Dynamics of Propagation and Interaction of Delta-Shock Waves in Conservation Law Systems

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Abstract. We introduce a *new definition of a δ-shock wave type solution* for a class of systems of conservation laws in the one-dimensional case. The *weak asymptotics method* developed by the authors is used to construct formulas describing the propagation and interaction of δ-shock waves. The dynamics of merging two δ-shocks is analytically described.

1. Introduction and the main results

1. Singular solutions to systems of conservation laws. In the papers [**3**], [**4**]– [**9**], [**28**], [**29**] (see also [**2**], [**27**]) the *weak asymptotics method* for studying the *dynamics of propagation and interaction* of different singularities (infinitely narrow δ-solitons, shocks, δ-shocks) of nonlinear equations and hyperbolic systems of conservation laws was developed. One of the main ideas of this method is based on the ideas of V. P. Maslov's approach that permits deriving the Rankine–Hugoniot conditions directly from the differential equations *considered in the weak sense* [**21**], [**24**], [**2**] (see also G. B. Whitham [**33**, 2.7.,5.6.]). Maslov's *algebras of singularities* [**22**], [**23**], [**2**] are essentially used in the our method.

In this paper we introduce a *new definition of a δ-shock wave type solution* for a class of hyperbolic systems of conservation laws. Using this definition, in the framework of the *weak asymptotics method* we describe the propagation and interaction of δ-shock waves. The subject of this paper was presented at the IXth International Conference on Hyperbolic Problems [**6**]. Here we give the full version of this work.

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Consider the system of conservation laws

$$\begin{aligned}\mathcal{L}_1[u,v] = u_t + \Big(F(u,v)\Big)_x &= 0,\\ \mathcal{L}_2[u,v] = v_t + \Big(G(u,v)\Big)_x &= 0,\end{aligned} \tag{1.1}$$

where $F(u,v)$ and $G(u,v)$ are smooth functions, such that $F(u,v)$, $G(u,v)$ are *linear* with respect to v, $u = u(x,t)$, $v = v(x,t) \in \mathbb{R}$, and $x \in \mathbb{R}$. As is well known, such a system, even in the case of smooth (and, moreover, in the case of discontinuous) initial data $(u^0(x), v^0(x))$, can have a discontinuous *shock wave* type solution. In this case, it is said that the pair of functions $(u(x,t), v(x,t)) \in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}^2\big)$ is a *generalized solution* of the Cauchy problem (1.1) with the initial data $(u^0(x), v^0(x))$ if the integral identities

$$\begin{aligned}\int_0^\infty\int \Big(u\varphi_t + F(u,v)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0,\\ \int_0^\infty\int \Big(v\varphi_t + G(u,v)\varphi_x\Big)\,dx\,dt + \int v^0(x)\varphi(x,0)\,dx &= 0\end{aligned} \tag{1.2}$$

hold for all compactly supported test functions $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\infty))$, where $\int \cdot\, dx$ denotes an improper integral $\int_{-\infty}^{\infty} \cdot\, dx$.

Let us consider the Cauchy problem for system (1.1) with the initial data

$$u^0(x) = u_0 + u_1 H(-x), \quad v^0(x) = v_0 + v_1 H(-x), \tag{1.3}$$

where u_0, u_1, v_0, v_1 are constants and $H(\xi)$ is the Heaviside function. It is well known [**6**]– [**9**], [**10**], [**11**], [**13**], [**15**], [**16**], [**18**], [**28**], [**29**], [**31**] that in order to solve this problem for some "nonclassical cases", it is necessary to introduce new *elementary singularities called δ-shock waves*. These are generalized solutions of the Cauchy problem of the form

$$\begin{aligned}u(x,t) &= u_0 + u_1 H(-x+ct),\\ v(x,t) &= v_0 + v_1 H(-x+ct) + e(t)\delta(-x+ct),\end{aligned} \tag{1.4}$$

where $e(0) = 0$ and $\delta(\xi)$ is the Dirac delta function.

We note that at present several approaches to constructing such solutions are known. An apparent difficulty in defining such solutions arises due to the fact that (as follows from (1.4)), to introduce a definition of the δ-shock wave type solution, we need to define the *product of the Heaviside fucntion and the δ-function*. As we see below, it is easy to overcome this difficulty (see also 1.3). Also we need to define *in which sense* the distributional solution (1.4) satisfies a nonlinear system (1.1).

In what follows, we present a short review of well-known methods used to solve problems close to those studied in this paper.

In [**15**], a δ-shock wave type solution of the system

$$u_t + (u^2/2)_x = 0, \quad v_t + (uv)_x = 0$$

(here $F(u,v) = u^2/2$, $G(u,v) = vu$) with the initial data (1.3), is defined as the weak limit of the solution $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ of the parabolic regularization

$$u_t + (u^2/2)_x = \varepsilon u_{xx}, \quad v_t + (uv)_x = \varepsilon v_{xx}$$

with the initial data (1.3), as $\varepsilon \to +0$.

In [**13**], to obtain a δ-shock wave type solution of system

$$L_1[u,v] = u_t + \big(f(u)\big)_x = 0, \quad L_2[u,v] = v_t + \big(g(u)v\big)_x = 0, \tag{1.5}$$

(here $F(u,v) = f(u)$, $G(u,v) = vg(u)$), this system is reduced to a system of Hamilton–Jacobi equations, and then the Lax formula is used. In [**18**], to construct a δ-shocks wave type solution for the case $g(u) = f'(u)$, the problem of multiplication of distributions is solved by using the definition of Volpert's averaged superposition [**32**]. In [**25**], using the ideas of A. Volpert the nonconservative product of singular functions is introduced. This product is used in theory of nonconservative hyperbolic systems.

In [**16**], the system

$$u_t + (u^2 - v)_x = 0, \quad v_t + (\frac{1}{3}u^3 - u)_x = 0, \tag{1.6}$$

with the initial data (1.3) is studied (here $F(u,v) = u^2 - v$, $G(u,v) = u^3 - u$). In order to construct *approximate δ-shock type solution* the Colombeau theory approach, as well as the Dafermos–DiPerna regularization, and the box approximations are used. But the notion of a *singular solution* of system (1.6) has *not been defined*.

In [**26**] in the framework of the Colombeau theory approach, for particular cases of system (1.1) *approximate δ-shock type solutions* were constructed.

In [**31**] for the system

$$L_{01}[u] = u_t + (u^2)_x = 0, \quad L_{02}[u,v] = v_t + (uv)_x = 0, \tag{1.7}$$

in [**1**], [**19**] for the "zero-pressure gas dynamics system"

$$v_t + \big(vu\big)_x = 0, \quad (vu)_t + \big(vu^2\big)_x = 0, \tag{1.8}$$

(here $v \geq 0$ is the density, u is the velocity), and in [**34**] for the system

$$v_t + \big(vf(u)\big)_x = 0, \quad (vu)_t + \big(vuf(u)\big)_x = 0, \tag{1.9}$$

with the initial data (1.3), the δ-shock wave type solutions are defined as a measure-valued solutions (see also [**30**]).

Let $BM(\mathbb{R})$ be the space of bounded Borel measures. A pair $(u,\, v)$, where $u(x,t) \in L^\infty\big([0,\,\infty), L^\infty(\mathbb{R})\big)$, $v(x,t) \in C\big([0,\infty), BM(\mathbb{R})\big)$, is said to be a measure-valued solution of the Cauchy problem (1.9), (1.3) if the integral identities

$$\begin{aligned} \int_0^\infty \int \Big(\varphi_t + f(u)\varphi_x\Big)\, v(dx,t) &= 0, \\ \int_0^\infty \int u\Big(\varphi_t + f(u)\varphi_x\Big)\, v(dx,t) &= 0, \end{aligned} \tag{1.10}$$

hold for all $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0,\,\infty))$.

Within the framework of this definition in [**31**], [**1**], and [**34**] for systems (1.7), (1.8), and (1.9), respectively, the following formulas for δ-shock waves were derived

$$
(u(x,t),\ v(x,t)) = \begin{cases} \big(u^-,\ v^-\big), & x < \phi(t), \\ \big(u_\delta,\ w(t)\delta(x-\phi(t))\big), & x = \phi(t), \\ \big(u^+,\ v^+\big), & x > \phi(t). \end{cases} \tag{1.11}
$$

Here u^-, u^+ and u_δ are the velocities before the discontinuity, after the discontinuity, and at the point of discontinuity, respectively, and $\phi(t) = \sigma_\delta t$ is the equation for the discontinuity line.

In [**10**], the global δ-shock wave type solution was obtained for system (1.8).

In [**14**], the interaction of (two)δ-shocks for system (1.9) is considered.

In the framework of the *weak asymptotics method*, in [**8**], the propagation of δ-shock waves was described for systems (1.5), (1.8), (1.6). In [**6**], [**7**], [**9**] a short review of our results on the propagation and interaction of δ-shock waves for system (1.5) was presented. In [**28**], [**29**] in the framework of the *weak asymptotics method* the δ-shock wave type solution of the system

$$
u_t + \big(f(u) - v\big)_x = 0, \quad v_t + \big(g(u)\big)_x = 0,
$$

was constructed, where $f(u)$ and $g(u)$ are polynomials of degree n and $n+1$, respectively, n is even. System (1.6) is a particular case of the last system. In the papers [**7**], [**8**] a *new definition of a δ-shock wave type solution* for systems (1.1), (1.8) was introduced. This definition is *close* to the standard definition of the shock type solutions (1.2) and *relevant* to the notion of δ-shocks.

The study of system (1.1), (1.9), which admit δ-shock wave type solutions is very important in applications, because systems of this type often arise in modeling physical processes in gas dynamics, magnetohydrodynamics, filtration theory, and cosmogony [**17**], [**12**], [**10**], [**35**].

In the present paper we apply the *weak asymptotics method* for studying the dynamics of *propagation and interaction* of δ-shock waves for system (1.5), i.e., we solve the Cauchy problem (1.5) with the initial data of the form

$$
\begin{aligned} u^0(x) &= u_0^0(x) + \textstyle\sum_{k=1}^2 u_k^0(x) H(-x + x_k^0), \\ v^0(x) &= v_0^0(x) + \textstyle\sum_{k=1}^2 \Big(v_k^0(x) H(-x + x_k^0) + e_k^0 \delta(-x + x_k^0)\Big), \end{aligned} \tag{1.12}
$$

where $u_0^0(x)$, $u_k^0(x)$, $v_0^0(x)$, and $v_k^0(x)$ are smooth functions, $u_k^0(x_k^0) > 0$, e_k^0 are constants, $k = 1, 2$, and $x_1^0 < x_2^0$. If we study only the dynamics of propagation of δ-shock waves, then we set $u_2^0(x) = v_2^0(x) = e_2^0 = 0$, $e_1^0 = e^0$, and $x_1^0 = 0$ in (1.12) and solve the Cauchy problem for system (1.5) with the initial data

$$
\begin{aligned} u^0(x) &= u_0^0(x) + u_1^0(x) H(-x), \\ v^0(x) &= v_0^0(x) + v_1^0(x) H(-x) + e^0 \delta(-x), \end{aligned} \tag{1.13}
$$

where $u_1^0(0) > 0$. The initial data (1.12), (1.13) *can contain* δ-function. But, as a rule, in the well-known papers on δ-shocks, the initial data without δ-function are considered.

2. δ-Shock wave type solutions. In what follows, we introduce a definition of a *generalized solution* [**7**], [**8**] for systems (1.1).

Suppose that $\Gamma = \{\gamma_i : i \in I\}$ is a connected graph in the upper half-plane $\{(x,t) : x \in \mathbb{R},\ t \in [0,\infty)\} \in \mathbb{R}^2$ containing smooth arcs γ_i, $i \in I$, and I is a finite set. By I_0 we denote a subset of I such that an arc γ_k for $k \in I_0$ starting from the points of the x-axis; $\Gamma_0 = \{x_k^0 : k \in I_0\}$ is the set of initial points of arcs γ_k, $k \in I_0$.

Consider the initial data $(u^0(x), v^0(x))$, where $v^0(x)$ has the following form

$$v^0(x) = V^0(x) + E^0\delta(\Gamma_0),$$

$E^0\delta(\Gamma_0) = \sum_{k\in I_0} e_k^0\delta(x - x_k^0)$, $u^0, V^0 \in L^\infty\big(\mathbb{R};\mathbb{R}\big)$, e_k^0 are constants, $k \in I_0$.

Definition 1.1. A pair of distributions $(u(x,t), v(x,t))$ and graph Γ, where $v(x,t)$ is represented in form of the sum

$$v(x,t) = V(x,t) + E(x,t)\delta(\Gamma),$$

$u, V \in L^\infty\big(\mathbb{R}\times(0,\,\infty);\mathbb{R}\big)$, $E(x,t)\delta(\Gamma) = \sum_{i\in I} e_i(x,t)\delta(\gamma_i)$, $e_i(x,t) \in C(\Gamma)$, $i \in I$, is called a *generalized δ-shock wave type solution* of system (1.1) with the initial data $(u^0(x), v^0(x))$ if the integral identities

$$\begin{aligned}
\int_0^\infty\!\!\int \Big(u\varphi_t + F(u,V)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty\!\!\int \Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt \qquad\qquad & \\
+\sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl \qquad & \\
+\int V^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} e_k^0\varphi(x_k^0,0) &= 0,
\end{aligned} \tag{1.14}$$

hold for all test functions $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\,\infty))$, where $\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}$ is the tangential derivative on the graph Γ, $\int_{\gamma_i}\cdot\,dl$ is a line integral over the arc γ_i.

For instance, the graph Γ containing only one arc $\{(x,t) : x = ct\}$, $\phi(0) = 0$ corresponds to solution (1.4).

Remark 1.1. The system of integral identities (1.14) generalizes the *usual system of integral identities* (1.2) which is the definition of a shock wave type solution. The integral identities (1.14) for *δ-shocks* differ from integral identities (1.2) by an additional term

$$\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl = \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl$$

in the second identity. This term reflects the fact that *the Rankine–Hugoniot conditions for δ-shocks* are defined by the *pair* of equations (fifth and sixth equations of (2.10)):

$$\dot{\phi}(t) = \frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}, \qquad \dot{e}(t) = \Big([vg(u)] - [v]\frac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)},$$

where the first equation is the *standard* Rankine–Hugoniot condition, $\dot{} = \frac{d}{dt}$. Moreover, the second relation appears due to the so-called *Rankine–Hugoniot deficit*, i.e., the right-hand side of the second equation of the last system.

According to Definition 1.1 a *generalized δ-shock wave type solution* is a pair of *distributions* $(u(x,t), v(x,t))$ unlike the Definition of measure-solutions given in [**1**], [**31**], [**34**], where $v(dx,t)$ is a *measure* and $u(x,t)$ is understood as a *measurable function* with respect to $v(dx,t)$.

Now we introduce the notion of a *weak asymptotic solution*, which is one of the most important in the *weak asymptotics method*.

We shall write $f(x,t,\varepsilon) = O_{\mathcal{D}'}(\varepsilon^\alpha)$, if $f(x,t,\varepsilon) \in \mathcal{D}'(\mathbb{R})$ is a distribution such that for any test function $\psi(x) \in \mathcal{D}(\mathbb{R}_x)$ we have

$$\langle f(x,t,\varepsilon), \psi(x)\rangle = O(\varepsilon^\alpha),$$

where $O(\varepsilon^\alpha)$ denotes a function continuous in t that admits the usual estimate $|O(\varepsilon^\alpha)| \le \text{const}\varepsilon^\alpha$ uniform in t. Relations of the form $o_{\mathcal{D}'}(1)$ are understood in the same way.

Definition 1.2. A pair of functions $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ smooth as $\varepsilon > 0$ is called a *weak asymptotic solution* of system (1.1) with the initial data $(u^0(x), v^0(x))$ if

$$\begin{aligned}
\int \mathcal{L}_1[u(x,t,\varepsilon), v(x,t,\varepsilon)]\psi(x)\,dx &= o(1),\\
\int \mathcal{L}_2[u(x,t,\varepsilon), v(x,t,\varepsilon)]\psi(x)\,dx &= o(1),\\
\int \Big(u(x,0,\varepsilon) - u^0(x)\Big)\psi(x)\,dx &= o(1),\\
\int \Big(v(x,0,\varepsilon) - v^0(x)\Big)\psi(x)\,dx &= o(1), \qquad \varepsilon \to +0,
\end{aligned}$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R})$. The last relations can be rewritten as

$$\begin{aligned}
\mathcal{L}_1[u(x,t,\varepsilon), v(x,t,\varepsilon)] &= o_{\mathcal{D}'}(1),\\
\mathcal{L}_2[u(x,t,\varepsilon), v(x,t,\varepsilon)] &= o_{\mathcal{D}'}(1),\\
u(x,0,\varepsilon) &= u^0(x) + o_{\mathcal{D}'}(1),\\
v(x,0,\varepsilon) &= v^0(x) + o_{\mathcal{D}'}(1),
\end{aligned} \tag{1.15}$$

where the first two estimates are uniform in t.

Within the framework of the *weak asymptotics method*, we find the *generalized solution* $(u(x,t), v(x,t))$ of the Cauchy problem (1.5), (1.12) as the weak limit (in the sense of the space of distributions $\mathcal{D}'(\mathbb{R}^2)$) Within the framework of the *weak asymptotics method*, we find the *generalized solution* $(u(x,t), v(x,t))$ of the Cauchy problem (1.5), (1.12) in the sense of Definition 1.1 as the weak limit (in the sense of the space of distributions $\mathcal{D}'(\mathbb{R}^2)$)

$$u(x,t) = \lim_{\varepsilon\to+0} u(x,t,\varepsilon), \quad v(x,t) = \lim_{\varepsilon\to+0} v(x,t,\varepsilon), \tag{1.16}$$

of the *weak asymptotic solution* of this problem.

3. The scheme of the method. Now for the case of δ-shocks we will describe the *typical* technique of our approach without paying attention to the algebraic aspects given in detail in [**2**], [**3**], [**27**].

a. To study the propagation of a *solitary* δ-shock wave, related to the hyperbolic system of conservation laws (for example, (1.5), we must solve the Cauchy problem (1.5), (1.13). To study the interaction of (two) δ-shocks, we solve the the Cauchy problem (1.5), (1.12).

According to our method, we will seek a *δ-shock wave type solution* of the Cauchy problem (1.5), (1.13) in the form of the *singular ansatz*

$$\begin{array}{rcl} u(x,t) & = & u_0(x,t) + u_1(x,t)H(-x+\phi(t)), \\ v(x,t) & = & v_0(x,t) + v_1(x,t)H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{array} \tag{1.17}$$

where $u_k(x,t)$, $v_k(x,t)$, $k=0,1$, $e(t)$, $\phi(t)$ are the desired functions.

We seek a *δ-shock wave type solution* of the Cauchy problem (1.5), (1.12) in the form of the *singular ansatz*

$$\begin{array}{rcl} u(x,t) & = & u_0(x,t) + \sum_{k=1}^2 u_k(x,t)H(-x+\phi_k(t)), \\ v(x,t) & = & v_0 + \sum_{k=1}^2 \Big(v_k(x,t)H(-x+\phi_k(t)) + e_k(t)\delta(-x+\phi_k(t)) \Big), \end{array} \tag{1.18}$$

which *corresponds* to the structure of the initial data (1.12). Here $u_0(x,t)$, $u_k(x,t)$, $v_0(x,t)$, $v_k(x,t)$, $e_k(t)$, $\phi_k(t)$ are the desired functions, $k=1,2$.

b. In the framework of our approach, we construct a *weak asymptotic solution* in the form of the *smooth ansatz*:

$$\begin{array}{rcl} u(x,t,\varepsilon) & = & \tilde{u}(x,t,\varepsilon) + R_u(x,t,\varepsilon), \\ v(x,t,\varepsilon) & = & \tilde{v}(x,t,\varepsilon) + R_v(x,t,\varepsilon), \end{array}$$

where a pair of functions $\big(\tilde{u}(x,t,\varepsilon), \tilde{v}(x,t,\varepsilon)\big)$ is a *regularization* of the singular ansatz (1.17) or (1.18) *with respect to singularities* $H(x)$, $\delta(x)$ and *with respect to phases amplitudes of δ-functions* $\phi_k(t)$, $e_2(t)$, $k=1,2$. Here the so-called *corrections* $R_u(x,t,\varepsilon)$, $R_v(x,t,\varepsilon)$ are desired functions which must admit the estimates:

$$R_j(x,t,\varepsilon) = o_{\mathcal{D}'}(1), \quad \frac{\partial R_j(x,t,\varepsilon)}{\partial t} = o_{\mathcal{D}'}(1), \qquad \varepsilon \to +0. \tag{1.19}$$

$j = u, v$.

In order to construct a regularization $f(x,\varepsilon)$ of the distribution $f(x) \in \mathcal{D}'(\mathbb{R})$ we use the representation

$$f(x,\varepsilon) = f(x) * \frac{1}{\varepsilon}\omega\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0, \tag{1.20}$$

where $*$ is a convolution, and a mollifier $\omega(\eta)$ has the following properties: (a) $\omega(\eta) \in C^\infty(\mathbb{R})$, (b) $\omega(\eta)$ has a compact support or decreases sufficiently rapidly, as $|\eta| \to \infty$, (c) $\int \omega(\eta)\,d\eta = 1$, (d) $\omega(\eta) \geq 0$, (e) $\omega(-\eta) = \omega(\eta)$. It is known that $\lim\limits_{\varepsilon\to+0} \langle f(x,\varepsilon), \varphi(x)\rangle = \langle f(x), \varphi(x)\rangle$ for all $\varphi(x) \in \mathcal{D}(\mathbb{R})$.

Thus, we will seek a *weak asymptotic solution* of the Cauchy problem (1.5), (1.12) in the following form:

$$
\begin{aligned}
u(x,t,\varepsilon) &= u_0(x,t) + \textstyle\sum_{k=1}^2 u_k(x,t) H_{uk}\big(-x+\phi_k(t,\varepsilon),\varepsilon\big) + R_u(x,t,\varepsilon),\\
v(x,t,\varepsilon) &= v_0(x,t) + \textstyle\sum_{k=1}^2 \Big(v_k(x,t) H_{vk}\big(-x+\phi_k(t,\varepsilon),\varepsilon\big)\\
&\quad + e_k(t,\varepsilon)\delta_{vk}\big(-x+\phi_k(t,\varepsilon),\varepsilon\big)\Big) + R_v(x,t,\varepsilon).
\end{aligned}
\tag{1.21}
$$

Here, according to the formula (1.20)

$$
\delta_{vk}(\xi,\varepsilon) = \frac{1}{\varepsilon}\omega_{\delta k}\big(\frac{\xi}{\varepsilon}\big)
\tag{1.22}
$$

are regularizations of the δ-function, and

$$
H_{jk}(\xi,\varepsilon) = \omega_{0jk}\big(\frac{\xi}{\varepsilon}\big) = \int_{-\infty}^{\frac{x}{\varepsilon}} \omega_{jk}(\eta)\,d\eta
\tag{1.23}
$$

are regularizations of the Heaviside function $H(\xi)$, where $\omega_{0jk}(z) \in C^\infty(\mathbb{R})$, and $\lim_{z\to+\infty}\omega_{0jk}(z) = 1$, $\lim_{z\to-\infty}\omega_{0jk}(z) = 0$, $j = u, v$, $k = 1, 2$; $\phi_k(t,\varepsilon)$, $e_k(t,\varepsilon)$ are desired functions such that

$$
\phi_k(t) = \lim_{\varepsilon\to+0}\phi_k(t,\varepsilon), \quad e_k(t) = \lim_{\varepsilon\to+0} e_k(t,\varepsilon), \qquad k = 1, 2.
$$

We construct a *weak asymptotic solution* of the Cauchy problem (1.5), (1.13) in the form (1.21) and set $u_2(x,t) = v_2(x,t) = e_2(t,\varepsilon) = 0$, $\phi_1(t,\varepsilon) \equiv \phi(t)$, $e_1(t,\varepsilon) \equiv e(t)$. Hence, we will seek a *weak asymptotic solution* of the Cauchy problem (1.5), (1.12) in the form

$$
\begin{aligned}
u(x,t,\varepsilon) &= u_0(x,t) + u_1(x,t) H_{u1}(-x+\phi(t),\varepsilon) + R_u(x,t,\varepsilon),\\
v(x,t,\varepsilon) &= v_0(x,t) + v_1(x,t) H_{v1}\big(-x+\phi(t),\varepsilon\big)\\
&\quad + e(t)\delta_{v1}\big(-x+\phi(t),\varepsilon\big) + R_v(x,t,\varepsilon).
\end{aligned}
\tag{1.24}
$$

The next step is to substitute the smooth ansatz (1.24) or (1.21) into the quasi-linear system $\mathcal{L}[u,v] = 0$ and to calculate the weak asymptotics of the left-hand side of $\mathcal{L}[u(x,t,\varepsilon), v(x,t,\varepsilon)]$ (in the sense of the space of distributions $\mathcal{D}'(\mathbb{R}_x)$) up to $o_{\mathcal{D}'}(1)$, as $\varepsilon \to +0$. We stress that in the framework of the weak asymptotics method, the discrepancy is assumed to be small in the sense of the space of functionals $\mathcal{D}'_x$ over test functions depending only on the “space” variable x. As we shall see below, this trivial trick allows us to reduce the problem of describing interaction of nonlinear waves to solving some system of ordinary differential equations (instead of solving partial differential equations).

In the construction of the weak asymptotics from $\mathcal{L}[u(x,t,\varepsilon),v(x,t,\varepsilon)]$, the *key role* is played by the construction of the weak asymptotics from superposition $f\big(u(x,t,\varepsilon),v(x,t,\varepsilon)\big)$, where $f(u,v)$ is a smooth function (for details, see Sec. 4). The weak asymptotics of $\mathcal{L}[u(x,t,\varepsilon),v(x,t,\varepsilon)]$ can be represented as *linear combinations* of the singularities $H(-x+\phi_k(t))$, $\delta(-x+\phi_k(t))$, $\delta'(-x+\phi_k(t))$, $k=1,2$ with smooth coefficients. That is why we can "separate" singularities and find a system of equations (in particular, the Rankine–Hugoniot conditions), which describes the dynamics of singularities and defines the desired functions $u_0(x,t)$, $u_k(x,t)$, $v_0(x,t)$, $v_k(x,t)$, $e_k(t,\varepsilon)$, $\phi_k(t,\varepsilon)$, $k=1,2$, and $R_u(x,t,\varepsilon)$, $R_v(x,t,\varepsilon)$. Thus, a *weak asymptotic solution* is constructed.

Since the *generalized δ-shock wave type solution* (1.18) is defined as a weak limit of (1.16), in view of the estimates (1.19), the corrections *do not make a contribution* to the generalized solution of the problem. However, these terms *make a contribution* to the weak asymptotics of the superposition $f\big(u(x,t,\varepsilon),v(x,t,\varepsilon)\big)$ (see (2.7) and (3.33) below) and hence plays an essential role in the construction of the generalized solution of the problem. Without introducing these terms, we cannot solve the Cauchy problem with arbitrary initial data (see Remark 2.2 below).

Note, that *choosing the corrections* is an *essential* part of the "right" construction of the *weak asymptotic solution.*

c. To describe the dynamics of interaction, we shall seek the phases of nonlinear waves $\phi_k(t,\varepsilon)=\widehat{\phi}_k(\tau,t)$ as functions of the "fast" variable $\tau=\psi_0(t)/\varepsilon$ and the "slow" variable t, where $\phi_{k0}(t)$ is the distance between the (solitary) wave fronts *before the instant of interaction.* Next, we obtain systems of equations for $\widehat{\phi}_k(\tau,t)$ and the differential equation with the boundary condition:

$$\frac{d\rho}{d\tau}=\mathcal{F}(\rho,t), \qquad \frac{\rho(\tau,t)}{\tau}\Big|_{\tau\to+\infty}=1, \tag{1.25}$$

where $\rho=\psi(t,\varepsilon)/\varepsilon$, $\psi(t,\varepsilon)=\phi_2(t,\varepsilon)-\phi_1(t,\varepsilon)$. Here the boundary condition shows that, before the interaction, the singularities propagate independently.

Finding the solution of the boundary value problem (1.25) and finding the limit values $\rho(\tau)\big|_{\tau\to-\infty}$ and $\phi_k(t)=\lim_{\tau\to-\infty}\widehat{\phi}_k(\tau,t)$, we can describe the dynamics of propagation and interaction of nonlinear waves and thus define the "result" of the interaction.

d. Constructing the *weak asymptotic solution* of the Cauchy problem and multiplying the first two relations (1.15) by a test function $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\infty))$, integrating these relations by parts and then passing to the limit as $\varepsilon\to+0$, we obtain that the pair of distributions (1.16) satisfy integral identities (1.14). Thus, we will prove that the limits of *weak asymptotic solutions* satisfy system (1.5), i.e. 'the pair of distributions (1.16) is a *δ-shock wave type solution* of the Cauchy problem (1.5), (1.12).

REMARK 1.2. To study the interaction of (two) *shock waves* for the scalar conservation law $u_t + (f(u))_x = 0$, we will seek a *weak asymptotic solution* of the problem in the form of the first relation (1.21), where we set $R_u(x,t,\varepsilon) = 0$ [**4**], [**5**].

To study the interaction of (two) *infinitely narrow* δ*-solitons* for the Korteweg-de Vries equation $v_t + (v^2)_x + \varepsilon^2 v_{xxx} = 0$, we will seek a *weak asymptotic solution* of the problem in the form of the second relation (1.21), where we set $R_v(x,t,\varepsilon) = 0$ and replace $H_{vk}(\xi,\varepsilon)$ by $\varepsilon H_{vk}(\xi,\varepsilon)$, and $\delta_{vk}(\xi,\varepsilon)$ by $\varepsilon\delta_{vk}(\xi,\varepsilon)$, $k = 1,2$ [**3**].

4. Example. We illustrate our approach by way of example of the specific case given by system (1.7). We solve the Cauchy problem (1.7), (1.3), i.e., we construct a solitary δ-shock wave type solution.

A. We seek a *weak asymptotic solution* of the Cauchy problem (1.7), (1.3) in the form

$$\begin{aligned} u(x,t,\varepsilon) &= u_0 + u_1 H_{u1}(-x+\phi(t),\varepsilon) + Q\Omega\big(\tfrac{-x+\phi(t)}{\varepsilon}\big), \\ v(x,t,\varepsilon) &= v_0 + v_1 H_{v1}\big(-x+\phi(t),\varepsilon\big) + e(t)\delta_{v1}\big(-x+\phi(t),\varepsilon\big), \end{aligned} \tag{1.26}$$

where u_k, v_k are constants, $k = 0,1$. We choose corrections in the form $R_u(x,t,\varepsilon) = Q\Omega\big(\frac{-x+\phi(t)}{\varepsilon}\big)$, $R_v(x,t,\varepsilon) = 0$, where $\varepsilon^{-1}\Omega\big(\xi/\varepsilon\big)$ is regularization (1.22) of the δ-function, Q is a constant. It is clear that

$$\int \Omega\big(\frac{x}{\varepsilon}\big)\psi(x)\,dx = \varepsilon\psi(0)\int \Omega(\xi)\,d\xi + O(\varepsilon^2), \quad \varepsilon \to +0,$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R})$, i.e., estimates (1.19) hold.

We note that in the *pointwise limit* we have

$$\lim_{\varepsilon\to+0} Q\Omega\Big(\frac{-x+\phi(t)}{\varepsilon}\Big) = \begin{cases} Q\Omega(0), & x = \phi(t), \\ 0, & x \neq \phi(t). \end{cases} \tag{1.27}$$

Hence the correction $R_u(x,t,\varepsilon)$ is a *regularization of the characteristic function of the point*, and if $Q\Omega(0) = u_\delta$, then, in the pointwise limit, our regularization $u(x,t,\varepsilon)$ converges to the expression obtained in [**31**] for the component $u(x,t)$ in solution (1.11). For our purposes, this similarity is not necessary. Moreover, in what follows, we shall construct *another* weak asymptotic solution, which does not possess this property, but satisfies the integral identities (1.14) in the limit. This weak asymptotic solution turns out to be more preferable for describing the δ-shock wave interaction studied in Section 3.

So we show how the *weak asymptotic solution* is constructed in our example.

We can show that (see [**3**], [**5**], [**6**] and Section 4), with accuracy $O_{\mathcal{D}'}(\varepsilon)$ as $\varepsilon \to +0$, we have

$$\big(u(x,t,\varepsilon)\big)^2 = u_0^2 + \Big(u_1^2 + 2u_0u_1\Big)H(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0, \tag{1.28}$$

$$\begin{aligned} u(x,t,\varepsilon)v(x,t,\varepsilon) &= u_0v_0 + \Big(u_0v_1 + u_1v_0 + u_1v_1\Big)H(-x+\phi(t)) \\ &\quad + \Big(u_0 + au_1 + bQ\Big)e(t)\delta(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0, \end{aligned} \tag{1.29}$$

where $a = \int \omega_{0u1}(\xi)\omega_{\delta 1}(\xi)\,d\xi$ and $b = \int \Omega(\xi)\omega_{\delta 1}(\xi)\,d\xi$.

Substituting regularization (1.26) and relations (1.28), (1.29) into the left-hand side of system (1.7), we see that

$$L_{01}[u(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon), \quad L_{02}[u(x,t,\varepsilon), v(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon) \tag{1.30}$$

if and only if

$$\begin{aligned} \Big(u_1\dot{\phi}(t) - (2u_0u_1 + u_1^2)\Big)\delta(-x+\phi(t)) &= 0, \\ \Big(v_1\dot{\phi}(t) + \dot{e}(t) - (u_0v_1 + u_1v_0 + u_1v_1)\Big)\delta(-x+\phi(t)) & \\ +\Big(\dot{\phi}(t) - (u_0 + au_1 + bQ)\Big)e(t)\delta'(-x+\phi(t)) &= 0. \end{aligned}$$

where $\dot{} = \frac{d}{dt}$. Hence we find the functions

$$\begin{aligned} \phi(t) &= \tfrac{[u^2]}{[u]}t = (2u_0 + u_1)t, \\ e(t) &= \Big([uv] - \tfrac{[u^2]}{[u]}[v]\Big)t = \big(u_1v_0 - u_0v_1\big)t, \end{aligned} \tag{1.31}$$

and the relation

$$Q = \frac{u_0 + (1-a)u_1}{b}, \tag{1.32}$$

which determines the constant Q, where $[\cdot]$ are jumps of the corresponding functions on the discontinuity curve $x = \phi(t)$. Thus, the *weak asymptotic solution* of the Cauchy problem (1.7), (1.3) is constructed.

Defining the generalized solution of our problem (1.7), (1.3) as the weak limit of regularizations (1.16), we obtain (1.4), where $c = \dot{\phi}(t)$, $\phi(t)$ and $e(t)$ are determined by system (1.31).

Relations (1.31) are the same as in [**31**].

We show that the weak limit (1.4) of the *weak asymptotic solution* (1.26) satisfies the integral identities (1.14). The integral identities (1.14) are derived in the same way as it is proved in [**5**] that the weak limit of the weak asymptotic solution satisfies the integral identity. Since $u(x,t,\varepsilon)$ and $v(x,t,\varepsilon)$ are smooth functions as $\varepsilon > 0$, applying the left- and right-hand sides of relations (1.30) to $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\,\infty))$ and integrating by parts the expression obtained in the left-hand side, we obtain relations (2.11), (2.12), where $f(u(x,t,\varepsilon)) = u^2(x,t,\varepsilon)$, $g(u(x,t,\varepsilon)) = u(x,t,\varepsilon)$, $T = \infty$. Next, passing to the limit in the last relations, as $\varepsilon \to +0$, and taking into account relations (1.26), (1.28), (1.29), and (2.13), (2.14), we obtain the following integral identities (1.14):

$$
(1.33)\qquad
\begin{aligned}
\int_0^{\infty}\int_{-\infty}^{\infty}\Big(u(x,t)\varphi_t+u^2(x,t)\varphi_x\Big)\,dx\,dt+\int_{-\infty}^{\infty}u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^{\infty}\int_{-\infty}^{\infty}\Big(V(x,t)\varphi_t+u(x,t)V(x,t)\varphi_x\Big)\,dx\,dt\qquad&\\
+\int_0^{\infty}e(t)\Big(\varphi_t(\phi(t),t)+\dot\phi(t)\varphi_x(\phi(t),t)\Big)\,dt\qquad&\\
+\int_{-\infty}^{\infty}V^0(x)\varphi(x,0)\,dx+e_0\varphi(0,0) &= 0,
\end{aligned}
$$

for all test functions $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\infty))$. Here, according to our notation,

$$
v(x,t)=V(x,t)+e(t)\delta(-x+\phi(t)),\quad V(x,t)=v_0+v_1H(-x+\phi(t)).
$$

We also use the Rankine–Hugoniot condition (1.31) to derive (1.33).

B. Now we will construct the solution of the Cauchy problem (1.7), (1.3), using the *weak asymptotic solution* of a different structure. Namely, we will seek the weak asymptotic solution in the form

$$
(1.34)\qquad
\begin{aligned}
u(x,t,\varepsilon) &= u_0+u_1H_{u1}(-x+\phi(t),\varepsilon),\\
v(x,t,\varepsilon) &= v_0+v_1H_{v1}\big(-x+\phi(t),\varepsilon\big)\\
&\quad+e(t)\delta_{v1}\big(-x+\phi(t),\varepsilon\big)+R(t)\tfrac{1}{\varepsilon}\Omega''\Big(\tfrac{-x+\phi(t)}{\varepsilon}\Big).
\end{aligned}
$$

We choose corrections in the form $R_u(x,t,\varepsilon)=0$, $R_v(x,t,\varepsilon)=R(t)\frac{1}{\varepsilon}\Omega''\Big(\frac{-x+\phi(t)}{\varepsilon}\Big)$, where $\varepsilon^{-3}\Omega''\Big(\xi/\varepsilon\Big)$ is a regularization of the distribution $\delta''(\xi)$. Since, for all $\psi(x)\in\mathcal{D}(\mathbb{R})$, we have

$$
\int\frac{1}{\varepsilon}\Omega''\Big(\frac{x}{\varepsilon}\Big)\varphi(x)\,dx=\varepsilon^2\varphi''(0)\int\Omega(\xi)\,d\xi+O(\varepsilon^3),\quad \varepsilon\to+0,
$$

it is clear that estimates (1.19) hold. We note that here, in the pointwise limit, as $\varepsilon\to+0$, the component $u(x,t)$ *does not contain* the characteristic function of the curve $x=\phi(t)$

As above, substituting (1.34) into the left-hand side of system (1.7), we see that (1.30) holds if and only if

$$
\begin{aligned}
\Big(u_1\dot\phi(t)-(2u_0u_1+u_1^2)\Big)\delta(-x+\phi(t)) &= 0,\\
\Big(v_1\dot\phi(t)+\dot e(t)-(u_0v_1+u_1v_0+u_1v_1)\Big)\delta(-x+\phi(t))\qquad&\\
+\Big(e(t)\big(\dot\phi(t)-(u_0+au_1)\big)-cR(t)\Big)\delta'(-x+\phi(t)) &= 0,
\end{aligned}
$$

where the constant $a=\int\omega_{0u1}(\xi)\omega_{\delta1}(\xi)\,d\xi$ is the same as in (1.29), and the constant $c=\int\omega_{0u1}(\xi)\Omega''(\xi)\,d\xi$. This allows us to find the functions $\phi(t)$ and $e(t)$, which, as before, are determined by relations (1.31), and to find the relation

$$R(t) = \frac{e(t)}{c}\Big(u_0 + (1-a)u_1\Big), \tag{1.35}$$

which determines the function $R(t)$. Obviously, the weak limit of the *weak asymptotic solution* (1.34) is the same, i.e., it is (1.4). As in the preceding case, it is easy to show that the weak limit (1.4) satisfies the integral identities (1.14).

In this paper we shall use the correction of the second kind (see (2.1) and (3.1)), because, from the analytic viewpoint, this simplifies describing the interaction of δ-shocks.

5. Main results. The eigenvalues of the characteristic matrix of system (1.5) are $\lambda_1(u) = f'(u)$, $\lambda_2(u) = g(u)$. We assume that

$$f''(u) > 0, \quad g'(u) > 0, \quad f'(u) \le g(u), \tag{1.36}$$

and that the "overcompression" conditions are satisfied

$$\lambda_1(u_+) \le \lambda_2(u_+) \le \dot\phi(t) \le \lambda_1(u_-) \le \lambda_2(u_-), \tag{1.37}$$

where $\dot\phi(t)$ is the speed of propagation of δ-shocks, and u_- and u_+ are respective left- and right-hand values of u on the discontinuity curve. Condition (1.37) serves as the admissibility condition for the δ-shocks and means that all characteristics on both sides of the discontinuity are in-coming.

In Subsection 1.2 we have defined a *generalized solution* of the δ-shock wave type for the Cauchy problem. In Subsection 1.3 we present the technique of the *weak asymptotics method* in the case of δ-shock waves, i.e., we construct the singular ansatz and the smooth ansatz, which are used to solve the Cauchy problems (1.5), (1.13) and (1.5), (1.12). In Subsection 4.1, we prove the main lemmas about the asymptotic expansions, which can be used for constructing the *weak asymptotic solution*. In Subsection 4.2, we prove a lemma from the theory of ordinary autonomous differential equations, which will be used for analyzing the process of interaction of δ-shock waves.

In Section 2, Theorem 2.1, we construct a weak asymptotic solution of the Cauchy problem (1.5), (1.13) in the form of a solitary δ-shock wave. Theorem 2.2 gives a generalized solution of our problem. We show that our solution satisfying the integral identity (1.14) *coincides* with the similar expression [**31**, (3.5)] (see [**34**]) treated as an element of the space $\mathcal{D}'(\mathbb{R}^2)$ (see also above). Corollary 2.1 gives the same results in the case of piesewise constant initial data.

In Section 3, in Theorem 3.1, we construct a weak asymptotic solution of the Cauchy problem (1.5), (1.12) with pointwise constant initial values. Next, in Corollary 3.1 and Theorem 3.2 we construct a generalized solution of this Cauchy problem, which describes the dynamics of propagation and interaction of two δ-shock waves. The formulas (3.43) describing the propagation and interaction of two δ-shock waves are constructed. Here the velocities $\dot\phi_k(t)$ and the Rankine–Hugoniot deficit $\dot e_k(t)$

of δ-shocks have the jumps (3.45). Systems (3.24)–(3.26) with the boundary conditions (3.9), (3.10)), obtained in the proof of Theorem 3.1, up to $O_{\mathcal{D}'}(\varepsilon)$, *describe the process of merging two δ-shock waves into one*.

REMARK 1.3. In the framework of the *weak asymptotics method* by relations (4.1), (2.7) and (4.2), (3.33), in fact, we define the *singular superposition of the Heaviside function and the delta function.* In the background of these relations there is the *construction of multiplication of distributions.* We can introduce the *"right" singular superpositions* by the following definition:

1) If $\big(u(x,t),v(x,t)\big)$ are given by (2.8), using (4.1), (2.7), we obtain

$$f\big(u(x,t)\big)\stackrel{def}{=}\lim_{\varepsilon\to+0}f\big(u(x,t,\varepsilon)\big)=f(u_0)+\big[f(u)\big]\Big|_{x=\phi(t)}H(-x+\phi(t)),$$

$$v(x,t)g\big(u(x,t)\big)\stackrel{def}{=}\lim_{\varepsilon\to+0}v(x,t,\varepsilon)g\big(u(x,t,\varepsilon)\big)=v_0g(u_0)$$

$$+\big[vg(u)\big]\Big|_{x=\phi(t)}H(-x+\phi(t))+e(t)\frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}\delta(-x+\phi(t)).$$

2) If $\big(u(x,t),v(x,t)\big)$ are given by (3.42), using (4.2), (3.33), and the limit properties of *interaction switches* $B_k((-1)^{k-1}\rho)$, $\widetilde{B}_2((-1)^{k-1}\rho)$, $k=1,2$ given by (4.4), (4.10), we obtain

$$f\big(u(x,t)\big)\stackrel{def}{=}\lim_{\varepsilon\to+0}f\big(u(x,t,\varepsilon)\big)$$

$$=f(u_0)+\big[f(u)\big]_1H(-x+\phi_1(t))+\big[f(u)\big]_2H(-x+\phi_2(t)),$$

$$v(x,t)g\big(u(x,t)\big)\stackrel{def}{=}\lim_{\varepsilon\to+0}v(x,t,\varepsilon)g\big(u(x,t,\varepsilon)\big)=g(u_0)v_0$$

$$+\big[vg(u)\big]_1H(-x+\phi_1(t))+\big[vg(u)\big]_2H(-x+\phi_2(t))$$

$$+e_1(t)\frac{[f(u)]_1}{[u]_1}\delta(-x+\phi_1(t))+e_2(t)\frac{[f(u)]_2}{[u]_2}\delta(-x+\phi_2(t)),$$

where $\phi_k(t)$ and $e_k(t)$ are given by (3.43). The jumps $[h(u,v)]_1$, $[h(u,v)]_2$ in function $h(u,v)$ are defined in Subsection 3.1. Here the limits are understood in the weak sense.

It is clear that, in general, the weak limits of $f\big(u(x,t,\varepsilon)\big)$ and $v(x,t,\varepsilon)g\big(u(x,t,\varepsilon)\big)$ *depend* on the regularization of the Heaviside function and delta function. But the above *unique "right" singular superpositions* can be obtained *only by the construction of a weak asymptotic solution.* In this paper we omit the algebraic aspects of our technique which is given in detail in [**2**], [**3**], [**27**].

By substituting *"right" singular superpositions* of $f\big(u(x,t)\big)$ and $v(x,t)g\big(u(x,t)\big)$ into system (1.5), Theorems 2.2, 3.2 *can be proved directly*.

2. Propagation of delta shocks

1. Let us consider the *propagation of a solitary δ-shock wave* of the system (1.5), i.e. we consider the Cauchy problem (1.5), (1.13).

In order to construct the *weak asymptotic solution* (1.24) of the problem we choose corrections in the form

$$\begin{aligned} R_u(x,t,\varepsilon) &= 0, \\ R_v(x,t,\varepsilon) &= R(t)\tfrac{1}{\varepsilon}\Omega''\Big(\tfrac{-x+\phi(t)}{\varepsilon}\Big), \end{aligned} \tag{2.1}$$

where $R(t)$ is a continuous function, $\varepsilon^{-3}\Omega''\big(x/\varepsilon\big)$ is a regularization of the distribution $\delta''(x)$, $\Omega(\eta)$ has the properties (a)–(c) (see Sec. 1). Since for any test function $\psi(x) \in \mathcal{D}(\mathbb{R}_x)$ we have

$$\begin{aligned} \int \frac{1}{\varepsilon}\Omega''\Big(\frac{x}{\varepsilon}\Big)\psi(x)\,dx &= \varepsilon^2\psi''(0)\textstyle\int \Omega(\xi)\,d\xi + O(\varepsilon^3), \\ \int \frac{\partial}{\partial x}\bigg(\frac{1}{\varepsilon}\Omega''\Big(\frac{x}{\varepsilon}\Big)\bigg)\psi(x)\,dx &= -\varepsilon^2\psi'''(0)\textstyle\int \Omega(\xi)\,d\xi + O(\varepsilon^3), \end{aligned} \tag{2.2}$$

relations (1.19) hold.

Theorem 2.1. *Let conditions* (1.36) *be satisfied. Then there exists* $T > 0$ *such that, for* $t \in [0,\,T)$*, the Cauchy problem* (1.5), (1.13) *has a weak asymptotic solution* (1.24), (2.1) *if and only if*

$$\begin{aligned} L_1[u_0] &= 0, \quad x > \phi(t), \\ L_1[u_0+u_1] &= 0, \quad x < \phi(t), \\ L_2[u_0,v_0] &= 0, \quad x > \phi(t), \\ L_2[u_0+u_1,v_0+v_1] &= 0, \quad x < \phi(t), \\ \dot\phi(t) &= \tfrac{[f(u)]}{[u]}\Big|_{x=\phi(t)}, \\ \dot e(t) &= \Big([vg(u)] - [v]\tfrac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)}, \\ R(t) &= \tfrac{e(t)}{c(t)}\bigg(\tfrac{[f(u)]}{[u]}\Big|_{x=\phi(t)} - a(t)\bigg), \end{aligned} \tag{2.3}$$

where

$$\big[h(u(x,t),v(x,t))\big]\Big|_{x=\phi(t)}$$

$$= \Big(h(u_0(x,t)+u_1(x,t),v_0(x,t)+v_1(x,t)) - h(u_0(x,t),v_0(x,t))\Big)\Big|_{x=\phi(t)}$$

is a jump in function $h(u(x,t),v(x,t))$ *across the discontinuity curve* $x = \phi(t)$*,*

$$\begin{aligned} a(t) &= \textstyle\int g\big(u_0(\phi(t),t) + u_1(\phi(t),t)\omega_{0u1}(\eta)\big)\omega_{\delta 1}(\eta)\,d\eta, \\ c(t) &= \textstyle\int g\big(u_0(\phi(t),t) + u_1(\phi(t),t)\omega_{0u1}(\eta)\big)\Omega''(\eta)\,d\eta \neq 0. \end{aligned} \tag{2.4}$$

The initial data for system (2.3) *are defined from* (1.13), *and*

$$\phi(0)=0,\quad R(0)=\frac{e^0}{c(0)}\Big(\frac{[f(u^0)]}{[u^0]}\Big|_{x=0}-a(0)\Big).$$

PROOF. Let us substitute (1.24), (2.1), and asymptotics $g(u(x,t,\varepsilon))v(x,t,\varepsilon)$ and $f(u(x,t,\varepsilon))$ given by formula (4.6) from Lemma 4.3 and formula (4.1) from Lemma 4.1, respectively, into system (1.5). Obviously, we obtain up to $O_{\mathcal{D}'}(\varepsilon)$ the following relations

$$\begin{aligned} L_1[u(x,t,\varepsilon)] &= L_1[u_0] \\ &+\Big\{\frac{\partial u_1}{\partial t}+\frac{\partial}{\partial x}\Big(f(u_0+u_1)-f(u_0)\Big)\Big\}H(-x+\phi(t)) \\ &+\Big\{u_1\dot\phi(t)-\Big(f(u_0+u_1)-f(u_0)\Big)\Big\}\delta(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon), \end{aligned}\tag{2.5}$$

$$\begin{aligned} L_2[u(x,t,\varepsilon),v(x,t,\varepsilon)] &= L_2[u_0,v_0] \\ &+\Big\{\frac{\partial v_1}{\partial t}+\frac{\partial}{\partial x}\Big(\big(v_0+v_1\big)g(u_0+u_1)-v_0g(u_0)\Big)\Big\}H(-x+\phi(t)) \\ &+\Big\{v_1\dot\phi(t)+\dot e(t)-\Big(\big(v_0+v_1\big)g(u_0+u_1)-v_0g(u_0)\Big)\Big\}\delta(-x+\phi(t)) \\ &+\Big\{e(t)\dot\phi(t)-e(t)a(t)-c(t)R(t)\Big\}\delta'(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon), \end{aligned}\tag{2.6}$$

where $a(t)$, $c(t)\neq 0$ are defined by formula (2.4) which follows from (4.7). Here we take into account the estimates (2.2), (1.19).

Setting the right-hand side of (2.5), (2.6) equal to zero, we obtain the necessary and sufficient conditions for the equalities

$$L_1[u(x,t,\varepsilon)]=O_{\mathcal{D}'}(\varepsilon),\quad L_2[u(x,t,\varepsilon),v(x,t,\varepsilon)]=O_{\mathcal{D}'}(\varepsilon),$$

i.e. system (2.3).

Now we consider the Cauchy problem

$$L_{11}[u]=u_t+\big(f(u)\big)_x=0,\quad u(x,0)=u^0(x).$$

Since, according to (1.36), $f(u)$ is convex and $u_1^0(0)>0$, according to the results [**20**, Ch.4.2.], we extend $u_+^0(x)=u_0^0(x)$ $(u_-^0(x)=u_0^0(x)+u_1^0(x))$ to $x\le 0$ $(x\ge 0)$ in a bounded C^1 fashion and continue to denote the extended functions by $u_\pm^0(x)$. By $u_\pm(x,t)$ we denote the C^1 solutions of the problems

$$L_{11}[u]=u_t+\big(f(u)\big)_x=0,\quad u_\pm(x,0)=u_\pm^0(x)$$

which exist for small enough time interval $[0,\ T_1]$ and are determined by integration along characteristics. The functions $u_\pm(x,t)$ determine a two-sheeted covering of the plane (x,t). Next, we define the function $x=\phi(t)$ as a solution of the problem

$$\dot\phi(t)=\frac{f\big(u_+(x,t)\big)-f\big(u_-(x,t)\big)}{u_+(x,t)-u_-(x,t)}\Big|_{x=\phi(t)},\quad \phi(0)=0.$$

It is clear that there exists a unique function $\phi(t)$ for sufficiently short times $[0,\ T_2]$. To this end, for $T=\min(T_1,T_2)$ we define the shock solution by

$$u(x,t)=\begin{cases} u_+(x,t), & x\ >\ \phi(t),\\ u_-(x,t), & x\ <\ \phi(t).\end{cases}$$

Thus the first, second and fifth equations of system (2.3) define a unique solution of the Cauchy problem $L_{11}[u]=u_t+\big(f(u)\big)_x=0$, $u(x,0)=u^0(x)$ for $t\in[0,\ T)$.

Solving this problem, we obtain $u(x,t)$, $\phi(t)$. Then substituting these functions into system (2.3), we obtain $V(x,t)=v_0(x,t)+v_1(x,t)H(-x+\phi(t))$, $e(t)$, and $v(x,t)=V(x,t)+e(t)\delta(-x+\phi(t))$.

It is clear that mollifiers $\omega_{0u1}(\xi)$, $\Omega(\xi)$ can be chosen such that $\int\omega_{u1}(\eta)\Omega'(\eta)\,d\eta>0$. Consequently, taking into account that $g'(u)>0$, $u_1^0(x)>0$ and integrating by parts, we obtain

$$c(t)=-\int g'\big(u_0+u_1\omega_{0u1}(\eta)\big)u_1\Big|_{x=\phi(t)}\omega_{u1}(\eta)\Omega'(\eta)\,d\eta\neq0.$$

So for any functions $u_0(x,t)$, $u_1(x,t)$, $e(t)$, $\phi(t)$, $t\in[0,\ T)$, there exists a function $R(t)$, which is defined by the last relation of (2.3). $\Box$

REMARK 2.1. By substituting the last relation (2.3), which determines $R(t)$, into the formula (4.6), we obtain

$$v(x,t,\varepsilon)g\big(u(x,t,\varepsilon)\big)=v_0g(u_0)+\big[vg(u)\big]\Big|_{x=\phi(t)}H(-x+\phi(t))$$

$$+e(t)\frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}\delta(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon),\quad\varepsilon\to+0, \tag{2.7}$$

2. We obtain a *generalized solution* of the Cauchy problem (1.5), (1.13) as a weak limit (1.16) of a *weak asymptotic solution* constructed by Theorem 2.1.

THEOREM 2.2. *Assume that conditions* (1.36) *are satisfied. Then, for* $t\in[0,\ T)$, *where* $T>0$ *is given by Theorem* 2.1, *the Cauchy problem* (1.5), (1.13), *has a unique generalized solution*

$$\begin{aligned} u(x,t)&=u_0(x,t)+u_1(x,t)H(-x+\phi(t)),\\ v(x,t)&=v_0(x,t)+v_1(x,t)H(-x+\phi(t))+e(t)\delta(-x+\phi(t)),\end{aligned} \tag{2.8}$$

which satisfies the integral identities cf. (1.14)*:*

$$\begin{aligned} \int_0^T\int\Big(u\varphi_t+f(u)\varphi_x\Big)\,dx\,dt+\int u^0(x)\varphi(x,0)\,dx&=0,\\ \int_0^T\int\Big(\varphi_t+g(u)\varphi_x\Big)V\,dx\,dt+\int V^0(x)\varphi(x,0)\,dx&\\ \int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl+e^0\varphi(0,0)&=0,\end{aligned} \tag{2.9}$$

for all $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0,\,T))$, *where* $\Gamma = \{(x,t) : x = \phi(t), \quad t \in [0,\,T)\}$,

$$\int_{\Gamma} e(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\,dl = \int_0^T e(t)\Big(\varphi_t(\phi(t),t) + \dot{\phi}(t)\varphi_x(\phi(t),t)\Big)\,dt,$$

$V(x,t) = v_0 + v_1 H(-x+\phi(t))$. *Here functions* $u_k(x,t)$, $v_k(x,t)$, $k = 0,1$, $\phi(t)$, $e(t)$ *are defined by the system*

$$\begin{aligned}
L_1[u_0] &= 0, \quad x > \phi(t),\\
L_1[u_0+u_1] &= 0, \quad x < \phi(t),\\
L_2[u_0,v_0] &= 0, \quad x > \phi(t),\\
L_2[u_0+u_1,v_0+v_1] &= 0, \quad x < \phi(t),\\
\dot{\phi}(t) &= \frac{[f(u)]}{[u]}\Big|_{x=\phi(t)},\\
\dot{e}(t) &= \Big([vg(u)] - [v]\frac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)}.
\end{aligned} \tag{2.10}$$

with the initial data defined from (1.13), $\phi(0) = 0$.

Proof. By Theorem 2.1 we have the following estimates:

$$L_1[u(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon), \quad L_2[u(x,t,\varepsilon), v(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon).$$

Let us apply the left-hand and right-hand sides of these relations to an arbitrary test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0,\,T))$. Then integrating by parts, we obtain

$$\int_0^T \int \Big(u(x,t,\varepsilon)\varphi_t(x,t) + f(u(x,t,\varepsilon)\varphi_x(x,t)\Big)\,dxdt + \int u(x,0,\varepsilon)\varphi(x,0)\,dx = O(\varepsilon), \tag{2.11}$$

$$\int_0^T \int \Big(v(x,t,\varepsilon)\varphi_t(x,t) + v(x,t,\varepsilon)g(u(x,t,\varepsilon))\varphi_x(x,t)\Big)\,dxdt + \int v(x,0,\varepsilon)\varphi(x,0)\,dx = O(\varepsilon), \quad \varepsilon \to +0. \tag{2.12}$$

Now let us substitute $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$, and the asymptotics $g(u(x,t,\varepsilon))v(x,t,\varepsilon)$ and $f(u(x,t,\varepsilon))$ given by (2.7) and (4.1), respectively, into the last relations. Now by passing to the limit as $\varepsilon \to +0$ in each of the integrals (2.11), (2.12), and taking into account that

$$\lim_{\varepsilon\to+0}\int_0^T \int e(t)\delta_{v1}\big(-x+\phi(t),\varepsilon\big)\varphi(x,t)\,dxdt = \int_0^T e(t)\varphi(\phi(t),t)\,dt, \tag{2.13}$$

$$\lim_{\varepsilon\to+0}\int e(0)\delta_{v1}\big(-x,\varepsilon\big)\varphi(x,0)\,dx = e(0)\varphi(0,0), \tag{2.14}$$

we obtain the integral identities (2.9).

In view of the above remark in Theorem 2.1, the Cauchy problem has a unique generalized solution. □

Let us consider the piecewise constant case of initial data (1.13), where $u_0^0 = u_0$, $u_1^0 = u_1 > 0$, $v_0^0 = v_0$, $v_1^0 = v_1$ are constants.

Corollary 2.1. *Assume that conditions* (1.36) *are satisfied. Then, for* $t \in [0, \infty)$*, the Cauchy problem* (1.5) *with the piecewise constant initial data* (1.13) *has a unique generalized solution*

$$\begin{aligned} u(x,t) &= u_0 + u_1 H(-x+\phi(t)), \\ v(x,t) &= v_0 + v_1 H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{aligned}$$

where

$$\phi(t) = \frac{[f(u)]}{[u]} t = \frac{f(u_0+u_1) - f(u_0)}{u_1} t, \quad e(t) = e^0 + \Big([g(u)v] - \frac{[f(u)]}{[u]}[v]\Big) t.$$

Remark 2.2. To find a *generalized solution* of the Cauchy problem (1.5), (1.13), we construct a weak asymptotic solution of the problem (1.24), (2.1), where the functions $u_k(x,t)$, $v_k(x,t)$, $e(t)$, $\phi(t)$, $k = 0,1$ are determined by system (2.10) and the functions $\omega_{0u1}(\eta)$, $\omega_{\delta 1}(\eta)$, $\Omega''(\eta)$ and the *correction* $R(t)$ are determined by the last relation of system (2.3) and system (2.4). In view of estimate (1.19), the *generalized solution* of the Cauchy problem *does not depend* on the functions $\omega_{0u1}(\eta)$, $\omega_{\delta 1}(\eta)$, $\Omega''(\eta)$ and the *correction* $R(t)$. But without introducing correction (2.1), i.e. setting $R(t) = 0$, we derive from the last relation (2.3) and (2.4) the relation

$$\frac{\big[f(u(x,t))\big]}{\big[u(x,t)\big]}\Big|_{x=\phi(t)} = \int g\big(u_0(\phi(t),t) + u_1(\phi(t),t)\omega_{0u1}(\eta)\big)\omega_{\delta 1}(\eta)\,d\eta, \tag{2.15}$$

which shows that we *cannot* solve the Cauchy problem with an *arbitrary* jump.

3. Interaction of delta shocks

1. Construction of a weak asymptotic solution. We describe the dynamics of propagation and interaction of two δ-shock waves for the system (1.5) with the piecewise constant initial data (1.12), where $u_0^0 = u_0$, $u_k^0 = u_k > 0$, $v_0^0 = v_0$, $v_k^0 = v_k$ are constants, $k = 1, 2$.

In order to construct a *weak asymptotic solution* (1.21) of the problem we choose corrections in the form

$$\begin{aligned} R_u(x,t,\varepsilon) &= 0, \\ R_v(x,t,\varepsilon) &= \textstyle\sum_{k=1}^2 R_k(t,\varepsilon)\frac{1}{\varepsilon}\Omega_k''\Big(\frac{-x+\phi_k(t,\varepsilon)}{\varepsilon}\Big), \end{aligned} \tag{3.1}$$

where $R_k(t,\varepsilon)$ are the desired functions, $\varepsilon^{-3}\Omega_k''\big(x/\varepsilon\big)$ are regularizations of the distribution $\delta''(x)$, $\Omega_k(\eta)$ has the properties (a)–(c) (see Sec. 1), $k = 1,2$. Relations (2.2) imply (1.19).

Thus, according to our approach, for problem (1.5), (1.12) we present a *two-δ-shocks weak asymptotic solution* in the form (1.21):

$$
\begin{aligned}
u(x,t,\varepsilon) &= u_0+\sum_{k=1}^{2} u_k H_{uk}\big(-x+\phi_k(t,\varepsilon),\varepsilon\big),\\
v(x,t,\varepsilon) &= v_0+\sum_{k=1}^{2}\bigg(v_k H_{vk}\big(-x+\phi_k(t,\varepsilon),\varepsilon\big)\\
&\quad +e_k(t,\varepsilon)\delta_{vk}\big(-x+\phi_k(t,\varepsilon),\varepsilon\big)+R_k(t,\varepsilon)\frac{1}{\varepsilon}\Omega''_k\Big(\frac{-x+\phi_k(t,\varepsilon)}{\varepsilon}\Big)\bigg).
\end{aligned}
\tag{3.2}
$$

Let $t=t^*>0$ be the *time instant of interaction*. In the interval $t\in[0,\ t^*)$ we have *two* δ-shock waves *propagating without interaction*. By Corollary 2.1, their phase functions $\phi_{k0}(t)$ and the amplitudes of δ-functions $e_{k0}(t)$ are defined by the system of equations

$$
\phi_{k0}(t)=\phi_{k0}(0)+\frac{[f(u)]_k}{[u]_k}t, \quad e_{k0}(t)=e_k^0+\Big([g(u)v]_k-[v]_k\frac{[f(u)]_k}{[u]_k}\Big)t,
\tag{3.3}
$$

where by

$$
\begin{aligned}
[h(u,v)]_1 &= h(u_0+u_1+u_2,v_0+v_1+v_2)-h(u_0+u_2,v_0+v_2),\\
[h(u,v)]_2 &= h(u_0+u_2,v_0+v_2)-h(u_0,v_0)
\end{aligned}
$$

we denote jumps in function $h(u,v)$ across the discontinuity curves $x=\phi_{10}(t)$, $x=\phi_{20}(t)$, respectively, $\phi_{k0}(0)=x_k^0$ are initial positions of singularities, $e_{k0}(0)=e_k^0$ are initial amplitudes of δ-functions, $k=1,2$.

By (3.3), before interaction, two δ-shock waves propagate across the lines $x_k=\phi_{k0}(t)$ which intersect at the point with the coordinates:

$$
\begin{aligned}
t^* &= u_1u_2\frac{x_2^0-x_1^0}{u_2f(u_0+u_1+u_2)-(u_1+u_2)f(u_0+u_2)+u_1f(u_0)},\\
x^* &= \frac{\Big(f(u_0+u_1+u_2)-f(u_0+u_2)\Big)u_2x_2^0-\Big(f(u_0+u_2)-f(u_0)\Big)u_1x_1^0}{u_2f(u_0+u_1+u_2)-(u_1+u_2)f(u_0+u_2)+u_1f(u_0)}.
\end{aligned}
\tag{3.4}
$$

Thus, we define the *time instant of interaction* $t=t^*>0$ as the time of intersection of the curves $x=\phi_{10}(t)$, $x=\phi_{20}(t)$, i.e. a root of the equation $\psi_0(t^*)=0$, where

$$
\psi_0(t)=\phi_{20}(t)-\phi_{10}(t)
$$

is the distance between the fronts of *non-interacting* δ-shock waves.

We write the *weak asymptotic solution* (3.2) in the form that potentially describes different scenarios of the processes that occur in the confluence of *two free δ-shocks*. Therefore, summarizing the above remarks, in order to describe the interaction dynamics, we will seek phases of δ-shocks and amplitudes of δ-functions as functions of the "fast" variable ("fast" time) $\tau=\frac{\psi_0(t)}{\varepsilon}\in\mathbb{R}$ and the "slow" variable $t\ge 0$:

$$
\begin{aligned}
\phi_k(t,\varepsilon) \overset{def}{=} \widehat{\phi}_k(\tau,t) &= \phi_{k0}(t)+\psi_0(t)\phi_{k1}(\tau)\Big|_{\tau=\frac{\psi_0(t)}{\varepsilon}},\\
e_k(t,\varepsilon) \overset{def}{=} \widehat{e}_k(\tau,t) &= e_{k0}(t)+\psi_0(t)e_{k1}(\tau)\Big|_{\tau=\frac{\psi_0(t)}{\varepsilon}},
\end{aligned}
\tag{3.5}
$$

where the functions $\phi_{k0}(t)$, $e_{k0}(t)$ are defined by equations (3.3) for $t \in [0,\ t^*)$; for $t \in [t^*,\ +\infty)$ these functions are defined by the *same equations* continuously extended for $t \ge t^*$. The desired functions $\phi_{k1}(\tau)$, $e_{k1}(\tau)$ are corrections to the phases and the amplitudes, respectively, rapidly varying during the time of interaction. We assume $\phi_{k1}(\tau)$, $e_{k1}(\tau)$ to be differentiable with respect to τ, $k = 1, 2$.

Analogously to (3.5), we will seek the corrections $R_k(t,\varepsilon)$ as functions of the fast variable τ and the slow variable t:

$$R_k(t,\varepsilon) \stackrel{def}{=} \widehat{R}_k(\tau,t) = R_{k0}(t) + R_{k1}(\tau,t)\Big|_{\tau=\frac{\psi_0(t)}{\varepsilon}}, \tag{3.6}$$

where, according to Theorem 2.1, the terms $R_{k0}(t)$ are determined by the relations

$$R_{k0}(t) = \frac{e_{k0}(t)}{c_k}\Big(\frac{[f(u)]_k}{[u]_k} - a_k\Big), \tag{3.7}$$

and $R_{k1}(\tau,t)$ are desired functions, $k = 1, 2$. Here by (2.4), we have

$$\begin{array}{rcl} a_1 &=& \int g(u_0 + u_1\omega_{0u1}(\eta) + u_2)\omega_{\delta 1}(\eta)\, d\eta, \\ c_1 &=& \int g(u_0 + u_1\omega_{0u1}(\eta) + u_2)\Omega_1''(\eta)\, d\eta \ne 0, \\ a_2 &=& \int g\big(u_0 + u_2\omega_{0u2}(\eta)\big)\omega_{\delta 2}(\eta)\, d\eta, \\ c_2 &=& \int g\big(u_0 + u_2\omega_{0u2}(\eta)\big)\Omega_2''(\eta)\, d\eta \ne 0. \end{array} \tag{3.8}$$

Before interaction, as $t < t^*$, we have $\phi_{10}(t) < \phi_{20}(t)$ and $\tau = \frac{\psi_0(t)}{\varepsilon} > 0$, *after interaction*, as $t > t^*$, we have $\phi_{10}(t) > \phi_{20}(t)$ and $\tau = \frac{\psi_0(t)}{\varepsilon} < 0$. We set the following boundary conditions for the corrections to the phases and the amplitudes:

$$\begin{array}{rclcrcl} \phi_{k1}(\tau)\Big|_{\tau\to+\infty} &=& 0, && e_{k1}(\tau)\Big|_{\tau\to+\infty} &=& 0, \\ \dfrac{d\phi_{k1}(\tau)}{d\tau}\Big|_{\tau\to-\infty} &=& o(\tau^{-1}), && \frac{de_{k1}(\tau)}{d\tau}\Big|_{\tau\to-\infty} &=& o(\tau^{-1}). \end{array} \tag{3.9}$$

This means that the derivatives of the phases and amplitudes with respect to the fast variable τ tend to zero as $|\tau| \to \infty$, while the phases tend to zero as $\tau \to \infty$, i.e. before interaction. We assume that, analogously to (3.9), the following boundary conditions hold:

$$R_{k1}(\tau,t)\Big|_{\tau\to+\infty} = 0, \quad R_{k1}(\tau,t)\Big|_{\tau\to-\infty} = R_{k1,-}(t), \tag{3.10}$$

and $R_{k1}(\tau,t)$, $\frac{\partial R_{k1}(\tau,t)}{\partial\tau}$ are bounded functions for all $t \ge 0$, $k = 1, 2$.

Finding the limit values of the corrections to the phases and the amplitudes, as $\tau \to -\infty$ (for $t > t^*$)

$$\phi_{k1}(\tau)\Big|_{\tau\to-\infty} = \phi_{k1,-}, \qquad e_{k1}(\tau)\Big|_{\tau\to-\infty} = e_{k1,-},$$

we find the limit values of the phases $\phi_k(t,\varepsilon)$ and amplitudes $e_k(t,\varepsilon)$:

$$\begin{array}{rcl} \widehat{\phi}_{k,-}(t) &=& \widehat{\phi}_k(\tau,t)\Big|_{\tau\to-\infty} = \phi_{k0}(t) + \psi_0(t)\phi_{k1,-}, \\ \widehat{e}_{k,-}(t) &=& \widehat{e}_k(\tau,t)\Big|_{\tau\to-\infty} = e_{k0}(t) + \psi_0(t)e_{k1,-}. \end{array} \tag{3.11}$$

Thus, in fact, we *determine "the result" of the interaction* of δ-shocks as $t > t^*$.

THEOREM 3.1. *Assume that conditions* (1.36) *are satisfied. Then for* $t \in [0, \infty)$, *the Cauchy problem* (1.5) *with the piecewise constant initial data (1.12), has a weak asymptotic solution* (3.2), (3.5), (3.6), *where functions* $\phi_{k0}(t)$, $e_{k0}(t)$, $R_{k0}(t)$ *are determined by the system of equations* (3.3), (3.7), *and desired corrections are defined by the system:*

$$\phi_{k1}(\tau) = \frac{(-1)^k}{u_k\Big(\frac{[f(u)]_2}{[u]_2} - \frac{[f(u)]_1}{[u]_1}\Big)\tau} \int\limits_0^\tau B_2(-\rho(\tau'))\,d\tau', \tag{3.12}$$

$$e_{k1}(\tau) = \frac{(-1)^k}{\Big(\frac{[f(u)]_2}{[u]_2} - \frac{[f(u)]_1}{[u]_1}\Big)\tau} \int\limits_0^\tau \widetilde{B}_2(-\rho(\tau'))\,d\tau' - v_k\phi_{k1}(\tau), \tag{3.13}$$

$$R_{k1}(\tau,t) = \frac{\widehat{e}_k(\tau,t)}{C_{Rk}\big((-1)^{k-1}\rho\big)}\Bigg(\frac{f(u_0+u_k) - f(u_0) + B_k\big((-1)^{k-1}\rho\big)}{u_k}$$

$$-A_k\big((-1)^{k-1}\rho\big)\Bigg) - R_{k0}(t), \tag{3.14}$$

where $B_2(-\rho)$ *and* $\widetilde{B}_k\big((-1)^{k-1}\rho\big)$, $A_k\big((-1)^{k-1}\rho\big)$, $C_{Rk}\big((-1)^{k-1}\rho\big)$ *are so-called interaction switch functions whose explicit form are given by* (4.3) *and* (4.9), *respectively,* $k = 1, 2$. *Here* $\rho = \rho(\tau)$ *is a solution of the differential equation with the boundary condition:*

$$\frac{d\rho}{d\tau} = F(\rho), \qquad \frac{\rho}{\tau}\Big|_{\tau\to+\infty} = 1, \tag{3.15}$$

where

$$F(\rho) = 1 + \frac{\Big(\frac{1}{u_1} + \frac{1}{u_2}\Big)B_2(-\rho)}{\frac{[f(u)]_2}{[u]_2} - \frac{[f(u)]_1}{[u]_1}}. \tag{3.16}$$

PROOF. 1. *Ansatz substitution.* Let us substitute the smooth ansatz (3.2) and the weak asymptotics $f\big(u(x,t,\varepsilon)\big)$, $g\big(u(x,t,\varepsilon)\big)v(x,t,\varepsilon)$, which are given by (4.2), (4.8), respectively, into the system (1.5). Obviously, we obtain up to $O_{\mathcal{D}'}(\varepsilon)$ the following relations

$$L_1[u(x,t,\varepsilon)] = \sum_{k=1}^{2}\Big\{u_k\dot{\phi}_k(t,\varepsilon)$$

$$-\Big(f(u_0+u_k) - f(u_0)\Big) - B_k\big((-1)^{k-1}\rho\big)\Big\}\delta(-x+\phi_k(t,\varepsilon)) + O_{\mathcal{D}'}(\varepsilon), \tag{3.17}$$

$$L_2[u(x,t,\varepsilon),v(x,t,\varepsilon)] = \sum_{k=1}^{2}\Big\{\Big(v_k\dot{\phi}_k(t,\varepsilon)+\dot{e}_k(t,\varepsilon)$$

$$-\Big(g(u_0+u_k)\big(v_0+v_k\big)-g(u_0)v_0+\widetilde{B}_k\big((-1)^{k-1}\rho\big)\Big)\Big)\delta(-x+\phi_k(t,\varepsilon))$$

$$+\Big(e_k(t,\varepsilon)\dot{\phi}_k(t,\varepsilon)-\Big(e_k(t,\varepsilon)A_k\big((-1)^{k-1}\rho\big)$$

$$+R_k(t,\varepsilon)C_{Rk}\big((-1)^{k-1}\rho\big)\Big)\Big)\delta'(-x+\phi_k(t,\varepsilon))\Big\}+O_{\mathcal{D}'}(\varepsilon), \tag{3.18}$$

where $\rho=\frac{\psi(t,\varepsilon)}{\varepsilon}$, $\psi(t,\varepsilon)=\phi_2(t,\varepsilon)-\phi_1(t,\varepsilon)$ is the distance between regularizations of the δ-shocks fronts $\phi_2(t,\varepsilon)$ and $\phi_1(t,\varepsilon)$. Here the estimate $O_{\mathcal{D}'}(\varepsilon)$ is uniform with respect to $\psi(t,\varepsilon)$.

By equating the coefficients of δ, δ' with zero in the right-hand side of (3.17), (3.18), we obtain the necessary and sufficient conditions for the relations

$$L_1[u(x,t,\varepsilon)]=O_{\mathcal{D}'}(\varepsilon), \quad L_2[u(x,t,\varepsilon),v(x,t,\varepsilon)]=O_{\mathcal{D}'}(\varepsilon),$$

i.e. the *generalized Rankine–Hugoniot type conditions*

$$u_k\dot{\phi}_k(t,\varepsilon)=\Big(f(u_0+u_k)-f(u_0)\Big)+B_k\big((-1)^{k-1}\rho\big), \tag{3.19}$$

and the system

$$\begin{aligned} v_k\dot{\phi}_k(t,\varepsilon)+\dot{e}_k(t,\varepsilon) &= g(u_0+u_k)\big(v_0+v_k\big)-g(u_0)v_0+\widetilde{B}_k\big((-1)^{k-1}\rho\big), \\ e_k(t,\varepsilon)\dot{\phi}_k(t,\varepsilon) &= e_k(t,\varepsilon)A_k\big((-1)^{k-1}\rho\big)+R_k(t,\varepsilon)C_{Rk}\big((-1)^{k-1}\rho\big). \end{aligned} \tag{3.20}$$

$k=1,2$. Systems (3.19) (3.20) describe functions $\phi_k(t,\varepsilon)$, $e_k(t,\varepsilon)$, $R_k(t,\varepsilon)$, which determine the weak asymtotics solution (3.2).

According to our assumption, we will seek functions $\phi_k(t,\varepsilon)$, $e_k(t,\varepsilon)$, $R_k(t,\varepsilon)$, in the form (3.5), (3.6) by introducing the dependence on ε into them, $k=1,2$. The form (3.5), (3.6) also reflects the structure of argument of *interaction switch functions* $\rho=\frac{\psi(t,\varepsilon)}{\varepsilon}$ and the structure of equations (3.19), (3.20). Let $\psi_1(\tau)=\phi_{21}(\tau)-\phi_{11}(\tau)$, then the full phase difference is $\psi(t,\varepsilon)=\psi_0(t)\big(1+\psi_1(\tau)\big)$, and

$$\rho(\tau)=\frac{\psi(t,\varepsilon)}{\varepsilon}=\tau\big(1+\psi_1(\tau)\big). \tag{3.21}$$

The derivatives of the phases and amplitudes with respect to time are given by the following equalities:

$$\begin{aligned} \frac{d\phi_k(t,\varepsilon)}{dt} \stackrel{def}{=} \frac{d\widehat{\phi_k}(\tau,t)}{dt} &= \dot{\phi}_{k0}(t)+\dot{\psi}_0(t)\tfrac{d}{d\tau}[\tau\phi_{k1}(\tau)], \\ \frac{de_k(t,\varepsilon)}{dt} \stackrel{def}{=} \frac{d\widehat{e}_k(\tau,t)}{dt} &= \dot{e}_{k0}(t)+\dot{\psi}_0(t)\tfrac{d}{d\tau}[\tau e_{k1}(\tau)]. \end{aligned} \tag{3.22}$$

Taking into account the boundary conditions (3.9), we find the limit values of the phases and their derivatives with respect to time as $\tau \to -\infty$ (for $t > t^*$):

$$\Big(\frac{d\widehat{\phi}_k(\tau,t)}{dt}\Big)_- = \frac{d\widehat{\phi}_{k,-}(t)}{dt}, \qquad \Big(\frac{d\widehat{e}_k(\tau,t)}{dt}\Big)_- = \frac{d\widehat{e}_{k,-}(t)}{dt}, \tag{3.23}$$

where $\widehat{\phi}_{k,-}(t)$, $\widehat{e}_{k,-}(t)$ are defined by (3.11).

2. By substituting (3.5), (3.22), (3.6) into (3.19) and (3.20), we obtain for all $t \geq 0$ and $\tau \in \mathbb{R}$ the *generalized Rankine–Hugoniot type conditions*:

$$\begin{aligned} \dot{\phi}_{10}(t) + \dot{\psi}_0(t)\frac{d}{d\tau}\Big(\tau\phi_{11}(\tau)\Big) &= \tfrac{f(u_0+u_1)-f(u_0)+B_1(\rho)}{u_1}, \\ \dot{\phi}_{20}(t) + \dot{\psi}_0(t)\frac{d}{d\tau}\Big(\tau\phi_{21}(\tau)\Big) &= \tfrac{f(u_0+u_2)-f(u_0)+B_2(-\rho)}{u_2}, \end{aligned} \tag{3.24}$$

and the following systems of equations:

$$\begin{aligned} \dot{e}_{10}(t) + \dot{\psi}_0(t)\frac{d}{d\tau}\Big(\tau e_{11}(\tau)\Big) &= \Big(v_0+v_1\Big)g(u_0+u_1) - v_0 g(u_0) \\ &\quad + \widetilde{B}_1(\rho) - v_1\Big(\tfrac{f(u_0+u_1)-f(u_0)+B_1(\rho)}{u_1}\Big), \\ \dot{e}_{20}(t) + \dot{\psi}_0(t)\frac{d}{d\tau}\Big(\tau e_{21}(\tau)\Big) &= \Big(v_0+v_2\Big)g(u_0+u_2) - v_0 g(u_0) \\ &\quad + \widetilde{B}_2(-\rho) - v_2\Big(\tfrac{f(u_0+u_2)-f(u_0)+B_2(-\rho)}{u_2}\Big), \end{aligned} \tag{3.25}$$

$$\begin{aligned} \Big(R_{10}(t) + R_{11}(\tau,t)\Big)C_{R1}(\rho) &= \widehat{e}_1(\tau,t)\bigg(\tfrac{f(u_0+u_1)-f(u_0)+B_1(\rho)}{u_1} \\ &\qquad - A_1(\rho)\bigg), \\ \Big(R_{20}(t) + R_{21}(\tau,t)\Big)C_{R2}(-\rho) &= \widehat{e}_2(\tau,t)\bigg(\tfrac{f(u_0+u_2)-f(u_0)+B_2(-\rho)}{u_2} \\ &\qquad - A_2(-\rho)\bigg), \end{aligned} \tag{3.26}$$

with the boundary conditions (3.9), (3.10). Here $\phi_{k0}(t)$, $e_{k0}(t)$, $R_{k0}(t)$ for all $t \geq 0$ are defined by systems (3.3), (3.7).

Subtracting the one Rankine–Hugoniot type condition (3.24) from the other, we reduce system (3.24) to the differential equation with the boundary condition (3.15):

$$\frac{d\rho}{d\tau} = F(\rho), \quad \frac{\rho}{\tau}\Big|_{\tau\to+\infty} = 1,$$

where

$$F(\rho) = \frac{1}{\dot{\psi}_0(t)}\bigg(\frac{f(u_0+u_2) - f(u_0) + B_2(-\rho)}{u_2} - \frac{f(u_0+u_1) - f(u_0) + B_1(\rho)}{u_1}\bigg),$$

and according to (3.3)

$$\dot{\psi}_0(t) = \frac{f(u_0+u_2) - f(u_0)}{u_2} - \frac{f(u_0+u_1+u_2) - f(u_0+u_2)}{u_1}. \tag{3.27}$$

Taking into account relation (4.5), the right-hand side $F(\rho)$ can be written in the equivalent forms

$$F(\rho)=\frac{\frac{f(u_0+u_1+u_2)-f(u_0+u_1)}{u_2}-\frac{f(u_0+u_1)-f(u_0)}{u_1}-\Big(\frac{1}{u_1}+\frac{1}{u_2}\Big)B_1(\rho)}{\frac{[f(u)]_2}{[u]_2}-\frac{[f(u)]_1}{[u]_1}}$$

or (3.16). Here the boundary condition (3.15) follows from the boundary conditions (3.9). This autonomous ordinary differential equation is typical for our approach.

Using the limit values (4.4) of the functions $B_1(\pm\infty)$, $B_2(\pm\infty)$, we obtain that

$$\begin{aligned} F(+\infty)=\lim_{\rho\to+\infty}F(\rho) &= 1,\\ F(-\infty)=\lim_{\rho\to-\infty}F(\rho) &= -\frac{\frac{f(u_0+u_1+u_2)-f(u_0+u_1)}{u_2}-\frac{f(u_0+u_1)-f(u_0)}{u_1}}{\frac{f(u_0+u_1+u_2)-f(u_0+u_2)}{u_1}-\frac{f(u_0+u_2)-f(u_0)}{u_2}} \end{aligned} \tag{3.28}$$

for any choice of mollifiers $\omega_{u1}(\xi)$, $\omega_{u2}(\xi)$. By using the inequality

$$\frac{f(x_2)-f(x)}{x_2-x}-\frac{f(x)-f(x_1)}{x-x_1}>0, \qquad x\in(x_1,x_2),$$

valid for any convex function $f(u)$ and $u_1,u_2>0$, we see that $F(-\infty)<0$ for any choice of $u_1,u_2>0$.

Thus, as $\rho\to\pm\infty$, the limit values of the right-hand side of the differential equation (3.15), (3.16) have opposite signs: $F(-\infty)<0$, $F(+\infty)=1>0$ for any $u_1,u_2>0$. According to (4.3), and (3.16), $B_1(\rho)$, $B_2(-\rho)$ and $F(\rho)$ are smooth functions. Therefore, the equation $F(\rho)=0$ has a root ρ_0 for any mollifiers $\omega_{uk}(\xi)$ and $u_k>0$, $k=1,2$. Since $f''(u)>0$ and $u_1,u_2>0$, according to (4.3), we have

$$B_1'(\rho)=u_1u_2\int f''\big(u_0+u_1\omega_{0u1}(-\eta)+u_2\omega_{0u2}(-\eta+\rho)\big)\omega_{u1}(-\eta)\omega_{u2}(-\eta+\rho)\,d\eta>0.$$

It follows from this inequality and (3.16) that

$$F'(\rho)=\frac{(u_1+u_2)B_1'(\rho)}{f(u_0+u_1+u_2)u_2-f(u_0+u_2)(u_1+u_2)+f(u_0)u_1}>0,$$

i.e. $F(\rho)$ is an increasing function. Therefore, ρ_0 is the maximal (simple) root of the right-hand side of the differential equation (3.15): $F(\rho)=0$.

In view of the above facts, according to Proposition 4.1, we have the limit relation

$$\rho(\tau)=\tau\big(1+\psi_1(\tau)\big)\to\rho_0, \quad \tau\to-\infty.$$

Thus, $\rho(\tau)$ is a function with values in the interval $[\rho_0,\ +\infty]$ for $\tau\in[-\infty,\ +\infty]$. Moreover, according to (3.16), (4.4), (4.5),

$$\begin{aligned} &\lim_{\tau\to-\infty}B_k\big((-1)^{k-1}\rho\big)=B_k\big((-1)^{k-1}\rho_0\big)\\ &\qquad=\frac{\big(f(u_0+u_1+u_2)-f(u_0+u_k)\big)u_k-\big(f(u_0+u_k)-f(u_0)\big)u_{3-k}}{u_1+u_2}, \end{aligned} \tag{3.29}$$

$k=1,2$. Taking into account relation (3.27), we have $B_2(-\rho_0)=-\frac{u_1u_2}{u_1+u_2}\dot\psi_0(t)$.

3. *Construction of corrections.* We obtain the function $\rho = \rho(\tau)$ by integrating the autonomous differential equation (3.15), (3.16). Substituting the phases $\phi_{k0}(t)$ from (3.3) into (3.24) and using (4.5), (3.27), we obtain the following equations:

$$\frac{d}{d\tau}\Big(\tau\phi_{k1}(\tau)\Big) = \frac{(-1)^k B_2(-\rho)}{u_k\Big(\frac{[f(u)]_2}{[u]_2} - \frac{[f(u)]_1}{[u]_1}\Big)}. \tag{3.30}$$

Substituting $e_{k0}(t)$ from (3.3) into (3.25), and using (4.11), (3.27), we obtain

$$\frac{d}{d\tau}\Big(\tau e_{k1}(\tau)\Big) = (-1)^k \frac{u_k\widetilde{B}_2(-\rho) - v_k B_2(-\rho)}{u_k\Big(\frac{[f(u)]_2}{[u]_2} - \frac{[f(u)]_1}{[u]_1}\Big)}.$$

We find the *phase corrections* (3.12) and the *amplitude corrections* (3.13) by integrating these equations.

Integrating the right-hand side of the last two relations (4.9) by parts, we obtain

$$\begin{aligned} C_{R1}(\rho) &= -\int g'\big(u_0 + u_1\omega_{0u1}(\eta) + u_2\omega_{0u2}(\eta+\rho)\big) \\ &\qquad \times\big(u_1\omega_{u1}(\eta) + u_2\omega_{u2}(\eta+\rho)\big)\Omega_1'(\eta)\,d\eta, \\ C_{R2}(-\rho) &= -\int g'\big(u_0 + u_1\omega_{0u1}(\eta-\rho) + u_2\omega_{0u2}(\eta)\big) \\ &\qquad \times\big(u_1\omega_{u1}(\eta-\rho) + u_2\omega_{u2}(\eta)\big)\Omega_2'(\eta)\,d\eta, \end{aligned} \tag{3.31}$$

where, as already mentioned, $\rho \in [\rho_0, +\infty]$. Since $\omega_{u1}(\eta), \omega_{u2}(\eta) \ge 0$, we *can choose* mollifiers $\Omega_1(\eta)$, $\Omega_2(\eta)$ such that

$$\begin{aligned} \omega_{u1}(\eta)\Omega_1'(\eta) &\ge 0, & \omega_{u2}(\eta+\rho)\Omega_1'(\eta) &\ge 0, \\ \omega_{u1}(\eta-\rho)\Omega_2'(\eta) &\ge 0, & \omega_{u2}(\eta)\Omega_2'(\eta) &\ge 0, \end{aligned}$$

for all $\rho \in [\rho_0, +\infty]$. Since, according to (1.36), $g'(u) > 0$ and $u_1, u_2 > 0$ then from (3.31) we have $C_{Rk}\big((-1)^{k-1}\rho\big) \ne 0$, $k = 1, 2$ for all $\rho \in [\rho_0, +\infty]$. Thus, from (3.26) one can obtain formulas (3.14), which describe *corrections* $R_{k1}(\tau, t)$, $k = 1, 2$.

Thus, corrections $\phi_{k1}(\tau)$, $e_{k1}(\tau)$, $R_{k1}(\tau, t)$, $k = 1, 2$ are constructed.

4. *Checking the a priori assumptions.* Let us check that the corrections $\phi_{k1}(\tau)$, $e_{k1}(\tau)$, $R_{k1}(\tau, t)$ found in (3.12)–(3.14) satisfy the a priori assumptions (3.9), (3.10).

Let $\tau \to +\infty$. According to (3.15), this means that $\rho(\tau) \to +\infty$. As was said in Remark 4.1, $B_2(-\rho) = O(|\rho|^{-N})$, $\widetilde{B}_2(-\rho) = O(|\rho|^{-N})$, as $\rho \to +\infty$ for all $N = 1, 2, \dots$. Hence, from (3.12), (3.13) we obtain

$$\begin{aligned} \phi_{k1}(\tau) &= O(\tau^{-1}), & \tau\frac{d\phi_{k1}(\tau)}{d\tau} &= O(\tau^{-1}), \\ e_{k1}(\tau) &= O(\tau^{-1}), & \tau\frac{de_{k1}(\tau)}{d\tau} &= O(\tau^{-1}), \qquad \tau \to +\infty. \end{aligned}$$

It follows from the last estimates and (3.14), (4.10), (3.7), (3.8) that

$$\lim_{\tau\to+\infty} R_{k1}(\tau, t) = 0.$$

As mentioned above, ρ_0 is a simple root of the right-hand side of the differential equation (3.15). Consequently, according to Proposition 4.1,

$$\rho(\tau) = \tau\big(1 + \psi_1(\tau)\big) - \rho_0 = O(|\tau|^{-N}), \quad \tau \to -\infty.$$

Using Taylor's formula, we obtain

$$B_k\big((-1)^{k-1}\rho\big) = B_k\big((-1)^{k-1}\rho_0\big) + O(|\tau|^{-N}), \quad \tau \to -\infty,$$

$$\widetilde{B}_k\big((-1)^{k-1}\rho\big) = \widetilde{B}_k\big((-1)^{k-1}\rho_0\big) + O(|\tau|^{-N}), \quad \tau \to -\infty,$$

for all $N = 1, 2, \dots$, where $B_k\big((-1)^{k-1}\rho_0\big)$ are defined by (3.29). Therefore, from (3.12), (3.13), (3.29) we have

$$\begin{array}{rcl} \phi_{k1}(\tau) & = & (-1)^{k-1}\frac{u_{3-k}}{u_1+u_2} + O(\tau^{-1}), \\ \psi_1(\tau) & = & -1 + O(\tau^{-1}), \\ e_{k1}(\tau) & = & (-1)^k\Big(\frac{\widetilde{B}_2(-\rho_0)}{\frac{[f(u)]_2}{[u]_2} - \frac{[f(u)]_1}{[u]_1}} + \frac{v_k u_{3-k}}{u_1+u_2}\Big) + O(\tau^{-1}), \quad \tau \to -\infty. \end{array} \tag{3.32}$$

To this end, we calculate the limit $\tau\frac{d\phi_{k1}(\tau)}{d\tau}$ as $\tau \to -\infty$. One can rewrite relation (3.30) as

$$\tau\frac{d\phi_{k1}(\tau)}{d\tau} = \frac{(-1)^k}{u_k\Big(\frac{[f(u)]_2}{[u]_2} - \frac{[f(u)]_1}{[u]_1}\Big)}\bigg(B_2\big(\rho(\tau)\big) - \frac{\int_0^\tau B_2\big(\rho(\tau')\big)\,d\tau'}{\tau}\bigg).$$

Calculating the limit of the second term in the brackets by using the L'Hospital rule, we find $\lim_{\tau\to-\infty}\tau\frac{d\phi_{k1}(\tau)}{d\tau} = 0$. It can be shown in an analogous way that $\lim_{\tau\to-\infty}\tau\frac{de_{k1}(\tau)}{d\tau} = 0$.

From (3.14), (3.32), (4.9) one can see that there exists the limit

$$R_{k1}(\tau,t)\Big|_{\tau\to-\infty} = \frac{\widehat{e}_{k,-}(t)}{C_{Rk}\big((-1)^{k-1}\rho_0\big)}\bigg(\frac{f(u_0+u_k) - f(u_0) + B_k\big((-1)^{k-1}\rho_0\big)}{u_k} - A_k\big((-1)^{k-1}\rho_0\big)\bigg) - R_{k0}(t).$$

Taking into account the properties of the *interaction switches*, we see $R_{k1}(\tau,t)$, $\frac{d}{d\tau}R_{k1}(\tau)$ are bounded functions.

Thus, the corrections $\phi_{k1}(\tau)$, $e_{k1}(\tau)$, $R_{k1}(\tau,t)$ defined in (3.12)–(3.14) satisfy our a priori assumptions of smoothness and estimates (3.9), (3.10). □

REMARK 3.1. By substituting the expression for $R_k(t,\varepsilon)$ given by (3.26) into (4.8), the last relation takes the form

$$\begin{aligned} v(x,t,\varepsilon)g\big(u(x,t,\varepsilon)\big) &= g(u_0)v_0 \\ &+\Big(g(u_0+u_1)\big(v_0+v_1\big) - g(u_0)v_0 + \widetilde{B}_1(\rho)\Big)H(-x+\phi_1) \\ &+\Big(g(u_0+u_2)\big(v_0+v_2\big) - g(u_0)v_0 + \widetilde{B}_2(-\rho)\Big)H(-x+\phi_2) \\ &+\widehat{e}_1(\tau,t)\frac{f(u_0+u_1) - f(u_0) + B_1(\rho)}{u_1}\delta(-x+\phi_1) \end{aligned}$$

$$+\widehat{e}_2(\tau,t)\frac{f(u_0+u_2)-f(u_0)+B_2(-\rho)}{u_2}\delta(-x+\phi_2)+O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon\to+0. \tag{3.33}$$

COROLLARY 3.1. *The weak asymptotic solution* $\big(u(x,t,\varepsilon),v(x,t,\varepsilon)\big)$ *constructed in Theorem* 3.1 *is independent of the choice of regularization* $H_{jk}(x,\varepsilon)$, $\delta_{vk}(x,\varepsilon)$, $j=u,v$, $k=1,2$ *and has the following properties:*

1) *for* $t\in(0,\ t^*)$

$$\begin{aligned} \phi_k(t) &= \lim_{\varepsilon\to+0}\phi_k(t,\varepsilon)=\phi_{k0}(t)=\phi_{k0}(0)+\tfrac{[f(u)]_k}{[u]_k}t, \\ e_k(t) &= \lim_{\varepsilon\to+0}e_k(t,\varepsilon)=e_{k0}(t)=e_{k0}(0)+\Big([vg(u)]_k-[v]_k\tfrac{[f(u)]_k}{[u]_k}\Big)t \end{aligned} \tag{3.34}$$

uniformly in t, $k=1,2$, *and the weak limit of the weak asymptotic solution is given by the relation*

$$\begin{aligned} \lim_{\varepsilon\to+0}u(x,t,\varepsilon) &= u_0+\textstyle\sum_{k=1}^2 u_kH(-x+\phi_{k0}(t)), \\ \lim_{\varepsilon\to+0}v(x,t,\varepsilon) &= v_0+\textstyle\sum_{k=1}^2\Big(v_kH(-x+\phi_{k0}(t))+e_{k0}(t)\delta(-x+\phi_{k0}(t))\Big); \end{aligned} \tag{3.35}$$

2) *for* $t\in(t^*,\ +\infty)$

$$\begin{aligned} \widehat{\phi}_-(t) &\stackrel{def}{=} \phi_k(t)=\lim_{\varepsilon\to+0}\phi_k(t,\varepsilon)=x^*+\tfrac{[f(u)]_-}{[u]_-}(t-t^*), \quad k=1,2, \\ \widehat{e}_-(t) &\stackrel{def}{=} \lim_{\varepsilon\to+0}\Big(e_1(t,\varepsilon)+e_2(t,\varepsilon)\Big) \\ &= \widehat{e}_-(t^*)+\Big([vg(u)]_--[v]_-\tfrac{[f(u)]_-}{[u]_-}\Big)(t-t^*) \end{aligned} \tag{3.36}$$

uniformly in t, *and the weak limit of the weak asymptotic solution is given by the relation*

$$\begin{aligned} \lim_{\varepsilon\to+0}u(x,t,\varepsilon) &= u_0+(u_1+u_2)H\big(-x+\widehat{\phi}_-(t)\big), \\ \lim_{\varepsilon\to+0}v(x,t,\varepsilon) &= v_0+(v_1+v_2)H\big(-x+\widehat{\phi}_-(t)\big)+e_-(t)\delta\big(-x+\widehat{\phi}_-(t)\big). \end{aligned} \tag{3.37}$$

Here $\widehat{e}_-(t^*)=e_{10}(t^*)+e_{20}(t^*)$, $[h(u,v)]_-=h(u_0+u_1+u_2,v_0+v_1+v_2)-h(u_0,v_0)$ *is a jump in function* $h(u,v)$ *across the discontinuity curve* $x=\widehat{\phi}_-(t)$, $(x^*,\ t^*)$ *is the point of intersection of the* δ*-shock waves trajectories defined by* (3.4).

PROOF. We remind that $\psi_0(t)>0$ for $t<t^*$, i.e., $\tau>0$ and $\psi_0(t)<0$ for $t>t^*$, i.e., $\tau<0$.

Let $\tau\to+\infty$ (for $t\in[0,\ t^*)$). In view of the boundary conditions (3.9), (3.10), it follows from (3.21) that $\rho\to+\infty$. Therefore, taking into account (3.23) and the limit properties of interaction switches $B_k((-1)^{k-1}\rho)$, $\widetilde{B}_2((-1)^{k-1}\rho)$, $k=1,2$ given by Lemmas 4.2, 4.4, for all $t\in(0,\ t^*)$, as $\tau\to+\infty$, from systems (3.24), (3.25) with the boundary conditions (3.9), (3.10) we derive the *limit system* of equations (3.34).

Pass to the limit in system (3.24) (for $t>t^*$), as $\tau\to-\infty$. Since $\frac{d\phi_{k1}(\tau)}{d\tau}\to0$ by (3.9), and $\phi_{k1}(\tau)\to\phi_{k1,-}=(-1)^{k-1}\frac{u_{3-k}}{u_1+u_2}$ by (3.32), taking into account (3.23) we derive the *limit* system of equations for phases:

$$\begin{aligned}\dot{\widehat{\phi}}_{1,-}(t) = \dot{\phi}_{10}(t) + \dot{\psi}_0(t)\phi_{11,-} &= \tfrac{f(u_0+u_1+u_2)-f(u_0)}{u_1},\\ \dot{\widehat{\phi}}_{2,-}(t) = \dot{\phi}_{20}(t) + \dot{\psi}_0(t)\phi_{21,-} &= \tfrac{f(u_0+u_1+u_2)-f(u_0)}{u_1},\end{aligned} \tag{3.38}$$

where $\widehat{\phi}_{k,-}(t^*) = \phi_{k0}(t^*) = x^*$ and $\phi_{k0}(t)$ are defined by (3.3) for all $t \ge 0$.

It can be seen from (3.38) that the phase limit values (3.11) of $\widehat{\phi}_k(\tau, t)$ *coincide*:

$$\widehat{\phi}_{2,-}(t) = \widehat{\phi}_{1,-}(t) \stackrel{def}{=} \widehat{\phi}_-(t), \qquad t \ge t^*, \tag{3.39}$$

i.e., we obtain the first relation (3.36).

Analogously, passing to the limit, as $\tau \to -\infty$, in system (3.25), taking into account (3.23), (3.9), (3.38), and limiting relations (see (3.32))

$$e_{k1,-} = \lim_{\tau\to-\infty} e_{k1}(\tau) = (-1)^k\Big(\frac{\widetilde{B}_2(-\rho_0)}{\frac{[f(u)]_2}{[u]_2} - \frac{[f(u)]_1}{[u]_1}} + \frac{v_k u_{3-k}}{u_1+u_2}\Big),$$

we obtain the *limit* system of equations for amplitudes of δ-functions:

$$\begin{aligned}\dot{\widehat{e}}_{1,-}(t) &= \dot{e}_{10}(t) + \dot{\psi}_0(t)e_{11,-}\\ &= \big(v_0+v_1+v_2\big)g(u_0+u_1+u_2) - \big(v_0+v_2\big)g(u_0+u_2)\\ &\qquad -\widetilde{B}_2(-\rho_0) - v_1\tfrac{f(u_0+u_1+u_2)-f(u_0)}{u_1+u_2},\\ \dot{\widehat{e}}_{2,-}(t) &= \dot{e}_{20}(t) + \dot{\psi}_0(t)e_{21,-}\\ &= \big(v_0+v_2\big)g(u_0+u_2) - v_0 g(u_0)\\ &\qquad +\widetilde{B}_2(-\rho_0) - v_2\tfrac{f(u_0+u_1+u_2)-f(u_0)}{u_1+u_2}, \quad t \ge t^*.\end{aligned} \tag{3.40}$$

Adding the first and the second equations (3.40), we obtain for $t \ge t^*$

$$\begin{aligned}\dot{\widehat{e}}_-(t) = \dot{\widehat{e}}_{1,-}(t) + \dot{\widehat{e}}_{2,-}(t) &= \big(v_0+v_1+v_2\big)g(u_0+u_1+u_2) - v_0 g(u_0)\\ &\quad -\big(v_1+v_2\big)\frac{f(u_0+u_1+u_2)-f(u_0)}{u_1+u_2},\end{aligned} \tag{3.41}$$

where, according to (3.39), $\widehat{e}_-(t^*) = e_{10}(t^*) + e_{20}(t^*)$. By solving (3.41), we obtain

$$\widehat{e}_-(t) = \widehat{e}_{1,-}(t) + \widehat{e}_{2,-}(t) = e_{10}(t) + e_{20}(t) + \big(e_{11,-} + e_{21,-}\big)\psi_0(t),$$

i.e., the second relation (3.36).

Taking into account that

$$\lim_{\varepsilon\to+0}\langle H_{uk}\big(-x+\phi_k(t,\varepsilon),\varepsilon\big), \varphi(x,t)\rangle = \langle H\big(-x+\phi_k(t)\big), \varphi(x,t)\rangle,$$

$$\lim_{\varepsilon\to+0}\langle \delta_{vk}\big(-x+\phi_k(t,\varepsilon),\varepsilon\big), \varphi(x,t)\rangle = \langle \delta\big(-x+\phi_k(t)\big), \varphi(x,t)\rangle,$$

for all $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\infty))$, $j = u, v$, $k = 1, 2$, we have (3.35), (3.37). □

2. Generalized solution of the problem. Theorem 3.1 and Corollary 3.1 imply the following theorem.

THEOREM 3.2. *Assume that conditions* (1.36) *are satisfied. Then, for* $t \in [0, \infty)$, *the Cauchy problem* (1.5) *with the piecewise constant initial data* (1.12) *has a unique generalized solution*

$$
\begin{aligned}
u(x,t) &= u_0 + \sum_{k=1}^{2} u_k H\big(-x+\phi_k(t),\\
v(x,t) &= v_0 + \sum_{k=1}^{2} \Big(v_k H\big(-x+\phi_k(t)\big) + e_k(t)\delta\big(-x+\phi_k(t)\big)\Big),
\end{aligned}
\tag{3.42}
$$

where

$$
\begin{aligned}
\phi_k(t) &= x_k^0 + \tfrac{[f(u)]_k}{[u]_k} t \\
&+ \tfrac{(-1)^{k-1}u_{3-k}}{u_1+u_2}\bigg(\tfrac{[f(u)]_2}{[u]_2} - \tfrac{[f(u)]_1}{[u]_1}\bigg)\cdot (t-t^*)H(t-t^*),\\
e_k(t) &= e_k^0 + \Big([vg(u)]_k - [v]_k \tfrac{[f(u)]_k}{[u]_k}\Big)t + (-1)^k\bigg(\widetilde{B}_2(-\rho_0) \\
&+ \tfrac{v_k u_{3-k}}{(u_1+u_2)u_k}\Big(\tfrac{[f(u)]_2}{[u]_2} - \tfrac{[f(u)]_1}{[u]_1}\Big)\bigg)\cdot (t-t^*)H(t-t^*),
\end{aligned}
\tag{3.43}
$$

$\phi(0) = x_k^0$, $k=1,2$. *Here* $\phi_k(t) = \phi_{k0}(t)$, $e_k(t) = e_{k0}(t)$, *for all* $t \in (0,\ t^*)$, *and*

$$
\begin{aligned}
\phi_1(t) = \phi_2(t) = \widehat{\phi}_-(t) &= x^* + \tfrac{[f(u)]_-}{[u]_-}(t-t^*), \quad k=1,2,\\
\widehat{e}_-(t) = e_1(t)+e_2(t) &= \widehat{e}_-(t^*) + \Big([vg(u)]_- - [v]_- \tfrac{[f(u)]_-}{[u]_-}\Big)(t-t^*),
\end{aligned}
$$

for all $t > t^*$.

This generalized solution satisfies the integral identities cf. (1.14)*:*

$$
\begin{aligned}
\int_0^\infty \int \Big(u\varphi_t + f(u)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty \int \Big(\varphi_t + g(u)\varphi_x\Big)V\,dx\,dt & \\
+ \sum_{k=1}^{2} \int_{\gamma_k} e(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\,dl + \int_{\gamma_-} e(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\,dl & \\
+ \int V^0(x)\varphi(x,0)\,dx + \sum_{k=1}^{2} e_k^0 \varphi(x_k^0, 0) &= 0,
\end{aligned}
\tag{3.44}
$$

for all $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\ \infty))$, *where* $\Gamma = \gamma_1 \cup \gamma_1 \cup \gamma_-$, $\gamma_k = \{(x,t) : x = \phi_{k0}(t), \quad t \in (0,\ t^*]\}$, $\gamma_- = \{(x,t) : x = \widehat{\phi}_-(t), \quad t \ge t^*\}$, $V(x,t) = v_0 + \sum_{k=1}^{2} v_k H(-x+\phi_k(t))$, *and*

$$
\int_{\gamma_k} e(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\,dl = \int_0^{t^*} e_{k0}(t)\Big(\varphi_t(\phi_{k0}(t),t) + \dot{\phi}_{k0}(t)\varphi_x(\phi_{k0}(t),t)\Big)\,dt, \quad k=1,2,
$$

$$
\int_{\gamma_-} e(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\,dl = \int_{t^*}^{\infty} \widehat{e}_-(t)\Big(\varphi_t(\widehat{\phi}_-(t),t) + \dot{\widehat{\phi}}_-(t)\varphi_x(\widehat{\phi}_-(t),t)\Big)\,dt.
$$

Thus, for $t \in (0,\ t^*)$ *we have two delta-shock waves which propagate independently till the time instant* t^* *and after interaction (at the time instant* $t = t^*$*) merge* *constituting one new delta-shock wave for $t > t^*$, where $t = t^*$ is defined by (3.4). In addition, the phases $\phi_k(t)$ and amplitudes $e_k(t)$ are continuous functions with respect to t; at the point $t = t^*$ the velocities and the Rankine–Hugoniot deficit have the jumps*

$$
\begin{aligned}
\dot{\widehat{\phi}}_{k,-}(t) - \dot{\phi}_{k0}(t) &= \tfrac{(-1)^{k-1}u_{3-k}}{u_1+u_2}\left(\tfrac{[f(u)]_2}{[u]_2} - \tfrac{[f(u)]_1}{[u]_1}\right), \\
\dot{\widehat{e}}_{k,-}(t) - \dot{e}_{k0}(t) &= (-1)^k\left(\widetilde{B}_2(-\rho_0) + \tfrac{v_k u_{3-k}}{u_1+u_2}\left(\tfrac{[f(u)]_2}{[u]_2} - \tfrac{[f(u)]_1}{[u]_1}\right)\right), \\
\dot{\widehat{e}}_{-}(t) - \dot{e}_{k0}(t) &= \dot{e}_{3-k0}(t) + \tfrac{u_1v_2-u_2v_1}{u_1+u_2}\left(\tfrac{[f(u)]_2}{[u]_2} - \tfrac{[f(u)]_1}{[u]_1}\right), \quad k=1,2.
\end{aligned}
\tag{3.45}
$$

Proof. By Theorem 3.1 we have

$$L_1[u(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon), \quad L_2[u(x,t,\varepsilon), v(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon)$$

uniformly with respect to $t \in (0,\ +\infty)$. Let us apply the left-hand and right-hand sides of these relations to an arbitrary test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0,\ \infty))$. Then integrating by parts, we obtain

$$
\begin{aligned}
&\left(\int_0^{t^*} + \int_{t^*}^{\infty}\right)\int \Big(u(x,t,\varepsilon)\varphi_t(x,t) + f(u(x,t,\varepsilon)\varphi_x(x,t)\Big)\,dxdt \\
&\qquad\qquad + \int u(x,0,\varepsilon)\varphi(x,0)\,dx = O(\varepsilon), \\
&\left(\int_0^{t^*} + \int_{t^*}^{\infty}\right)\int \Big(v(x,t,\varepsilon)\varphi_t(x,t) + v(x,t,\varepsilon)g(u(x,t,\varepsilon))\varphi_x(x,t)\Big)\,dxdt \\
&\qquad\qquad + \int v(x,0,\varepsilon)\varphi(x,0)\,dx = O(\varepsilon), \quad \varepsilon \to +0.
\end{aligned}
$$

Let us substitute the ansatz $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ and the asymptotics $f(u(x,t,\varepsilon))$, $g(u(x,t,\varepsilon))v(x,t,\varepsilon)$ which are given by (3.2) and (4.2), (3.33), respectively, into the last relations. Next, passing to the limit, as $\varepsilon \to +0$, and taking into account Corollary 3.1 and (2.13), (2.14), by easy calculations, we obtain the integral identities (3.44).

According to Corollary 3.1, *before the interaction* for $t \in (0,\ t^*)$ we have the system of equations (3.34) describing *two propagating δ-shock waves*. *After the interaction* for $t > t^*$ we have the system of equations (3.36) describing a *single solitary δ-shock wave* which appears as the result of the interaction of two δ-shocks.

For thus constructed generalized solution the stability conditions $u_1 > 0,\ u_2 > 0$ hold, hence by the Oleinik uniqueness theorem, this solution is unique.

Formulas for $\widehat{\phi}_-(t)$, $\widehat{e}_-(t)$ follow from (3.36). Formulas (3.45) follow from (3.34), (3.36), (3.40). □

3. Example. Consider the simplest case of system (1.5):

$$u_t + (u^2)_x = 0, \quad v_t + 2(uv)_x = 0.$$

Using Theorem 3.2, we obtain the following results.

A. Suppose $e_2^0 = 0$ and $v_2 = -2v_0$. Then for $t \in (0,\ t^*)$ we have one delta-shock

$$\phi_{10}(t) = x_1^0 + (2u_0 + u_1 + 2u_2)t, \quad e_{10}(t) = e_1^0 + u_1(v_1 + v_2)t,$$

and one shock

$$\phi_{20}(t) = x_2^0 + (2u_0 + u_2)t, \quad (e_{20}(t) = 0),$$

which propagate independently. After interaction for $t > t^*$ we have one new delta-shock

$$\begin{array}{rcl} \widehat{\phi}_-(t) & = & x^* + (2u_0 + u_1 + u_2)(t - t^*), \\ \widehat{e}_-(t) & = & e_{10}(t^*) + v_1(u_1 + u_2)(t - t^*). \end{array}$$

B. Suppose $e_1^0 = e_2^0 = 0$ and $v_1 = 2v_0$, $v_2 = -2v_0$. Then for $t \in (0,\ t^*)$ we have two shocks

$$\begin{array}{rclrcl} \phi_{10}(t) & = & x_1^0 + (2u_0 + u_1 + 2u_2)t, & (e_{10}(t) & = & 0), \\ \phi_{20}(t) & = & x_2^0 + (2u_0 + u_2)t, & (e_{20}(t) & = & 0). \end{array}$$

After interaction for $t > t^*$ we have one new delta-shock

$$\begin{array}{rcl} \widehat{\phi}_-(t) & = & x^* + (2u_0 + u_1 + u_2)(t - t^*), \\ \widehat{e}_-(t) & = & 2v_0(u_1 + u_2)(t - t^*). \end{array}$$

C. Suppose $v_1 = -2v_0 - v_2$. Then for $t \in (0,\ t^*)$ we have two delta-shocks

$$\begin{array}{rclrcl} \phi_{10}(t) & = & x_1^0 + (2u_0 + u_1 + 2u_2)t, & e_{10}(t) & = & e_1^0 + u_1 v_2 t, \\ \phi_{20}(t) & = & x_2^0 + (2u_0 + u_2)t, & e_{20}(t) & = & e_2^0 - u_2 v_1 t. \end{array}$$

After interaction for $t > t^*$ we have one new shock and delta-function at the point $(x^*,\ t^*)$

$$\begin{array}{rcl} \widehat{\phi}_-(t) & = & x^* + (2u_0 + u_1 + u_2)(t - t^*), \\ \widehat{e}_-(t) & = & e_{10}(t^*) + e_{20}(t^*). \end{array}$$

4. Some auxiliary results

1. In this subsection we present formulas for weak asymptotic expansions of some expressions. These formulas are used to construct solutions describing propagation and interaction of δ-shock waves.

LEMMA 4.1. ([**5**, Corollary 1.1.], [**3**, 1.3.]) *Let $f(u)$ be a smooth function, let $u_0(x,t)$, $u_1(x,t)$ be bounded functions. If $u(x,t,\varepsilon)$ is defined by* (1.24), (2.1) *then*

$$f\big(u(x,t,\varepsilon)\big) = f(u_0)$$

$$+\Big(f(u_0(x,t) + u_1(x,t)) - f(u_0(x,t))\Big)H(-x + \phi) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0, \tag{4.1}$$

LEMMA 4.2. ([**5**, Lemma 1.1.], [**3**, Lemma 1.1.]) *Let $f(u)$ be a smooth function, let $u_k(x,t)$, $k = 0,1,2$ be bounded functions. If $u(x,t,\varepsilon)$ is defined by* (1.21), (3.1) *then*

$$f\big(u(x,t,\varepsilon)\big) = f(u_0(x,t))$$
$$+\Big(f(u_0(x,t) + u_1(x,t)) - f(u_0(x,t))\Big)H(-x + \phi_1)$$

$$+\Big(f(u_0(x,t)+u_2(x,t))-f(u_0(x,t))\Big)H(-x+\phi_2)$$

$$+B_1\Big(x,t,\frac{\Delta\phi}{\varepsilon}\Big)H(-x+\phi_1)+B_2\Big(x,t,-\frac{\Delta\phi}{\varepsilon}\Big)H(-x+\phi_2)+O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon\to+0, \tag{4.2}$$

where $\Delta\phi=\phi_2-\phi_1$ *and the estimate* $O_{\mathcal{D}'}(\varepsilon)$ *is uniform with respect to* $\Delta\phi$.

The functions $B_k(x,t,\rho)$, $k=1,2$ *called "interaction switch functions" have the following form:*

$$\begin{aligned} B_1(x,t,\rho) &= \int\Big\{f'\big(u_0(x,t)+u_1(x,t)\omega_{0u1}(-\eta)+u_2(x,t)\omega_{0u2}(-\eta+\rho)\big) \\ &\qquad -f'\big(u_0(x,t)+u_1(x,t)\omega_{0u1}(-\eta)\big)\Big\}u_1(x,t)\omega_{u1}(-\eta)\,d\eta, \\ B_2(x,t,-\rho) &= \int\Big\{f'\big(u_0(x,t)+u_1(x,t)\omega_{0u1}(-\eta-\rho)+u_2(x,t)\omega_{0u2}(-\eta)\big) \\ &\qquad -f'\big(u_0(x,t)+u_2(x,t)\omega_{0u2}(-\eta)\big)\Big\}u_2(x,t)\omega_{u2}(-\eta)\,d\eta. \end{aligned} \tag{4.3}$$

In addition, the "interaction switch functions" satisfy the relations

$$\begin{aligned} \lim_{\rho\to+\infty}B_k(x,t,\rho) &= f(u_0+u_1+u_2)-f(u_0+u_1)-f(u_0+u_2)+f(u_0), \\ \lim_{\rho\to-\infty}B_k(x,t,\rho) &= 0, \qquad k=1,2, \end{aligned} \tag{4.4}$$

and for any $\rho\in\mathbb{R}$

$$B_1(x,t,\rho)+B_2(x,t,-\rho)=f(u_0(x,t)+u_1(x,t)+u_2(x,t))$$

$$-f(u_0(x,t)+u_1(x,t))-f(u_0(x,t))+u_2(x,t))+f(u_0(x,t)). \tag{4.5}$$

LEMMA 4.3. *Let* $g(u)$ *be a smooth function, let* $u_k(x,t)$, $v_k(x,t)$, $k=0,1$, $e(t)$ *be bounded functions. If* $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ *are defined by* (1.24), (2.1) *then*

$$v(x,t,\varepsilon)g\big(u(x,t,\varepsilon)\big)=g\big(u_0(x,t)\big)v_0(x,t)$$

$$+\Big(g\big(u_0(x,t)+u_1(x,t)\big)\big(v_0(x,t)+v_1(x,t)\big)-g\big(u_0(x,t)\big)v_0(x,t)\Big)H(-x+\phi)$$

$$+\Big(e(t)a(t)+R(t)c(t)\Big)\delta(-x+\phi)+O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon\to+0, \tag{4.6}$$

where

$$\begin{aligned} a(t) &= \int g\big(u_0(0,t)+u_1(0,t)\omega_{0u1}(\eta)\big)\omega_{\delta1}(\eta)\,d\eta, \\ c(t) &= \int g\big(u_0(0,t)+u_1(0,t)\omega_{0u1}(\eta)\big)\Omega''(\eta)\,d\eta. \end{aligned} \tag{4.7}$$

PROOF. Using Lemma 4.1, it is easy to obtain the weak asymptotics

$$\Big(v_0(x,t)+v_1(x,t)H_{v1}(-x,\varepsilon)\Big)g\big(u(x,t,\varepsilon)\big)=g\big(u_0(x,t)\big)v_0(x,t)$$

$$+\Big(g\big(u_0(x,t)+u_1(x,t)\big)\big(v_0(x,t)+v_1(x,t)\big)-g\big(u_0(x,t)\big)v_0(x,t)\Big)H(-x)$$

$$+O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon\to+0.$$

Next, after the change of variables $x = -\varepsilon\eta$, we have

$$J(\varepsilon) = \Big\langle \Big(e(t)\delta_{v1}(-x,\varepsilon) + R(t)\frac{1}{\varepsilon}\Omega''\Big(\frac{-x}{\varepsilon}\Big)\Big) g\big(u(x,t,\varepsilon)\big), \psi(x) \Big\rangle$$

$$= \psi(0)\Big(e(t)a(t) + R(t)c(t) \Big) + O(\varepsilon), \quad \varepsilon \to +0, \quad \forall \psi(x) \in \mathcal{D}(\mathbb{R}).$$

□

LEMMA 4.4. *Let $g(u)$ be a smooth function, let u_0, u_k, v_0, v_k be constants, $e_k(t)$ be bounded functions. If $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ are defined by* (3.2) *then*

$$v(x,t,\varepsilon)g\big(u(x,t,\varepsilon)\big) = g(u_0)v_0$$

$$+\Big(g(u_0+u_1)\big(v_0+v_1\big) - g(u_0)v_0 \Big) H(-x+\phi_1) + \widetilde{B}_1\Big(\frac{\Delta\phi}{\varepsilon}\Big) H(-x+\phi_1)$$

$$+\Big(g(u_0+u_2)\big(v_0+v_2\big) - g(u_0)v_0 \Big) H(-x+\phi_2) + \widetilde{B}_2\Big(-\frac{\Delta\phi}{\varepsilon}\Big) H(-x+\phi_2)$$

$$+\Big(e_1(t)A_1\Big(\frac{\Delta\phi}{\varepsilon}\Big) + R_1(t,\varepsilon)C_{R1}\Big(\frac{\Delta\phi}{\varepsilon}\Big)\Big)\delta(-x+\phi_1)$$

$$+\Big(e_2(t)A_2\Big(-\frac{\Delta\phi}{\varepsilon}\Big) + R_2(t,\varepsilon)C_{R2}\Big(-\frac{\Delta\phi}{\varepsilon}\Big)\Big)\delta(-x+\phi_2) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0, \tag{4.8}$$

where $\Delta\phi = \phi_2 - \phi_1$ and the estimate $O_{\mathcal{D}'}(\varepsilon)$ is uniform with respect to $\Delta\phi$.

Here "interaction switch functions" have the following form:

$$
\begin{aligned}
\widetilde{B}_1(\rho) &= \int \Big\{ \Big[g'\big(u_0 + u_1\omega_{0u1}(\eta) + u_2\omega_{0u2}(\eta+\rho)\big) \\
&\quad \times \Big(v_0 + v_1\omega_{0v1}(\eta) + v_2\omega_{0v2}(\eta+\rho) \Big) \\
&\quad - g'\big(u_0 + u_1\omega_{0u1}(\eta)\big) \Big(v_0 + v_1\omega_{0v1}(\eta) \Big) \Big] u_1\omega_{u1}(\eta) \\
&\qquad + \Big[g\big(u_0 + u_1\omega_{0u1}(\eta) + u_2\omega_{0u2}(\eta+\rho)\big) \\
&\qquad\quad - g\big(u_0 + u_1\omega_{0u1}(\eta)\big) \Big] v_1\omega_{v1}(\eta) \Big\} \, d\eta, \\
\widetilde{B}_2(-\rho) &= \int \Big\{ \Big[g'\big(u_0 + u_1\omega_{0u1}(\eta-\rho) + u_2\omega_{0u2}(\eta)\big) \\
&\quad \times \Big(v_0 + v_1\omega_{0v1}(\eta-\rho) + v_2\omega_{0v2}(\eta) \Big) \\
&\quad - g'\big(u_0 + u_2\omega_{0u2}(\eta)\big) \Big(v_0 + v_2\omega_{0v2}(\eta) \Big) \Big] u_2\omega_{u2}(\eta) \\
&\qquad + \Big[g\big(u_0 + u_1\omega_{0u1}(\eta-\rho) + u_2\omega_{0u2}(\eta)\big) \\
&\qquad\quad - g\big(u_0 + u_2\omega_{0u2}(\eta)\big) \Big] v_2\omega_{v2}(\eta) \Big\} \, d\eta, \\
A_1(\rho) &= \int g\big(u_0 + u_1\omega_{0u1}(\eta) + u_2\omega_{0u2}(\eta+\rho)\big)\omega_{\delta 1}(\eta)\, d\eta, \\
A_2(-\rho) &= \int g\big(u_0 + u_1\omega_{0u1}(\eta-\rho) + u_2\omega_{0u2}(\eta)\big)\omega_{\delta 2}(\eta)\, d\eta, \\
C_{R1}(\rho) &= \int g\big(u_0 + u_1\omega_{0u1}(\eta) + u_2\omega_{0u2}(\eta+\rho)\big)\Omega''_1(\eta)\, d\eta, \\
C_{R2}(-\rho) &= \int g\big(u_0 + u_1\omega_{0u1}(\eta-\rho) + u_2\omega_{0u2}(\eta)\big)\Omega''_2(\eta)\, d\eta.
\end{aligned} \tag{4.9}
$$

In addition, the "interaction switch functions" satisfy the relations

$$
\begin{aligned}
\lim_{\rho\to+\infty} \widetilde{B}_k(\rho) &= \big(v_0 + v_1 + v_2\big) g(u_0 + u_1 + u_2) \\
&\quad - \big(v_0 + v_1\big) g(u_0 + u_1) \\
&\quad - \big(v_0 + v_2\big) g(u_0 + u_2) + v_0 g(u_0), \\
\lim_{\rho\to-\infty} \widetilde{B}_k(\rho) &= 0, \\
\lim_{\rho\to+\infty} A_k\big((-1)^{k-1}\rho\big) &= a_k, \\
\lim_{\rho\to+\infty} C_{Rk}\big((-1)^{k-1}\rho\big) &= c_k, \qquad k = 1, 2,
\end{aligned} \tag{4.10}
$$

where a_k, c_k are constants defined by (3.8). *Moreover, for any $\rho \in \mathbb{R}$*

$$
\begin{aligned}
\widetilde{B}_1(\rho) + \widetilde{B}_2(-\rho) &= \big(v_0 + v_1 + v_2\big) g(u_0 + u_1 + u_2) \\
&\quad - \big(v_0 + v_1\big) g(u_0 + u_1) - \big(v_0 + v_2\big) g(u_0 + u_2) + v_0 g(u_0).
\end{aligned} \tag{4.11}
$$

Proof. First, we construct the weak asymptotics of the following expression

$$
\begin{aligned}
J(a,\varepsilon) &= \Big\langle g\big(u_0 + u_1 H_{u1}(-x,\varepsilon) + u_2 H_{u2}(-x+a,\varepsilon)\big) \\
&\quad \times \Big(v_0 + v_1 H_{v1}(-x,\varepsilon) + v_2 H_{v2}(-x+a,\varepsilon) \Big), \psi(x) \Big\rangle, \quad \forall \psi(x) \in \mathcal{D}(\mathbb{R}).
\end{aligned}
$$

Since

$$J(a,\varepsilon)=\Big\langle g\big(u_0+u_1H_{u1}(-x,\varepsilon)+u_2H_{u2}(-x+a,\varepsilon)\big)\times$$
$$\Big(v_0+v_1H_{v1}(-x,\varepsilon)+v_2H_{v2}(-x+a,\varepsilon)\Big),\frac{d}{dx}\psi^{(-1)}(x)\Big\rangle,\quad \psi^{(-1)}(x)=\int_{-\infty}^{x}\psi(\xi)\,d\xi,$$

integrating by parts and taking into account that $\psi^{(-1)}(-\infty)=0$, $\psi^{(-1)}(+\infty)=\langle 1,\psi(x)\rangle$, and

$$g\big(u_0+u_1H_{u1}(-x,\varepsilon)+u_2H_{u2}(-x+a,\varepsilon)\big)\Big(v_0+v_1H_{v1}(-x,\varepsilon)$$
$$+v_2H_{v2}(-x+a,\varepsilon)\Big)\psi^{(-1)}(x)\Big|_{-\infty}^{\infty}=\langle v_0g\big(u_0\big),\psi(x)\rangle,$$

we obtain $J(a,\varepsilon)=\langle v_0g\big(u_0\big),\psi(x)\rangle+J_1(a,\varepsilon)+J_2(a,\varepsilon)$, where

$$J_1(a,\varepsilon)=\int\Big\{g'\big(u_0+u_1H_{u1}(-x,\varepsilon)+u_2H_{u2}(-x+a,\varepsilon)\big)$$
$$\times\Big(v_0+v_1H_{v1}(-x,\varepsilon)+v_2H_{v2}(-x+a,\varepsilon)\Big)u_1\frac{1}{\varepsilon}\omega_{u1}\Big(\frac{-x}{\varepsilon}\Big)$$
$$+g\big(u_0+u_1H_{u1}(-x,\varepsilon)+u_2H_{u2}(-x+a,\varepsilon)\big)v_1\frac{1}{\varepsilon}\omega_{v1}\Big(\frac{-x}{\varepsilon}\Big)\Big\}\psi^{(-1)}(x)\,dx,$$

$$J_2(a,\varepsilon)=\int\Big\{g'\big(u_0+u_1H_{u1}(-x,\varepsilon)+u_2H_{u2}(-x+a,\varepsilon)\big)$$
$$\times\Big(v_0+v_1H_{v1}(-x,\varepsilon)+v_2H_{v2}(-x+a,\varepsilon)\Big)u_2\frac{1}{\varepsilon}\omega_{u2}\Big(\frac{-x+a}{\varepsilon}\Big)$$
$$+g\big(u_0+u_1H_{u1}(-x,\varepsilon)+u_2H_{u2}(-x+a,\varepsilon)\big)v_2\frac{1}{\varepsilon}\omega_{v2}\Big(\frac{-x+a}{\varepsilon}\Big)\Big\}\psi^{(-1)}(x)\,dx.$$

After the change of variables $x=-\varepsilon\eta$, we transform $J_1(a,\varepsilon)$ to the form

$$J_1(a,\varepsilon)=\int\Big\{g'\big(u_0+u_1\omega_{0u1}(-\eta)+u_2\omega_{0u2}\Big(-\eta+\frac{a}{\varepsilon}\Big)\big)$$
$$\times\Big(v_0+v_1\omega_{0v1}(-\eta)+v_2\omega_{0v2}\Big(-\eta+\frac{a}{\varepsilon}\Big)\Big)u_1\omega_{u1}(-\eta)$$
$$+g\big(u_0+u_1\omega_{0u1}(-\eta)+u_2\omega_{0u2}\Big(-\eta+\frac{a}{\varepsilon}\Big)\big)v_1\omega_{v1}(-\eta)\Big\}\Big(\int_{-\infty}^{\varepsilon\eta}\psi(\xi)\,d\xi\Big)\,d\eta$$
$$=\widehat{B}_1\Big(\frac{a}{\varepsilon}\Big)\langle H(-x),\psi(x)\rangle+O(\varepsilon),$$

where the estimate $O(\varepsilon)$ is uniform with respect to a, $\langle H(-x),\psi(x)\rangle=\int_{-\infty}^{0}\psi(\xi)\,d\xi$, and

$$\widehat{B}_1(\rho)=\int\Big\{g'(u_0+u_1\omega_{0u1}(-\eta)+u_2\omega_{0u2}(-\eta+\rho))$$
$$\times\big(v_0+v_1\omega_{0v1}(-\eta)+v_2\omega_{0v2}(-\eta+\rho)\big)u_1\omega_{u1}(-\eta)$$

$$+g(u_0+u_1\omega_{0u1}(-\eta)+u_2\omega_{0u2}(-\eta+\rho))v_1\omega_{v1}(-\eta)\Big\}\,d\eta. \tag{4.12}$$

Analogously, making the change of variables $x=\varepsilon\eta+a$ we obtain

$$J_2(a,\varepsilon)=\widehat{B}_2\Big(-\frac{a}{\varepsilon}\Big)\langle H(-x+a),\psi(x)\rangle+O(\varepsilon),$$

where the estimate $O(\varepsilon)$ is uniform in a, $\langle H(-x+a),\psi(x)\rangle=\int_{-\infty}^{a}\psi(\xi)\,d\xi$, and

$$\widehat{B}_2(-\rho)=\int\Big\{g'(u_0+u_1\omega_{0u1}(-\eta-\rho)+u_2\omega_{0u2}(-\eta))$$

$$\times\big(v_0+v_1\omega_{0v1}(-\eta-\rho)+v_2\omega_{0v2}(-\eta)\big)u_2\omega_{u2}(-\eta)$$

$$+g(u_0+u_1\omega_{0u1}(-\eta-\rho)+u_2\omega_{0u2}(-\eta))v_2\omega_{0v2}(-\eta)\Big\}\,d\eta. \tag{4.13}$$

Adding derivatives of the terms $g\big(u_0+u_1\omega_{0u1}(\eta)\big)\Big(v_0+v_1\omega_{0v1}(\eta)\Big)$ and $-g\big(u_0+u_2\omega_{0u2}(\eta)\big)\Big(v_0+v_2\omega_{0v2}(\eta)\Big)$ to the integrands (4.12) and (4.13), respectively, we obtain

$$\begin{aligned}\widehat{B}_1(\rho)&=\Big(g(u_0+u_1)\big(v_0+v_1\big)-g(u_0)v_0\Big)+\widetilde{B}_1(\rho),\\ \widehat{B}_2(-\rho)&=\Big(g(u_0+u_2)\big(v_0+v_2\big)-g(u_0)v_0\Big)+\widetilde{B}_2(-\rho),\end{aligned} \tag{4.14}$$

where "interaction switch functions" $\widetilde{B}_k((-1)^{k-1}\rho)$, $k=1,2$ are defined by (4.9).

Next, we construct the weak asymptotics of the expression $J^0(a,\varepsilon)=J^0_1(a,\varepsilon)+J^0_2(a,\varepsilon)$, where

$$J^0_1(a,\varepsilon)=\langle g\big(u_0+u_1H_{u1}(-x,\varepsilon)+u_2H_{u2}(-x+a,\varepsilon)\big)$$
$$\times\bigg(e_1(t)\delta_{v1}(-x,\varepsilon)+R_1(t,\varepsilon)\frac{1}{\varepsilon}\Omega''_1\Big(\frac{-x}{\varepsilon}\Big)\bigg),\psi(x)\rangle,$$
$$J^0_2(a,\varepsilon)=\langle g\big(u_0+u_1H_{u1}(-x,\varepsilon)+u_2H_{u2}(-x+a,\varepsilon)\big)$$
$$\times\bigg(e_2(t)\delta_{v2}(-x+a,\varepsilon)+R_2(t,\varepsilon)\frac{1}{\varepsilon}\Omega''_2\Big(\frac{-x+a}{\varepsilon}\Big)\bigg),\psi(x)\rangle.$$

Making the change of variables $x=\varepsilon\eta$ we transform $J^0_1(a,\varepsilon)$ to the following form

$$J^0_1(a,\varepsilon)=e_1(t)\psi(0)A_1\Big(\frac{a}{\varepsilon}\Big)+R_1(t,\varepsilon)\psi(0)C_{R1}\Big(\frac{a}{\varepsilon}\Big)+O(\varepsilon);$$

making the change of variables $x=\varepsilon\eta+a$, we obtain from $J^0_2(a,\varepsilon)$ the following expression

$$J^0_2(a,\varepsilon)=e_2(t)\psi(a)A_2\Big(-\frac{a}{\varepsilon}\Big)+R_2(t,\varepsilon)\psi(a)C_{R2}\Big(-\frac{a}{\varepsilon}\Big)+O(\varepsilon),$$

where the estimates $O(\varepsilon)$ is uniform with respect to a and "interaction switch functions" $A_k((-1)^{k-1}\rho)$, $C_{Rk}((-1)^{k-1}\rho)$, $k=1,2$ are defined by (4.9).

Adding $J(a,\varepsilon)=\langle v_0g\big(u_0\big),\psi(x)\rangle+J_1(a,\varepsilon)+J_2(a,\varepsilon)$ and $J^0(a,\varepsilon)=J^0_1(a,\varepsilon)+J^0_2(a,\varepsilon)$, we obtain the weak asymptotics (4.8).

Passing to the limit as $\rho \to \pm\infty$ in (4.9) by integrating the limit expressions, we obtain the first and the second relations (4.10). The other relations (4.10) are obtained from (4.9), (3.8).

Next, after the change of variables $\eta - \rho \to -\eta$, we obtain from (4.9)

$$\widetilde{B}_2(-\rho) = \int \Big\{ g'\big(u_0 + u_1\omega_{0u1}(\eta) + u_2\omega_{0u2}(\eta+\rho)\big)\Big(v_0 + v_1\omega_{0v1}(\eta)$$

$$+v_2\omega_{0v2}(\eta+\rho)\Big)u_2\omega_{u2}(\eta+\rho) + g\big(u_0 + u_1\omega_{0u1}(\eta) + u_2\omega_{0u2}(\eta+\rho)\big)v_2\omega_{v2}(\eta+\rho)$$

$$-g'\big(u_0 + u_2\omega_{0u2}(\eta)\big)\Big(v_0 + v_2\omega_{0v2}(\eta)\Big)u_2\omega_{u2}(\eta) - g\big(u_0 + u_2\omega_{0u2}(\eta)\big)v_2\omega_{v2}(\eta)\Big\}\, d\eta.$$

Adding $\widetilde{B}_2(-\rho)$ and $\widehat{B}_1(\rho)$, we have

$$\widetilde{B}_1(\rho) + \widetilde{B}_2(-\rho)$$

$$= \int \Big(g\big(u_0 + u_1\omega_{0u1}(\eta) + u_2\omega_{0u2}(\eta+\rho)\big)\Big(v_0 + v_1\omega_{0v1}(\eta) + v_2\omega_{0v2}(\eta+\rho)\Big)\Big)'\, d\eta$$

$$-\int \Big(g\big(u_0 + u_1\omega_{0u1}(\eta)\big)\Big(v_0 + v_1\omega_{0v1}(\eta)\Big) + g\big(u_0 + u_2\omega_{0u2}(\eta)\big)\Big(v_0 + v_2\omega_{0v2}(\eta)\Big)\Big)'\, d\eta.$$

Formula (4.11) is obtained by integrating the last relation. □

REMARK 4.1. (see [**5**, Remark 1.1.]) Mollifiers $\omega_{uk}(\tau)$, $k = 1, 2$ have compact supports or decrease sufficiently fast as $|z| \to \infty$. Therefore, we have

$$\begin{aligned} \omega_{0uk}(z) &= \int_{-\infty}^{z} \omega_{uk}(\eta)\, d\eta = 1 + O(z^{-N}), \quad z \to +\infty, \\ \omega_{0uk}(z) &= O(|z|^{-N}), \quad z \to -\infty. \end{aligned}$$

Consequently, using the Lagrange theorem, we obtain the estimate

$$\Big(f'\big(u_0 + u_1\omega_{0u1}(-\eta) + u_2\omega_{0u2}(-\eta+\rho)\big) - f'\big(u_0 + u_1\omega_{0u1}(-\eta)\big)\Big)u_1\omega_{u1}(-\eta)$$

$$= f''\big(u_0 + u_1\omega_{0u1}(-\eta) + \Theta u_2\omega_{0u2}(-\eta+\rho)\big)u_1u_2\omega_{u1}(-\eta)\omega_{0u2}(-\eta+\rho),$$

where $0 < \Theta < 1$.

It follows from this estimate and (4.3), (4.4), (4.9), (4.10) that

$$\begin{aligned} B_1(x,t,\rho) &= f(u_0(x,t) + u_1(x,t) + u_2(x,t)) - f(u_0(x,t) + u_1(x,t)) \\ &\quad -f(u_0(x,t) + u_2(x,t)) + f(u_0(x,t)) + O(\rho^{-N}), \quad \rho \to +\infty, \\ B_1(x,t,\rho) &= O(|\rho|^{-N}), \quad \rho \to -\infty, \\ \widetilde{B}_k(\rho) &= \big(v_0 + v_1 + v_2\big)g(u_0 + u_1 + u_2) - \big(v_0 + v_1\big)g(u_0 + u_1) \\ &\quad -\big(v_0 + v_2\big)g(u_0 + u_2) + v_0 g(u_0) + O(\rho^{-N}), \quad \rho \to +\infty, \\ \widetilde{B}_1(\rho) &= O(|\rho|^{-N}), \quad \rho \to -\infty, \quad N = 1, 2, \dots. \end{aligned}$$

2. To analyse the dynamics of interaction of δ-shocks we need a result concerned to autonomous ordinary differential equations. This differential equation is typical for our approach (see (3.15), (3.16)).

Proposition 4.1. ([**5**, Proposition 4.1.]) *For the autonomous differential equation*

$$\frac{d\rho}{d\tau} = F(\rho), \qquad F(\rho) \in C^1(\mathbb{R})$$

to have a solution such that

$$\frac{\rho(\tau)}{\tau}\Big|_{\tau\to+\infty} = 1, \quad \rho(\tau)\Big|_{\tau\to-\infty} = \rho_0,$$

where ρ_0 is a constant, it is necessary and sufficient that the following conditions hold:

$$\begin{aligned} F(\rho)\Big|_{\rho\to+\infty} &= 1, \\ F(\rho_0) &= 0, \\ F(\rho) &> 0 \quad \textit{for} \quad \rho > \rho_0, \end{aligned}$$

where ρ_0 is the maximal root of the equation $F(\rho) = 0$.

In addition, if ρ_0 is an ordinary (nonmultiple) root of the equation $F(\rho) = 0$ then for any $N = 1, 2, \dots$ we have $\rho(\tau) - \rho_0 = O(|\tau|^{-N})$, $\tau \to -\infty$.

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