather dissipative at contact discontinuities.

While Deshpande used the exact Maxwellian g, Perthame replaced it with an approximate Maxwellian $\hat{g}(\boldsymbol{U}, \underline{v}, \xi)$ which is chosen in a way such that (3.4) still holds, i. e.

$$\boldsymbol{U} = \iint \boldsymbol{\psi} \hat{g} \,\mathrm{d}\underline{v}\,\mathrm{d}\xi \qquad \text{and} \qquad \underline{\boldsymbol{F}}(\boldsymbol{U}) = \iint \boldsymbol{\psi} \underline{v} \hat{g} \,\mathrm{d}\underline{v}\,\mathrm{d}\xi. \tag{3.5}$$

3.3 Derivation of flux decompositions

For the present paper, the following remark is essential:

Remark 3.2 *Note that* (3.4) *respectively* (3.5) *may be interpreted as continuous flux decompositions, e. g. with parameter* $\underline{v} \in \mathbb{R}^d$ *by setting*

$$\boldsymbol{S}_{\underline{v}}(\boldsymbol{U}) = \int_{\mathbb{R}^K} \boldsymbol{\psi}(\underline{v}, \xi) g(\boldsymbol{U}, \underline{v}, \xi) \,\mathrm{d}\xi \qquad \textit{and} \qquad \underline{a}_{\underline{v}}(\boldsymbol{U}) = \underline{v}.$$

Since we do not care about the integral with respect to ξ, which we carry out exactly, we introduce the notation

$$\overline{\varphi}_{\boldsymbol{U}}(\underline{v}) := \int_{\mathbb{R}^K} \varphi(\underline{v}, \xi) \rho M_K^{\lambda}(\xi) \,\mathrm{d}\xi$$

for any (vector- or scalar-valued) function $\varphi(\underline{v}, \xi)$. Now we look at the weighted integral with respect to $\underline{v}$, which we abbreviate by

$$I_{\boldsymbol{U}}(\chi) := \int_{\mathbb{R}^d} \chi(\underline{v}) M_d^{\lambda}(\underline{v} - \underline{u}) \,\mathrm{d}\underline{v}. \tag{3.6}$$

Now, (3.4) can be rewritten as

$$\boldsymbol{U} = I_{\boldsymbol{U}}(\overline{\boldsymbol{\psi}}_{\boldsymbol{U}}), \qquad \underline{\boldsymbol{F}}(\boldsymbol{U}) = I_{\boldsymbol{U}}(\overline{\boldsymbol{\psi}}_{\boldsymbol{U}} \underline{v}). \tag{3.7}$$

The idea which will lead to our discrete flux decompositions is to replace the exact integral $I_{\boldsymbol{U}}$ with a quadrature rule $\hat{I}_{\boldsymbol{U}}$ in such a way that the consistency relations (3.7) are still satisfied.

Lemma 3.3 *Let $\hat{I}_{\boldsymbol{U}}$ be a quadrature rule for the integral $I_{\boldsymbol{U}}$. Then, the consistency relations*

$$\boldsymbol{U} = \hat{I}_{\boldsymbol{U}}(\overline{\boldsymbol{\psi}}_{\boldsymbol{U}}), \qquad\qquad \underline{\boldsymbol{F}}(\boldsymbol{U}) = \hat{I}_{\boldsymbol{U}}(\overline{\boldsymbol{\psi}}_{\boldsymbol{U}}\underline{v}) \tag{3.8}$$

are satisfied if and only if $\hat{I}_{\boldsymbol{U}}$ is exact for $\chi(\underline{v}) \in \{1, \underline{v}, \underline{v}^{\mathrm{T}}\underline{v}, \underline{v}|\underline{v}|^2\}$.

PROOF. We simply note that all of the functions $\boldsymbol{\psi}$ and $\boldsymbol{\psi}\underline{v}$ occuring in (3.4) are linear combinations of the polynomials $\varphi(\underline{v})$. (Note that $\underline{v}^{\mathrm{T}}\underline{v}$ is a $d \times d$ matrix.) $\square$

Corollary 3.4 *Equation* (3.8) *holds if any only if*

$$\begin{aligned}
\hat{I}_{\boldsymbol{U}}(1) &= 1, &(3.9a)\\
\hat{I}_{\boldsymbol{U}}(\underline{v}) &= \underline{u}, &(3.9b)\\
\hat{I}_{\boldsymbol{U}}(\underline{v}^{\mathrm{T}}\underline{v}) &= \underline{u}^{\mathrm{T}}\underline{u} + \frac{p}{\rho}\mathbf{1}, &(3.9c)\\
\hat{I}_{\boldsymbol{U}}(\underline{v}|\underline{v}|^2) &= \frac{\underline{u}p}{\rho}(d+2) + \underline{u}|\underline{u}|^2. &(3.9d)
\end{aligned}$$

Alternatively, equation (3.9d) *can be replaced with*

$$\hat{I}_{\boldsymbol{U}}((\underline{v} - \underline{u})|\underline{v}|^2) = \frac{2p}{\rho}\underline{u}. \tag{3.10}$$

PROOF. The right hand sides of (3.9) are just the results of $I_{\boldsymbol{U}}(\varphi(\underline{v}))$ for those functions φ given in the lemma. Equation (3.10) can easily be seen to be equivalent to (3.10) when (3.9a) and (3.9c) hold. $\square$

We now specialize to fully discrete decompositions.

Lemma 3.5 *Let*

$$\hat{I}_{\boldsymbol{U}}(\varphi) = \sum_{l=0}^{L} \hat{\omega}_l \varphi(\hat{\underline{u}}_l) \tag{3.11}$$

where $\hat{\omega}_l = \hat{\omega}_l(\boldsymbol{U})$ and $\hat{\underline{u}}_l = \hat{\underline{u}}_l(\boldsymbol{U})$ are called weights and nodes of the quadrature formula. Then, the expressions

$$\boldsymbol{S}_l(\boldsymbol{U}) = \hat{\omega}_l \overline{\boldsymbol{\psi}}_{\boldsymbol{U}}(\hat{\underline{u}}_l), \qquad\qquad \underline{a}_l = \hat{\underline{u}}_l$$

form a flux decomposition if and only if

$$\sum_{l=0}^{L} \hat{\omega}_l = 1, \tag{3.12a}$$

$$\sum_{l=0}^{L} \hat{\omega}_l \underline{\hat{u}}_l = \underline{u}, \tag{3.12b}$$

$$\sum_{l=0}^{L} \hat{\omega}_l \underline{\hat{u}}_l^{\mathrm{T}} \underline{\hat{u}}_l = \underline{u}^{\mathrm{T}} \underline{u} + \frac{p}{\rho} \underline{\mathbf{1}}, \tag{3.12c}$$

$$\sum_{l=0}^{L} \hat{\omega}_l \underline{\hat{u}}_l^2 \underline{\hat{u}}_l = \frac{up}{\rho}(d+2) + \underline{u}^2 \underline{u}. \tag{3.12d}$$

Again, equation (3.12d) *can be replaced with*

$$\sum_{l=0}^{L} \hat{\omega}_l (\underline{\hat{u}}_l \underline{\hat{u}}_l^{\mathrm{T}})(\underline{\hat{u}}_l - \underline{u}) = \frac{2p}{\rho} \underline{u}. \tag{3.13}$$

PROOF. We have that

$$\sum_{l=0}^{L} \boldsymbol{S}_l(\boldsymbol{U}) = \sum_{l=0}^{L} \hat{\omega}_l \overline{\boldsymbol{\psi}}_{\boldsymbol{U}}(\underline{\hat{u}}_l) = \hat{I}_{\boldsymbol{U}}(\overline{\boldsymbol{\psi}}_{\boldsymbol{U}})$$

and

$$\sum_{l=0}^{L} \boldsymbol{S}_l(\boldsymbol{U}) \underline{a}_l = \sum_{l=0}^{L} \hat{\omega}_l \overline{\boldsymbol{\psi}}_{\boldsymbol{U}}(\underline{\hat{u}}_l) \underline{\hat{u}}_l = \hat{I}_{\boldsymbol{U}}(\overline{\boldsymbol{\psi}}_{\boldsymbol{U}} \underline{v}).$$

Thus, the defining equtions (2.1) and (2.2) of a flux decomposition hold if and only if (3.8) holds. The assertion thus follows from inserting (3.11) into corollary 3.4. □

In our examples below, we always choose one component $\boldsymbol{S}_0$ of the flux decomposition traveling with the macroscopic velocity $\underline{u}$ (roughly corresponding to the entropy wave). The other components discretize the mach cone.

Lemma 3.6 *Let $L \in \mathbb{N}$ be fixed and $\alpha \in \mathbb{R}$ arbitrary. For $l = 1, \dots, L$ let $\underline{n}_l \in \mathbb{R}^d$ be any vectors of length $\sqrt{d}$ which satisfy*

$$\sum_{l=1}^{L} \underline{n}_l = \underline{0}, \qquad \sum_{l=1}^{L} \underline{n}_l^{\mathrm{T}} \underline{n}_l = L\underline{\mathbf{1}}.$$

Then, the following choice of $\underline{\hat{u}}_l$ *and* $\hat{\omega}_l$ *satisfies condition* (3.5)*:*

$$\begin{aligned}
&\hat{\omega}_0 = 1-\alpha, && \hat{\omega}_l = \frac{\alpha}{L}, && l = 1,\dots,L,\\
&\underline{\hat{u}}_0 = \underline{u}, && \underline{\hat{u}}_l = \underline{u} + \frac{c\underline{n}_l}{\sqrt{\alpha\gamma}}, && l = 1,\dots,L,
\end{aligned}$$

where $c = \sqrt{\gamma p/\rho}$ *is the speed of sound.*

The proof just consists of checking conditions (3.12).

Example 3.7 ($\alpha = 1/\gamma$) *Let* $L = 2^d$ *and*

$$\begin{aligned}
&\hat{\omega}_0 = \frac{\gamma-1}{\gamma}, && \hat{\omega}_l = \frac{1}{L\gamma}, && l = 1,\dots,L,\\
&\underline{\hat{u}}_0 = \underline{u}, && \underline{\hat{u}}_l = \underline{u} + c\underline{n}_l, && l = 1,\dots,L, && \underline{n}_l \in \{(\pm1,\dots,\pm1)\}.
\end{aligned}$$

Example 3.8 ($\alpha = 1$) *Let* $L = 2^d$ *and*

$$\begin{aligned}
&\hat{\omega}_0 = 0, && \hat{\omega}_l = \frac{1}{L}, && l = 1,\dots,L,\\
&\underline{\hat{u}}_0 = \underline{u}, && \underline{\hat{u}}_l = \underline{u} + \frac{c\underline{n}_l}{\sqrt{\gamma}}, && l = 1,\dots,L, && \underline{n}_l \in \{(\pm1,\dots,\pm1)\}.
\end{aligned}$$

We will now introduce a slight generalization of our approach by which we can also derive Fey's [10] flux decomposition. The idea, which was already used by Zimmermann [34] and Perthame [27], is to split $\boldsymbol{\psi}$ into an interior and an exterior part,

$$\boldsymbol{\psi}(\underline{v},\xi) = \boldsymbol{\psi}^{\text{int}}(\xi) + \boldsymbol{\psi}^{\text{ext}}(\underline{v}) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}\xi^2 \end{pmatrix} + \begin{pmatrix} 1 \\ \underline{v}^{\mathrm{T}} \\ \frac{1}{2}\underline{v}^2 \end{pmatrix},$$

and then to use different quadrature formulas $\hat{I}_{\boldsymbol{U}}^{\text{int}}$ and $\hat{I}_{\boldsymbol{U}}^{\text{ext}}$ for integration of these two parts. The condition for the resulting expression $\boldsymbol{S}_l$ and $\underline{a}_l$ to form a flux decomposition is that (3.12a)–(3.12d) hold for $\hat{\omega}_l^{\text{ext}}$ and $\underline{\hat{u}}_l^{\text{ext}}$ whereas only (3.12a) and (3.12b) have to be satiesfied for $\hat{\omega}_l^{\text{int}}$ and $\underline{\hat{u}}_l^{\text{int}}$. This enables us to set $\underline{\hat{u}}_l^{\text{int}} = \underline{\hat{u}}_l^{\text{ext}}$ even when $\hat{\omega}_l^{\text{int}}$ and $\hat{\omega}_l^{\text{ext}}$ are obtained from lemma 3.6 with different values of α:

Example 3.9 ($\alpha^{\text{int}} = 1$, $\alpha^{\text{ext}} = 1/\gamma$) *Let* $L = 2^d$ *and*

$$\begin{aligned}
\hat{\omega}_0^{\text{int}} &= 0, & \hat{\omega}_l^{\text{int}} &= \frac{1}{L}, & l &= 1, \dots, L, \\
\hat{\omega}_0^{\text{ext}} &= \frac{\gamma - 1}{\gamma}, & \hat{\omega}_l^{\text{ext}} &= \frac{1}{L\gamma}, & l &= 1, \dots, L, \\
\underline{\hat{u}}_0^{\text{int,ext}} &= \underline{u}, & \underline{\hat{u}}_l^{\text{int,ext}} &= \underline{u} + c\underline{n}_l, & l &= 1, \dots, L, \quad \underline{n}_l \in \{(\pm 1, \dots, \pm 1)\}.
\end{aligned}$$

This example results in Fey's discrete state decomposition.

Remarks 3.10

(1) Using the same ideas, one can also derive partially or fully continuous flux decompositions. A physically motivated idea would be to have one component with speed $\underline{u}$ *and additionally an integral over a sphere around this point. However, as lemma 3.6 already suggests, the radius of the sphere would be by a factor of* $\sqrt{d}$ *larger than that of the Mach cone—or (compensating this by a different choice of* α*) the weight of the middle point would become unphysically (especially negative for* $d > \gamma$*). For details see Kröger [15].*

(2) The results of this section extend work of Zimmermann [33,34]. See also Junk [14] for related ideas.

4 Exact and approximate evolution operators for smooth solutions

We will now return to an arbitrary system of hyperbolic conservation laws and assume that a state decomposition is given. Based on the state decomposition, we will derive an exact evolution operator for the system. If further more the state decomposition is flux-consistent, we will discretize the exact evolution operator by the use of quadrature rules in time and space such that the resulting scheme is just the MoT-ICE.

If a system (1.1) and a state decomposition (2.1) are given, we can write

$$\partial_t \boldsymbol{S}_l(\boldsymbol{U}) + \boldsymbol{\nabla} \cdot (\boldsymbol{S}_l(\boldsymbol{U})\underline{a}_l) = -\boldsymbol{S}_l'(\boldsymbol{U})\boldsymbol{\nabla} \cdot \underline{\boldsymbol{F}}(\boldsymbol{U}) + \boldsymbol{\nabla} \cdot (\boldsymbol{S}_l(\boldsymbol{U})\underline{a}_l) =: \boldsymbol{T}_l(\underline{x}, t), \tag{4.1}$$

where

$$\underline{a}_l(\underline{x}, t) := \underline{a}_l(\boldsymbol{U}(\underline{x}, t)).$$

The representation (4.1) can be considered as an advection equation for each $\boldsymbol{S}_l(\boldsymbol{U})$ with a right hand side.

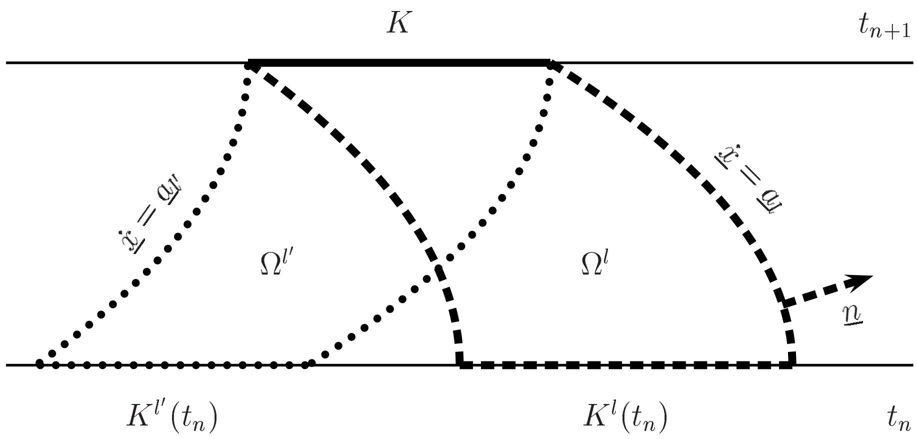

Fig. 4.1. The cell K is transported backwards in time along characteristics.

Lemma 4.1 *For a flux-consistent state decomposition, we have that* $\sum_l \boldsymbol{T}_l = 0$.

PROOF.

$$\sum_{l=1}^{L} \boldsymbol{T}_l = -\frac{\partial}{\partial \boldsymbol{U}} \sum_{l=1}^{L} \boldsymbol{S}_l(\boldsymbol{U}) \boldsymbol{\nabla} \cdot \underline{\boldsymbol{F}}(\boldsymbol{U}) + \boldsymbol{\nabla} \cdot \sum_{l=1}^{L} (\boldsymbol{S}_l(\boldsymbol{U}) \underline{a}_l)$$
$$= -\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{U}} \boldsymbol{\nabla} \cdot \underline{\boldsymbol{F}}(\boldsymbol{U}) + \boldsymbol{\nabla} \cdot \underline{\boldsymbol{F}}(\boldsymbol{U}) = 0.$$

□

Define the l-th characteristic $\xi^l_{\underline{x}}$ through the point $(\underline{x}, t_{n+1})$ by

$$\partial_\tau \xi^l_{\underline{x}}(\tau) = \underline{a}_l(\xi^l_{\underline{x}}(\tau), \tau), \qquad \xi^l_{\underline{x}}(t_{n+1}) = \underline{x}, \tag{4.2}$$

and for any $K \subset \mathbb{R}^d$ (with sufficiently smooth boundary) define

$$K^l(\tau) = \{\xi^l_{\underline{x}}(\tau) : \underline{x} \in K\}$$

and

$$\Omega^l = \{(\xi^l_{\underline{x}}(\tau), \tau) : \underline{x} \in K,\ t_n \le \tau \le t_{n+1}\} \subset \mathbb{R}^d \times \mathbb{R},$$

see figure 4.1. Then, integration of (4.1) over Ω^l yields

$$\int_{\Omega^l} \boldsymbol{T}_l(\underline{x}, \tau)\, \mathrm{d}\underline{x}\, \mathrm{d}\tau = \int_{\Omega^l} \Big(\partial_t \boldsymbol{S}_l(\boldsymbol{U}) + \boldsymbol{\nabla} \cdot (\boldsymbol{S}_l(\boldsymbol{U}) \underline{a}_l)\Big)\, \mathrm{d}\underline{x}\, \mathrm{d}\tau$$
$$= \int_{\partial\Omega^l} \boldsymbol{S}_l(\boldsymbol{U})(\underline{a}_l, 1) \cdot \underline{n}\, \mathrm{d}s.$$

The boundary $\partial\Omega^l$ of the tube Ω^l consists of the top $K \times \{t_{n+1}\}$, the bottom $K^l(t_n) \times \{t_n\}$ and the surface $S := \{(x, t) \in \partial\Omega^l : t_n < t < t_{n+1}\}$. At each point

$(\underline{x}, \tau) \in S$, we have that $\underline{n}$ is perpendicular to the characteristic $(\xi^l_{\underline{x},\tau}(\tau), \tau)$. Due to (4.2), this means that the integral over S vanishes. Thus, we have

$$\int_{\Omega^l} \boldsymbol{T}_l(\underline{x}, \tau)\,\mathrm{d}\underline{x}\,\mathrm{d}\tau = \int_K \boldsymbol{S}_l(\boldsymbol{U}(\underline{x}, t_{n+1}))\,\mathrm{d}\underline{x} - \int_{K^l(t_n)} \boldsymbol{S}_l(\boldsymbol{U}(\underline{x}, t_n))\,\mathrm{d}\underline{x}. \tag{4.3}$$

Summing over all l yields

$$\sum_{l=1}^{L} \int_{\Omega^l} \boldsymbol{T}_l(\underline{x}, \tau)\,\mathrm{d}\underline{x}\,\mathrm{d}\tau = \int_K \underbrace{\sum_{l=1}^{L} \boldsymbol{S}_l(\boldsymbol{U}(\underline{x}, t_{n+1}))}_{\boldsymbol{U}(\underline{x}, t_{n+1})}\,\mathrm{d}\underline{x} - \sum_{l=1}^{L} \int_{K^l(t_n)} \boldsymbol{S}_l(\boldsymbol{U}(\underline{x}, t_n))\,\mathrm{d}\underline{x}$$

and thus gives rise to the following representation of the solution $\boldsymbol{U}$:

Lemma 4.2 (Exact evolution operator) *Let $\boldsymbol{U}$ be a smooth solution of* (1.1) *and $(\boldsymbol{S}_l, \underline{a}_l)_l$ be a (not necessarily flux-consistent) state decomposition. Then,*

$$\int_K \boldsymbol{U}(\underline{x}, t_{n+1})\,\mathrm{d}\underline{x} = \sum_{l=1}^{L} \int_{K^l(t_n)} \boldsymbol{S}_l(\boldsymbol{U}(\underline{x}, t_n))\,\mathrm{d}\underline{x} + \sum_{l=1}^{L} \int_{\Omega^l} \boldsymbol{T}_l(\underline{x}, \tau)\,\mathrm{d}\underline{x}\,\mathrm{d}\tau. \tag{4.4}$$

□

Remark 4.3 *Choosing $L = 1$, $\boldsymbol{S}_1(\boldsymbol{U}) = \boldsymbol{U}$ and $\underline{a}_1(\boldsymbol{U}) = 0$ leads to the standard integral form of the conservation law,*

$$\int_K \boldsymbol{U}(\underline{x}, t_{n+1})\,\mathrm{d}\underline{x} = \int_K \boldsymbol{U}(\underline{x}, t_n)\,\mathrm{d}\underline{x} - \int_{t_n}^{t_{n+1}} \int_K \boldsymbol{\nabla} \cdot \underline{\boldsymbol{F}}(\boldsymbol{U})\,\mathrm{d}\underline{x}\,\mathrm{d}\tau,$$

which is the basis of classical finite volumne discretizations.

Note that this decomposition is not flux-consistent.

In section 5, we will show that lemma 4.2 also includes the EG operator, which is derived from the classical characteristic theory.

4.1 Approximate evolution operators

The exact evolution operator cannot be used in practice because it contains a time integral over $[t_n, t_{n+1}]$ and in a practical scheme one does not know anything about the solution at those intermediate time levels. Application of the trapezoidal rule to the time integral in the left hand side of (4.3) yields

$$\begin{aligned}\frac{\Delta t}{2} \int_K \boldsymbol{T}_l(\underline{x}, t_{n+1})\,\mathrm{d}\underline{x} + \frac{\Delta t}{2} \int_{K^l(t_n)} \boldsymbol{T}_l(\underline{x}, t_n)\,\mathrm{d}\underline{x} \\ = \int_K \boldsymbol{S}_l(\boldsymbol{U}(\underline{x}, t_{n+1}))\,\mathrm{d}\underline{x} - \int_{K^l(t_n)} \boldsymbol{S}_l(\boldsymbol{U}(\underline{x}, t_n))\,\mathrm{d}\underline{x} + O(\Delta t^3 |K|),\end{aligned}$$

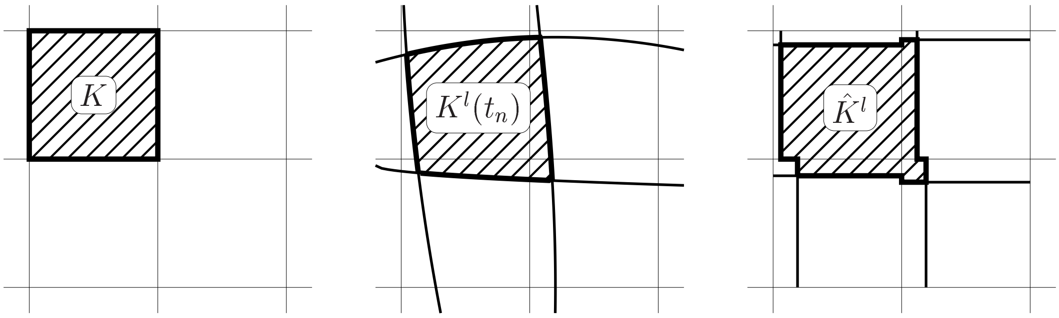

Fig. 4.2. The cell edges are transported backwards in time and approximated by cellwise axiparallel lines. These define the cells $\hat{K}^l$ which consist of a union of finitely many rectangles.

or

$$\int_K \left(\boldsymbol{S}_l(\boldsymbol{U}) - \frac{\Delta t}{2}\boldsymbol{T}_l\right)(\underline{x}, t_{n+1})\,\mathrm{d}\underline{x} \\ = \int_{K^l(t_n)} \left(\boldsymbol{S}_l(\boldsymbol{U}) + \frac{\Delta t}{2}\boldsymbol{T}_l\right)(\underline{x}, t_n)\,\mathrm{d}\underline{x} + O(\Delta t^3|K|).$$

If we again sum over all l and now assume the state decomposition to be flux-consistent, the term $\boldsymbol{T}_l$ on the left hand side cancels and we get:

Lemma 4.4 (Approximate evolution operator I) *For a smooth solution* $\boldsymbol{U}$ *and a flux-consistent state decomposition of* (1.1),

$$\int_K \boldsymbol{U}(\underline{x}, t_{n+1})\,\mathrm{d}\underline{x} = \sum_{l=1}^{L} \int_{K^l(t_n)} \left(\boldsymbol{S}_l(\boldsymbol{U}) + \frac{\Delta t}{2}\boldsymbol{T}_l\right)(\underline{x}, t_n)\,\mathrm{d}\underline{x} + O(\Delta t^3|K|). \quad (4.5)$$

□

In general, $K^l(t_n)$ has a curvilinear boundary and cannot be determined exactly. In [23, section 3], we approximated the cell by a set $\hat{K}^l$ which is a union of finitely many rectangles [2], see figure 4.2. For this $\hat{K}^l$, the following estimate holds:

Lemma 4.5 *Suppose that the velocity field* $\underline{a}_l$ *is smooth. Let* $\hat{K}^l$ *be the approximation to* $K^l(t_n)$ *used by the second order MoT-ICE (cf. [23, section 3]). Then,*

$$\int_{K^l(t_n)} \phi(\underline{x})\,\mathrm{d}\underline{x} = \int_{\hat{K}^l} \phi(\underline{x})\,\mathrm{d}\underline{x} + O(\Delta t^3|K|)$$

for any smooth function ϕ.

[2] Similar approximations were developed by M. Fey and coworkers (private communication).

PROOF. Suppose that ϕ is the initial function of an advection problem at time t_n and ψ is the exact solution at t_{n+1}, then $\int_{K^l(t_n)} \phi = \int_K \psi$ and as shown in [23, theorem 3.2], $\int_K \psi = \int_{\hat{K}^l} \phi + O(\Delta t^3 |K|)$. This implies the assertion. $\square$

Especially, we have

$$\int_K \Big(\boldsymbol{S}_l(\boldsymbol{U}) - \frac{\Delta t}{2} \boldsymbol{T}_l \Big)(\underline{x}, t_{n+1})\, \mathrm{d}\underline{x} = \int_{\hat{K}^l} \Big(\boldsymbol{S}_l(\boldsymbol{U}) + \frac{\Delta t}{2} \boldsymbol{T}_l \Big)(\underline{x}, t_n)\, \mathrm{d}\underline{x} + O(\Delta t^3 |K|).$$

Using this approximation we obtain:

Lemma 4.6 (Approximate evolution operator II) *Let*

$$\hat{\boldsymbol{S}}_l := \frac{1}{|K|} \int_{\hat{K}^l} \Big(\boldsymbol{S}_l(\boldsymbol{U}) + \frac{\Delta t}{2} \boldsymbol{T}_l \Big)(\underline{x}, t_n)\, \mathrm{d}\underline{x}. \tag{4.6}$$

Then,

$$\frac{1}{|K|} \int_K \boldsymbol{U}(\underline{x}, t_{n+1})\, \mathrm{d}\underline{x} = \sum_{l=1}^{L} \hat{\boldsymbol{S}}_l + O(\Delta t^3) \tag{4.7}$$

for a smooth solution $\boldsymbol{U}$ *and a flux-consistent state decomposition of* (1.1).

PROOF. We have that

$$\hat{\boldsymbol{S}}_l = \frac{1}{|K|} \int_K \Big(\boldsymbol{S}_l(\boldsymbol{U}) - \frac{\Delta t}{2} \boldsymbol{T}_l \Big)(\underline{x}, t_{n+1})\, \mathrm{d}\underline{x} + O(\Delta t^3), \tag{4.8}$$

thus,

$$\begin{aligned}\hat{\boldsymbol{U}} = \sum_{l=1}^{L} \hat{\boldsymbol{S}}_l &= \frac{1}{|K|} \int_K \Big(\underbrace{\sum_{l=1}^{L} \boldsymbol{S}_l(\boldsymbol{U})}_{\boldsymbol{U}} - \frac{\Delta t}{2} \underbrace{\sum_{l=1}^{L} \boldsymbol{T}_l}_{0} \Big)(\underline{x}, t_{n+1})\, \mathrm{d}\underline{x} + O(\Delta t^3) \\ &= \frac{1}{|K|} \int_K \boldsymbol{U}(\underline{x}, t_{n+1})\, \mathrm{d}\underline{x} + O(\Delta t^3). \qquad \square\end{aligned}$$

We would like to point out that we are now able to determine the coupling terms $\boldsymbol{T}_l$ at the new time level implicitely from the numerical data: Suppose $\boldsymbol{T}_l(\underline{x}, t_n)$ is known. First, use (4.7) to approximate $\boldsymbol{U}(\underline{x}, t_{n+1})$. Then, from (4.8) we get

$$\frac{\Delta t}{2|K|} \int_K \boldsymbol{T}_l(\underline{x}, t_{n+1})\, \mathrm{d}\underline{x} = \frac{1}{|K|} \int_K \boldsymbol{S}_l(\boldsymbol{U})(\underline{x}, t_{n+1})\, \mathrm{d}\underline{x} - \hat{\boldsymbol{S}}_l + O(\Delta t^3).$$

The advantage of this procedure is that the explicit formula 4.1 does not have to be coded. That is to say, this formula can become extremely complicated, see for example von Törne et al [31,11] for the equations of magnetohydrodynamics. However, although this idea in principal works, there are several unsolved problems with this construction which we observed in our numerical experiments, for example the question of limiting of the coupling terms or their treatment at the boundary. For details see [15]. The numerical results shown in our companion paper [16] are therefore based on the explicit formula (4.1).

5 Comparison with the EG evolution operator

In this section, we will show that the evolution operator derived in lemma 4.2 is closely related to the one developed by Ostkamp [24,25] and later used by Lukáčová, Morton and Warnecke [20] for the derivation of their new EG 3 scheme. This evolution operator, which we call EG operator in the following, is based on the classical characteristic theory, see [3,5,28].

Ostkamp already showed that there is a close connection between her characteristic Galerkin scheme and Fey's version of the Method of Transport. We will now show that there is a canonical continuous state decomposition (derived from the classical characteristic theory) for which our exact evolution operator (4.4) becomes identical to the EG operator. In other words, the connection between the two approaches is lifted up from the level of schemes to the level of exact evolution operators, and lemmas 5.1 and 5.3 below can be considered as the fundamental interface between the two ideas.

Note that in [24,25,20,22,21] the EG operator has only been used for linear or linearized systems. Thus, to compare both evolution operators, we will restrict ourselves to the linear, constant coefficient case. Hence, let

$$\partial_t \boldsymbol{U} + \sum_{s=1}^{d} \boldsymbol{A}_s \partial_{x_s} \boldsymbol{U} = 0 \tag{5.1}$$

be a linear system of hyperbolic conservation laws. Here, $\boldsymbol{A}_s$ are $m \times m$ matrices.

The consistency condition of a state decomposition, for a linear system, reads

$$\sum_{l=1}^{L} \boldsymbol{S}_l(\boldsymbol{U}) = \boldsymbol{U}, \tag{5.2}$$

and flux-consistency would additionally require

$$\sum_{l=1}^{L} \boldsymbol{S}_l(\boldsymbol{U}) \cdot a_{l,s}(\boldsymbol{U}) = \boldsymbol{A}_s \boldsymbol{U}, \tag{5.3}$$

where $a_{l,s}$, $s = 1, \dots, d$ denote the single components of the speeds $\underline{a}_l$. It seems to be sensible for linear systems that $a_{l,s}(\boldsymbol{U})$ should not depend on $\boldsymbol{U}$ (i. e. only on l and s). As a consequence, the characteristics $\xi^l_{\underline{x}}$ are straight lines, $\xi^l_{\underline{x}}(\tau) = \underline{x} + (\tau - t_{n+1})\underline{a}_l$. In this case, by considering the limit of a one-point set K, i. e. $K \to \{\underline{x}\}$, the exact evolution operator (4.4) can be stated pointwise as

$$\boldsymbol{U}(\underline{x}, t_{n+1}) = \sum_{l=1}^{L} \boldsymbol{S}_l(\boldsymbol{U}(\underline{x} - \Delta t \cdot \underline{a}_l, t_n)) + \sum_{l=1}^{L} \int_{t_n}^{t_{n+1}} \boldsymbol{T}_l(\underline{x} + (\tau - t_{n+1})\underline{a}_l, \tau)\,\mathrm{d}\tau. \tag{5.4}$$

Here the state decomposition still has to be specified. One can construct a *continuous state decomposition* (i. e. the sum in (2.1) is replaced with an integral or, more general, a sum of integrals) in which $\boldsymbol{S}_l(\boldsymbol{U})$ are the eigenvectors of the Jacobian of $\underline{\boldsymbol{F}}(\boldsymbol{U})$ (when a certain normal direction in the $\underline{x}$ space is given) and the velocities $\underline{a}_l$ are the associated ray velocities arising from the characteristic theory (see Courant and Hilbert [5] or Jeffry and Taniuti [13]): Consider a characteristic surface of the linearized system in the $(\underline{x}, t)$ space which at a given time level (say, at $t = t_0$) concentrates in one point. Then construct the intersection of this surface with the $t = t_0 + 1$ plane, see figure 5.1 for an example (such a diagram is called Friedrichs diagram). If a normal direction is given, the ray velocity is the position vector of the point on the Friedrichs diagram in which the tangential plane is orthogonal to the given normal direction. It must not be confused with the normal velocity. The normal velocity always points into the normal direction, the ray velocity doesn't, but the projection of the ray velocity onto the normal direction equals the normal velocity.

In symbols, the state decomposition is defined as follows: Let $\underline{p} \in \mathbb{R}^d$ be the normal direction, $\underline{p} \neq 0$. From the hyperbolicity of the system we know that the matrix

$$\boldsymbol{A}(\underline{p}) := \sum_{s=1}^{d} p_s \boldsymbol{A}_s$$

is diagonalizable with real eigenvalues. Let $\boldsymbol{r}^k_{\underline{p}}$ be the right (column) eigenvectors, $\boldsymbol{l}^k_{\underline{p}}$ the left (row) eigenvectors and $\lambda^k_{\underline{p}}$ the eigenvalues—where $k = 1, \dots, m$. We only consider these terms for $|\underline{p}| = 1$, but it is important to understand that they are defined as well for other values of $\underline{p}$ so that derivatives of $\lambda^k_{\underline{p}}$ with respect to a component p_s are defined. The eigenvectors are assumed to be normalized such that $\boldsymbol{R}_{\underline{p}} = (\boldsymbol{L}_{\underline{p}})^{-1}$ where $\boldsymbol{R}_{\underline{p}}$ and $\boldsymbol{L}_{\underline{p}}$ are the matrices whose columns are $\boldsymbol{r}^k_{\underline{p}}$ or whose rows are $\boldsymbol{l}^k_{\underline{p}}$, respectively.

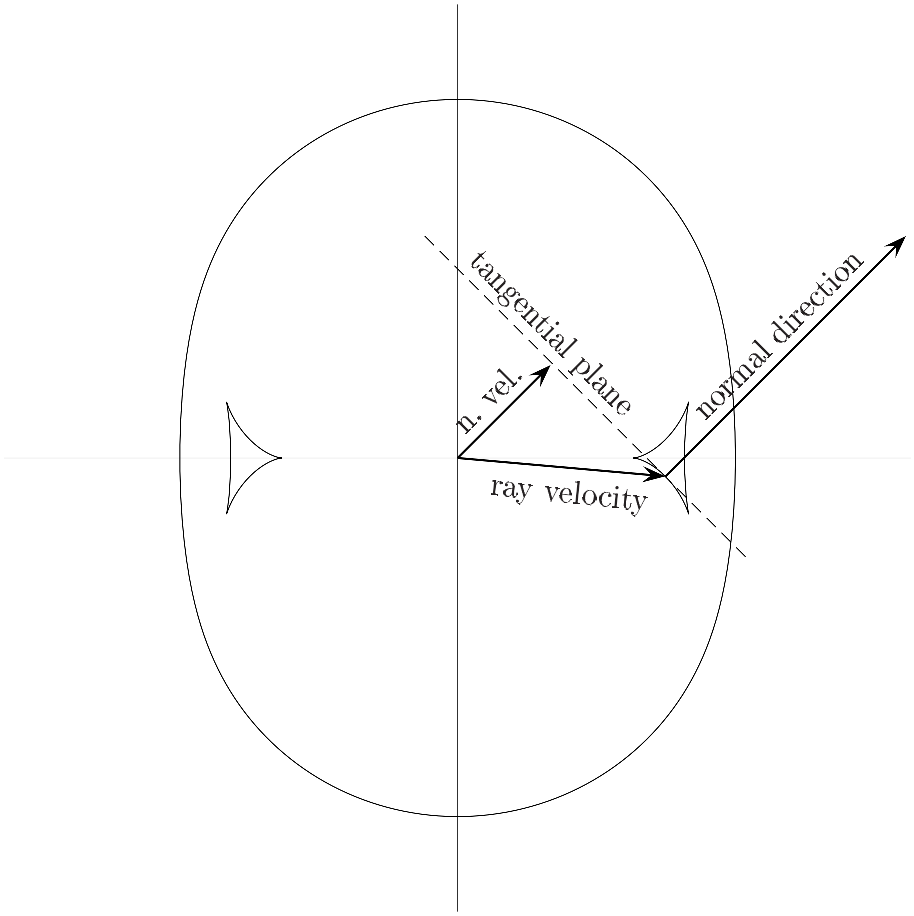

Fig. 5.1. Example for a Friedrichs diagram for the linearized equations of magneto-hydrodynamics.

Now, set

$$\boldsymbol{S}_l(\boldsymbol{U}) := \boldsymbol{S}_{\underline{p}}^k(\boldsymbol{U}) := \frac{1}{|S^{d-1}|}\boldsymbol{r}_{\underline{p}}^k \boldsymbol{l}_{\underline{p}}^k \boldsymbol{U}, \tag{5.5}$$

where $S^{d-1} \subset \mathbb{R}^d$ is the unit sphere. These $\boldsymbol{S}_{\underline{p}}^k(\boldsymbol{U})$ are the components of a continuous state decomposition. In fact, (5.2) is satisfied:

$$\begin{aligned}\int_{S^{d-1}} \sum_{k=1}^m \boldsymbol{S}_{\underline{p}}^k(\boldsymbol{U})\,\mathrm{d}\underline{p} &= \frac{1}{|S^{d-1}|}\int_{S^{d-1}} \sum_{k=1}^m \boldsymbol{r}_{\underline{p}}^k \boldsymbol{l}_{\underline{p}}^k\,\mathrm{d}\underline{p}\,\boldsymbol{U} = \frac{1}{|S^{d-1}|}\int_{S^{d-1}} \boldsymbol{R}_{\underline{p}} \boldsymbol{L}_{\underline{p}}\,\mathrm{d}\underline{p}\,\boldsymbol{U} \\ &= \boldsymbol{U}.\end{aligned}$$

Then set

$$\underline{a}_l := \underline{a}_{\underline{p}}^k := \nabla_{\underline{p}} \lambda_{\underline{p}}^k, \tag{5.6}$$

this is the gradient of $\lambda_{\underline{p}}^k$ with respect to $\underline{p}$. These are the ray velocities that arise from the Friedrich diagrams. Note that the normal velocities in our notation are $\underline{p} \cdot \lambda_{\underline{p}}^k$. Although this may seem somewhat artificial, we would like to emphasize that it is in fact the most natural choice of the velocities: Recall that the ray velocities are precisely those velocities with which a point disturbance propagates (see figure 5.1).

The so-defined state decomposition is in general *not* flux-consistent in the sense of definiton 2.2. This can be seen, for example, for the Euler equations in the case $d > 1$, by an elementary but involved calculation. See Kröger [15] for details. Nevertheless, the exact evolution operator (4.4) applies anyway, since lemma 4.2 does not require flux-consistency.

Using the state decomposition (5.5) and (5.6) in lemma 4.2, we will now identify our exact evolution operator with the EG operator.

Inserting (5.5) and (5.6) into (5.4), we get:

Lemma 5.1 *For a smooth solution $\boldsymbol{U}$ of the linear system* (5.1) *of hyperbolic conservation laws, we have*

$$\boldsymbol{U}(\underline{x}, t_{n+1}) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} \sum_{k=1}^{m} \boldsymbol{r}^k_{\underline{p}} \boldsymbol{l}^k_{\underline{p}} \boldsymbol{U}(\underline{x} - \Delta t \cdot \nabla_{\underline{p}} \lambda^k_{\underline{p}}, t_n)\, \mathrm{d}\underline{p} \tag{5.7}$$

$$+ \int_{S^{d-1}} \sum_{k=1}^{m} \int_{t_n}^{t_{n+1}} \boldsymbol{T}^k_{\underline{p}}(\underline{x} + (\tau - t_{n+1}) \nabla_{\underline{p}} \lambda^k_{\underline{p}}, \tau)\, \mathrm{d}\tau\, \mathrm{d}\underline{p} \tag{5.8}$$

where $\boldsymbol{T}^k_{\underline{p}}$ is given in (5.9). □

We now translate the EG evolution operator (cf. [20, section 3, especially equation (3.6)]) into our notation. First, we note the following result:

Lemma 5.2 *The term $\underline{b}_{jj}(\underline{n})$ in the notation of Lukáčová, Morton, Warnecke [20, (3.4)] corresponds to $\nabla_{\underline{p}} \lambda^k_{\underline{p}} = \underline{a}^k_{\underline{p}}$ in our notation.*

PROOF. Just translating the notation, we get that the $b^k_{jj}(\underline{n})$ in [20] (where we use k instead of j, s instead of k and $\underline{p}$ instead of $\underline{n}$) equals the k-th diagonal element of the matrix $\boldsymbol{L}_{\underline{p}} \boldsymbol{A}_s \boldsymbol{R}_{\underline{p}}$, and that is $\boldsymbol{l}^k_{\underline{p}} \boldsymbol{A}_s \boldsymbol{r}^k_{\underline{p}}$. We thus have to show that

$$\boldsymbol{l}^k_{\underline{p}} \boldsymbol{A}_s \boldsymbol{r}^k_{\underline{p}} = \partial_{p_s} \lambda^k_{\underline{p}}.$$

But differentiating the identity

$$0 = \boldsymbol{l}^k_{\underline{p}} \left(\lambda^k_{\underline{p}} \cdot \mathbf{1} - \sum_{s=1}^{d} p_s \boldsymbol{A}_s \right) \boldsymbol{r}^k_{\underline{p}}$$

with respect to p_s yields

$$\begin{aligned}0 &= (\partial_{p_s}\boldsymbol{l}^k_{\underline{p}})\underbrace{\left(\lambda^k_{\underline{p}}\cdot\mathbf{1}-\sum_{s=1}^d p_s\boldsymbol{A}_s\right)\boldsymbol{r}^k_{\underline{p}}}_{\mathbf{0}}+\underbrace{\boldsymbol{l}^k_{\underline{p}}\left(\lambda^k_{\underline{p}}\cdot\mathbf{1}-\sum_{s=1}^d p_s\boldsymbol{A}_s\right)}_{\mathbf{0}}(\partial_{p_s}\boldsymbol{r}^k_{\underline{p}})\\ &\quad+\boldsymbol{l}^k_{\underline{p}}\left(\partial_{p_s}\lambda^k_{\underline{p}}\cdot\mathbf{1}-\boldsymbol{A}_s\right)\boldsymbol{r}^k_{\underline{p}}\\ &= \partial_{p_s}\lambda^k_{\underline{p}}-\boldsymbol{l}^k_{\underline{p}}\boldsymbol{A}_s\boldsymbol{r}^k_{\underline{p}}. \qquad \square\end{aligned}$$

Using this result, we get that the EG operator in our notation takes the form

$$\begin{aligned}\boldsymbol{U}(\underline{x},t_{n+1}) &= \frac{1}{|S^{d-1}|}\int_{S^{d-1}}\sum_{k=1}^m(\boldsymbol{l}^k_{\underline{p}}\boldsymbol{U}(\underline{x}-\Delta t\cdot\nabla_{\underline{p}}\lambda^k_{\underline{p}},t_n))\boldsymbol{r}^k_{\underline{p}}\,\mathrm{d}\underline{p}\\ &\quad+\int_{S^{d-1}}\sum_{k=1}^m\int_{t_n}^{t_{n+1}}\boldsymbol{Z}^k_{\underline{p}}(\underline{x}+(\tau-t_{n+1})\nabla_{\underline{p}}\lambda^k_{\underline{p}},\tau)\,\mathrm{d}\tau\,\mathrm{d}\underline{p}\end{aligned}$$

where $\boldsymbol{Z}^k_{\underline{p}}$ is some coupling term to be studied more precisely in a moment.

Since $\boldsymbol{l}^k_{\underline{p}}\boldsymbol{U}(\dots)$ is a scalar value and therefore commutes with the vector $\boldsymbol{r}^k_{\underline{p}}$, we see that the first term is the same in both evolution operators. If we now show that $\boldsymbol{Z}^k_{\underline{p}}=\boldsymbol{T}^k_{\underline{p}}$ then both evolution operators coincide. We have that

$$\begin{aligned}\boldsymbol{T}^k_{\underline{p}}(\underline{x},t) &= -\frac{1}{|S^{d-1}|}\boldsymbol{r}^k_{\underline{p}}\boldsymbol{l}^k_{\underline{p}}\sum_{s=1}^d\boldsymbol{A}_s\partial_{x_s}\boldsymbol{U}(\underline{x},t)+\frac{1}{|S^{d-1}|}\sum_{s=1}^d\partial_{p_s}\lambda^k_{\underline{p}}\boldsymbol{r}^k_{\underline{p}}\boldsymbol{l}^k_{\underline{p}}\partial_{x_s}\boldsymbol{U}(\underline{x},t)\\ &= \frac{1}{|S^{d-1}|}\boldsymbol{r}^k_{\underline{p}}\boldsymbol{l}^k_{\underline{p}}\sum_{s=1}^d(\partial_{p_s}\lambda^k_{\underline{p}}\cdot\mathbf{1}-\boldsymbol{A}_s)\partial_{x_s}\boldsymbol{U}(\underline{x},t).\end{aligned}\tag{5.9}$$

For $\boldsymbol{Z}^k_{\underline{p}}$, if we translate everything in our notation, we get

$$\boldsymbol{Z}^k_{\underline{p}}(\underline{x},t)=\frac{1}{|S^{d-1}|}\boldsymbol{r}^k_{\underline{p}}\sum_{s=1}^d\Big((\boldsymbol{\Lambda}_{\underline{p},s}-\boldsymbol{L}_{\underline{p}}\boldsymbol{A}_s\boldsymbol{R}_{\underline{p}})_k\boldsymbol{L}_{\underline{p}}\partial_{x_s}\boldsymbol{U}(\underline{x},t)\Big)$$

where

$$\boldsymbol{\Lambda}_{\underline{p},s}=\begin{pmatrix}\partial_{p_s}\lambda^1_{\underline{p}} & & & \\ & \partial_{p_s}\lambda^2_{\underline{p}} & & \\ & & \ddots & \\ & & & \partial_{p_s}\lambda^m_{\underline{p}}\end{pmatrix}$$

and $(\cdot)_k$ means the k-th row of a matrix-valued term. Computing this gives

$$\begin{aligned}\boldsymbol{Z}_{\underline{p}}^k(\underline{x},t) &= \frac{1}{|S^{d-1}|}\boldsymbol{r}_{\underline{p}}^k\sum_{s=1}^d(\partial_{p_s}\lambda_{\underline{p}}^k\boldsymbol{l}_{\underline{p}}^k - \boldsymbol{l}_{\underline{p}}^k\boldsymbol{A}_s\boldsymbol{R}_{\underline{p}}\boldsymbol{L}_{\underline{p}})\partial_{x_s}\boldsymbol{U}(\underline{x},t)\\ &= \frac{1}{|S^{d-1}|}\boldsymbol{r}_{\underline{p}}^k\boldsymbol{l}_{\underline{p}}^k\sum_{s=1}^d(\partial_{p_s}\lambda_{\underline{p}}^k\cdot\mathbf{1} - \boldsymbol{A}_s)\partial_{x_s}\boldsymbol{U}(\underline{x},t).\end{aligned}$$

Hence, $\boldsymbol{T}_{\underline{p}}^k = \boldsymbol{Z}_{\underline{p}}^k$, and thus we have:

Lemma 5.3 *For a smooth solution $\boldsymbol{U}$ of the linear system* (5.1) *of hyperbolic conservation laws, we have that our evolution operator* (5.7) *where $\boldsymbol{S}_{\underline{p}}^k$ and $\underline{a}_{\underline{p}}^k$ are given by* (5.5) *and* (5.6) *is identical with the EG operator.* □

We would like to recall that an overview of these connections has been given in figure 1.1.

Remarks 5.4

(1) The continuous state decomposition (5.5)–(5.6) *can also be introduced for the non-linear case. Also, the exact evolution operator* (4.4) *remains valid in that case. We have therefore found a generalisation of the EG operator to non-linear systems.*

(2) Since the continuous state decomposition is in general not flux-consistent, a discretisation as in lemmas 4.4 and 4.6 would not lead to a consistent scheme for (1.1)*. Indeed, we are convinced that this scheme (for the linear, constant coefficient case) would coincide with Ostkamp's 'inconsistent' scheme.*

(3) Perhaps the techniques of Lukáčová, Morton and Warnecke [20] can be used to construct yet another EG scheme out of this generalized evolution operator in the nonlinear case. Unfortunately, for the non-linear case the coupling terms $\boldsymbol{T}_l = \boldsymbol{T}_{\underline{p}}^k$ would become extremely complicated due to the fact that $\boldsymbol{r}_{\underline{p}}^k$, $\boldsymbol{l}_{\underline{p}}^k$ and $\lambda_{\underline{p}}^k$ in general also depend on $\boldsymbol{U}$. The coupling terms now read

$$\begin{aligned}\boldsymbol{T}_{\underline{p}}^k = \frac{1}{|S^{d-1}|}\Bigg(&\Big(\boldsymbol{r}_{\underline{p}}^{k,\prime}(\boldsymbol{U})(\boldsymbol{l}_{\underline{p}}^k(\boldsymbol{U})\boldsymbol{U}) + \boldsymbol{r}_{\underline{p}}^k(\boldsymbol{U})\Big((\boldsymbol{l}_{\underline{p}}^{k,\mathrm{T},\prime}(\boldsymbol{U})\boldsymbol{U})^{\mathrm{T}} + \boldsymbol{l}_{\underline{p}}^k(\boldsymbol{U})\Big)\Big)\\ &\cdot\Big(\Big(\nabla_{\underline{p}}\lambda_{\underline{p}}^k(\boldsymbol{U})(\nabla_{\underline{x}}\boldsymbol{U})^{\mathrm{T}}\Big)^{\mathrm{T}} - \nabla_{\underline{x}}\cdot\underline{\boldsymbol{F}}(\boldsymbol{U})\Big)\\ &+\boldsymbol{r}_{\underline{p}}^k(\boldsymbol{U})\boldsymbol{l}_{\underline{p}}^k(\boldsymbol{U})\boldsymbol{U}\cdot\operatorname{tr}\Big(\nabla_{\boldsymbol{U}}(\nabla_{\underline{p}}\lambda_{\underline{p}}^k)^{\mathrm{T}}(\boldsymbol{U})\nabla_{\underline{x}}\boldsymbol{U}\Big)\Bigg)\end{aligned}$$

where $\operatorname{tr} A$ denotes the trace of a matrix A, and A^{T} denotes the transposed of A, and $'$ denotes the derivative with respect to $\boldsymbol{U}$.

As Ostkamp pointed out, her consistent scheme coincides with Fey's original MoT [8] for the linearized, constant coefficient Euler equations. Note that for

Euler's equations, however, we have that $m = d + 2$ and we get the very special situation that (for appropriate numbering) $\underline{a}^1_{\underline{p}} = \underline{a}^m_{-\underline{p}}$ and all other $\underline{a}^k_{\underline{p}}$ $(k = 2, \ldots, m-1)$ are equal and do not depend on $\underline{p}$. Using these facts the continuous state decomposition (5.5)–(5.6) can be simplified by combining $\boldsymbol{S}^1_{\underline{p}}$ and $\boldsymbol{S}^m_{-\underline{p}}$ to one component and integrating all the $\boldsymbol{S}^k_{\underline{p}}$ components (for $k = 2, \ldots, m-1$ and $\underline{p} \in S^{d-1}$) to one single component. If this is done and further more the $\underline{p}$-dependent part of $\boldsymbol{S}^1_{\underline{p}} = \boldsymbol{S}^m_{-\underline{p}}$ is multiplied by the space dimension d, one gets exactly Fey's continuous state decomposition used in [8], and therefore, one get's Ostkamp's consistent scheme. This state decomposition is flux-consistent, but only because some special part of $\boldsymbol{S}^k_{\underline{p}}$ has been multiplied by a factor d in some places. Ostkamp [24] derived this factor d systematically, but this derivation relies on a number of rather restrictive assumptions which seem to be tailored to the equations of gas dynamics. For a general system of conservation laws, it does not seem to be able to make the continuous state decomposition flux-consistent this way.

Even though Ostkamp was able to identify her EG scheme and Fey's original MoT, we would like to suggest that the two approaches are fundamentally different. [3] As we see it, the main difference between the MoT and the EG approach is that the state decomposition which leads to the EG operator is not flux-consistent. The MoT approximates the evolution operator in lemma 4.2 exploiting the flux-consistency of a given state decomposition (see especially the approximate evolution operator of lemma 4.4), while the EG schemes use an approximation technique which does not rely on the flux-consistency. However, this has the consequence that the coupling terms $\boldsymbol{T}_l$ become more essential in the EG approach. In the MoT-ICE, if the $\boldsymbol{T}_l$ are neglected, one still gets a first order scheme; this can be seen in (4.5) by recognizing that the sets $K^l(t_n)$ differ from K only by sets of Lebesque measure $O(\Delta t|K|)$, and $\sum_l \int_K \boldsymbol{T}_l \, \mathrm{d}\underline{x} = 0$ by lemma 4.1. So, the error made if $\boldsymbol{T}_l$ is neglected in (4.5) is of order $O(\Delta t^2|K|)$, thus the scheme would still be of first order. In distinction to that, if the coupling terms are neglected in the EG operator, one gets an inconsistent scheme. In fact, this is just what Ostkamp [24,25] did in her 'inconsistent' scheme (which would be first order consistent on Ostkamp's terminology).

Neglecting the coupling terms in the first order MoT seems to be closely related to dropping the interaction terms $Q(f, f)$ in Boltzmann's equation (3.2) during a time step, which leads to the first order version of the kinetic schemes. One could suggest that the second order MoT corresponds to approximating $Q(f, f)$ in a suitable way.

[3] Quoting D. Adams' 'The Hitch Hiker's Guide to the Galaxy' (Random House Value Publishing Inc., 1979), we are tempted to claim that the MoT is 'almost, but not quite, entirely unlike' the EG schemes.

6 Conclusion

We presented some new (as we hope) connections between a number of Riemann-solver free approaches to numerics for systems of hyperbolic conservation laws. The concept of state decompositions, flux decompositions and evolution operators, which is the natural framework for the Method of Transport, can be derived from the kinetic theory by just applying some quadrature rule to the moment integral. This establishes a connection between the MoT and the kinetic schemes—especially there is a close relation between the kinetic schemes and the first order MoT-ICE, in which the coupling terms $\boldsymbol{T}_l$ are neglected.

On the other hand, we illustrated that the choice of a special, physically reasonable, but continuous state decomposition lets our exact evolution operator become identical to the EG evolution operator used by Ostkamp [24,25] and by Lukáčová, Morton and Warnecke [20] in the linear, constant coefficient case. Thus, we introduced a new connection between the MoT and the EG schemes at the level of exact evolution operators. However, there is still an essential difference, namely the concept of flux-consistency: To construct a MoT-like scheme out of our exact evolution operator the state decomposition is required to be flux-consistent, and also, state decompositions obtained from the kinetic theory automatically are flux-consistent. But the EG approach is based on a non-flux-consistent state decomposition obtained from the characteristic theory.

Thus, the concepts of state decompositions and flux decompositions help to clarify similarities and differences between the kinetic approach and the MoT on one hand and the EG approach (and even standard finite volume schemes, see remark 4.3) on the other hand.

Extensive numerical tests of the MoT-ICE are presented in [16].

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