COMPOSITE WAVES IN THE DAFERMOS REGULARIZATION

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Abstract. We show that composite-wave Riemann solutions of scalar conservation laws have nearby scale-invariant solutions of the Dafermos regularization.

1. Introduction

The result of this paper is that certain composite-wave solutions of scalar conservation laws have nearby scale-invariant solutions of the Dafermos regularization. This result supports the conjecture that all structurally stable Riemann solutions have scale-invariant solutions of the Dafermos regularization nearby. It thereby supports the validity of approximating Riemann solutions by numerically computing scale-invariant solutions of the Dafermos regularization, as advocated in [10].

A *system of conservation laws* in one space dimension is a partial differential equation of the form

$$u_t + f(u)_x = 0, \tag{1.1}$$

with $t \geq 0$, $x \in \mathbb{R}$, $u(x,t) \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ a smooth map. The simplest discontinuous solutions of (1.1) are the centered, piecewise constant *shock waves* defined by

$$u(x,t) = \begin{cases} u_- & \text{for } x < st, \\ u_+ & \text{for } x > st. \end{cases} \tag{1.2}$$

The triple (u_-, s, u_+) is required to satisfy the *Rankine-Hugoniot condition*

$$f(u_+) - f(u_-) - s(u_+ - u_-) = 0. \tag{1.3}$$

This condition follows from the requirement that (1.2) be a weak solution of (1.1) [15]. We shall require in addition that the shock wave (1.2) satisfy the *viscous profile criterion* for the viscosity u_{xx}: The differential equation

$$u_t + f(u)_x = u_{xx} \tag{1.4}$$

must have a traveling wave solution $u(x - st)$ that satisfies the boundary conditions

$$u(-\infty) = u_-, \quad u(+\infty) = u_+. \tag{1.5}$$

More general viscosities can be considered, as in [13] and [10], but we shall not do so here.

A traveling wave solution $u(x - st)$ of (1.4) that satisfies (1.5) exists if and only if the ODE

$$\dot{u} = f(u) - f(u_-) - s(u - u_-) \tag{1.6}$$

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has an equilibrium at u_+ (it automatically has one at u_-) and a connecting orbit from u_- to u_+. The condition that (1.6) have an equilibrium at u_+ is just the Rankine-Hugoniot condition (1.3).

A *Riemann problem* for (1.1) is (1.1) together with the initial condition

$$u(x,0) = \begin{cases} u_L & \text{for } x < 0, \\ u_R & \text{for } x > 0. \end{cases} \tag{1.7}$$

One seeks piecewise continuous weak solutions of Riemann problems in the scale-invariant form $u(x,t) = \hat{u}(\xi)$, $\xi = \frac{x}{t}$. Usually one requires that the solution consist of a finite number of constant parts, continuously changing parts (rarefaction waves), and jump discontinuities (shock waves). Shock waves occur when

$$\lim_{\xi \to s-} \hat{u}(\xi) = u_- \neq u_+ = \lim_{\xi \to s+} \hat{u}(\xi).$$

The triple (u_-, s, u_+) is required to satisfy the viscous profile criterion for u_{xx}.

The importance of Riemann problems is twofold. First, an understanding of Riemann problems leads to a more general understanding of initial value problems for systems of conservation laws. Second, solutions of viscous conservation laws such as (1.4), with boundary conditions $u(-\infty, t) = u_L$, $u(+\infty, t) = u_R$, when expressed in variables $(\frac{x}{t}, t)$, often approach, as $t \to \infty$, solutions of the corresponding Riemann problem.

In [1], Dafermos proposed a different regularization of (1.1) which has less physical motivation than (1.4) but which has a close relation to Riemann solutions:

$$u_t + f(u)_x = \epsilon t u_{xx}. \tag{1.8}$$

Like the Riemann problem, but unlike (1.4), (1.8) has many scale-invariant solutions $u(x,t) = \hat{u}(\xi)$, $\xi = \frac{x}{t}$. They satisfy the nonautonomous second-order ODE

$$(Df(u) - \xi I)\frac{du}{d\xi} = \epsilon \frac{d^2u}{d\xi^2}, \tag{1.9}$$

where we have written u instead of $\hat{u}$. Corresponding to the initial condition (1.7), we use the boundary conditions

$$u(-\infty) = u_L, \quad u(+\infty) = u_R. \tag{1.10}$$

Dafermos conjectured that solutions of the boundary value problem (1.9)–(1.10) should converge to Riemann solutions in the L^1 sense as $\epsilon \to 0$. This has been proved for u_R close to u_L by Tzavaras [16].

Recently Szmolyan [17] has taken the opposite point of view. He regards (1.9)–(1.10) as a singular perturbation problem that has a given Riemann solution $\hat{u}(\frac{x}{t})$ of (1.1), (1.7) as a singular solution when $\epsilon = 0$. Shock waves are assumed to satisfy the viscous profile criterion. For a Riemann solution $\hat{u}(\frac{x}{t})$ that consists of n compressive shock waves and rarefactions, Szmolyan uses geometric singular perturbation theory to show that for small $\epsilon > 0$, (1.9)–(1.10) has a solution near $\hat{u}(\xi)$. The result allows u_R far from u_L. A novel aspect of the singular perturbation problem is that normal hyperbolicity is lost along rarefactions. Szmolyan deals with this difficulty by a blowing-up construction.

In [12], Schecter, Marchesin, and Plohr studied *structurally stable* Riemann solutions. These are Riemann solutions that are stable to perturbation of u_L, u_R, and f: The nearby Riemann problem has a solution with the same number of waves, of the same types; shock waves must satisfy the viscous profile criterion. For example, Riemann solutions that consists of n compressive shock waves and rarefactions are structurally stable. (This use of the term "structurally stable" is consistent with its use in dynamical systems theory, but differs from Majda's use of the term in [9].) In [13] Schecter showed that structurally stable Riemann solutions consisting entirely of shock waves, including undercompressive shock waves, have solutions of the Dafermos regularization nearby. We conjecture that for any structurally stable Riemann solution $\hat{u}(\frac{x}{t})$, the Dafermos boundary value problem (1.9)–(1.10) has a solution near $\hat{u}(\xi)$ for small $\epsilon > 0$. For some non-structurally stable Riemann solutions, see [8].

The correspondence between solutions of the boundary value problem (1.9)–(1.10) and Riemann solutions of (1.1), (1.7) whose shock waves satisfy the viscous profile criterion for u_{xx} suggests that one can approximate Riemann solutions by numerically solving the boundary value problem (1.9)–(1.10) for a small $\epsilon > 0$ [10]. In order to justify such an approach to interesting Riemann problems, one must show in greater generality that Riemann solutions of (1.1), (1.7), are close to solutions of (1.9)–(1.10).

Structurally stable Riemann solutions can contain composite waves, which are combinations of rarefactions and adjacent shock waves [11, 18, 7, 12]. In this paper we study the simplest composite wave, a combination of one rarefaction and one adjacent shock wave, in the simplest situation, that of a scalar conservation law, for the simplest viscosity, u_{xx}. We do not require that u_L and u_R be close. We show that for small $\epsilon > 0$, the Dafermos boundary value problem has nearby solutions. We believe that the method of this paper applies to all structurally stable Riemann solutions with composite waves.

The proof is based on Szmolyan's geometric approach to the Dafermos regularization and his application of the blowing-up construction to rarefactions. We must analyze the flow past a normally hyperbolic "corner equilibrium" of the blown-up vector field that was not relevant to [17]. At such an equilibrium, the vector field cannot be viewed as a parameterized family, so the exchange lemma [6, 5] is not relevant. We use instead the "corner lemma," proved in [14], which plays of role the exchange lemma in tracking the flow past such points.

The rest of the paper is organized as follows. Precise assumptions are given in Section 2, and our result is stated in Section 3. The blow-up construction for rarefactions is reviewed in Section 4, and relevant invariant manifolds are identified. The result is proved in Section 5. For the reader's convenience, the corner lemma is reviewed in Section 6.

2. Problem Statement

We consider the scalar conservation law

$$u_t + f(u)_x = 0. \tag{2.1}$$

A *rarefaction wave* is a smooth solution $u(\xi)$, $\xi = \frac{x}{t}$, of (2.1), with $\frac{du}{d\xi} \neq 0$. It therefore satisfies $f'(u(\xi)) = \xi$, from which we see that $\frac{du}{d\xi} = \frac{1}{f''(u)}$. In particular, if $f''(u) > 0$ for $u_1 \leq u \leq u_2$, then there is a rarefaction $u(\xi)$, $f'(u_1) \leq \xi \leq f'(u_2)$, from u_1 to u_2.

We assume that the solution of the Riemann problem consisting of (2.1) and the initial conditions

$$u(x,0) = \begin{cases} u_L & \text{for } x < 0, \\ u_R & \text{for } x > 0, \end{cases} \tag{2.2}$$

is a composite wave consisting of a shock wave followed by a rarefaction. More precisely, we assume that $u_L < u_R$ and that there exist s and u_M, with $u_L < u_M < u_R$, such that the following assumptions are satisfied.

(A1) $f(u_M) - f(u_L) - s(u_M - u_L) = f(u_R) - f(u_L) - s(u_R - u_L) = 0$.
(A2) $f'(u_L) > s = f'(u_M)$.
(A3) $f(u) - f(u_L) - s(u - u_L) > 0$ for $u_L < u < u_M$.
(A4) $f''(u) > 0$ for $u_M \leq u \leq u_R$.

See Figure 2.1.

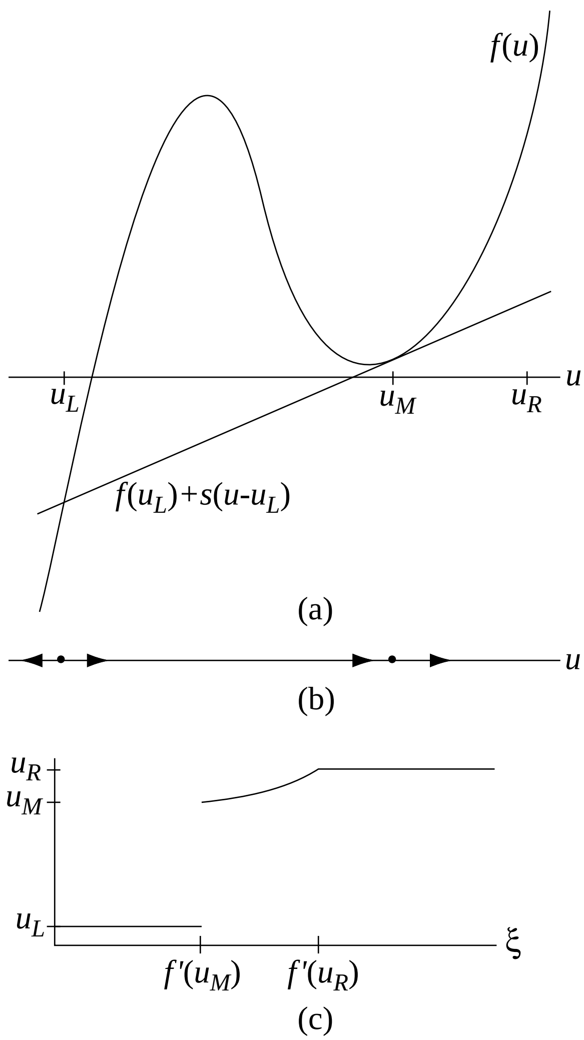

FIGURE 2.1. (a) Graphs of the curve $y = f(u)$ and the line $y = f(u_L) + s(u - u_L)$. (b) Phase portrait of $\dot{u} = f(u) - f(u_L) - s(u - u_L)$. (c) Riemann solution.

These assumptions imply (see Figure 2.1):

(W1) There is a shock wave with speed s and viscous profile from u_L to u_M. The corresponding connecting orbit of $\dot{u} = f(u) - f(u_L) - s(u - u_L)$ has a hyperbolic repeller at u_L and a nonhyperbolic equilibrium at u_M.
(W2) There is a rarefaction $\tilde{u}(\xi)$, $s = f'(u_M) \leq \xi \leq f'(u_R)$, from u_M to u_R.

The solution of the Riemann problem (2.1)–(2.2) is the first of these waves followed by the second. More precisely, it is $\hat{u}(\xi)$, $\xi = \frac{x}{t}$, with

$$\hat{u}(\xi) = \begin{cases} u_L & \text{for } \xi < f'(u_M), \\ \tilde{u}(\xi) & \text{for } f'(u_M) \le \xi \le f'(u_R), \\ u_R & \text{for } f'(u_R) < \xi. \end{cases} \tag{2.3}$$

See Figure 2.1.

Without loss of generality we may assume

(1) $u_M = 0$,
(2) $f(0) = 0$
(3) $s = 0$.
(4) $f''(0) = \frac{1}{2}$.

Thus we have $u_L < 0 < u_R$ such that

(A1′) $f(u_L) = f(0) = f(u_R) = 0$.
(A2′) $f'(u_L) > 0 = f'(0)$.
(A3′) $f(u) > 0$ for $u_L < u < 0$.
(A4′) $f''(u) > 0$ for $0 \le u \le u_R$.
(A5′) $f''(0) = \frac{1}{2}$.

See Figure 2.2.

Therefore:

(W1′) There is a shock wave with speed 0 and viscous profile from u_L to 0. The corresponding connecting orbit of $\dot{u} = f(u)$ has a hyperbolic repeller at u_L and a nonhyperbolic equilibrium at 0.
(W2′) There is a rarefaction $\tilde{u}(\xi)$, $0 = f'(0) \le \xi \le f'(u_R)$, from 0 to u_R.

The solution of the Riemann problem (2.1)–(2.2)

$$\hat{u}(\xi) = \begin{cases} u_L & \text{for } \xi < 0, \\ \tilde{u}(\xi) & \text{for } 0 \le \xi \le f'(u_R), \\ u_R & \text{for } f'(u_R) < \xi. \end{cases} \tag{2.4}$$

See Figure 2.2.

The Dafermos ODE (1.9) in the scalar case is

$$(f'(u) - \xi)\frac{du}{d\xi} = \epsilon \frac{d^2u}{d\xi^2}. \tag{2.5}$$

Let $\xi = \epsilon\zeta$, and let a dot denote differentiation with respect to ζ. Then (2.5) can be rewritten as an autonomous first-order system as follows.

$$\dot{u} = v, \tag{2.6}$$

$$\dot{v} = (f'(u) - \xi)v, \tag{2.7}$$

$$\dot{\xi} = \epsilon. \tag{2.8}$$

For $\epsilon = 0$, each plane $\xi = k$ is invariant, and the plane $v = 0$ consists of equilibria. The equilibria are normally repelling for $\xi < f'(u)$, not normally hyperbolic for $\xi = f'(u)$, and normally attracting for $\xi > f'(u)$.

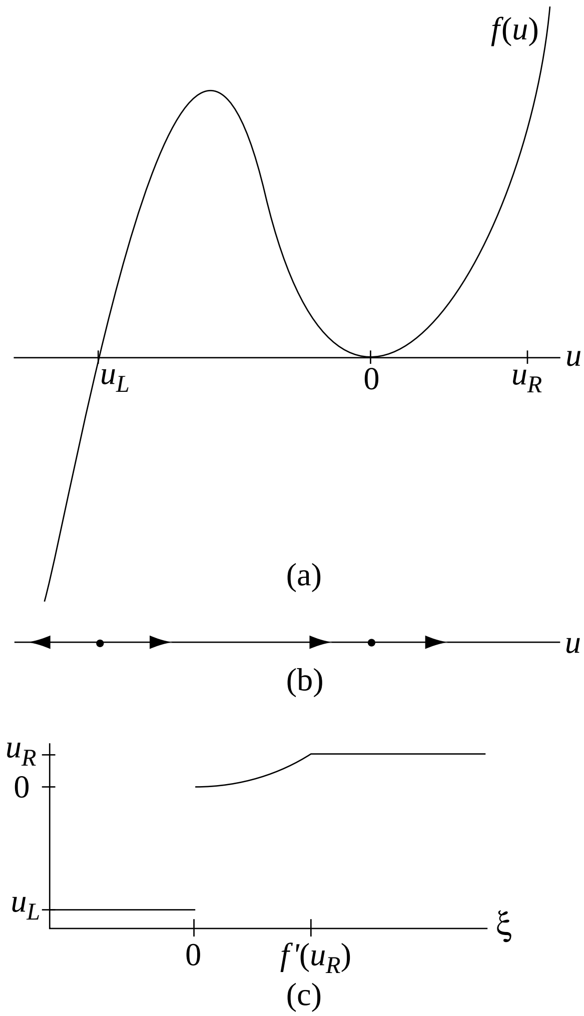

FIGURE 2.2. After the simplifying assumptions (1)–(4): (a) Graph of f. (b) Phase portrait of $\dot{u} = f(u)$. (c) Riemann solution.

Figure 2.3 shows part of the flow for $\epsilon = 0$. The Riemann solution (2.3) corresponds to the union of the following sets:

- S_1 The line of equilibria $\{(u, v, \xi) : u = u_L, v = 0, \xi \leq 0\}$.
- S_2 A heteroclinic orbit of (2.6)–(2.8) with $\epsilon = 0$ from $(u_L, 0, 0)$ to $(0, 0, 0)$. This orbit corresponds to the connection of $\dot{u} = f(u)$ from u_L to 0. It is an open subset of the invariant curve $\{(u, v, \xi) : v = f(u) - f(u_L) - \xi(u - u_L), \xi = 0\}$.
- S_3 The curve of nonhyperbolic equilibria $\{(u, v, \xi) : 0 \leq u \leq u_R, v = 0, \xi = f'(u)\}$, which corresponds to the rarefaction $\tilde{u}(\xi)$, $0 \leq \xi \leq f'(u_R)$.
- S_4 The line of equilibria $\{(u, v, \xi) : u = u_R, v = 0, f'(u_R) \leq \xi\}$.

We shall show that for small $\epsilon > 0$ there is a solution of the Dafermos system (2.6)–(2.8) that lies near this set and is asymptotic to it as $\xi \to \pm\infty$.

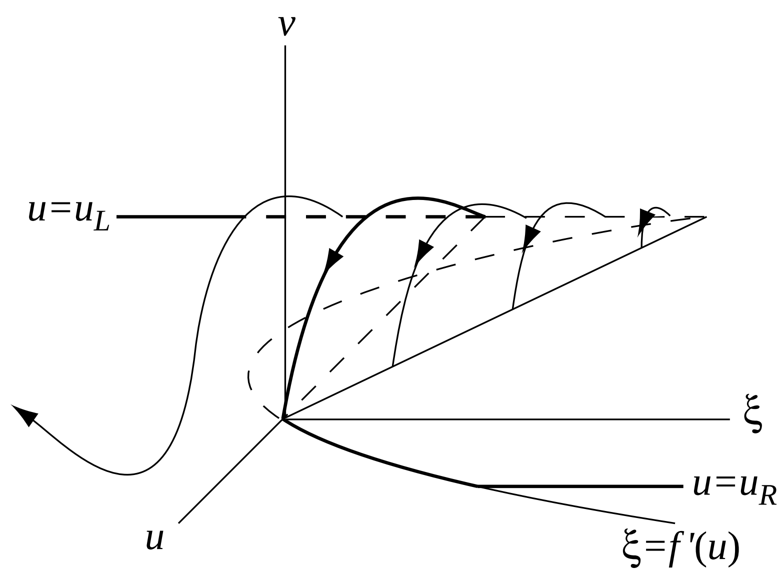

FIGURE 2.3. The Dafermos system (2.6)–(2.8) with $\epsilon = 0$. The plane $v = 0$ consists of equilibria, which are normally repelling for $\xi < f'(u)$, not normally hyperbolic for $\xi = f'(u)$, and normally attracting for $\xi > f'(u)$. The union of the sets $S_1, \ldots, S_4$ is thickened.

3. Main Result

We add to the system (2.6)–(2.8) the differential equation

$$\dot{\epsilon} = 0, \tag{3.1}$$

and we regard the system (2.6)–(3.1) as an ODE on $\mathbb{R}^4$. We make the change of variables $\xi = f'(u) + \sigma$. The system (2.6)– (3.1) becomes

$$\dot{u} = v, \tag{3.2}$$

$$\dot{v} = -\sigma v, \tag{3.3}$$

$$\dot{\sigma} = \epsilon - f''(u)v, \tag{3.4}$$

$$\dot{\epsilon} = 0. \tag{3.5}$$

Each 3-dimensional space $\epsilon = k$ is invariant, and in the 3-dimensional space $\epsilon = 0$, the plane $v = 0$ consists of equilibria. Within the space $\epsilon = 0$, the equilibria are normally repelling for $\sigma < 0$, not normally hyperbolic for $\sigma = 0$, and normally attracting for $\sigma > 0$. However, in the space $\epsilon = 0$, the planes $\sigma = k$ are not invariant. See Figure 3.1.

In these coordinates, we wish to show that for small $\epsilon > 0$ there is a solution of (3.2)–(3.5) that lies near the union of the following sets:

- T_1 The line of equilibria $\{(u, v, \sigma, \epsilon) : u = u_L, v = 0, \sigma \leq -f'(u_L), \epsilon = 0\}$.
- T_2 The heteroclinic orbit from $(u_L, 0, -f'(u_L), 0)$ to $(0, 0, 0, 0)$.
- T_3 The line $\{(u, v, \sigma, \epsilon) : 0 \leq u \leq u_R, v = \sigma = \epsilon = 0\}$.
- T_4 The line of equilibria $\{(u, v, \sigma, \epsilon) : u = u_R, v = 0, 0 \leq \sigma, \epsilon = 0\}$.

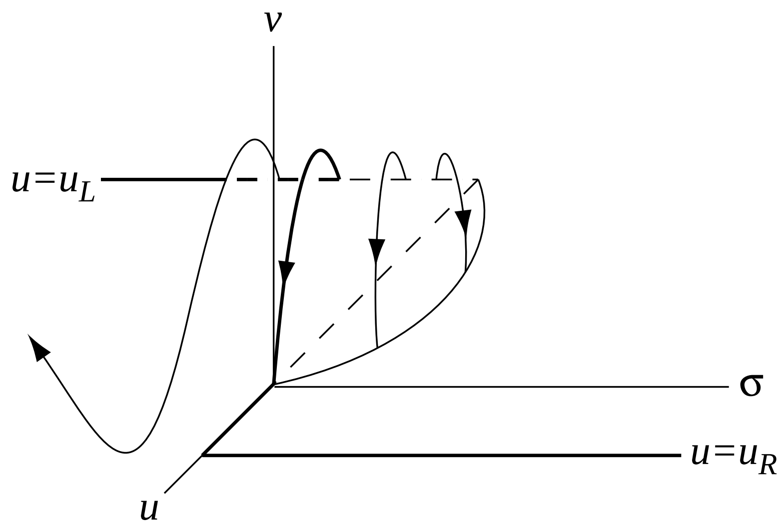

FIGURE 3.1. The transformed Dafermos system (3.2)–(3.4) with $\epsilon = 0$. The plane $v = 0$ consists of equilibria, which are normally repelling for $\sigma < 0$, not normally hyperbolic for $\sigma = 0$, and normally attracting for $\sigma > 0$. The union of the sets $T_1, \ldots, T_4$ is thickened.

Fix a small $\delta > 0$. For the system (3.2)–(3.5), the sets

$$M_-^0 = \{(u, v, \sigma, 0) : |u - u_L| < \delta, v = 0, \sigma < -\delta\},$$
$$M_+^0 = \{(u, v, \sigma, 0) : |u - u_R| < \delta, v = 0, \delta < \sigma\}$$

are 2-dimensional manifolds of equilibria that are uniformly normally hyperbolic within the space $\epsilon = 0$. (M_-^0 is repelling, M_+^0 is attracting.) By [3, 4] they perturb to 2-dimensional invariant manifolds M_-^ϵ and M_+^ϵ that are normally hyperbolic within the 3-dimensional space $\epsilon =$ constant. In fact, for fixed ϵ,

$$M_-^\epsilon = \{(u, v, \sigma, \epsilon) : v = 0, \sigma < -\delta, \epsilon \text{ fixed}\},$$
$$M_+^\epsilon = \{(u, v, \sigma, \epsilon) : v = 0, \delta < \sigma, \epsilon \text{ fixed}\}.$$

The flow on $M_\pm^\epsilon$ is

$$\dot{u} = 0,$$
$$\dot{\sigma} = \epsilon.$$

Thus each line

$$M_-^\epsilon(u) = \{(u, v, \sigma, \epsilon) : u \text{ fixed}, v = 0, \sigma < -\delta, \epsilon \text{ fixed}\},$$
$$M_+^\epsilon(u) = \{(u, v, \sigma, \epsilon) : u \text{ fixed}, v = 0, \delta < \sigma, \epsilon \text{ fixed}\}$$

is invariant. Note that $T_1 \subset M_-^0(u_L)$ and $T_4 \subset M_+^0(u_R)$.

From Fenichel's theory of normally hyperbolic invariant manifolds [3, 4] the lines $M_-^\epsilon(u)$ have 2-dimensional unstable manifolds $W^u(M_-^\epsilon(u))$, and the lines $M_+^\epsilon(u)$ have 2-dimensional stable manifolds $W^s(M_+^\epsilon(u))$. Both depend smoothly on (u, ϵ). Of course, $W^u(M_-^0(u))$ is just the union of the unstable manifolds of the equilibria that comprise it, and $W^s(M_+^0(u))$

is just the union of the stable manifolds of the equilibria that comprise it. For small $\epsilon > 0$, we seek a solution in $W^u(M_-^\epsilon(u_L)) \cap W^s(M_+^\epsilon(u_R))$.

The result of this paper is:

Theorem 3.1. *For small* $\epsilon > 0$, $W^u(M_-^\epsilon(u_L)) \cap W^s(M_+^\epsilon(u_R))$ *contains an orbit of* (3.2)–(3.5) *that is near the set* $T_1 \cup \ldots \cup T_4$ *and that is asymptotic to it as* $\epsilon \to 0$.

4. Blow-up

Following [17], we shall blow up the u-axis in $uv\sigma\epsilon$-space, which consists of non-normally hyperbolic equilibria of (3.2)–(3.5), to a spherical cylinder, *i.e.*, the product of $\mathbb{R}$ with a 2-sphere. The 2-sphere is a blow-up of the origin in $v\sigma\epsilon$-space.

Let $\mathbb{R}_+ = [0, \infty)$. The blow-up transformation is a map from $\mathbb{R} \times S^2 \times \mathbb{R}_+$ to $uv\sigma\epsilon$-space defined as follows. Let $(u, (\bar{v}, \bar{\sigma}, \bar{\epsilon}), \bar{r})$ be a point of $\mathbb{R} \times S^2 \times \mathbb{R}_+$; we have $\bar{v}^2 + \bar{\sigma}^2 + \bar{\epsilon}^2 = 1$. Then the blow-up transformation is

$$u = u, \tag{4.1}$$

$$v = \bar{r}^2 \bar{v}, \tag{4.2}$$

$$\sigma = \bar{r}\bar{\sigma}, \tag{4.3}$$

$$\epsilon = \bar{r}^2 \bar{\epsilon}. \tag{4.4}$$

Under this transformation the system (3.2)–(3.5) becomes one for which the spherical cylinder $\bar{r} = 0$ consists entirely of equilibria. The system we shall study is this one divided by $\bar{r}$. Division by $\bar{r}$ desingularizes the system on the spherical cylinder $\bar{r} = 0$ but leaves it invariant.

We shall use three charts.

4.1. Chart for $\bar{\epsilon} > 0$.

This chart uses the coordinates u, $v_3 = \frac{\bar{v}}{\bar{\epsilon}}$, $\sigma_3 = \frac{\bar{\sigma}}{\sqrt{\bar{\epsilon}}}$ and $r_3 = \bar{r}\sqrt{\bar{\epsilon}}$ on the set of points in $\mathbb{R} \times S^2 \times \mathbb{R}_+$ with $\bar{\epsilon} > 0$. Thus we have

$$u = u, \tag{4.5}$$

$$v = r_3^2 v_3, \tag{4.6}$$

$$\sigma = r_3 \sigma_3, \tag{4.7}$$

$$\epsilon = r_3^2, \tag{4.8}$$

with $r_3 \geq 0$. After division by r_3 (equivalent to division by $\bar{r}$ up to multiplication by a positive function), the system (3.2)–(3.5) becomes

$$\dot{u} = r_3 v_3, \tag{4.9}$$

$$\dot{v_3} = -\sigma_3 v_3, \tag{4.10}$$

$$\dot{\sigma_3} = 1 - f''(u) v_3, \tag{4.11}$$

$$\dot{r_3} = 0. \tag{4.12}$$

Each 3-dimensional space $r_3 = k$ is invariant. The space $r_3 = 0$ corresponds to the u-axis crossed with the top of the 2-sphere, with $v_3\sigma_3$-coordinates. It contains all equilibria of (4.9)–(4.12). Within the space $r_3 = 0$, each plane $u = k$ is invariant. For a fixed u with $f''(u) > 0$, the flow in this plane is pictured in Figure 4.1. In this figure, the line $v_3 = 0$ is invariant, and there is a hyperbolic saddle at $(v_3, \sigma_3) = (\frac{1}{f''(u)}, 0)$.

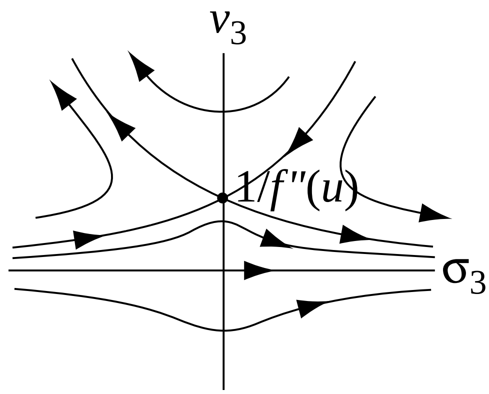

FIGURE 4.1. Chart for $\bar{\epsilon} > 0$. Flow of (4.9)–(4.12) in the invariant plane $r_3 = 0$, $u = k$.

4.2. **Chart for $\bar{v} > 0$.** This chart uses the coordinates u, $r_1 = \bar{r}\sqrt{\bar{v}}$, $\sigma_1 = \frac{\bar{\sigma}}{\sqrt{\bar{v}}}$ and $\epsilon_1 = \frac{\bar{\epsilon}}{\bar{v}}$ on the set of points in $\mathbb{R} \times S^2 \times \mathbb{R}_+$ with $\bar{v} > 0$. Thus we have

$$u = u, \tag{4.13}$$

$$v = r_1^2, \tag{4.14}$$

$$\sigma = r_1\sigma_1, \tag{4.15}$$

$$\epsilon = r_1^2\epsilon_1, \tag{4.16}$$

with $r_1 \geq 0$. After division by r_1 (equivalent to division by $\bar{r}$ up to multiplication by a positive function), the system (3.2)–(3.5) becomes

$$\dot{u} = r_1, \tag{4.17}$$

$$\dot{r_1} = -\frac{1}{2}r_1\sigma_1, \tag{4.18}$$

$$\dot{\sigma_1} = \epsilon_1 - f''(u) + \frac{1}{2}\sigma_1^2, \tag{4.19}$$

$$\dot{\epsilon_1} = \sigma_1\epsilon_1. \tag{4.20}$$

Each set $r_1^2\epsilon_1 = k$ is invariant. For $k = 0$, the 3-dimensional spaces $r_1 = 0$ and $\epsilon_1 = 0$ are each invariant. The space $r_1 = 0$ corresponds to the u-axis crossed with the back of the 2-sphere, with $\sigma_1\epsilon_1$-coordinates. It contains all equilibria of (4.17)–(4.20). Within the space $r_1 = 0$, each plane $u = k$ is invariant. For a fixed u with $f''(u) > 0$, the flow in this plane is pictured in Figure 4.2. In this figure, the line $\epsilon_1 = 0$ is invariant, and there are a hyperbolic saddle at $(\sigma_1, \epsilon_1) = (0, f''(u))$, a hyperbolic attractor at $(\sigma_1, \epsilon_1) = (-\sqrt{2f''(u)}, 0)$, and a hyperbolic repeller at $(\sigma_1, \epsilon_1) = (\sqrt{2f''(u)}, 0)$.

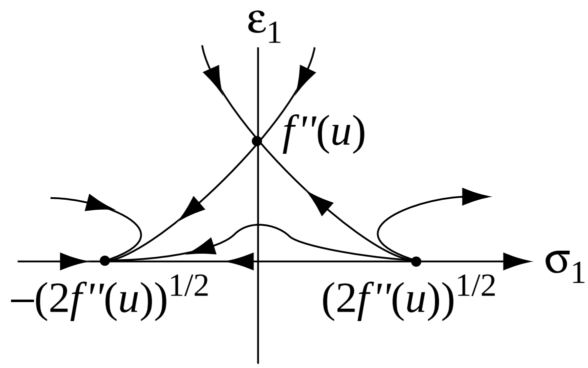

FIGURE 4.2. Chart for $\bar{v} > 0$. Flow of (4.17)–(4.20) in the invariant plane $r_1 = 0$, $u = k$.

4.3. **Chart for $\bar{\sigma} > 0$.** This chart uses the coordinates u, $v_2 = \frac{\bar{v}}{\bar{\sigma}^2}$, $r_2 = \bar{r}\bar{\sigma}$, and $\epsilon_2 = \frac{\bar{\epsilon}}{\bar{\sigma}^2}$ on the set of points in $\mathbb{R} \times S^2 \times \mathbb{R}_+$ with $\bar{\sigma} > 0$. Thus we have

$$u = u, \tag{4.21}$$

$$v = r_2^2 v_2, \tag{4.22}$$

$$\sigma = r_2, \tag{4.23}$$

$$\epsilon = r_2^2 \epsilon_2, \tag{4.24}$$

with $r_2 \geq 0$. After division by r_2 (equivalent to division by $\bar{r}$ up to multiplication by a positive function), the system (3.2)–(3.5) becomes

$$\dot{u} = r_2 v_2, \tag{4.25}$$

$$\dot{v_2} = -v_2(1 + 2\epsilon_2 - 2f''(u)v_2), \tag{4.26}$$

$$\dot{r_2} = r_2(\epsilon_2 - f''(u)v_2), \tag{4.27}$$

$$\dot{\epsilon_2} = -2\epsilon_2(\epsilon_2 - f''(u)v_2). \tag{4.28}$$

Each set $r_2^2\epsilon_2 = k$ is invariant. For $k = 0$, the 3-dimensional spaces $r_2 = 0$ and $\epsilon_2 = 0$ are each invariant. The space $\epsilon_2 = 0$ contains the plane of equilibria $v_2 = \epsilon_2 = 0$. The space $r_2 = 0$ corresponds to the u-axis crossed with the right side of the 2-sphere, with $v_2\epsilon_2$-coordinates. It contains all other equilibria of the system (4.25)–(4.28). Within the space $r_2 = 0$, each plane $u = k$ is invariant. For a fixed u with $f''(u) > 0$, the flow in this plane is pictured in Figure 4.3. In this figure, the line $\epsilon_2 = 0$ and $v_2 = 0$ are invariant. There are a hyperbolic repeller at $(v_2, \epsilon) = (\frac{1}{2f''(u)}, 0)$ and a nonhyperbolic equilibrium at the origin. The latter's stable manifold is the line $\epsilon_2 = 0$, and one center manifold is the line $v_2 = 0$. The origin is quadratically attracting on the portion of this line with $\epsilon_2 > 0$.

Figure 4.4 shows the flow in the portion of blow-up space $\mathbb{R} \times S^2 \times \mathbb{R}_+$ with $\bar{\epsilon} \geq 0$, as reconstructed from these coordinate charts and the corresponding ones for $\bar{v} < 0$ and $\bar{\sigma} < 0$. A value of u is fixed, with $f''(u) > 0$; in the figure we look straight down the ϵ-axis. We see the top of the sphere $u = k$, $r = 0$, and, outside it, the plane $u = k$, $\bar{\epsilon} = 0$, in which the origin has been blown up to a circle. There are equilibria $e(u, \sigma)$ along the u-axis. However, we distinguish $e_\pm(u, 0)$ where the σ-axis meets the circle; $e_+(u, 0)$ is the origin in the Figure 4.3.

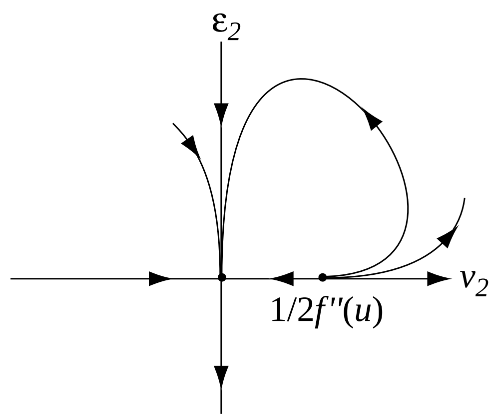

FIGURE 4.3. Chart for $\bar{\sigma} > 0$. Flow of (4.25)–(4.28) in the invariant plane $r_2 = 0$, $u = k$.

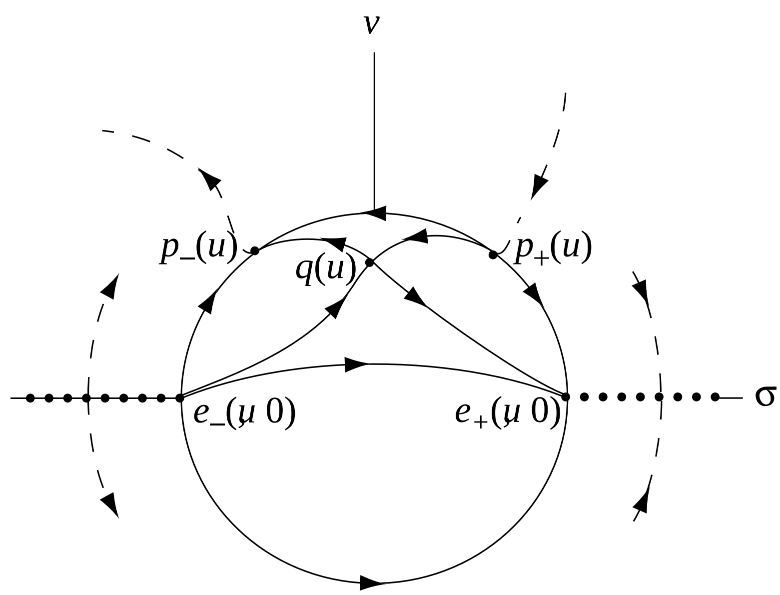

FIGURE 4.4. Flow in blow-up space.

There are also two equilibria $p_\pm(u)$ elsewhere on the circle—they are the equilibria on the σ_1-axis in Figure 4.2, and $p_+(u)$ is also the equilibrium to the right of the origin in Figure 4.3; and an equilibrium $q(u)$ on the sphere—it is seen in Figures 4.1 and 4.2. The eigenvalues are given in Table 4.1. Figure 4.4 shows the curves $W^u(p_-(u))$, $W^s(p_+(u))$, unstable manifolds of $e(u,\sigma)$ for $\sigma = \bar{r}\bar{\sigma} < 0$, and stable manifolds of $e(u,\sigma)$ for $\sigma = \bar{r}\bar{\sigma} > 0$, although none of these curves lies in $u = k$.

We shall often use the same symbol to denote a subset of blow-up space and its representation in different charts.

We note:

1. The set $P_+ = \{p_+(u) : f''(u) > 0\}$ is a normally hyperbolic curve of equilibria. Its 3-dimensional unstable manifold is an open subset of the spherical cylinder $\bar{r} = 0$, and its 2-dimensional stable manifold is contained in $\epsilon = 0$. The unstable manifold of P_+ is the

$e_-(u,0)$	$+$	0	0	0
$e_+(u,0)$	$-$	0	0	0
$p_-(u)$	$-$	$-$	$+$	0
$p_+(u)$	$+$	$+$	$-$	0
$q(u)$	$+$	$-$	0	0

TABLE 4.1. Eigenvalues of equilibria.

union of the 2-dimensional unstable manifolds of the points $p_+(u)$, and the stable manifold of P_+ is the union of the 1-dimensional stable manifolds of the points $p_+(u)$.

2. The set $Q^0 = \{q(u) : f''(u) > 0\}$ is a curve of equilibria that is normally hyperbolic within the spherical cylinder $\bar{r} = 0$, with 2-dimensional unstable manifold and 2-dimensional stable manifold. Using the chart for $\bar{\epsilon} > 0$, we see from [3, 4] that any compact portion of Q^0 perturbs to a family Q^ϵ of invariant curves that are normally hyperbolic within $\epsilon = \bar{r}^2\bar{\epsilon}$ = constant (corresponding to r_3=constant in the chart for $\bar{\epsilon} > 0$). The unstable and stable manifolds of Q^ϵ depend smoothly on ϵ.

3. The point $e_+(u_R, 0)$ has a 3-dimensional center manifold, which we denote E. In the chart for $\bar{\sigma} > 0$, E can be taken to be an open subset of $\{(u, v_2, r_2, \epsilon_2) : v_2 = 0\}$. E is foliated by 2-dimensional invariant manifolds

$$\tilde{M}_+(u) = \{(u, v_2, r_2, \epsilon_2) \in E : u \text{ fixed}\},$$

Each $\tilde{M}_+(u)$ contains the line of equilibria

$$\tilde{M}^0_+(u) = \{(u, v_2, r_2, \epsilon_2) : u \text{ fixed}, v_2 = 0, 0 \le r_2, \epsilon_2 = 0\},$$

and the invariant curves

$$\tilde{M}^\epsilon_+(u) = \{(u, v_2, r_2, \epsilon_2) : u \text{ fixed}, v_2 = 0, 0 < r, r_2^2\epsilon_2 = \epsilon\}.$$

Using the invariant foliation [3, 4] of the stable manifold of E, which is an open set, we see that E is foliated by the 3-dimensional stable manifolds of each $\tilde{M}_+(u)$. Moreover, the stable manifold of $\tilde{M}_+(u)$ contains 2-dimensional stable manifolds of each curve $\tilde{M}^\epsilon_+(u)$.

The subsets $M^\epsilon_\pm(u)$ of $uv\sigma\epsilon$-space defined in the Section 3 correspond to subsets of blow-up space, which we continue to denote $M^\epsilon_\pm(u)$. Note that for $\epsilon \ge 0$, $\tilde{M}^\epsilon_+(u)$ and $W^s(\tilde{M}^\epsilon_+(u))$ extend $M^\epsilon_+(u)$ and $W^s(M^\epsilon_+(u))$ respectively.

5. PROOF OF THEOREM 3.1

We work in blow-up space $\mathbb{R} \times S^2 \times \mathbb{R}_+$.

By (W1$'$), the 2-dimensional manifold $W^u(M^0_-(u_L))$ contains a branch of the curve $W^s(p_+(0))$.

To prove Theorem 3.1, we shall show:

(1) Within the 3-dimensional space $\bar{\epsilon} = 0$, $W^u(M^0_-(u_L))$ is transverse to the 2-dimensional manifold $W^s(P_+)$ along $W^s(p_+(0))$.

(2) Let $\tilde{q}_0$ be a point in the curve $W^s(q(0))$ that is near $p_+(0)$. Near $\tilde{q}_0$, for small $\epsilon > 0$, the 2-dimensional manifold $W^u(M^\epsilon_-(u_L))$ meets the 2-dimensional manifold $W^s(Q^\epsilon)$ transversally within the 3-dimensional manifold $\epsilon = \bar{r}^2\bar{\epsilon}$ = constant. The transversality is uniform as $\epsilon \to 0$.

(3) Let $\tilde{q}_R$ be a point in the curve $W^u(q_R)$ that is near $e_+(u_R, 0)$. Near $\tilde{q}_R$, for small $\epsilon > 0$, $W^u(M^\epsilon_-(u_L))$ meets the 3-dimensional manifold $W^s(\tilde{M}_+(u_R))$ transversally.

(4) Since $\epsilon = \bar{r}^2\bar{\epsilon}$ is constant on solutions, the 1-dimensional intersection must lie in $W^s(\tilde{M}_+^\epsilon(u_R))$, which completes the proof.

We shall prove (1) at the end of this section.

To show that (1) implies (2), let U_0 be a small neighborhood of $\tilde{q}_0$ in blow-up space. By (1) and the Corner Lemma, for small $\epsilon > 0$, $W^u(M_-^\epsilon(u_L)) \cap U_0$ is C^1-close to $W^u(p_+(0)) \cap U_0$, *i.e.*, to $\{0\} \times S^2 \times \{0\}$. Since $\{0\} \times S^2 \times \{0\}$ is transverse at $\tilde{q}_0$ to the 2-dimensional manifold $W^s(Q^0)$ within the 3-dimensional space $\bar{r} = 0$, (2) follows.

To show that (2) implies (3), let U_R be a small neighborhood of $\tilde{q}_R$ in blow-up space. By (2) and the Exchange Lemma, for small $\epsilon > 0$, $W^u(M_-^\epsilon(u_L)) \cap U_R$ is C^1-close to $W^u(Q^0) \cap U_R$. Since $W^u(Q^0)$ is transverse to $W^s(\tilde{M}_+(u_R))$ near $\tilde{q}_R$, (3) follows.

Since (4) is self-explanatory, this completes the proof.

In the remainder of this section, we give the proof of (1).

The Dafermos system (2.6)–(2.8) in $uv\xi$-space with $\epsilon = 0$ has the first integrals ξ and $f(u) - \xi u - v$. Thus the unstable manifold of the point $(u_L, 0, \xi)$, $\xi < f'(u_L)$, is an open subset of the curve

$$f(u) - f(u_L) - \xi(u - u_L) - v = 0, \tag{5.1}$$

$$\xi = \text{constant}, \tag{5.2}$$

in $uv\xi$-space. $W^u(M_-^0(u_L))$ is the union of these curves over all $\xi < -\delta$, an open subset of the surface (5.1) in $uv\xi$-space. By (W1$'$), the curve defined by (5.1) and $\xi = 0$ contains a solution of the Dafermos system (2.6)–(2.8) with $\epsilon = 0$ from $(u_L, 0, 0)$ to $(0, 0, 0)$. In blow-up space this solution is the branch of $W^s(p_+(0))$ that is contained in $W^u(M_-^0(u_L))$.

In the chart for $\bar{v} > 0$, we must show transversality of $W^s(P_+)$ and $W^u(M_-^0(u_L))$ along $W^s(p_+(0))$ in the space $\epsilon_1 = 0$. In this chart, $W^u(M_-^0(u_L))$ is an open subset of the surface

$$f(u) - f(u_L) - (f'(u) + r_1\sigma_1)(u - u_L) - r_1^2 = 0, \tag{5.3}$$

$$\epsilon_1 = 0. \tag{5.4}$$

The tangent space to $W^u(M_-^0(u_L))$ at a point $(u, r_1, \sigma_1, 0)$ is the set of all $(\bar{u}, \bar{r}_1, \bar{\sigma}_1, \bar{\epsilon}_1)$ that satisfy the equations

$$(f''(u)(u - u_L) + r_1\sigma_1)\bar{u} + (2r_1 + \sigma_1(u - u_L))\bar{r}_1 + r_1(u - u_L)\bar{\sigma}_1 = 0, \tag{5.5}$$

$$\bar{\epsilon}_1 = 0. \tag{5.6}$$

In the chart for $\bar{v} > 0$, the equilibrium $p_+(u)$ is the point $(u, r_1, \sigma_1, \epsilon_1) = (u, 0, \sqrt{2f''(u)}, 0)$. Thus

$$P_+ = \{(u, 0, \sqrt{2f''(u)}, 0) : f''(u) > 0\}.$$

Recall that $f''(0) = \frac{1}{2}$. Therefore $p_+(0) = (0, 0, 1, 0) \in P_+$.

The linearization of the system (4.17)–(4.20) at $p_+(u)$ has the matrix representation

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2}\sqrt{2f''(u)} & 0 & 0 \\ -f'''(u) & 0 & \sqrt{2f''(u)} & 1 \\ 0 & 0 & 0 & \sqrt{2f''(u)} \end{pmatrix}, \tag{5.7}$$

with eigenvalues 0, $-\frac{1}{2}\sqrt{2f''(u)}$, and $\sqrt{2f''(u)}$ twice. An eigenvector for $-\frac{1}{2}\sqrt{2f''(u)}$ is $(-\sqrt{\frac{2}{f''(u)}}, 1, -\frac{2f'''(u)}{3f''(u)}, 0)$.

The tangent space to $W^s(P_+)$ at $p_+(0) = (0, 0, 1, 0)$ is spanned by a stable eigenvector of the equilibrium and a tangent vector to P_+, *i.e.*, by

$$(-2, 1, -\frac{4}{3}f'''(0), 0) \text{ and } (1, 0, f'''(0), 0).$$

As $(u, r_1, \sigma_1, \epsilon_1)$ approaches $p_+(0) = (0, 0, 1, 0)$ along the branch of its stable manifold that is contained in $W^u(M^0_-(u_L))$, $TW^u(M^0_-(u_L))$ approaches the linear space

$$\frac{1}{2}\bar{u} + \bar{r}_1 = 0, \quad \bar{\epsilon}_1 = 0.$$

Since the vector $(1, 0, f'''(0), 0)$ is not in this space, we see that $W^s(P_+)$ and $W^u(M^0_-(u_L))$ are transverse within the space $\epsilon_1 = 0$.

6. Corner Lemma

In blown-up geometric singular perturbation problems, at manifolds of normally hyperbolic corner equilibria such as P_+, the following problem arises: Given a normally hyperbolic manifold P of equilibria and a manifold N that is transverse to $W^s(P)$, track the flow of N past P. At corner equilibria the differential equation cannot be regarded as a parameterized family, so the exchange lemma [6, 5] is not relevant. The following lemma, proved in [14], plays the role of the exchange lemma for such points. Like the exchange lemma, it is a consequence of a result of Deng [2] about solutions of Silnikov problems near nonhyperbolic points.

The notation of this section is independent of that of the remainder of the paper.

Consider a differential equation $\dot{w} = f(w)$ on a neighborhood of 0 in $\mathbb{R}^p$ that is C^{r+4} , $r \geq 1$, and:

(1) The origin is an equilibrium.
(2) There are integers $k \geq 0$, $\ell \geq 0$, $m \geq 1$, and $n \geq 1$ such that $Df(0)$ has $k + \ell$ eigenvalues equal to 0, m eigenvalues with negative real part, and n eigenvalues with positive real part, with $k + \ell + m + n = p$.
(3) A codimension one subspace S of $\mathbb{R}^p$ is invariant.
(4) The restriction of $Df(0)$ to S has $k + \ell$ eigenvalues equal to 0, m eigenvalues with negative real part, and $n - 1$ eigenvalues with positive real part.
(5) The origin is part of a $k + \ell$-dimensional manifold of equilibria P.

P is a normally hyperbolic manifold of equilibria. Each point of P has a stable manifold of dimension m and an unstable manifold of dimension n. The union of the stable manifolds of points of P is $W^s(P)$, which has dimension $k+\ell+m$; the union of the unstable manifolds of points of P is $W^u(P)$, which has dimension $k + \ell + n$. P and $W^s(P)$ are necessarily contained in S.

Let N be a C^{r+4} manifold of dimension $k + n$ that is transverse to $W^s(P)$ at a point p in $W^s(0) \setminus \{0\}$ and such that $T_pN \cap T_pW^s(0) = \{0\}$. Then the intersection of N and $W^s(P)$ is a manifold of dimension k that projects along fibers to a k-dimensional submanifold R of P. Let y_n be a coordinate on $\mathbb{R}^p$ that vanishes on S, and, for $\delta > 0$, let $N_\delta = N \cap \{y_n = \delta\}$, a manifold of dimension $k + n - 1$. Let q be a point in $W^u(R)$ with $y_n(q) > 0$. Notice that $W^u(R)$ has dimension $k + n$. Under the flow of $\dot{w} = f(w)$, N_δ becomes a manifold $\tilde{N}_\delta$ of dimension $k + n$ that passes near q Let U be a small neighborhood of q.

Theorem 6.1 (Corner Lemma). *As $\delta \to 0$, $\tilde{N}_\delta \cap U \to W^u(R) \cap U$ in the C^r topology.*

See Figure 6.1.

In the application of the corner lemma in Section 5, we work in the chart for $\bar{v} > 0$. We have $k = 0$, $\ell = m = 1$, and $n = 2$; S is the space $\epsilon_1 = 0$, and y_n is ϵ_1. P is P_+ and R is the origin. N is the union of the unstable manifolds of $M_-^\epsilon(u_L)$, ϵ near 0, intersected with a plane $r_1 =$ constant.

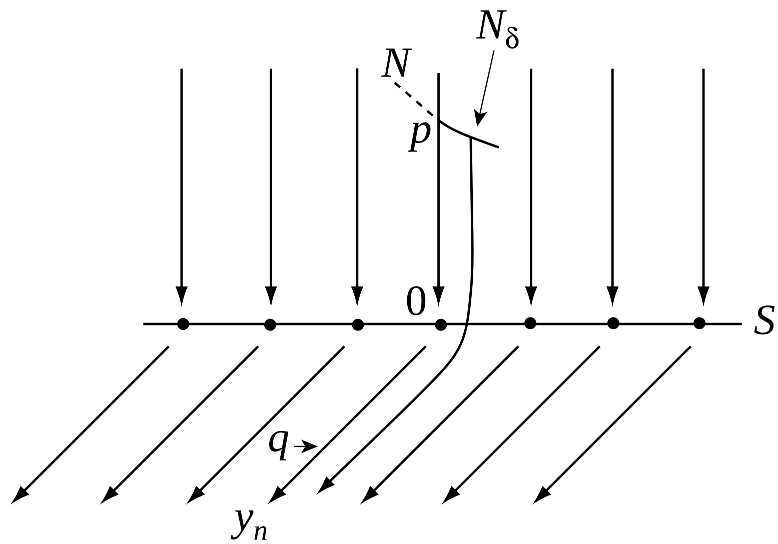

FIGURE 6.1. Corner lemma with $k = 0$ and $\ell = m = n = 1$. Thus $Q = \{0\}$, N is 1-dimensional and N_δ is a point. In this simple situation, the corner lemma just says that the solution through this point passes near q and is C^r-close to the 1-dimensional unstable manifold of the origin near q. In the application of the corner lemma in this paper, $n = 2$.

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