INITIAL BOUNDARY VALUE PROBLEMS FOR A QUASILINEAR PARABOLIC SYSTEM IN THREE-PHASE CAPILLARY FLOW IN POROUS MEDIA

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Abstract. We study two types of initial boundary value problems for a quasilinear parabolic system motivated by three-phase flow in porous medium in the presence of capillarity effects. The first type of problem prescribes a boundary condition of mixed type involving a combination of the value of the solution and its normal derivative at the boundary. The second type of problem prescribes Dirichlet boundary conditions and its solution is obtained as a limit case of the first type. The main assumption about the "viscosity" matrix of the system is that it is triangular with strictly positive diagonal elements. Another interesting feature is concerned specifically with the application to three-phase capillary flow in porous medium. Namely, we derive an important practical consequence of the assumption that the diffusion term in the equation of one of the phases, say gas, depends only on the saturation of the corresponding phase. We show that this mathematical assumption in turn provides an efficient method for the definition of the capillary pressures in the interior of the triangle of saturations through the solution of a well posed boundary value problem for a linear hyperbolic system. As an example, we include the analysis of a very special model of three-phase capillary flow where the capillarity matrix results to be degenerate, but we are still able to solve it, due to the particular form of the flux functions.

1. Introduction

We consider initial boundary value problems for 2×2 quasilinear parabolic systems of the form

$$u_t + f(u)_x = (B(u)u_x)_x, \quad 0 < t < T, \quad x \in (-1,1) \equiv \Omega, \tag{1.1}$$

motivated by one-dimensional three-phase capillary flows in a petroleum reservoir. Here,

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

and equality (1.1) is a short version of

$$\frac{\partial u_i}{\partial t} + \frac{\partial f_i(u)}{\partial x} = \frac{\partial}{\partial x}(B_{ij}(u)\frac{\partial u_j}{\partial x}). \tag{1.2}$$

Our main results concerning system (1.1) (see Theorems 1.1 and 1.2 below) assume a nondegeneracy condition (see (1.14) below) besides some structural ones (see (1.3) and (1.8) below). As a matter of fact, in the application to three-phase capillary flow the matrix B is, in general, degenerate, being singular at the boundary of the triangle of saturations. As an example, we also include the analysis of a special case

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(see Theorem 1.3 below) in which B is degenerate, but we are able to prove that the solution never "sees" the set of degeneracy, that is, the boundary of the saturations triangle. Mathematically, passing to a regularized system with a non-degenerate matrix B close to the physical one does not trivialize the subject at all. The reason is that the theory of quasilinear parabolic systems is still quite incomplete and, in particular, the case of a general non-degenerate matrix B, yielding a well established parabolicity condition, remains an open problem as far as global in time existence is concerned.

Relatively to progresses on the theory of parabolic quasilinear systems, it is shown in the book of Ladyzhenskaya, Solonnikov and Ural'tseva [6] that the machinery developed for one quasilinear parabolic equation can be applied to system (1.1) in the case where one has $B_{11} = B_{22}, \quad B_{12} = B_{21} = 0$. We also recall the results of H.Amann [2] which, when restricted to 2×2 systems, require that $\frac{\partial f_2}{\partial u_1} \equiv 0$, $\frac{\partial B_{22}}{\partial u_1} \equiv 0$ and

$$B_{21} \equiv 0, \tag{1.3}$$

but, for such systems, existence is conditioned by the assumption that some norms of the solutions are finite. Here, we also impose (1.3) in our results, Theorems 1.1, 1.2 and 1.3, but, for the first two cases, both f_2, B_{22}, also depend, in general, on both variables, u_1, u_2.

We consider two initial boundary value problems for system (1.1):

$$\delta u_n + u = u_\partial \quad \text{at} \quad |x| = 1, \quad u|_{t=0} = u_0(x), \tag{1.4}$$

and

$$u = u_\partial \quad \text{at} \quad |x| = 1, \quad u|_{t=0} = u_0(x), \tag{1.5}$$

where $\delta = \text{const} > 0$ and

$$u_n|_{x=\pm 1} = \pm u_x|_{x=\pm 1}, \quad u_\partial|_{x=\pm 1} = u_\pm(t).$$

Motivated by the application to capillary three-phase flow, where the functions u_i stand for fluid saturations, u_1, u_2 must verify the constraints

$$0 \le u_i \le 1, \quad u_1 + u_2 \le 1. \tag{1.6}$$

We write the constraints (1.6) as

$$u \in \Delta = \{u : u \in \mathbb{R}^2, \quad 0 \le u_i \le 1, \quad u_1 + u_2 \le 1\}. \tag{1.7}$$

Observe that the triangle Δ can be defined as an intersection

$$\Delta = \cap_1^3 \{G_i(u) \le 0\}, \quad G_1 = -u_1, \quad G_2 = -u_2, \quad G_3 = u_1 + u_2 - 1.$$

The fulfilment of (1.7) will be a consequence of the following assumptions on the matrices $B^\top$ and $(f')^\top$, with $(f')_{ij} = \frac{\partial f_i}{\partial u_j}$:

$$(f')^\top \langle \nabla_u G_i \rangle = \alpha_i \nabla_u G_i, \quad B^\top \langle \nabla_u G_i \rangle = \mu_i \nabla_u G_i, \qquad \forall u \in \partial\Delta, \ \text{s.t.}\ G_i(u) = 0, \ i = 1, 2, 3. \tag{1.8}$$

for some scalar functions of u, α_i and $\mu_i \ge 0$, eigenvalues of $(f')^\top$ and $B^\top$, respectively, defined on $\partial\Delta \cap \{G_i(u) = 0\}$, $i = 1, 2, 3$. Observe that conditions (1.8) are equivalent to

$$f_1|_{u_1=0} = \text{const}, \quad f_2|_{u_2=0} = \text{const}, \quad (f_1+f_2)|_{u_1+u_2=1} = \text{const}, \tag{1.9}$$

$$B_{12}|_{u_2=0} = 0, \quad B_{21}|_{u_1=0} = 0, \quad (B_{11}+B_{21}-B_{12}-B_{22})|_{u_1+u_2=1} = 0, \tag{1.10}$$

due to the simple structure of the functions G_i. The conditions (1.8) allow the use of the theory of Chuey, Conley, and Smoller [11] on positively invariant regions for nonlinear parabolic systems (see also [10]). We note that conditions (1.9) and (1.10) are naturally verified in the application to three-phase capillary flow in porous medium.

We are interested in classical solutions of the initial boundary value problem (1.1),(1.4),(1.6). So, we assume that

$$f_i, \quad \frac{\partial f_i}{\partial u_j}, \quad \frac{\partial B_{ij}}{\partial u_k}, \quad \frac{\partial^2 B_{ik}}{\partial u_s \partial u_k} \in H^\beta(\Delta), \tag{1.11}$$

where $H^\beta(\Delta)$ is the space of Hölder continuous functions on Δ with $\beta \in (0,1)$.

The initial and boundary data are assumed to be also in Hölder spaces:

$$u_0 \in H^{2+\beta}(\overline{\Omega}), \quad u_\pm(t) \in H^{1+\beta}([0,T]). \tag{1.12}$$

We can now state our result concerning the problem (1.1),(1.4),(1.6).

Theorem 1.1. *Assume that the data $B(u)$, $f(u)$, $u_0(x)$, and $u_\pm(t)$ satisfy the conditions* (1.3),(1.8),(1.11), *and* (1.12), *and*

$$u_0(x), \quad u_\pm(t) \in \Delta \quad \textit{for each} \quad x \in \Delta \quad \textit{and} \quad t \in [0,T]. \tag{1.13}$$

Let ν be a positive number and assume that

$$B_{ii} \geq \nu \quad i = 1,2, \ \textit{for each } u \quad \in \Delta. \tag{1.14}$$

Suppose that the compatibility conditions

$$\pm\delta u_0'(\pm 1) + u_0(\pm 1) = u_\pm(0)$$

are satisfied. Then problem (1.1),(1.4),(1.6) *has a unique solution $u(t,x)$ such that $u \in H^{2+\beta,1+\beta/2}(\overline{Q})$.*

The heart of the proof is the Leray-Schauder fixed-point argument and strong a priori estimates in Hölder spaces. This was also the approach applied in [4], where we considered the less difficult case of periodic boundary conditions.

As for the Dirichlet boundary-value problem our result does not includes uniqueness and, in this sense, it is weaker. The obstacle, which is yet to be overcome, is the a priori estimate for $|u_x|$ at $\partial\Omega$. In the case of the boundary conditions (1.4) such an estimate is a simple consequence of (1.7). We now state our result concerning problem (1.1),(1.5),(1.6).

Theorem 1.2. *Let $B(u)$ and $f(u)$ be as in Theorem 1.1. Let the functions $u_0(x)$ and $u_\pm(t)$ satisfy* (1.13) *and $u_0 \in L^2(\Omega)$, $u_\pm \in W^{1,1}(0,T)$. Then problem* (1.1),(1.5), (1.6) *has a solution $u(t,x)$ in the sense that u is a classical solution of* (1.1) *in $Q := (0,T) \times (-1,1)$, and u satisfies* (1.5) *in the sense of $L^2(0,T)$, for the boundary condition, and $L^2(-1,1)$, for the initial condition. Moreover, if $u_\pm(t) \equiv 0$, $t \in [0,T]$, and $u_0 \in H^{2+\beta}(\overline{\Omega})$, with $u_0(\pm 1) = 0$, then $u(t,x)$ is a classical solution of* (1.1) *in Q, $u \in H^{\beta,\beta/2}(\overline{Q})$ and the initial and boundary conditions are assumed in the usual sense for continuous functions.*

The proof involves the passage to the limit when $\delta \downarrow 0$ in a sequence of solutions to the problem (1.1),(1.4),(1.6). We prove Theorem 1.1, in sections 2 and 3. Theorem 1.2 is proved in section 4.

In section 5 we recall the basic facts about capillary multiphase flow in a porous medium. To motivate considerations, we discuss the mobility laws and capillary pressure laws which lead to the hypothesis

$$B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}, \quad \frac{\partial B_{22}}{\partial u_1} = 0. \tag{1.15}$$

and to the condition $\frac{\partial f_2}{\partial u_1} = 0$. The capillary-pressure laws are given by the equations

$$p_i - p_j = P_{ij}(u), \quad \text{with} \quad i, j \in \{1, 2, 3\}.$$

Here, p_i is the pressure in the i-th phase. The capillary pressure functions P_{ij} are assumed as defined somehow on Δ, but this knowledge is very poor both experimentally and theoretically [3, 12].

The main feature of section 5 is a remarkable outcome of the mathematical assumptions (1.15) on the capillarity matrix, in the application to three-phase capillary flow. It is related with the problem of defining the capillary pressures in the interior of the triangle of saturations. Experimentally, capillary pressures are only known, as functions of the saturations, in two-phase fluid flows [9, 5]. This allows us to take the functions

$$P_{ij}(u)|_{u_k=0}, \quad k \neq i, \quad k \neq j, \tag{1.16}$$

as given by the two-phase flow experiments. The hypothesis (1.15) amounts to a linear hyperbolic system of partial differential equations for the functions $P_i \equiv P_{i3}$ (see (5.7) below). When the mobilities are linear functions (see formulas (5.8)), we prove that the two-phase capillary pressures

$$p_{13}(u_1) \equiv P_{13}(u)|_{u_2=0}, \quad p_{23}(u_2) \equiv P_{23}(u)|_{u_1=0}$$

give rise to functions P_i obeying system (5.7) and such that

$$P_1|_{u_2=0} = p_{13}, \quad P_2|_{u_1=0} = p_{23}. \tag{1.17}$$

We call the process yielding resulting formulas (see (5.11) below) the *method of physical interpolation for capillary pressures*, since these formulas define the capillary pressures in the interior of the saturations triangle from the two-phase capillary pressures given in (1.17). In other words, system (5.7) and conditions (1.17) comprise a boundary-value problem for P_j with the two-phase capillary pressures p_{13} and p_{23} as boundary data.

Finally, in section 6, the techniques developed in sections 2 to 4 are applied to the study of a particular degenerate reservoir fluid flow problem. This problem reduces to the Dirichlet initial boundary value problem for a degenerate system of the form (1.1), with constraints given in (1.6), where (1.15) as well as $\frac{\partial f_2}{\partial u_1} = 0$ hold. In this case the second equation in (1.2) reads

$$\frac{\partial u_2}{\partial t} + \frac{\partial f_2(u_2)}{\partial x} = \frac{\partial}{\partial x}(B_{22}(u_2)\frac{\partial u_2}{\partial x}),$$

but, due to (1.6), equations (1.2) are not completely decoupled. Here, f and B in (1.1) have the form

$$f = \frac{1}{ku_2 + 1} \begin{pmatrix} u_1 \\ (1+k)u_2 \end{pmatrix}, \quad B = \begin{pmatrix} \alpha k_1 \xi(1-\xi) & \xi(B_{11} - B_{22}) \\ 0 & \beta u_2(1-u_2) \end{pmatrix}, \tag{1.18}$$

where α, β, k, and $k_1(\geq 0)$ are constants and $\xi = u_1(1-u_2)^{-1}$.

The matrix B is degenerate at $\partial\Omega$. This is the main difficulty. The introduction of the variable ξ, so called relative saturation, saves the situation. The variable ξ appears naturally when one is looking for invariant solutions of the homogeneous system corresponding to (5.7). With the change of variables $(u_1, u_2) \to (\xi, u_2)$, system (1.1) writes

$$\xi_t + \frac{\xi_x}{ku_2+1} = (B_{11}\xi_x)_x - \frac{\xi_x u_{2x}(B_{11}+B_{22})}{1-u_2} ,$$
$$\frac{\partial u_2}{\partial t} + \frac{\partial f_2(u_2)}{\partial x} = \frac{\partial}{\partial x}(B_{22}(u_2)\frac{\partial u_2}{\partial x}).$$

It enables us to apply the maximum principle and guarantee that

$$0 < \delta \leq u_2 \leq 1-\delta, \quad \delta \leq \xi \leq 1-\delta, \tag{1.19}$$

provided that these estimates are valid at $t=0$ and $|x|=1$. In this way one avoids degeneracy. We then obtain the following result proved in section 6.

Theorem 1.3. *Assume*

$$u_0 \in L^\infty(\Delta), \quad u_\pm \in W^{1,1}(0,T), \quad 0<\delta \leq u_{2,\pm}(t) \leq 1-\delta, \quad \delta \leq \xi_\pm(t) \leq 1-\delta,$$
$$\delta \leq u_{2,0}(x) \leq 1-\delta, \quad \delta \leq \xi_0(x) \leq 1-\delta,$$

for some $\delta \in (0,1)$, where

$$\xi_0 = \frac{u_{1,0}}{1-u_{2,0}} , \quad \xi_\pm = \frac{u_{1,\pm}}{1-u_{2,\pm}} .$$

Then, with f and B given by (1.18)*, problem* (1.1),(1.5),(1.6) *has a weak solution $u(t,x)$ such that*

$$u \in L^\infty(Q), \quad u_x \in L^2(Q), \quad u_t \in L^2(0,T; W^{-1,2}(\Omega)),$$

and the estimates (1.19) *hold.*

2. A mixed type initial boundary value problem

In this section, for given $\varepsilon, \delta > 0$, we consider the initial boundary value problem

$$u_t + f(u)_x = (B(u)u_x)_x + \varepsilon h, \quad (t,x) \in Q \equiv (0,T)\times\Omega, \quad \Omega = (-1,1), \tag{2.1}$$
$$\delta u_n + u = u_{\partial,\varepsilon} \quad \text{at} \quad |x|=1, \quad u|_{t=0} = u_{0,\varepsilon}(x). \tag{2.2}$$

Here

$u_n|_{x=\pm1} = \pm u_x$, $u_\partial|_{x=\pm1} = u_\pm(t)$, $u_{i\pm,\varepsilon} = (1-\varepsilon)(\frac{\varepsilon}{2}+u_{i\pm})$, $u_{i0,\varepsilon} = (1-\varepsilon)(\frac{\varepsilon}{2}+u_{i0})$, $i=1,2$. In the following lemmas, the constants c do not depend on ε, and we omit the subscript ε when referring to the functions $u_{0,\varepsilon}(x)$, $u_{\pm,\varepsilon}(t)$.

Lemma 2.1. *The solution u takes values in $int(\Delta)$ whenever u_0 and $u_\pm$ take values in Δ.*

Proof. We follow the method of positively invariant regions [11] (see also [10]). Denoting $z_i = G_i(u)$, we prove that $z_i < 0$ for each i. Clearly,

$$\max_{x\in\Omega} z_i(0,x) < 0, \quad i \in \{1,2,3\}.$$

Suppose, there is a first moment $t_1 > 0$ such that

$$\max_{x\in\Omega} z_i(t_1,x) = z_i(t_1,x_0) = 0$$

for some i. There are two possibilities: $|x_0| < 1$ and $|x_0| = 1$. The case $x_0 = 1$ is impossible. Indeed, it follows from the equality

$$\delta z^i_x + z^i = -u^i_{+\varepsilon} \tag{2.3}$$

that $z^i_x(t_1, 1) < 0$, which gives a contradiction since $z^i(t_1, x) \le 0$, $x \in [-1, 1]$. The case $x_0 = -1$ can be treated similarly.

Let us consider the case $|x_0| < 1$. Multiplying (2.1) by $\nabla_u G_i$ and using (1.8), one arrives at the equality

$$z^i_t + \alpha_i z^i_x = (\mu_i z^i_x)_x + \varepsilon h \cdot \nabla_u G_i \quad \text{at} \quad (t_1, x_0). \tag{2.4}$$

By the assumption,

$$z^i(t_1, x_0) = \max z^i(\tau, y),$$

where the max is taken over

$$0 \le \tau \le t_1, \quad |y| \le 1.$$

Hence, we must have

$$z^i_x(t_1, x_0) = 0, \quad z^i_{xx}(t_1, x_0) \le 0, \quad z^i_t(t_1, x_0) \ge 0. \tag{2.5}$$

Due to the choice of h, we have

$$h \cdot \nabla_u G_i < 0 \quad \text{at} \quad (t_1, x_0).$$

Now, it follows from (2.4) that $z^i_t(t_1, x_0) < 0$, contradicting to (2.5). □

Lemma 2.2. *The estimate*

$$\|u_x\|_{L^2(Q)} + \delta \sum_{\pm} \int_0^T |u_x(t, \pm 1)|^2 dt \le c \tag{2.6}$$

holds with a constant c depending on ν and the norms $\|\dot{u}_\pm\|_{L^1(0,T)}$ and $\|h\|_{L^1(Q)}$. In particular, by (2.1), it follows that

$$\|u_t\|_{L^2(0,T;W^{-1,2}(\Omega))} \le c \tag{2.7}$$

uniformly in ε and δ.

Proof. Denote

$$w = \frac{1-x}{2} u_- + \frac{1+x}{2} u_+ , \quad v = u - w.$$

Then we have from equation $(2.1)_2$ that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_2^2 dx + \int_\Omega B_{22}|v_{2x}|^2 dx = v_2(B_{22}(v_{2x} + w_{2x}) - f_2)|_{-1}^{+1}$$

$$+ \int_\Omega v_{2x}(f_2 - B_{22}w_{2x}) - w_{2t}v_2 + \varepsilon h_2 v_2 \; dx.$$

Since

$$v|_{x=\pm 1} = \mp\delta(v_x + w_x)|_{x=\pm 1}$$

and $B_{22} \ge \nu$, estimate (2.6) for u_2 follows by the Cauchy inequality.

From equation $(2.1)_1$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_1^2 dx + \int_\Omega B_{11}|v_{1x}|^2 dx = v_1(B_{11}(v_{1x} + w_{1x}) + B_{12}u_{2x} - f_1)|_{-1}^{+1}$$

$$-\int_{\Omega} v_{1x}(B_{11}w_{1x}+B_{12}u_{2x}-f_1)-w_{1t}v_1+\varepsilon h_1 v_1\,dx.$$

By the same argument, one can derive the claim of the lemma for the function u_1, using estimate (2.6) for u_2. $\square$

The following estimates depend, in general, on δ.

Lemma 2.3. *There are constants c and $\alpha\in(0,1)$ such that*

$$|u_2|_Q^{(\alpha)}\equiv\|u_2\|_{H^{\alpha,\alpha/2}(\overline{Q})}\le c. \tag{2.8}$$

Moreover, if $u_\pm\equiv 0$, the estimate (2.8) *holds uniformly in δ.*

Proof. Let $\zeta(t,x)$ be a test function with values between 0 and 1 and that is different from zero only for $x\in K_\rho$, the ball of radius ρ centered at $x^0\in\overline{\Omega}$. Denote

$$\Omega_\rho=\overline{\Omega}\cap K_\rho=[x^0_-,x^0_+],\quad x^0_+=\min\{1,x_0+\rho\},\quad x^0_-=\max\{-1,x_0-\rho\}.$$

Given $\delta'>0$, we multiply equation $(2.1)_1$ by

$$\zeta^2\max\{u_2-k,0\}\equiv\zeta^2u_2^{(k)},\quad k\ge-\delta',$$

and integrate over Ω_ρ. We have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_\rho}\zeta^2|u_2^{(k)}|^2\,dx+\int_{\Omega_\rho}\zeta^2B_{22}|u_{2x}^{(k)}|^2\,dx=\zeta^2B_{22}u_{2x}u_2^{(k)}\Big|_{x^0_-}^{x^0_+}-\zeta^2f_2u_2^{(k)}\Big|_{x^0_-}^{x^0_+}$$
$$-\int_{\Omega_\rho}2\zeta\zeta_xB_{22}u_{2x}u_2^{(k)}-\zeta\zeta_t|u_2^{(k)}|^2-f_2(2\zeta\zeta_xu_2^{(k)}+\zeta^2u_{2x}^{(k)})-\varepsilon h_2\zeta^2u_2^{(k)}\,dx.$$

Observe that $B_{22}\ge\nu$,

$$\delta u_x|_{x=\pm1}=\pm(u_\pm-u)|_{x=\pm1},$$
$$\zeta^2B_{22}u_{2x}u_2^{(k)}\Big|_{x^0_-}^{x^0_+}\le\frac{1}{\delta}\zeta^2B_{22}u_2^{(k)}u_{2+}|_{x=1}+\frac{1}{\delta}\zeta^2B_{22}u_2^{(k)}u_{2-}|_{x=-1}\ ,$$
$$|\zeta^2v^{(k)}|_{|x|=1}\le\Big|\int_{\Omega_\rho}\zeta^2v_x^{(k)}+2\zeta\zeta_xv^{(k)}\,dx\Big| \tag{2.9}$$

for small ρ. Thus,

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_\rho}\zeta^2|u_2^{(k)}|^2\,dx+\nu\int_{\Omega_\rho}\zeta^2|u_{2x}^{(k)}|^2\,dx\le\frac{\nu}{2}\int_{\Omega_\rho}\zeta^2|u_{2x}^{(k)}|^2\,dx \tag{2.10}$$
$$+c_1\int_{\Omega_\rho}|u_2^{(k)}|^2(|\zeta_x|^2+|\zeta\zeta_t|+\zeta^2)+\zeta^2\mathbf{1}_{A_{k,\rho}(t)}\,dx,$$

where $A_{k,\rho}(t)$ is the intersection of the support of $u_2^{(k)}$ with K_ρ, and $\mathbf{1}_A$ is the characteristic function of the set A. Proceeding in an analogous way, we prove that (2.9) also hods with u_2 replaced by $-u_2$, for $k\le1+\delta'$. These inequalities imply that u_2 belongs to a class $\mathcal{B}_2(Q,M,\gamma,r,\delta',\kappa)$ [6] (Chapter II, §7), with $M=1$, $r=6$, $\kappa=2$, and, hence, $u_2\in H^{\alpha,\alpha/2}(\overline{Q})$ for some $\alpha\in(0,1)$.

As for the last statement, indeed, in this case,

$$\zeta^2B_{22}u_{2x}u_2^{(k)}\Big|_{x^0_-}^{x^0_+}=-\sum_{x=\pm1}\frac{1}{\delta}\zeta^2B_{22}u_2u_2^{(k)}\le0,$$

and the constant c_1 in (2.10) does not depend on δ. $\square$

Lemma 2.4. *There is a constant c such that*

$$\max_{0\le t\le T}\{\int_\Omega u_{2x}^2\,dx+\delta(\sum_{x=\pm1}u_{2x}^2)\}+\int_Q u_{2xx}^2+u_{2x}^4+u_{2t}^2\,dx\,dt\le c\ . \tag{2.11}$$

Moreover, if $u_\pm\equiv 0$, the constant c in (2.11) *does not depend on δ.*

Proof. Let $\zeta(x)$ be the test function like above. We multiply equation $(2.1)_2$ by $(\zeta^2u_{2x})_x$ and integrate over K_ρ. Using the equality

$$\delta\dot u_n+\dot u=\dot u_\partial\quad\text{at}\quad x=\pm1,$$

the Young inequality, and inequality (2.9), we obtain that

$$\frac12\frac{d}{dt}\{\int_{\Omega_\rho}\zeta^2u_{2x}^2\,dx+\delta(\sum_{x=x_\pm^0}\zeta^2u_{2x}^2)\}+\nu\int_{\Omega_\rho}\zeta^2u_{2xx}^2\,dx\le J,$$

$$J=\|\dot u_\partial\|_{C([0,T])}\times\sum_{x=x_\pm^0}|\zeta^2u_{2x}|+\frac{\nu}{2}\int_{\Omega_\rho}\zeta^2u_{2xx}^2\,dx$$

$$+c_*\int_{\Omega_\rho}\zeta^2u_{2x}^4dx+c\int_{\Omega_\rho}\zeta^2(u_{2x}^2+u_{1x}^2)+u_{2x}^2(\zeta_x^2+\zeta^2)+\zeta^2\,dx.$$

Observe [6] that

$$\int_{K_\rho}\zeta^2v_x^4dx\le 16\mathrm{osc}^2\{v,K_\rho\}\int_{K_\rho}2\zeta^2v_{xx}^2+\zeta^2v_x^2\,dx. \tag{2.12}$$

By Lemma 2.3,

$$\mathrm{osc}^2\{u_2,K_\rho\}\le c\rho^{\alpha_1},\quad \alpha_1<\alpha.$$

Now, the assertion of the lemma follows if we take ρ such that $32c_*\rho^\alpha<\nu/4$, where α is the constant from Lemma 2.3. $\square$

Lemma 2.5. *There are constants c and $\alpha\in(0,1)$ such that $|u_1|_Q^{(\alpha)}\le c$. Moreover, if $u_\pm\equiv0$, the constant c in Lemma 2.5 does not depend on δ.*

Proof. Let $\zeta(t,x)$ be a function like in Lemma 2.3. Then, given $\delta'>0$, for $k\ge-\delta'$,

$$\frac12\frac{d}{dt}\int_{\Omega_\rho}\zeta^2|u_1^{(k)}|^2\,dx+\int_{\Omega_\rho}\zeta^2B_{11}|u_{1x}^{(k)}|^2\,dx=J_1+J_2\ ,\quad J_1=\zeta^2B_{11}u_{1x}u_1^{(k)}|_{x_-^0}^{x_+^0},$$

$$J_2=\int_{\Omega_\rho}\zeta\zeta_tu_1^{(k)}-2\zeta\zeta_xB_{11}u_{1x}u_1^{(k)}+u_1^{(k)}\zeta^2[(B_{12}u_{2x})_x-\frac{\partial f_1}{\partial x}+\varepsilon h_1]\,dx.$$

We have

$$J_1\le\frac1\delta\sum_{x=\pm1}\zeta^2B_{11}u_{1,\pm}u_1^{(k)}\overset{(2.9)}{\le}\int_{\Omega_\rho}\frac{\nu}{4}\zeta^2|u_{1x}^{(k)}|^2+c\zeta_x^2|u_1^{(k)}|^2+\zeta^2\mathbf{1}_{A_{k,\rho}(t)}\,dx,$$

$$J_2\le\int_{\Omega_\rho}|u_1^{(k)}|^2(|\zeta\zeta_t|+\zeta^2)+c(\int_{\Omega_\rho}\zeta\mathbf{1}_{A_{k,\rho}(t)})^{1/2}(\int_{\Omega_\rho}\mathbf{1}_{A_{k,\rho}(t)}(1+u_{2x}^4+u_{2xx}^2))^{1/2}.$$

As in Lemma 2.3, we can prove analogously that the above inequality also holds with u_1 replaced by $-u_1$ and $k\le1+\delta'$. Hence, (see [6] Ch. II, §7) the function u_1

belongs to a class $\mathcal{B}_2(Q, M, \gamma, r, \delta', \kappa)$, with $M = 1$, $r = 6$, $\kappa = 2$, and the lemma is proved. □

Lemma 2.6. *There is a constant c such that*

$$\max_{0\le t\le T}\{\int_\Omega u_{1x}^2\,dx + \delta \sum_{x=\pm 1} |u_{1x}|^2\} + \int_Q u_{1xx}^2 + u_{1x}^4 + u_{1t}^2\,dx\,dt \le c.$$

Moreover, if $u_\pm \equiv 0$, the constant c in the above lemma does not depend on δ.

Proof. Let $\zeta(x)$ be a test function like in Lemma 2.5. Then it follows from equation $(2.1)_1$ that

$$\frac{1}{2}\frac{d}{dt}\{\int_{\Omega_\rho} \zeta^2 u_{1x}^2\,dx + \delta(\sum_{x=x_\pm^0} \zeta^2 u_{1x}^2)\} + \nu \int_{\Omega_\rho} \zeta^2 u_{1xx}^2\,dx \le cJ,$$

$$J = \|\dot{u}_\partial\|_{C([0,T])} \sum_{x=x_\pm^0} |\zeta^2 u_{1x}| + \frac{\nu}{2}\int_{\Omega_\rho} \zeta^2 u_{1xx}^2\,dx + \int_{\Omega_\rho} \zeta_x^2 (u_{2x}^2 + u_{1x}^2 + u_{2xx}^2 + u_{2x}^4)\,dx$$

$$+ \int_{\Omega_\rho} \zeta^2 (u_{1x}^4 + u_{2x}^4 + u_{1x}^2\zeta_x^2 + u_{2xx}^2 + u_{1x}^2 + 1 + u_{2x}^4\zeta_x^4 + u_{2x}^2\zeta_x^2).$$

Applying inequality (2.12), one arrives at the conclusion of the lemma. □

Lemma 2.7. *There is a constant c such that*

$$\int_Q |u_{ix}|^6\,dx\,dt \le c, \quad \int_Q |u_{ix}u_{ixx}|^2\,dx\,dt \le c, \quad \int_Q |u_{ixx}|^3\,dx\,dt \le c. \tag{2.13}$$

Proof. We start with the simple inequality

$$\int_Q |u_{ix}|^6\,dx\,dt \le \int_0^T \max_{x\in\Omega} |u_{ix}|^4 \int_\Omega |u_{ix}|^2\,dx\,dt.$$

Observe that for any x and y,

$$v_x^2(x) - v_x^2(y) = 2\int_x^y v_{xx}v_x dz\ ,$$

so

$$\max_{|x|<1} v_x^4 \le \frac{1}{2}\|v_x\|_{L^2(\Omega)}^2 + 8\|v_{xx}\|_{L^2(\Omega)}^2 \|v_x\|_{L^2(\Omega)}^2.$$

Hence,

$$\|u_{ix}\|_{L^6(Q)}^6 \le \frac{1}{2}\|u_{ix}\|_{L^\infty(0,T;L^2(\Omega))}^4 (1 + \|u_{ixx}\|_{L^2(Q)}^2) \le c,$$

and the first estimate of the lemma is proved.

Let us write equation $(2.1)_2$ as

$$\begin{cases} u_{2t} = B_{22}u_{2xx} + F, \\ F = \frac{\partial B_{22}}{\partial u_2}|u_{2x}|^2 - (\frac{\partial f_2}{\partial u_1}u_{1x} + \frac{\partial f_2}{\partial u_2}u_{2x}) + \frac{\partial B_{22}}{\partial u_1}u_{1x}u_{2x} + \varepsilon h_2 \end{cases} . \tag{2.14}$$

By Lemmas 2.4 and 2.6, $\|F\|_{L^3(Q)} \leq c$. Now, the theory of linear parabolic equations (see [6], Ch. IV,§9) can be applied to derive the estimate

$$\int_Q |u_{2xx}|^3 \, dx \, dt \leq c. \tag{2.15}$$

The second estimate in (2.13) for u_2 follows because of the inequality

$$\int_Q |uv|^2 \, dx \, dt \leq (\int_Q |u|^6 \, dx \, dt)^{1/3} (\int_Q |v|^3 \, dx \, dt)^{2/3}. \tag{2.16}$$

Equation $(2.1)_1$ writes

$$u_{1t} = B_{11} u_{1xx} + F, \tag{2.17}$$

$$F = u_{1x}(\frac{\partial B_{11}}{\partial u_1} u_{1x} + \frac{\partial B_{11}}{\partial u_2} u_{2x}) + u_{2x}(\frac{\partial B_{12}}{\partial u_1} u_{1x} + \frac{\partial B_{12}}{\partial u_2} u_{2x}) + B_{12} u_{2xx}$$
$$- (\frac{\partial f_1}{\partial u_1} u_{1x} + \frac{\partial f_1}{\partial u_2} u_{2x}) + \varepsilon h_1,$$

with $\|F\|_{L^3(Q)} \leq c$. Hence, the estimate (2.15) is also valid for u_{1xx}. Now, the second estimate in (2.13) for u_1 follows due to inequality (2.16).

□

Lemma 2.8. *There are constants c and $\alpha \in (0,1)$ such that $|u_{2x}|_Q^{(\alpha)} \leq c$.*

Proof. Let us differentiate equation $(2.1)_2$ with respect to x. The function $u_{2x} = v$ solves the linear equation

$$v_t = (B_{22} v_x)_x + F + g_x,$$
$$F = \frac{\partial^2 B_{22}}{\partial u_2^2}(u_{2x})^3 + 2\frac{\partial B_{22}}{\partial u_2} u_{2x} u_{2xx} + \frac{\partial B_{22}}{\partial u_1}(u_{1xx} u_{2x} + u_{1x} u_{2xx})$$
$$+ \frac{\partial^2 B_{22}}{\partial u_1^2} u_{1x}^2 u_{2x} + 2\frac{\partial^2 B_{22}}{\partial u_1 \partial u_2} u_{1x} u_{2x}^2,$$
$$g = -\frac{\partial f_2}{\partial u_1} u_{1x} - \frac{\partial f_2}{\partial u_2} u_{2x} + \varepsilon h_2.$$

By the above lemmas,

$$\|F\|_{q,r,Q} \equiv (\int (\int_\Omega F^q)^{r/q} dt)^{1/r} \leq c, \quad \|g^2\|_{q,r,Q} \leq c,$$

when $q = 2$, $r = 2$. Clearly, the constants q and r satisfy the conditions

$$\frac{1}{r} + \frac{1}{2q} = 1 - \kappa, \quad 0 < \kappa < \frac{1}{2}, \quad q \in [1, \infty], \quad r \in [\frac{1}{1-\kappa}, \frac{2}{1-2\kappa}],$$

with $\kappa = 1/4$. Moreover, it follows from the boundary conditions (3.2) and Lemma 3.3, that

$$\|v(t, \pm 1)\|_{H^{\alpha/2}([0,T])} \leq c.$$

Thus, by the theory of linear parabolic equations [LSU, Ch.III,§10]

$$|v|_Q^{(\alpha')} \leq c$$

for some $\alpha' \leq \alpha$. □

Lemma 2.9. *There are constants c and $\alpha \in (0,1)$ such that $|u_{1x}|_Q^{(\alpha)} \le c$.*

Proof. Denoting $u_{1x} = v$ and differentiating equation $(2.1)_1$ with respect to x, we have

$$v_t = (B_{11}(u_1,u_2)v_x)_x + F_1 + g_{1x}, \tag{2.18}$$

$$F_1 = u_{1xx}(\frac{\partial B_{11}}{\partial u_1}u_{1x} + \frac{\partial B_{11}}{\partial u_2}u_{2x}) + u_{1x}\{\frac{\partial B_{11}}{\partial u_1}u_{1xx} + u_{1x}(\frac{\partial^2 B_{11}}{\partial u_1^2}u_{1x} + \frac{\partial^2 B_{11}}{\partial u_1 \partial u_2}u_{2x})$$

$$+\frac{\partial B_{11}}{\partial u_2}u_{2xx} + u_{2x}(\frac{\partial^2 B_{11}}{\partial u_1\partial u_2}u_{1x} + \frac{\partial^2 B_{11}}{\partial u_2^2}u_{2x})\},$$

$$g_1 = u_{2x}(\frac{\partial B_{12}}{\partial u_1}u_{1x} + \frac{\partial B_{12}}{\partial u_2}u_{2x}) + B_{12}u_{2xx} - (\frac{\partial f_1}{\partial u_1}u_{1x} + \frac{\partial f_1}{\partial u_2}u_{2x}) + \varepsilon h_1.$$

By Lemma 2.7, $\|F_1\|_{L^2(Q)} \le c$. With the estimate of Lemma 2.8 for u_{2x} at hand, the function F in equation (2.14) meets the estimate $\|F\|_{L^4(Q)} \le c$. It implies that

$$\|u_{2xx}\|_{L^4(Q)} \le c, \quad \|g_1\|_{L^4(Q)} \le c.$$

Now, one may treat equation (2.18) as a linear parabolic one for v, with

$$\|F_1, g_1^2\|_{2,2,Q} \le c, \quad \|v(t,\pm 1)\|_{H^{\alpha/2}([0,T])} \le c.$$

By the same argument like in Lemma 2.8, we conclude that

$$\|u_{1x}\|_{H^{\alpha',\alpha'/2}(\overline{Q})} \le c$$

for some $\alpha' < \alpha$. □

Lemma 2.10. *Let*

$$u_0 \in H^{2+\beta}(\overline{\Omega}), \quad u_\pm \in H^{1+\beta/2}([0,T]), \quad 0<\beta<1,$$

and the compatibility conditions

$$\pm\delta u_0'(\pm 1) + u_0(\pm 1) = u_\pm(0) \tag{2.19}$$

be satisfied. Then there is a constant c such that solutions to problem (2.1),(2.2) *satisfy the estimate $|u|_Q^{(2+\beta)} \le c$. The constant c does not depend on ε but depends on T, $\|u_0|H^{2+\beta}(\overline{\Omega})\|$, $\|u_\pm|H^{1+\beta/2}([0,T])\|$, and the L^∞–norms of $f(u)$, $\nabla_u f$, $B_{ij}(u)$, $\nabla_u B_{ij}$, and $\frac{\partial^2 B_{ij}}{\partial u_i \partial u_j}$.*

Proof. First, we observe that the data $u_{0\varepsilon}$ and $u_{\partial\varepsilon}$ also satisfy the compatibility conditions (2.19). We know from the above lemmas that there are constants c and $\alpha \in (0,1)$ such that $|u_i, u_{ix}|_Q^{(\alpha)} \le c$. If $\gamma = \min\{\alpha,\beta\}$, it follows from the linear equation (2.14) that $|u_2|_Q^{(2+\gamma)} \le c$ [LSU, Ch.IV, §5]. By the same argument, we conclude from equation (3.17) that $|u_1|_Q^{(2+\gamma)} \le c$. To increase γ up to β, one should return to problem (2.14), which now ensures that $|u_2|_Q^{(2+\beta)} \le c$. Next, one should pass to equation (2.17) to make sure that $|u_1|_Q^{(2+\beta)} \le c$. □

3. Existence and uniqueness

To prove the solvability of problem (2.1),(2.2), we apply a fixed point argument in the form of the Leray-Schauder principle as in [6]. Let $\mathbf{B}$ be a Banach space of vector-functions $\mathbf{u}(t,x) \in \mathbb{R}^2$, having the bounded norm

$$\|\mathbf{u}\|_{\mathbf{B}} = |\mathbf{u}|_Q^{(\beta)} + |\mathbf{u}_x|_Q^{(\beta)}.$$

Given $\mathbf{a} \equiv (a_1, a_2) \in \mathbf{B}$ and $\lambda \in [0,1]$, we define $\mathbf{u} = (u, v)$ as a solution to the linear problem

$$v_t + \lambda[\frac{\partial f_2(\mathbf{a})}{\partial x} - (B_{22}(\mathbf{a})v_x)_x - \varepsilon h_2(\mathbf{a})] = (1-\lambda)v_{xx},$$

$$u_t + \lambda[\frac{\partial f_1(\mathbf{a})}{\partial x} - (B_{11}(\mathbf{a})u_x)_x - (B_{12}(\mathbf{a})v_x)_x - \varepsilon h_1(\mathbf{a})] = (1-\lambda)u_{xx},$$

$$\delta \mathbf{u}_n + \mathbf{u} = \mathbf{u}_{\partial\varepsilon}, \quad \mathbf{u}|_{t=0} = \mathbf{u}_{0\varepsilon} = (1-\varepsilon)(\frac{\varepsilon}{2} + u_{01}, \frac{\varepsilon}{2} + u_{02}).$$

The first equation does not involve the function u. So, by the theory of linear parabolic equations, the operator $\mathbf{a} \mapsto \mathbf{u} \equiv A_\lambda(\mathbf{a})$ is well-defined, and his fixed points are solutions to problem (2.1),(2.2) when $\lambda = 1$. By repeating the arguments of the lemmas in Section 2, one arrives at the a priori estimates for the fixed points $\mathbf{u}_\lambda$ of the operator A_λ:

$$\mathbf{u}_\lambda \in \Delta, \quad |\mathbf{u}_\lambda, \mathbf{u}_{\lambda x}|_Q^{(\beta)} \le M, \quad |\mathbf{u}_\lambda|_Q^{(2+\beta)} \le M_1,$$

where the constants M, M_1 are independent of λ. We restrict A_λ to the set

$$\mathcal{U} = \{\mathbf{u} \in \mathbf{B} : \mathbf{u} \in \Delta', \quad |\mathbf{u}_\lambda, \mathbf{u}_{\lambda x}|_Q^{(\beta)} \le M', \quad \mathbf{u}|_{t=0} = \mathbf{u}_{0\varepsilon}(x), \quad \delta\mathbf{u}_n + \mathbf{u} = \mathbf{u}_{\partial\varepsilon}\},$$

where $\mathrm{int}(\Delta') \supset \bar{\Delta}$ and $M' > M$. Clearly, $\mathcal{U}$ is a bounded convex set in B, and all the fixed-points $\mathbf{u}_\lambda$ of A_λ are strictly inside of $\mathcal{U}$.

As in [6], one can prove that the other conditions of the Leray-Schauder theorem are also verified. Namely,

(i) The set $A_\lambda(\mathcal{U})$ is compact in $\mathbf{B}$ for each $\lambda \in [0,1]$;
(ii) the mapping $\mathbf{a} \to A_\lambda(\mathbf{a})$ is continuous on $\mathcal{U}$ uniformly in $(\mathbf{a}, \lambda) \in \mathcal{U} \times [0,1]$;
(iii) the mapping $\lambda \to A_\lambda(\mathbf{a})$ is continuous in $(\mathbf{a}, \lambda) \in \mathcal{U} \times [0,1]$;
(iv) the operator A_0 has a unique fixed point inside of $\mathcal{U}$, and the mapping $\mathbf{a} \mapsto \mathbf{a} - A_0(\mathbf{a})$ has an inverse near this fixed point.

Hence, problem (2.1),(2.2) has at least one solution in the Hölder space $H^{2+\beta,1+\beta/2}(\bar{Q})$. Uniqueness can be established in the same manner as in [6]. Thus, we have proved the following.

Theorem 3.1. *Let the functions $f(u)$, $\nabla_u f$, $B_{ij}(u)$, $\nabla_u B_{ij}$, $\frac{\partial^2 B_{ij}}{\partial u_i \partial u_j}$, and the function $h(u)$ be Hölder continuous with the Hölder exponent $\beta \in (0,1)$. Let the conditions of Lemma 2.10 be satisfied. Then problem (2.1),(2.2) has a unique solution $u(t,x) \in H^{2+\beta,1+\beta/2}(\bar{Q})$ such that $\mathbf{u}(t,x) \in \Delta$ for each $(t,x) \in Q$.*

Proof of Theorem 1.1. Since the estimate of Lemma 2.10 does not depend on ε there is a sequence $\varepsilon_k \downarrow 0$, such that the corresponding sequence u_k of solutions of problem (2.1),(2.2) converges to a function $u \in H^{2+\beta,1+\beta/2}(\overline{Q})$ in the norm $|\cdot|_Q^{(2+\gamma)}$ for any $\gamma < \beta$. Clearly, u solves the problem (1.1),(1.4),(1.6). Thus, Theorem 1.1 is proved. □

4. Dirichlet boundary conditions

In this section we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let us consider the Dirichlet problem (1.1),(1.5),(1.6). The estimates (2.6) and (2.7) are uniform with respect to $\delta \downarrow 0$. By the Aubin-Lions compactness theorem [7], they imply that there are a sequence u_k, $\delta_k \downarrow 0$, of solutions to problem (1.1),(1.4),(1.6) and a function u such that

$$u \in L^\infty(Q), \quad u_x \in L^2(Q), \quad u_t \in L^2(0,T; W^{-1,2}(\Omega)), \tag{4.1}$$

$$u_k \to u \quad \text{in} \quad L^2(Q), \quad u(t,x) \in \Delta \quad \text{for each} \quad (t,x) \in Q.$$

Clearly, the function u solves equation (1.1) weakly. Now, given any open set Q' with $\overline{Q'} \subseteq Q$, we have that u_k is uniformly bounded in $H^{2+\beta,1+\beta/2}(\overline{Q'})$, so that $u \in H^{2+\beta,1+\beta/2}(\overline{Q'})$. In particular, u is a classical solution of (1.1) in Q. Due to estimate (2.6), the boundary condition $u|_{x=\pm1} = u_\pm$ holds in $L^2(0,T)$. The inclusions (4.1) imply that $u \in C(0,T; L^2(\Omega))$, so the function u satisfies the initial condition $u|_{t=0} = u_0$ weakly in $L^2(\Omega)$. This proves the first part of Theorem 1.2. The last part is a consequence of the fact that the estimates of Lemmas 2.2-2.5 are uniform in δ. Therefore, we have $u \in H^{\beta,\beta/2}(\overline{Q})$, and, so, the last assertion follows. □

5. Basic equations of three-phase flow

For the reader's convenience we recall the underlying laws of multiphase flows in a porous medium [1]. We consider one-dimensional horizontal flows of three incompressible immiscible fluids formed in phases. The balance of masses is governed by the mass conservation equations

$$\frac{\partial}{\partial t}(m u_i \rho_i) + \frac{\partial}{\partial x}(\rho_i v_i) = 0, \tag{5.1}$$

where m denotes porosity of the porous medium, u_i, ρ_i, and v_i are the saturation, density, and seepage velocity of the i-th phase. The functions u_i satisfy the volume-balance equation

$$u_1 + u_2 + u_3 = 1. \tag{5.2}$$

The theory of multiphase flows in porous media is based on the following form of Darcy's law

$$v_i = -k\lambda_i p_{ix}, \quad \lambda_i = \lambda_i(u_1, u_2), \tag{5.3}$$

where k stands for the absolute permeability, λ_i is the mobility of the i-th phase, and p_i is the pressure of the i-th phase.

The capillary pressures are defined as the pressure differences (cf.,e.g.,[9, 1]), and we assume here that they are functions of the saturations u_1, u_2, that is,

$$P_1(u_1,u_2) = p_1 - p_3, \quad P_2(u_1,u_2) = p_2 - p_3. \tag{5.4}$$

Denote

$$\lambda = \sum_1^3 \lambda_i, \quad f_i = \frac{\lambda_i}{\lambda}, \qquad i = 1,2,3.$$

For

$$v = \sum_1^3 v_i,$$

we find from (5.1) and (5.2) that $v_x = 0$, so v depends on t only. We assume for simplicity that $v \equiv 1$ and $k = m = 1$ as well.

Eliminating the pressure derivative p_{3x}, we have from (2.3)

$$v_1 = f_1(1 + \lambda_2 P_{2x} - (\lambda_2 + \lambda_3)P_{1x}), \quad v_2 = f_2(1 + \lambda_1 P_{1x} - (\lambda_1 + \lambda_3)P_{2x}).$$

When we substitute these velocities into the first two equations in (5.1) we obtain the system (1.1), where the 2×2-matrix B is given by

$$B_{11} = \frac{\lambda_1(\lambda_2 + \lambda_3)}{\lambda}\frac{\partial P_1}{\partial u_1} - \frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_2}{\partial u_1}, \; B_{12} = -\frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_2}{\partial u_2} + \frac{\lambda_1(\lambda_2 + \lambda_3)}{\lambda}\frac{\partial P_1}{\partial u_2},$$

$$B_{21} = \frac{\lambda_2(\lambda_1 + \lambda_3)}{\lambda}\frac{\partial P_2}{\partial u_1} - \frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_1}{\partial u_1}, \quad B_{22} = -\frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_1}{\partial u_2} + \frac{\lambda_2(\lambda_1 + \lambda_3)}{\lambda}\frac{\partial P_2}{\partial u_2}.$$

Since u is a saturation vector, we should impose the restriction $u \in \Delta$.

The mobilities λ_i are subject to natural restrictions [1]

$$\lambda_i \geq 0, \quad \lambda_i|_{u_i=0} = 0, \quad i \in \{1, 2, 3\}. \tag{5.5}$$

Let us formulate hypotheses on the capillary pressure laws (5.4).

Whatever the function $P_i(u_1, u_2)$ are, they manifest itself in equations (1.1) only through the matrix B. We assume that

$$B_{21} = 0, \quad B_{22} = B_{22}(u_2) \geq 0, \quad B_{11} \geq 0 \quad \text{in} \quad \Delta. \tag{5.6}$$

The capillarity-diffusion hypothesis (5.6) means that the first and the third phases are not responsible for the amount of diffusion in the equation for the second phase. The first two conditions in (5.6) read

$$A\frac{\partial P_1}{\partial u_1} = \frac{\partial P_2}{\partial u_1}, \quad \frac{\partial P_2}{\partial u_2} = A\frac{\partial P_1}{\partial u_2} + \frac{\lambda D_{22}}{\lambda_2(\lambda_1 + \lambda_3)}, \quad A = \frac{\lambda_1}{\lambda_1 + \lambda_3}. \tag{5.7}$$

We study these equations for $P_i(u_1, u_2)$ in the case when the mobilities λ_i are linear functions:

$$\lambda_i = k_i u_i, \quad k_i = \text{const}. \tag{5.8}$$

A symmetry group analysis (see [8]), performed for system (5.7), suggests to look for solutions of the form

$$P_i = q_i(\xi) + Q_i(u_2), \quad \xi = \frac{u_1}{1 - u_2} \equiv \frac{u_1}{u_1 + u_3}. \tag{5.9}$$

It follows from (5.7) that the functions q_i and Q_i solve the system

$$q_2'(\xi) = q_1'(\xi)A(\xi), \quad A = \frac{k_1\xi}{(k_1 - k_3)\xi + k_3}, \quad Q_1'(u_2) = -\frac{k_0 B_{22}(u_2)}{1 - u_2},$$

$$Q_2'(u_2) = B_{22}(u_2)\left(\frac{1}{k_3(1 - u_2)} + \frac{1}{k_2 u_2}\right), \quad k_0 = \frac{k_3 - k_1}{k_1 k_3}. \tag{5.10}$$

Assume that the capillary pressure

$$p_1 - p_3 \equiv P_1(u)$$

is a given function at the two-phase boundary $u_2 = 0$ of the triangle Δ:

$$P_1|_{u_2=0} = p_{13}(u_1).$$

Assume also that the capillary pressure

$$p_2 - p_3 \equiv P_2(u)$$

is a given function at the two-phase boundary $u_1 = 0$ of the triangle Δ:

$$P_2|_{u_1=0} = p_{23}(u_2).$$

It follows from (5.9) that

$$p_{13}(u_1) = q_1(u_1) + Q_1(0), \quad p_{23}(u_2) = q_2(0) + Q_2(u_2).$$

It is naturally to set

$$q_1(\xi) = p_{13}(\xi), \quad Q_2(u_2) = p_{23}(u_2).$$

Then the other functions $Q_1(u_2)$ and $q_2(\xi)$ are defined from (5.10) as follows:

$$q_2(\xi) = \int_0^\xi A(\xi)p_{13}'(\xi)d\xi, \quad B_{22}(u_2) = \frac{k_2k_3u_2(1-u_2)p_{23}'(u_2)}{k_2u_2 + k_3(1-u_2)},$$

$$Q_1'(u_2) = -\frac{k_0}{1-u_2}B_{22}(u_2).$$

Thus, we arrive at the formulas for the capillary pressures:

$$P_1(u_1,u_2) = p_{13}(\xi) - \int_0^{u_2} \frac{k_0k_2k_3u_2p_{23}'(u_2)}{k_2u_2 + k_3(1-u_2)}\, du_2 + \text{const},$$

$$P_2(u_1,u_2) = \int_0^\xi A(\xi)p_{13}'(\xi)\; d\xi + p_{23}(u_2) + \text{const}. \tag{5.11}$$

We call the process yielding formulas (5.11) *the method of physical interpolation* since these formulas define the pressure differences $p_i - p_j$ in Δ starting from the values of

$$p_1 - p_3 \quad \text{at} \quad \{u_2 = 0\} \cap \Delta \quad \text{and} \quad p_2 - p_3 \quad \text{at} \quad \{u_1 = 0\} \cap \Delta.$$

Due to the definition (5), we have

$$B_{11} = k_1(1 - A(\xi))p_{13}'(\xi), \quad B_{22} = \frac{k_2k_3u_2(1-u_2)p_{23}'(u_2)}{k_2u_2 + k_3(1-u_2)},$$

$$B_{21} = 0, \quad B_{12} = \xi(B_{11} - B_{22}) \tag{5.12}$$

When

$$p_{13}'(\xi) \geq 0, \quad p_{23}'(u_2) \geq 0,$$

system (1.1) is parabolic and degenerate at $\partial\Delta$. An important property of the matrix B given by (5.12) is that the vector $\nabla_u G_i$ is an eigenvector of $B^\top$ at $\{G_i(u) = 0\} \cap \Delta$:

$$B^\top\langle\nabla_u G_i\rangle = \mu_i \nabla_u G_i, \qquad \mu_i \geq 0. \tag{5.13}$$

This property is easily verified if one observes that equalities (5.13) are equivalent to

$$B_{12} = 0, \quad B_{21} = 0, \quad B_{11} - B_{12} - B_{22} = 0,$$

at the sets $G_1 = 0$, $G_2 = 0$, and $G_3 = 0$ respectively.

We observe also that $\nabla_u G_i(u)$ is an eigenvector of the matrix $(f')^\top$ for any u such that $G_i(u) = 0$:

$$(f')^\top\langle\nabla_u G_i\rangle = \alpha_i \nabla_u G_i.$$

We verify this property only for the function G_3 since the other cases require less calculations. The fact that $\nabla_u G_3$ is an eigen-vector of $(f')^\top$ writes

$$\frac{\partial f_1}{\partial u_1} + \frac{\partial f_2}{\partial u_1} = \frac{\partial f_1}{\partial u_2} + \frac{\partial f_2}{\partial u_2} \quad \text{if} \quad u_1 + u_2 = 1.$$

When $\lambda_3 = 0$ this equality is equivalent to

$$\frac{\partial \lambda_3}{\partial u_1} = \frac{\partial \lambda_3}{\partial u_2} \quad \text{if} \quad u_1 + u_2 = 1.$$

It really holds since, by the hypothesis (5.5),

$$\lambda_3(u_1, u_2) = 0 \quad \text{if} \quad u_1 + u_2 = 1.$$

6. A degenerate problem

Here, we study a particular system arising in petroleum reservoir fluid flows. We assume that mobilities are linear functions

$$\lambda_i = k_i u_i, \quad i \in \{1,2,3\}, \quad k_1 = k_3,$$

and the capillary pressures are given by the formulas

$$P_1 = \alpha\xi, \quad P_2 = \frac{\alpha}{2}\xi^2 + \frac{\beta}{2}u_2^2\left(\frac{1}{k_1} - \frac{1}{k_2}\right) + \frac{\beta}{k_2}u_2\ , \quad \xi = \frac{u_1}{1-u_2}.$$

In this case, the flow is governed by the degenerate parabolic system

$$\begin{gathered} u_{1t} + f_1(u_1,u_2)_x = (B_{11}(u_1,u_2)u_{1x})_x + (B_{12}(u_1,u_2)u_{2x})_x, \\ u_{2t} + f_2(u_2)_x = (B_{22}(u_2)u_{2x})_x, \end{gathered} \tag{6.1}$$

with

$$f_1 = \frac{u_1}{ku_2+1}, \quad f_2 = (1+k)\frac{u_2}{ku_2+1}, \quad k = \frac{k_2}{k_1} - 1,$$

$$B_{11} = \alpha k_1 \xi(1-\xi), \quad B_{22} = \beta u_2(1-u_2), \quad B_{12} = \xi(B_{11} - B_{22}), \quad B_{21} = 0.$$

Equations $(6.1)_1$ and $(6.1)_2$ are coupled through the condition $u(t,x) \in \Delta$, which can be written as

$$0 \le u_i(t,x) \le 1, \quad u_2(t,x) \le 1 - u_1(t,x). \tag{6.2}$$

We consider the Dirichlet initial-boundary value problem

$$u|_{x=\pm1} = u_\pm(t), \quad u|_{t=0} = u_0(x). \tag{6.3}$$

Proof of Theorem 1.3. Consider the approximate non-degenerate problem

$$\begin{cases} u_t + f(u)_x = (B^\nu(u)u_x)_x, \\ \nu u_n + u = u_\partial^\nu \quad \text{at} \quad |x| = 1, \quad u|_{t=0} = u_0^\nu(x), \end{cases} \tag{6.4}$$

with

$$B_{11}^\nu = \nu + \chi_\nu(u_2)B_{11}, \quad B_{22}^\nu = \nu + \chi_\nu(u_2)B_{22}, \quad B_{12}^\nu = \chi_\nu(u_2)\xi(B_{11}^\nu - B_{22}^\nu),$$

$$u_0^\nu \in H^{2+\beta}(\overline{\Omega}), \quad u_0^\nu(x) \in \Delta, \quad u_\pm^\nu \in H^{1+\beta/2}([0,T]), \quad u_\pm^\nu(t) \in \Delta,$$

$$\pm\nu u_0^\nu(\pm1) + u_0^\nu(\pm1) = u_\pm^\nu(0),$$

$$\|u_\pm^\nu - u_\pm\|_{W^{1,1}(0,T)} \to 0, \quad \|u_0^\nu - u_0\|_{L^2(\Omega)} \to 0, \quad \text{as} \quad \nu \downarrow 0.$$

Here, $\chi_\nu(u_2)$ is a smooth function such that

$$\chi_\nu(u_2) = 1 \quad \text{if} \quad 0 \le u_2 \le 1-\nu, \quad \chi_\nu(u_2) = 0, \quad \text{if} \quad 1 - \frac{\nu}{2} \le u_2 \le 1.$$

Clearly, the matrix B^ν satisfies the hypotheses of Theorem 1.1, and so we have the unique solvability of problem (6.4). We also observe that, under the conditions of Theorem 1.3 on the data u_0^ν and $u_\pm^\nu$, any smooth solution of problem (6.4) satisfies the a priori estimate

$$\delta \leq u_2(t,x) \leq 1-\delta. \tag{6.5}$$

With this estimate at hand, the matrix B^ν, for small ν, reads

$$B_{11}^\nu = \nu + B_{11}, \quad B_{22}^\nu = \nu + B_{22}, \quad B_{12}^\nu = \xi(B_{11}^\nu - B_{22}^\nu). \tag{6.6}$$

We then have $B_{22}^\nu(u^\nu) \geq \delta^2$ uniformly in $\nu \downarrow 0$. Thus, the constant c in Lemma 2.2 does not depend on ν, and

$$\|u_{2x}^\nu\|_{L^2(Q)} \leq c, \quad \|u_{2t}^\nu\|_{L^2(0,T;W^{-1,2}(\Omega))} \leq c, \tag{6.7}$$

uniformly in ν.

Taking into account the last equality in (6.6) for the entry B_{12}^ν, one can calculate from equations (6.4) that the function $\xi = u_1^\nu/(1-u_2^\nu)$ solves the problem

$$\xi_t + \frac{\xi_x}{ku_2^\nu + 1} = (B_{11}^\nu \xi_x)_x - \frac{\xi_x u_{2x}^\nu (B_{11}^\nu + B_{22}^\nu)}{1-u_2^\nu}\ ,$$

$$\frac{\nu(1-u_2^\nu)}{1-u_\pm^\nu}\xi_n + \xi = \xi_\pm \quad \text{at} \quad x = \pm 1, \quad \xi|_{t=0} = \xi_0(x).$$

By the maximum principle,

$$\delta \leq \xi(t,x) \leq 1-\delta\ , \tag{6.8}$$

uniformly in ν. Now, it is a consequence of (6.5) and (6.8) that

$$\delta^2 \leq u_1^\nu(t,x) \leq (1-\delta)^2, \quad B_{11}^\nu \geq \delta^2.$$

By the same argument as in Lemma 2.2, we have

$$\|u_{1x}^\nu\|_{L^2(Q)} \leq c, \quad \|u_{1t}^\nu\|_{L^2(0,T;W^{-1,2}(\Omega))} \leq c, \tag{6.9}$$

uniformly in ν.

Estimates (6.7) and (6.9) imply by the Aubin-Lions compactness theorem that there are a sequence $u^n \equiv u^{\nu_n}$ and a function u such as described in Theorem 1.3 and such that

$$u^n(t,x) \to u(t,x) \quad \text{a.e. in} \quad Q, \quad u_x^n \to u_x \quad \text{weakly in} \quad L^2(Q).$$

Clearly, u is a weak solution of problem (6.1)-(6.3). Theorem 1.3 is proved. □

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