# Existence of Global Entropy Solutions to Euler Equations of Compressible Fluid Flow

Yun-guang Lu
Department of Mathematics
University of Science and Technology of China, Hefei
and
Department of Mathematicas
National University of Colombia, Bogota
email yglu@matematicas.unal.edu.co

**Abstract**

In this paper we use the theory of compensated compactness coupled with some basic ideas of the kinetic formulation by Lions, Perthame, Souganidis and Tadmor [LPS, LPT] to establish an existence theorem for global entropy solutions to the Euler equations of compressible fluid flow.

## 1 Introduction

In this paper, we consider the Cauchy problem for the nonlinear, nonstrictly hyperbolic system

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + (\frac{u^2}{2} + P(\rho))_x = 0, \end{cases} \tag{1.1}$$

with bounded measurable initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \quad \rho_0(x) \geq 0, \tag{1.2}$$

where the nonlinear function $P(\rho) = \frac{\theta}{2}\rho^{\gamma-1}, \theta = \frac{\gamma-1}{2}$ and $\gamma > 3$ is a constant.

System (1.1) was first derived by S. Earnshaw in 1858 for isentropic flow (cf. [Ea], [Wh]) and is also called the Euler equations of one-dimensional, compressible fluid flow (cf. [KM]), where $\rho$ denotes the density, $u$ the velocity, and $P(\rho)$ the pressure of the fluid.

System (1.1) has other different physical backgrounds. For instance, it is a scaling limit system of a Newtonian dynamics with long-range interaction for a continuous distribution of mass in R (cf. [Oe1, Oe2]) and also a hydrodynamic limit for the Vlasov equation (cf. [CEMP]).

By simple calculations, two eigenvalues of system (1.1) are

$$\lambda_1 = u - \theta\rho^\theta, \quad \lambda_2 = u + \theta\rho^\theta \tag{1.3}$$

with corresponding right eigenvectors

$$r_1 = (1, -\theta\rho^{\theta-1})^T, \quad r_2 = (1, \theta\rho^{\theta-1})^T; \tag{1.4}$$

the two corresponding Riemann invariants are

$$z = u - \rho^\theta, \quad w = u + \rho^\theta; \tag{1.5}$$

and

$$\nabla\lambda_1 \cdot r_1 = \theta(\theta+1)\rho^{\theta-1}, \quad \nabla\lambda_2 \cdot r_2 = \theta(\theta+1)\rho^{\theta-1}. \tag{1.6}$$

Therefore, it follows from (1.3) that $\lambda_1 = \lambda_2$ at the line $\rho = 0$ at which the strict hyperbolicity fails to hold, and from (1.6) that both characteristic fields are linearly degenerate on $\rho = 0$ if $\gamma > 3$ and on $\rho = \infty$ if $1 < \gamma < 3$.

The study of the existence of global weak solutions for the Cauchy problem (1.1), (1.2) was started by DiPerna [Di] for the case of $1 < \gamma < 3$ by using the Glimm's scheme method (cf. [Gl]). Since system (1.1) has two different invaraint regions (cf. [CCS]), which induce two different $L^\infty$ estimates as follows:

$$\rho \geq \min_{-\infty<x<\infty} \rho_0(x), \qquad \text{for } 1 < \gamma < 3 \tag{1.7}$$

and

$$0 \leq \rho \leq M, \quad |u| \leq M, \qquad \text{for } \gamma > 3, \tag{1.8}$$

then the Glimm's scheme method works for the case of $1 < \gamma < 3$ since system (1.1) is strictly hyperbolic in the condition $\min_{-\infty<x<\infty} \rho_0(x) > 0$.

However, for the case $\gamma > 3$, the strict hyperbolicity of system (1.1) fails since $\rho$ could be zero at a finite time.

When we use the theory of compensated compactness to study system (1.1), the main difficulty is that system (1.1) has no a strictly convex entropy, which restrains the use of weak entropy-entropy flux pairs. To overcome this difficulty, in [Lu2, Lu3], we add a small perturbation $\delta$ to the nonlinear function $P(\rho)$ : $P(\rho) = \int_0^\rho s(s+\delta)^{\gamma-3}ds$ so that system (1.1) has a strictly convex entropy for any fixed $\delta > 0$ and hence, both strong and weak entropy-entropy flux pairs of the perturbation system of (1.1) satisfy the $H^{-1}$ compactness condition. Therefore the existence of entropy solutions is obtained for this perturbation system.

In this paper, we study the Cauchy problem (1.1)-(1.2) by using the kinetic formulation of systems of conservation laws developed by Lions, Perthame, Souganidis and Tadmor [LPS, LPT] and obatin the main result as follows:

**<u>Theorem</u> 1** *Let the initial data $(\rho_0(x), u_0(x))$ be bounded measurable and $\rho_0(x) \geq 0$, then the Cauchy problem (1.1)-(1.2) has a global entropy solution.*

# 2 Proof of theorem 1

To prove Theorem 1, we first consider the following perturbation system

$$\begin{cases} \rho_t + ((\rho-\delta)u)_x = 0 \\ u_t + (\frac{1}{2}u^2 + \int_\delta^\rho \theta^2 (t-\delta) t^{\gamma-3} dt)_x = 0, \end{cases} \tag{2.1}$$

where $\delta > 0$ is the perturbation constant.

By simple calculations, two eigenvalues of system (2.1) are

$$\lambda_1 = u - \theta\rho^{\theta-1}(\rho-\delta), \quad \lambda_2 = u + \theta\rho^{\theta-1}(\rho-\delta) \tag{2.2}$$

and the two corresponding Riemann invariants are same to system (1.1):

$$z = u - \rho^\theta, \quad w = u + \rho^\theta. \tag{2.3}$$

Any entropy-entropy flux pair $(\eta(\rho,u), q(\rho,u))$ of system (2.1) satisfies the additional system

$$q_\rho = u\eta_\rho + \theta^2(\rho-\delta)\rho^{\gamma-3}\eta_u, \quad q_u = (\rho-\delta)\eta_\rho + u\eta_u. \tag{2.4}$$

Eliminating the $q$ from (2.4), we have

$$\eta_{\rho\rho} = \theta^2 \rho^{\gamma-3}\eta_{uu}. \tag{2.5}$$

Therefore, system (2.1) and system (1.1) have the same entropies.

Now consider the Cauchy problem for the related parabolic system

$$\begin{cases} \rho_t + ((\rho-\delta)u)_x = \varepsilon\rho_{xx} \\ u_t + (\frac{1}{2}u^2 + \int_\delta^\rho \theta^2(t-\delta)t^{\gamma-3}dt)_x = \varepsilon u_{xx}, \end{cases} \tag{2.6}$$

with initial data

$$(\rho^\delta(x,0), u^\delta(x,0)) = (\rho_0(x)+\delta, u_0(x)). \tag{2.7}$$

We multiply (2.6) by $(w_\rho, w_u)$ and $(z_\rho, z_u)$, respectively to obtain

$$w_t + \lambda_2 w_x = \varepsilon w_{xx} - \theta(\theta-1)\rho^{\theta-2}\rho_x^2 \le \varepsilon w_{xx} \tag{2.8}$$

and

$$z_t + \lambda_1 z_x = \varepsilon z_{xx} + \theta(\theta-1)\rho^{\theta-2}\rho_x^2 \ge \varepsilon z_{xx}. \tag{2.9}$$

If we consider (2.8) and (2.9) as inequalities about the variables $w$ and $z$, then we can get the estimates $w(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}) \le M, z(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}) \ge -M$ by applying the maximum principle to (2.8) and (2.9). Using the first equation in (2.6), we get $\rho \ge \delta$. These show that the region

$$\Sigma = \{(\rho,u) : w(\rho,u) \le M,\ z(\rho,u) \ge -M, \rho \ge \delta\}$$

is an invariant region. Thus we obtain the estimates $\delta \le \rho^{\varepsilon,\delta} \le M_1$ and $\|u^{\varepsilon,\delta}\| \le M_1$ for a suitable constant $M_1$, being independent of $\varepsilon, \delta$.

Using entropy equation (2.5), we obtain a convex entropy of system (2.1)

$$\eta_\star = \frac{1}{2}u^2 + \frac{\gamma-1}{2(\gamma-2)}\rho^{\gamma-1}. \tag{2.10}$$

We multiply (2.6) by $(\eta_\rho^\star, \eta_u^\star)$ to obtain the boundedness of

$$\varepsilon(\rho_x, u_x) \cdot \nabla^2\eta^\star(\rho,u) \cdot (\rho_x, u_x)^T \tag{2.11}$$

in $L^1_{loc}(R \times R^+)$. Then it follows that

$$\varepsilon 2\theta^2\rho^{\gamma-3}\rho_x^2 + \varepsilon u_x^2 \tag{2.12}$$

is bounded in $L^1_{loc}(R \times R^+)$.

One family of weak entropies of system (1.1) (also system (2.1)) is given by

$$\eta_0(\rho,u) = \int_R g(\xi)G_0(\rho, \xi-u)d\xi, \tag{2.13}$$

and the weak entropy flux $q_0$ of system (1.1) associated with $\eta_0$ is

$$q_0(\rho, u) = \int_R g(\xi)[\theta\xi + (1-\theta)u]G_0(\rho, \xi - u)d\xi; \tag{2.14}$$

two families of strong entropies of system (1.1) are given as follows (cf. [LPS, LPT]):

$$\eta_\pm(\rho, u) = \int_R g(\xi)G_\pm(\rho, \xi - u)d\xi, \tag{2.15}$$

and the strong entropy fluxes $q_\pm$ of system (1.1) associated with $\eta_\pm$ are

$$q_\pm(\rho, u) = \int_R g(\xi)[\theta\xi + (1-\theta)u]G_\pm(\rho, \xi - u)d\xi, \tag{2.16}$$

where $g(\xi)$ is a smooth function with a compact support set in $(-\infty, \infty)$ and the fundamental solutions

$$\begin{cases} G_0(\rho, u) = [(w-\xi)(\xi - z)]_+^\lambda, \\ G_+(\rho, u) = (\xi - z)^\lambda(\xi - w)_+^\lambda, \\ G_-(\rho, u) = (w-\xi)^\lambda(z-\xi)_+^\lambda \end{cases} \tag{2.17}$$

and $\lambda = \frac{3-\gamma}{2(\gamma-1)} > -\frac{1}{2}$. Here we use the notation $x_+ = max(0, x)$.

**<u>Lemma</u> 2** *For the viscosity solutions $(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))$ of the Cauchy problem (2.6) and (2.7), if the entropy $\eta(\rho, u)$ of system (1.1) satisfies that*

$$\eta_\rho(0, u) = 0, \quad \frac{\partial^i \eta(\rho, u)}{\partial u^i} \quad i = 0, 1, 2, 3, \tag{2.18}$$

*are bounded in $0 \le \rho \le M_1, |u| \le M_1$, then*

$$\eta(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))_t + q(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))_x \tag{2.19}$$

*is compact in $H^{-1}_{loc}(R \times R^+)$ as $\varepsilon$ and $\delta$ tends to zero, where $q$ is the entropy flux of system (1.1) associated with $\eta$.*

**Proof of Lemma 2.** Using entropy equation (2.5) and the condition $\eta_\rho(0, u) = 0$, we have

$$\eta_\rho(\rho, u) = \int_0^\rho \theta^2 s^{\gamma-3}\eta_{uu}(s, u)ds \tag{2.20}$$

and

$$\eta_{\rho u}(\rho, u) = \int_0^\rho \theta^2 s^{\gamma-3}\eta_{uuu}(s, u)ds. \tag{2.21}$$

Thus

$$|\eta_\rho(\rho, u)| \le M\int_0^\rho \theta^2 s^{\gamma-3}ds \le M\rho^{\gamma-2} \le M_2\rho^{\frac{\gamma-3}{2}}, \quad |\eta_{\rho u}(\rho, u)| \le M_2\rho^{\frac{\gamma-3}{2}}, \tag{2.22}$$

where $M, M_2$ are positive constants.

Let the entropy flux of system (2.1) associated with $\eta(\rho, u)$ be $\bar{q}(\rho, u, \delta)$. Then multiplying system (2.6) by $(\eta(\rho, u)_\rho, \eta(\rho, u)_u)$, we have

$$\begin{aligned}
& \eta(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})_t + q(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})_x \\
= \;& \varepsilon\eta(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})_{xx} - (\bar{q}(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \delta) - q(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}))_x \\
- \;& \varepsilon(\eta(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})_{\rho\rho}(\rho_x^{\varepsilon,\delta})^2 + 2\eta(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})_{\rho u}\rho_x^{\varepsilon,\delta}u_x^{\varepsilon,\delta} + \eta(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})_{uu}(u_x^{\varepsilon,\delta})^2).
\end{aligned} \tag{2.23}$$

Since $\bar{q}(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \delta) - q(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}) \to 0$ as $\delta \to 0$, then $(\bar{q}(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \delta) - q(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}))_x$ is compact in $H^{-1}_{loc}(R \times R^+)$. Using the first estimate in (2.22), (2.12) and the boundedness of $\eta_u$, we have the compactness of

$$\varepsilon\eta(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})_{xx} \quad \text{in } H^{-1}_{loc}(R \times R^+), \tag{2.24}$$

as $\varepsilon, \delta$ tend to zero. Using the second estimate in (2.22), (2.12), the boundedness of $\eta_{uu}$ and the relation $\eta_{\rho\rho} = \theta^2\rho^{\gamma-3}\eta_{uu}$, we have the boundedness of

$$\varepsilon(\eta(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})_{\rho\rho}(\rho_x^{\varepsilon,\delta})^2 + 2\eta(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})_{\rho u}\rho_x^{\varepsilon,\delta}u_x^{\varepsilon,\delta} + \eta(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})_{uu}(u_x^{\varepsilon,\delta})^2) \tag{2.25}$$

in $L^1_{loc}(R \times R^+)$, and hence the compactness in $W^{-1,\alpha}$ for some $\alpha \in (1, 2)$ by Sobolev's embedding Theorem.

Therefore the right-hand side of (2.23) is compact in $W^{-1,\alpha}_{loc}(R \times R^+)$ for some $\alpha \in (1, 2)$, but the left-hand side is bounded in $W^{-1,\infty}(R \times R^+)$. This implies the compactness of $\eta(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})_t + q(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})_x$ in $H^{-1}_{loc}(R \times R^+)$ by Murat's Theorem [Mu], and hence the proof of Lemma 2.

**Lemma 3** *For the viscosity solutions $(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))$ of the Cauchy problem (2.6) and (2.7),*

$$\eta_j(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))_t + q_j(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))_x, \quad j = 1, 2, 3, \tag{2.26}$$

*are compact in $H^{-1}_{loc}(R \times R^+)$ as $\varepsilon$ and $\delta$ tends to zero, where*

$$C = -\frac{2\lambda\theta\int_0^\infty (s+2)^{\lambda-1}s^\lambda ds}{\int_{-1} 1(1-s^2)^\lambda ds} > 0, \tag{2.27}$$

$$\eta_1 = \eta_+ + C\eta_0, \quad \eta_2 = \eta_- + C\eta_0, \quad \eta_3 = \eta_+ - \eta_-, \tag{2.28}$$

*$\eta_\pm, \eta_0$ being given by (2.13),(2.15), and $q_j$ are corresponding entropy fluxes of $\eta_j$.*

**Proof of Lemma 3.** We only give the proof of Lemma 3 for $(\eta_1, q_1)$. A similar treatment gives the proof for $(\eta_j, q_j), j = 2, 3$.

Let $\tau = \xi - w$. Then

$$\eta_+(\rho, u) = \int_w^\infty g(\xi)(\xi - z)^\lambda (\xi - w)^\lambda d\xi = \int_0^\infty g(\tau + w)(\tau + 2\rho^\theta)^\lambda \tau^\lambda d\tau \tag{2.29}$$

and hence

$$\begin{aligned} \eta_+(\rho, u)_\rho \ &= \textstyle\int_0^\infty g'(\tau + w)(\tau + 2\rho^\theta)^\lambda \tau^\lambda d\tau \theta \rho^{\theta-1} \\ &+ \textstyle\int_0^\infty g(\tau + w)(\tau + 2\rho^\theta)^{\lambda-1} \tau^\lambda d\tau 2\lambda\theta\rho^{\theta-1}. \end{aligned} \tag{2.30}$$

Since $-1 < 2\lambda < 0$, the first term on the right-hand side tends to zero as $\rho$ tends zero. Let $\tau = \rho^\theta s$ in the second part of the right hand-side of (2.30). Then

$$\begin{aligned} \eta_+(\rho, u)_\rho \ &= \textstyle\int_0^\infty g'(\tau + w)(\tau + 2\rho^\theta)^\lambda \tau^\lambda d\tau \theta \rho^{\theta-1} \\ &+ \textstyle\int_0^\infty g(\rho^\theta s + w)(s + 2)^{\lambda-1} s^\lambda ds 2\lambda\theta, \end{aligned} \tag{2.31}$$

since $(\rho^\theta)^{2\lambda+1}\rho^{-1} = 1$. Thus

$$\eta_+(\rho, u)_\rho|_{\rho=0} = 2\lambda\theta g(u) \int_0^\infty (s + 2)^{\lambda-1} s^\lambda ds. \tag{2.32}$$

Similarly

$$\eta_0(\rho, u) = \rho \int_{-1}^1 g(u + \rho^\theta s)(1 - s^2)^\lambda ds \tag{2.33}$$

and hence

$$\eta_0(\rho, u)_\rho = \int_{-1}^1 g(u + \rho^\theta s)(1 - s^2)^\lambda ds + \theta\rho^\theta \int_{-1}^1 g'(u + \rho^\theta s)(1 - s^2)^\lambda ds \tag{2.34}$$

or

$$\eta_0(\rho, u)_\rho|_{\rho=0} = g(u) \int_{-1}^1 (1 - s^2)^\lambda ds. \tag{2.35}$$

Combining (2.32) and (2.35), we have $\eta_1(\rho, u)_\rho|_{\rho=0} = 0$. It is easy to see that $\eta_1$ is smooth on the variable $u$, hence the proof of Lemma 3 is ended by using Lemma 2.

**Proof of Theorem 1.** we apply the measure equation to $(\eta_i, q_i)$ and $(\eta_j, q_j)$ to obtain

$$\begin{aligned}
&\int g(\xi_1)\overline{G_i(\xi_1)}d\xi_1 \int h(\xi_2)\overline{[\theta\xi_2+(1-\theta)u]G_j(\xi_2)}d\xi_2 \\
&\quad -\int h(\xi_2)\overline{G_j(\xi_2)}d\xi_2 \int g(\xi_1)\overline{[\theta\xi_1+(1-\theta)u]G_i(\xi_1)}d\xi_1 \\
&\quad = \int g(\xi_1)h(\xi_2)\overline{G_i(\xi_1)[\theta\xi_2+(1-\theta)u]G_j(\xi_2)}d\xi_1 d\xi_2 \\
&\quad -\int g(\xi_1)h(\xi_2)\overline{G_i(\xi_1)[\theta\xi_1+(1-\theta)u]G_j(\xi_2)}d\xi_1 d\xi_2.
\end{aligned} \tag{2.36}$$

where $G_i$ are fundamental solutions corresponding to the entropies $\eta_i$.

The last equality holds for arbitrary functions $g, h$ with compact support sets, and this yields

$$\begin{aligned}
&\overline{G_i(\xi_1)}\,\overline{[\theta\xi_2+(1-\theta)u]G_j(\xi_2)} - \overline{G_j(\xi_2)}\,\overline{[\theta\xi_1+(1-\theta)u]G_i(\xi_1)} \\
&= \overline{G_i(\xi_1)[\theta\xi_2+(1-\theta)u]G_j(\xi_2)} - \overline{G_j(\xi_2)[\theta\xi_1+(1-\theta)u]G_i(\xi_1)} \\
&= \theta(\xi_2-\xi_1)\overline{G_i(\xi_1)G_j(\xi_2)}.
\end{aligned} \tag{2.37}$$

Here and below we use the overbar to indicate the usual integration with respect to the Young measure; for instance

$$\overline{G(\xi)} = \int G(\rho, u-\xi)d\nu_{x,t}(\rho, u). \tag{2.38}$$

Let

$$\xi_+ = \inf_{(\rho,u)\in\ \mathrm{supp}\,\nu_{x,t}} w(\rho, u), \quad \xi_- = \sup_{(\rho,u)\in\ \mathrm{supp}\,\nu_{x,t}} z(\rho, u). \tag{2.39}$$

**Case I:** $\xi_- \le \xi_+$.

If $\xi_- \le \xi_+$, we choose $G_i = G_j = G_3$ and $\xi_1, \xi_2 \in (\xi_+, \infty)$. Since $G_-(\xi_1) = G_-(\xi_2) = 0$, we may rewrite (2.37) as

$$\frac{\theta}{1-\theta}\left[\frac{\overline{G_+(\xi_1)G_+(\xi_2)}}{\overline{G_+(\xi_1)}\,\overline{G_+(\xi_2)}} - 1\right] = \frac{1}{\xi_2-\xi_1}\left[\frac{\overline{uG_+(\xi_2)}}{\overline{G_+(\xi_2)}} - \frac{\overline{uG_+(\xi_1)}}{\overline{G_+(\xi_1)}}\right]. \tag{2.40}$$

Similarly

$$\frac{\theta}{1-\theta}\left[\frac{\overline{G_-(\xi_1)G_-(\xi_2)}}{\overline{G_-(\xi_1)}\,\overline{G_-(\xi_2)}} - 1\right] = \frac{1}{\xi_2-\xi_1}\left[\frac{\overline{uG_-(\xi_2)}}{\overline{G_-(\xi_2)}} - \frac{\overline{uG_-(\xi_1)}}{\overline{G_-(\xi_1)}}\right] \tag{2.41}$$

for $\xi_1, \xi_2 \in (-\infty, \xi_-)$.

As doen in [LPT], let $f_0^{\pm}(\xi)$ abbreviate $f_0^{\pm}(\xi) = \frac{G_{\pm}(\xi) - \overline{G_{\pm}(\xi)}}{\overline{G_{\pm}(\xi)}}$, so that (2.40) and (2.41) take the equivalent form

$$\frac{\theta}{1-\theta}\overline{f_0^{\pm}(\xi_1)f_0^{\pm}(\xi_2)} = \frac{1}{\xi_2 - \xi_1}\left[\frac{\overline{uG_{\pm}(\xi_2)}}{\overline{G_{\pm}(\xi_2)}} - \frac{\overline{uG_{\pm}(\xi_1)}}{\overline{G_{\pm}(\xi_1)}}\right] \tag{2.42}$$

Let $I_\alpha(\xi)$ be a nonnegative, smooth function with compact support set in $(-\frac{1}{\alpha}, \frac{1}{\alpha})$ and $I_\alpha(\xi) \to 1$ as $\alpha \to 0^+, \psi_\alpha(\xi) \geq 0$ be a unit mass mollifier, and denote $f_\alpha^{\pm} = (f_0^{\pm} I_\alpha) * \psi_\alpha$. Then we have from (2.42) that

$$\frac{\theta}{1-\theta}\overline{f_\alpha^{\pm}(\xi_1)f_\alpha^{\pm}(\xi_2)} = \frac{1}{\xi_2 - \xi_1}\left[\frac{\overline{uG_{\pm}(\xi_2)}}{\overline{G_{\pm}(\xi_2)}} - \frac{\overline{uG_{\pm}(\xi_1)}}{\overline{G_{\pm}(\xi_1)}}\right] I_\alpha(\xi_1)I_\alpha(\xi_2) * \psi_\alpha(\xi_1) * \psi_\alpha(\xi_2). \tag{2.43}$$

Thanks to the boundedness of the left-hand side and the smoothness of the right-hand side, we may now take $\xi_2 = \xi_1 = \xi$, to find out that

$$\begin{aligned} &\tfrac{\theta}{1-\theta}\overline{(f_\alpha^{\pm}(\xi))^2} \\ &= \tfrac{1}{\xi_2-\xi_1}\left[\tfrac{\overline{uG_{\pm}(\xi_2)}}{\overline{G_{\pm}(\xi_2)}} - \tfrac{\overline{uG_{\pm}(\xi_1)}}{\overline{G_{\pm}(\xi_1)}}\right] I_\alpha(\xi_1)I_\alpha(\xi_2) * \psi_\alpha(\xi_1) * \psi_\alpha(\xi_2)\ |_{\xi_2=\xi_1=\xi}\,. \end{aligned} \tag{2.44}$$

If we now let $\alpha$ tend to zero, then the left-hand side of (2.44) yields a negative measure since $1-\theta < 0$, whereas the right-hand side tends to

$$\frac{\partial}{\partial \xi}\left(\frac{\overline{uG_{\pm}(\xi)}}{\overline{G_{\pm}(\xi)}}\right).$$

Therefore $\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}}$ and $\frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}}$ are both nonincresing.

However,

$$\begin{aligned} \overline{u} = \lim_{\xi\to\infty} \tfrac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} \leq \lim_{\xi\to\xi_+} \tfrac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} \leq \tfrac{\xi_+ + \xi_-}{2} \\ \leq \lim_{\xi\to\xi_-} \tfrac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} \leq \lim_{\xi\to-\infty} \tfrac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} = \overline{u}, \end{aligned} \tag{2.45}$$

then $\frac{\overline{uG_+(\xi_2)}}{\overline{G_+(\xi_2)}}$ and $\frac{\overline{uG_-(\xi_1)}}{\overline{G_-(\xi_1)}}$ are both constant on $\xi_2 \in (\xi_+, \infty), \xi_1 \in (-\infty, \xi_-)$.

Using (2.40) and (2.41) again, we have

$$\overline{G_\alpha^{\pm}(\xi_1)G_\alpha^{\pm}(\xi)} = \overline{G_\alpha^{\pm}(\xi_1)}\,\overline{G_\alpha^{\pm}(\xi)}. \tag{2.46}$$

Letting $\xi_1 \to \xi$ in (2.46), we get

$$\overline{(G_\alpha^{\pm}(\xi))^2} = (\overline{G_\alpha^{\pm}(\xi)})^2, \tag{2.47}$$

which implies that

$$\overline{(G_\alpha^\pm(\xi) - \overline{G_\alpha^\pm(\xi)})^2} = 0 \tag{2.48}$$

and hence,

$$(G_\alpha^\pm(w,z,\xi) - \overline{G_\alpha^\pm(\xi)})^2 = 0, \tag{2.49}$$

on $(w, z) \in supp\, \nu_{x,t}$. Hence $G_\alpha^\pm(w,z,\xi) - \overline{G_\alpha^\pm(\xi)} = 0$ on the support set of $\nu_{x,t}$ and, by letting $\alpha \to 0$, so does $G_\pm(w,z,\xi) = \overline{G_\pm(\xi)}$. This shows that $\nu_{x,t}$ is a Dirac mass.

**Case II:** $\xi_- > \xi_+$.

If $\xi_- > \xi_+$, by the same treatment as above, we have that

$$\frac{\overline{uG_+(\xi_2)}}{\overline{G_+(\xi_2)}}, \quad \frac{\overline{uG_-(\xi_1)}}{\overline{G_-(\xi_1)}} \quad \text{are both nonincresing} \tag{2.50}$$

for $\xi_2 \in (\xi_-, \infty), \xi_1 \in (-\infty, \xi_+)$.

However since the following estimates from (2.45),

$$\lim_{\xi\to\infty} \frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} = \overline{u}, \quad \lim_{\xi\to-\infty} \frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} = \overline{u},$$

we have the following inequality

$$\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}} \geq \frac{\overline{uG_-(\xi_+)}}{\overline{G_-(\xi_+)}}. \tag{2.51}$$

Now we choose $G_i = G_1, G_j = G_2$ and $\xi_1 = \xi_2 = \xi$ in (2.37) to obtain

$$\begin{aligned} &(\overline{uG_+(\xi) + CuG_0(\xi)})(\overline{G_-(\xi) + CG_0(\xi)}) \\ &= (\overline{uG_-(\xi) + CuG_0(\xi)})(\overline{G_+(\xi) + CG_0(\xi)}) \end{aligned} \tag{2.52}$$

or

$$\begin{aligned} &\overline{uG_+(\xi)}\;\overline{G_-(\xi)} + C\overline{uG_0(\xi)}\;\overline{G_-(\xi)} + C\overline{uG_+(\xi)}\;\overline{G_0(\xi)} \\ &= \overline{uG_-(\xi)}\;\overline{G_+(\xi)} + C\overline{uG_0(\xi)}\;\overline{G_+(\xi)} + C\overline{uG_-(\xi)}\;\overline{G_0(\xi)}. \end{aligned} \tag{2.53}$$

Let

$$w_+ = \sup_{(\rho,u)\in\, \mathrm{supp}\, \nu_{x,t}} w(\rho,u), \quad z_- = \inf_{(\rho,u)\in\, \mathrm{supp}\, \nu_{x,t}} z(\rho,u).$$

If we choose $\xi \in (\xi_-, w_+)$, then $\overline{uG_-(\xi)} = \overline{G_-(\xi)} = 0$ and hence by (2.53)

$$\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} = \frac{\overline{uG_0(\xi)}}{\overline{G_0(\xi)}} \tag{2.54}$$

for $\xi \in (\xi_-, w_+)$.

In particular,

$$\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}} = \frac{\overline{uG_0(\xi_-)}}{\overline{G_0(\xi_-)}}. \tag{2.55}$$

Similarly, if chossing $\xi \in (z_-, \xi_+)$, we have

$$\frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} = \frac{\overline{uG_0(\xi)}}{\overline{G_0(\xi)}} \tag{2.56}$$

for $\xi \in (z_-, \xi_+)$.

In particular,

$$\frac{\overline{uG_-(\xi_+)}}{\overline{G_-(\xi_+)}} = \frac{\overline{uG_0(\xi_+)}}{\overline{G_0(\xi_+)}}. \tag{2.57}$$

Using (2.53), we have

$$\begin{aligned}&\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} + C\frac{\overline{uG_0(\xi)}}{\overline{G_+(\xi)}} + C\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}}\,\frac{\overline{G_0(\xi)}}{\overline{G_-(\xi)}}\\ &= \frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} + C\frac{\overline{uG_0(\xi)}}{\overline{G_-(\xi)}} + C\frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}}\,\frac{\overline{G_0(\xi)}}{\overline{G_+(\xi)}}.\end{aligned} \tag{2.58}$$

Below we use $\xi^{+0}$ to indicate the right limit and $\xi^{-0}$ the left limit at $\xi$. Letting $\xi \to \xi_-$ in (2.58) and using (2.55), we have

$$\begin{aligned}&\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}} + C\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}}\,\frac{\overline{G_0(\xi_-)}}{\overline{G_+(\xi_-)}} + C\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}}\,\frac{\overline{G_0(\xi_-)}}{\overline{G_-(\xi_-^{-0})}}\\ &= \frac{\overline{uG_-(\xi_-^{-0})}}{\overline{G_-(\xi_-^{-0})}} + C\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}}\,\frac{\overline{G_0(\xi_-)}}{\overline{G_-(\xi_-^{-0})}} + C\frac{\overline{uG_-(\xi_-^{-0})}}{\overline{G_-(\xi_-^{-0})}}\,\frac{\overline{G_0(\xi_-)}}{\overline{G_+(\xi_-)}}.\end{aligned} \tag{2.59}$$

or

$$\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}}\Big(1 + C\frac{\overline{G_0(\xi_-)}}{\overline{G_+(\xi_-)}}\Big) = \frac{\overline{uG_-(\xi_-^{-0})}}{\overline{G_-(\xi_-^{-0})}}\Big(1 + C\frac{\overline{G_0(\xi_-)}}{\overline{G_+(\xi_-)}}\Big), \tag{2.60}$$

which implies

$$\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}} = \frac{\overline{uG_-(\xi_-^{-0})}}{\overline{G_-(\xi_-^{-0})}}. \tag{2.61}$$

Similarly, letting $\xi \to \xi_+$ in (2.58) and using (2.57), we have

$$\frac{\overline{uG_+(\xi_+^{+0})}}{\overline{G_+(\xi_+^{+0})}} = \frac{\overline{uG_-(\xi_+)}}{\overline{G_-(\xi_+)}}. \tag{2.62}$$

Let $G_i = G_j = G_1$ in (2.37). By the same treatment in Case I, we have that

$$\frac{\overline{uG_1(\xi)}}{\overline{G_1(\xi)}} \quad \text{is nonincreasing for} \quad \xi \in (\xi_+, \xi_-). \tag{2.63}$$

However
$$\lim_{\xi\to\xi_-}\frac{\overline{uG_1(\xi)}}{\overline{G_1(\xi)}}=\lim_{\xi\to\xi_-}\frac{\overline{uG_+(\xi)+CuG_0(\xi)}}{\overline{G_+(\xi)+CG_0(\xi)}}$$

$$=\lim_{\xi\to\xi_-}\Big(\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}}+C\frac{\overline{uG_0(\xi)}}{\overline{G_0(\xi)}}\ \frac{\overline{G_0(\xi)}}{\overline{G_+(\xi)}}\Big)\frac{1}{1+C\frac{\overline{G_0(\xi)}}{\overline{G_+(\xi)}}} \tag{2.64}$$

$$=\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}},$$

where (2.55) is used in the last equality, and
$$\lim_{\xi\to\xi_+}\frac{\overline{uG_1(\xi)}}{\overline{G_1(\xi)}}=\lim_{\xi\to\xi_+}\frac{\overline{uG_+(\xi)+CuG_0(\xi)}}{\overline{G_+(\xi)+CG_0(\xi)}}$$

$$=\lim_{\xi\to\xi_+}\Big(\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}}+C\frac{\overline{uG_0(\xi)}}{\overline{G_0(\xi)}}\ \frac{\overline{G_0(\xi)}}{\overline{G_+(\xi)}}\Big)\frac{1}{1+C\frac{\overline{G_0(\xi)}}{\overline{G_+(\xi)}}} \tag{2.65}$$

$$=\frac{\overline{uG_-(\xi_+)}}{\overline{G_-(\xi_+)}},$$

where (2.57) and (2.62) are used in the last equality. Thus
$$\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}}\le\frac{\overline{uG_-(\xi_+)}}{\overline{G_-(\xi_+)}}. \tag{2.66}$$

(2.66) and (2.51) imply that
$$\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}}\quad\text{is a constant for}\quad \xi\in(\xi_+,\infty) \tag{2.67}$$

and
$$\frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}}\quad\text{is a constant for}\quad \xi\in(-\infty,\xi_-). \tag{2.68}$$

Hence Young measure $\nu$ is also a Dirac mass from the proof in Case I. This is contrary to the assumption $\xi_+<\xi_-$ since $w\ge z$. Therefore only Case I, i.e., $\xi_+\ge\xi_-$ is permitted, and $\nu$ is a Dirac mass. So we end the proof of Theorem 1.
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