Onsager Relations and Eulerian Hydrodynamic Limit for Systems with Several Conservation Laws

Bálint Tóth Benedek Valkó

Institute of Mathematics
Technical University Budapest

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Abstract

We present the derivation of the hydrodynamic limit under Eulerian scaling for a general class of one-dimensional interacting particle systems with two or more conservation laws. Following Yau's relative entropy method it turns out that in case of more than one conservation laws, in order that the system exhibit hydrodynamic behaviour, some particular identities reminiscent of Onsager's reciprocity relations must hold. We check validity of these identities whenever a stationary measure with *product structure* exists. It also follows that, as a general rule, *the equilibrium thermodynamic entropy* (as function of the densities of the conserved variables) *is a globally convex Lax entropy* of the hyperbolic systems of conservation laws arising as hydrodynamic limit. As concrete examples we also present a number of models modeling deposition (or domain growth) phenomena. The Onsager relations arising in the context of hydrodynamic limits under hyperbolic scaling seem to be novel. The fact that equilibrium thermodynamic entropy is Lax entropy for the arising Euler equations was noticed earlier in the context of Hamiltonian systems with weak noise, see [7].

1 Introduction

We investigate the hydrodynamic behaviour of a very general class of one dimensional interacting particle systems with two or more conserved observables. The systems are not reversible and the hydrodynamic limit under Eulerian scaling is investigated. We apply Yau's relative entropy method and obtain validity of the hydrodynamic limit up to the occurence of the first shock wave in the solution of the limiting pde. There is no novelty in the standard steps of the relative entropy proof, so we only sketch these. The real novelty appears when it turns out that, in case of more than one conserved quantity, in order to complete the relative entropy proof, a class of identities should hold, relating the macroscopic fluxes appearing in the hydrodynamic pdes. These identities are much reminiscent of Onsager's reciprocity relations. As far as we know these relations have not been pointed out in the context mathematically rigorous Eulerian hydrodynamics. We check the validity of these relations assuming only the existence of a stationary measure with product structure. As a consequence of the Onsager relations it follows that the systems of partial differential equations (systems of conservation laws) arising as hydrodynamic limit are by force of hyperbolic type and the equilibrium thermodynamic entropy of the system (as function of the densities of the conserved quantities) is globally convex Lax entropy of the hydrodynamic equations. This fact may be not so surprising, as it is commonly accepted physical fact. So much so that hyperbolic systems of conservation laws possessing a globally convex Lax entropy are commonly called *of physical type*, see [11]. In the context of Hamiltonian systems perturbed by a weak noise a similar result was established in [7]. Nevertheless, as far as we know, this fact has not been emphasized in its full generality in the context of mathematically rigorous derivation of hydrodynamic behaviour. It is worth noting that given a hyperbolic system of conservation laws the existence of convex Lax entropies is far from trivial: in the case of two component systems the local existence of convex Lax entropies was established in the very technical work [6]. In case of more than two components in general the pdes defining Lax entropies are overdetermined, so in general Lax entropies do not exist at all. It turns out from our result that the hyperbolic systems of conservation laws arising as hydrodynamic limit are of very special type: they always possess a globally convex Lax entropy, namely the equilibrium thermodynamic entropy of the system.

Beside the general framework we also present a number of concrete examples of deposition models with two conserved quantities to which the general result applies, deriving in this way systems of pdes (hyperbolic systems of conservation laws) which describe macroscopically domain growth phenomena in 1+1 dimension.

Our general results are easily extended to more than one dimensions, however the formalism becomes more complicated. We were mostly motivated by the (one dimensional) deposition models presented as concrete examples.

The paper is organized as follows: In section 2 we present the general formalism and the conditions under which the hydrodynamic limit is derived. In section 3 we state the main results of the paper. In section 4 we present a number of concrete examples to which the general framework applies. We hope that the models introduced in this section could be of interest in the context of deposition/domain growth phenomena. In section 5 we sketch the proof of the main result formulated in section 3. The sketchy proof is broken up into several parts. We only hint at the standard steps of the relative entropy proof, referring the reader to the original work [15] or the monographs [4] or [3]. The essential parts of this section are subsections 5.2 and 5.3 where the Onsager relations and their consequences are derived. Finally, in section 6 we extend the results formulated in the previous sections from two to arbitrary number of conserved quantities.

2 Microscopic models

2.1 State space, conserved quantities

Throughout this paper we denote by $\mathbb{T}^N$ the discrete tori $\mathbb{Z}/N\mathbb{Z}$, $N \in \mathbb{N}$, and by $\mathbb{T}$ the continuous torus $\mathbb{R}/\mathbb{Z}$. We will denote the local spin state by S, we only consider the case when S is finite. The state space of the interacting particle system is

$$\Omega^N := S^{\mathbb{T}^N}.$$

Configurations will be denoted

$$\underline{\omega} := (\omega_j)_{j\in\mathbb{T}^N} \in \Omega^N,$$

For sake of simplicity we consider discrete (integer valued) conserved quantities only. The two conserved quantities are denoted by

$$\xi : S \to \mathbb{Z},$$
$$\eta : S \to \mathbb{Z},$$

we also use the notations $\xi_j = \xi(\omega_j)$, $\eta_j = \eta(\omega_j)$. This means that the sums $\sum_j \xi_j$ and $\sum_j \eta_j$ are conserved by the dynamics. We assume that the conserved quantities are different and non-trivial, i.e. the functions ξ, η and the constant function 1 on S are linearly independent.

2.2 Rate function, infinitesimal generator

We consider the *rate function* $r : S \times S \times S \times S \to \mathbb{R}_+$. The dynamics of the system consists of elementary jumps effecting nearest neighbour spins, $(\omega_j, \omega_{j+1}) \longrightarrow (\omega'_j, \omega'_{j+1})$, performed with rate $r(\omega_j, \omega_{j+1}; \omega'_j, \omega'_{j+1})$.

We require that the rate function r satisfy the following conditions.

(A) If $r(\omega_1, \omega_2; \omega'_1, \omega'_2) > 0$ then

$$\begin{array}{rcl} \xi(\omega_1) + \xi(\omega_2) & = & \xi(\omega'_1) + \xi(\omega'_2), \\ \eta(\omega_1) + \eta(\omega_2) & = & \eta(\omega'_1) + \eta(\omega'_2). \end{array} \tag{1}$$

(B) For every $K \in [\min \xi, \max \xi], L \in [\min \eta, \max \eta]$ the set

$$\Omega^N_{K,L} := \left\{ \underline{\omega} \in \Omega^N : \sum_{j \in \mathbb{T}^N} \xi_j = K, \sum_{j \in \mathbb{T}^N} \eta_j = L \right\}$$

is an irreducible component of Ω^N, i.e. if $\underline{\omega}, \underline{\omega}' \in \Omega^N$ then there exists a series of elementary jumps with positive rates transforming $\underline{\omega}$ into $\underline{\omega}'$.

(C) There exists a probability measure π on S such that for any ω_1, ω_2, $\omega_3 \in S$

$$Q(\omega_1, \omega_2) + Q(\omega_2, \omega_3) + Q(\omega_3, \omega_1) = 0,$$

where

$$Q(\omega_1, \omega_2) := \sum_{\omega'_1, \omega'_2 \in S} \left\{ \frac{\pi(\omega'_1)\pi(\omega'_2)}{\pi(\omega_1)\pi(\omega_2)} r(\omega'_1, \omega'_2; \omega_1, \omega_2) - r(\omega_1, \omega_2; \omega'_1, \omega'_2) \right\}$$

For a precise formulation of the infinitesimal generator on Ω^N we first define the map $\Theta_j^{\omega', \omega''} : \Omega^N \to \Omega^N$ for every $\omega', \omega'' \in S$, $j \in \mathbb{T}^N$:

$$\left(\Theta_j^{\omega', \omega''} \underline{\omega} \right)_i = \begin{cases} \omega' & \text{if} \quad i = j \\ \omega'' & \text{if} \quad i = j+1 \\ \omega_i & \text{if} \quad i \neq j, j+1. \end{cases}$$

The infinitesimal generator of the process defined on Ω^N is

$$L^N f(\underline{\omega}) = \sum_{j \in \mathbb{T}^N} \sum_{\omega', \omega'' \in S} r(\omega_j, \omega_{j+1}; \omega', \omega'') (f(\Theta_j^{\omega', \omega''} \underline{\omega}) - f(\underline{\omega})).$$

We denote by X_t^N the Markov process on the state space Ω^N with infinitesimal generator L^N.

Remarks:

(1) Condition (A) implies that $\sum_j \xi_j$ and $\sum_j \eta_j$ are indeed conserved quantities of the dynamics, while condition (B) ensures that there are no other hidden conservation laws.

(2) Condition (B) is somewhat implicit. It seems to us that it is far not trivial (if not impossible) to formulate explicit conditions involving the rate functions which would be necessary and sufficient for irreducibility. However, in the concrete examples treated in section 4 one can easily check that irreducibility holds.

(3) Condition (C) implies that the stationary measures of the process X_t^N are computable and have the structure required for hydrodynamic behaviour. Actually, it is equivalent to the property that X_t^N has a stationary measure with product structure with identical marginals. See the next subsection for details. Another consequence of this condition is Lemma 1 which turns out to be of crucial importance for hydrodynamic behaviour.

(4) Conditions (A), (B) and (C) determine the measure $\pi(\omega)$ *up to an exponential distortion*, that is the probability measures satisfying these conditions are of the form (2) of the next subsection.

2.3 Stationary measures, reversed process

For every $\theta, \tau \in \mathbb{R}$ let $G(\theta, \tau)$ be the moment generating function defined below:

$$G(\theta, \tau) := \log \sum_{\omega \in S} e^{\theta\xi(\omega)+\tau\eta(\omega)}\pi(\omega).$$

In thermodynamic terms $G(\theta, \tau)$ corresponds to the Gibbs free energy, see [9]. We define the probability measures

$$\pi_{\theta,\tau}(\omega) := \pi(\omega)\exp(\theta\xi(\omega) + \tau\eta(\omega) - G(\theta, \tau)) \tag{2}$$

on S.

Using condition (C), by very similar considerations as in [1], [2], [10] or [14] one can show that for any θ and τ the product measure

$$\pi^N_{\theta,\tau} := \prod_{j\in\mathbb{T}^N} \pi_{\theta,\tau}$$

is stationary for the Markov process on X_t^N on Ω^N with infinitesimal generator L^N. We will refer to these measures as the *canonical* measures. Since $\sum_j \xi_j$ and $\sum_j \eta_j$ are conserved the canonical measures on Ω^N are not ergodic. The conditioned measures defined on $\Omega^N_{K,L}$ by:

$$\pi^N_{K,L}(\underline{\omega}) := \pi^N_{\theta,\tau}\left(\underline{\omega}\,\middle|\,\sum_j \xi_j = K,\ \sum_j \eta_j = L\right) = \frac{\pi^N_{\theta,\tau}(\underline{\omega})1\!\!1\{\underline{\omega}\in\Omega^N_{K,L}\}}{\pi^N_{\theta,\tau}(\Omega^N_{K,L})}$$

are also stationary and due to condition (B) satisfied by the rate functions they are also ergodic. We shall call these measures the *microcanonical measures* of our system. (It is easy to see that the measure $\pi^N_{K,L}$ does not depend on the values of θ, τ.)

The elementary movements of the reversed stationary process are $(\omega_{j-1}, \omega_j) \longrightarrow (\omega'_{j-1}, \omega'_j)$ with rate $r(\omega_j, \omega_{j-1}; \omega'_j, \omega'_{j-1})$. The reversed generator is

$$L^{*N} f(\underline{\omega}) = \sum_{j\in\mathbb{T}^N}\sum_{\omega',\omega''\in S} r(\omega_j, \omega_{j-1}; \omega'', \omega')(f(\Theta^{\omega',\omega''}_{j-1}\underline{\omega}) - f(\underline{\omega})).$$

This is the adjoint of the operator L^N with respect to all microcanonical (and canonical) measures. I.e. the reversed process is the same for any $\pi^N_{\theta,\tau}$ or $\pi^N_{K,L}$.

2.4 Expectations

Expectation, variance, covariance with respect to the measures $\pi^N_{\theta,\tau}$ will be denoted by $\mathbf{E}_{\theta,\tau}(.)$, $\mathbf{Var}_{\theta,\tau}(.)$, $\mathbf{Cov}_{\theta,\tau}(.)$.

We compute the expectations of the conserved quantities with respect to the canonical measures, as functions of the parameters θ and τ:

$$\begin{aligned} u(\theta,\tau) &:= \mathbf{E}_{\theta,\tau}(\xi) = \sum_{\omega\in S}\xi(\omega)\pi_{\theta,\tau}(\omega) = G'_\theta(\theta,\tau), \\ v(\theta,\tau) &:= \mathbf{E}_{\theta,\tau}(\eta) = \sum_{\omega\in S}\eta(\omega)\pi_{\theta,\tau}(\omega) = G'_\tau(\theta,\tau). \end{aligned}$$

Elementary calculations show, that the matrix-valued function

$$\begin{pmatrix} u'_\theta & u'_\tau \\ v'_\theta & v'_\tau \end{pmatrix} = \begin{pmatrix} G''_{\theta\theta} & G''_{\theta\tau} \\ G''_{\theta\tau} & G''_{\tau\tau} \end{pmatrix} =: G''(\theta,\tau)$$

is equal to the covariance matrix $\mathbf{Cov}_{\theta,\tau}(\xi,\eta)$ and therefore it is strictly positive definit. It follows that the function $(\theta,\tau) \mapsto (u(\theta,\tau), v(\theta,\tau))$ is invertible. We denote the inverse function by $(u,v) \mapsto (\theta(u,v), \tau(u,v))$. Actually, denoting by $(u,v) \mapsto S(u,v)$ the convex conjugate (Legendre transform) of the strictly convex function $(\theta,\tau) \mapsto G(\theta,\tau)$:

$$S(u,v) := \sup_{\theta,\tau} \big(u\theta + v\tau - G(\theta,\tau) \big), \tag{3}$$

we have

$$\theta(u,v) = S'_u(u,v), \qquad \tau(u,v) = S'_v(u,v).$$

In probabilistic terms: $S(u,v)$ is the rate function for joint large deviations of $(\sum_j \xi_j, \sum_j \eta_j)$. In thermodynamic terms: $S(u,v)$ corresponds to the equilibrium thermodynamic entropy, see [9]. Let

$$\begin{pmatrix} \theta'_u & \theta'_v \\ \tau'_u & \tau'_v \end{pmatrix} = \begin{pmatrix} S''_{uu} & S''_{uv} \\ S''_{uv} & S''_{vv} \end{pmatrix} =: S''(u,v).$$

It is obvious that the matrices $G''(\theta,\tau)$ and $S''(u,v)$ are strictly positive definit and are inverse of each other:

$$G''(\theta,\tau) S''(u,v) = I, \tag{4}$$

where either $(\theta,\tau) = (u(\theta,\tau), v(\theta,\tau))$ or $(u,v) = (\theta(u,v), \tau(u,v))$. With slight abuse of notation we shall denote: $\pi_{\theta(u,v),\tau(u,v)} =: \pi_{u,v}$, $\pi^N_{\theta(u,v),\tau(u,v)} =: \pi^N_{u,v}$, $\mathbf{E}_{\theta(u,v),\tau(u,v)} =: \mathbf{E}_{u,v}$, etc.

We introduce the flux of the conserved quantities. The infinitesimal generator L^N acts on the conserved quantities as follows:

$$\begin{aligned} L^N \xi_i &= -\phi(\omega_i, \omega_{i+1}) + \phi(\omega_{i-1}, \omega_i) =: -\phi_i + \phi_{i-1}, \\ L^N \eta_i &= -\psi(\omega_i, \omega_{i+1}) + \psi(\omega_{i-1}, \omega_i) =: -\psi_i + \psi_{i-1}, \end{aligned}$$

where

$$\begin{aligned} \phi(\omega_1,\omega_2) &:= \sum_{\omega'_1,\omega'_2 \in S} r(\omega_1,\omega_2;\omega'_1,\omega'_2)(\xi(\omega'_2) - \xi(\omega_2)) \\ &= \sum_{\omega'_1,\omega'_2 \in S} r(\omega_1,\omega_2;\omega'_1,\omega'_2)(\xi(\omega_1) - \xi(\omega'_1)), \\ \psi(\omega_1,\omega_2) &:= \sum_{\omega'_1,\omega'_2 \in S} r(\omega_1,\omega_2;\omega'_1,\omega'_2)(\eta(\omega'_2) - \eta(\omega_2)) \\ &= \sum_{\omega'_1,\omega'_2 \in S} r(\omega_1,\omega_2;\omega'_1,\omega'_2)(\eta(\omega_1) - \eta(\omega'_1)). \end{aligned} \tag{5}$$

We shall denote the expectations of these functions with respect to the canonical measure $\pi^2_{u,v}$ by

$$
\begin{aligned}
\Phi(u,v) &:= \mathbf{E}_{u,v}(\phi) \\
&= \sum_{\substack{\omega_1,\omega_2,\\ \omega_1',\omega_2' \in S}} r(\omega_1,\omega_2;\omega_1',\omega_2')(\xi(\omega_2') - \xi(\omega_2))\pi_{u,v}(\omega_1)\pi_{u,v}(\omega_2), \\
\Psi(u,v) &:= \mathbf{E}_{u,v}(\psi) \\
&= \sum_{\substack{\omega_1,\omega_2,\\ \omega_1',\omega_2' \in S}} r(\omega_1,\omega_2;\omega_1',\omega_2')(\eta(\omega_2') - \eta(\omega_2))\pi_{u,v}(\omega_1)\pi_{u,v}(\omega_2).
\end{aligned}
\tag{6}
$$

The first derivative matrix of the fluxes Φ and Ψ will be denoted

$$
D(u,v) := \begin{pmatrix} \Phi_u' & \Phi_v' \\ \Psi_u' & \Psi_v' \end{pmatrix} \tag{7}
$$

As a general convention, if $\delta : S^m \to \mathbb{R}$ is a local function then its expectation with respect to the canonical measure $\pi^m_{u,v}$ is denoted by

$$
\Delta(u,v) := \mathbf{E}_{u,v}(\delta) = \sum_{\omega_1,\ldots,\omega_m \in S^m} \delta(\omega_1,\ldots,\omega_m)\pi_{u,v}(\omega_1)\cdots\pi_{u,v}(\omega_m).
$$

3 The hydrodynamic limit

We will show, applying Yau's relative entropy method, that under Eulerian scaling the local densities of the conserved quantities $u(t,x)$, $v(t,x)$ evolve according to the following system of partial differential equations:

$$
\left\{ \begin{aligned} \partial_t u + \partial_x \Phi(u,v) &= 0 \\ \partial_t v + \partial_x \Psi(u,v) &= 0. \end{aligned} \right. \tag{8}
$$

It also turns out from our proof (more precisely as a consequence of the Onsager relations proved in Lemma 1) that the *systems of conservation laws* (8) arising as hydrodynamic limits are necessarily *of hyperbolic type* and the equilibrium thermodynamic entropy function $(u,v) \mapsto S(u,v)$ is a (very special) globally *convex Lax entropy* for the system (8). (See [11] or [12] for the pde notions used.) This may be not so surprising, as it is commonly accepted fact and drops out automatically, without any computations in some particular model systems investigated so far. Nevertheless, we have not found a general statement or proof of this fact in the hydrodynamic limit literature.

3.1 Notations

For the proper formulation of our results we need some more notations. Let $u(t,x), v(t,x), t \in [0,T], x \in \mathbb{T}$ be a smooth solution of (8) (more precisely: let it be twice continuously differentiable in both variables). We shall use the notations

$$\begin{aligned} \theta(t,x) &:= \theta(u(t,x), v(t,x)) \\ \tau(t,x) &:= \tau(u(t,x), v(t,x)). \end{aligned}$$

The *true distribution* of the Markov process X_s^N at macroscopic time t, i.e., at microscopic time Nt is

$$\mu_t^N := \mu_0^N \exp\left\{NtL^N\right\}. \tag{9}$$

The true distribution will be compared to the following *time dependent reference measure* (also called local equilibrium) on Ω^N:

$$\nu_t^N := \prod_{j \in \mathbb{T}^N} \pi_{u(t,\frac{j}{N}), v(t,\frac{j}{N})}. \tag{10}$$

This measure is not stationary (unless u and v are constant), and the local densities of the conserved quantities are discrete approximations of the functions $u(t,x), v(t,x)$.

We shall use a stationary measure $\pi^N := \pi_{0,0}^N$ on Ω^N as an *absolute reference measure.* The Radom-Nikodym derivatives of the true distribution and the time dependent reference measure with respect to the absolute reference measure are denoted as follows:

$$\begin{aligned} h_t^N &:= \frac{d\mu_t^N}{d\pi^N}(\underline{\omega}) = \exp\{NtL^{*N}\} h_0^N. \\ f_t^N &:= \frac{d\nu_t^N}{d\pi^N}(\underline{\omega}) \qquad (11) \\ &= \prod_{j \in \mathbb{T}^N} \exp\{\xi(\omega_j)\theta(t,\frac{j}{N}) + \eta(\omega_j)\tau(t,\frac{j}{N}) - G(\theta(t,\frac{j}{N}), \tau(t,\frac{j}{N}))\} \end{aligned}$$

3.2 The main result

Our aim is to prove that if μ_0^N is close to ν_0^N in the sense of relative entropy, then μ_t^N stays close to ν_t^N in the same sense uniformly for $t \in [0,T]$. If we consider two different pairs of smooth solutions $(u_i(t,x), v_i(t,x))$, $i = 1,2$

of (8) it is a simple exercise to show that the relative entropy of the two time dependent reference measures is of order $\asymp N$. This suggests that one should prove

$$H^N(t) := H(\mu_t^N | \nu_t^N) = o(N), \tag{12}$$

uniformly for $t \in [0, T]$.

Theorem. *Consider an interacting particles system model defined as in the previous section which satisfies conditions (A), (B) and (C). Let $\Phi(u, v)$ and $\Psi(u, v)$ be defined as in (6).*
(i) The system of conservation laws (8) is hyperbolic in the domain $(u, v) \in (\min \xi, \max \xi) \times (\min \eta, \max \eta)$. Furthermore, the equilibrium thermodynamic entropy $(u, v) \mapsto S(u, v)$ is a globally convex Lax entropy for the system (8).
(ii) Let $[0, T] \times \mathbb{T} \ni (t, x) \mapsto (u(t, x), v(t, x))$ be a smooth solution of (8), and let μ_t^N and ν_t^N be the measures defined in (9), respectively, (10). Then, if

$$H(\mu_0^N | \pi^N) = \mathcal{O}(N),$$

and (12) holds for $t = 0$ then it will hold uniformly for $t \in [0, T]$.

Remark: Part (i) of the Theorem is commonly accepted fact. In the context of Hamiltonian systems with weak random noise a similar result was established in [7] (see the end of section 3 of that paper). However, we do not know about any explicit formulation (or proof) of the *general fact* stated here.

From part (ii) of the Theorem, by applying the entropy inequality in a standard way (comparing the true measure μ_t^N with the local equilibrium reference measure ν_t^N) one gets the following corollary:

Corollary. *Under the conditions of the Theorem, for any $t \in [0, T]$, the following limits hold as $N \to \infty$:*
(i) For any smooth test function $g : \mathbb{T} \to \mathbb{R}$

$$\frac{1}{N} \sum_{j \in \mathbb{T}^N} g(j/N) \xi_j(t) \xrightarrow{\mathbf{P}} \int_{\mathbb{T}} g(x) u(t, x)\, dx,$$
$$\frac{1}{N} \sum_{j \in \mathbb{T}^N} g(j/N) \eta_j(t) \xrightarrow{\mathbf{P}} \int_{\mathbb{T}} g(x) v(t, x)\, dx.$$

(ii) The asymptotics of the relative entropy of the true distribution μ_t^N *with respect to the absolute reference measure* π_{u_0,v_0}^N *is*

$$N^{-1}H(\mu_t^N \mid \pi_{u_0,v_0}^N) \to \int_{\mathbb{T}} \big(S(u(t,x),v(t,x)) - S(u_0,v_0)\big)\,dx, \tag{13}$$

where $S(u,v)$ *is the thermodynamic entropy defined in (3).*

Remark: (1) Note that since $S(u,v)$ is Lax entropy of the pde (8) the right hand side of (13) does not change in time as long as the solution $(u(t,x),v(t,x))$ of (8) is smooth, and starts to decrease when the first shock appears. This means that the relative entropy $H(\mu_t^N \mid \pi_{u_0,v_0}^N)$ decreases by $o(N)$ before the appearance of the first shock in the system.
(2) A result formally similar to (13) *for one component systems* (in particular: asymmetric simple exclusion process) was established in [5] for all times — even after the appearence of discontinuities (shocks) in the solution of the hydrosdynamic pde. In this paper it is also proved that for a.s.e.p. relative entropy with respect to the local equilibrium measure ν_t^N remains $o(N)$ even after appearence of the shocks.

4 Examples: deposition models

If we fix the size of the spin space, then we have finitely many equations from the conditions on the rate function, thus we can get a finite-parameter family of models. The smallest value of $|S|$, for which there exists a proper model is 3, since we need to have two different non-trivial conserved quantities. We present two concrete examples: one with $|S| = 3$, one with $|S| = 4$. A third example with $|S| = \infty$, to which the Theorem applies with some modifications, is described in [13].

Our concrete examples are *deposition models.* $\eta : S \to \mathbb{N}$, and $\xi : S \to \mathbb{Z}$. η_j, respectively, ξ_j are interpreted as particle occupation number, respectively, (negative) discrete gradient of deposition height. The dynamical driving mechanism is such that
(i) The deposition height growth is influenced by the local particle density. Typically: growth is enhanced by higher particle densities.
(ii) The particle motion is itself influenced by the deposition profile. Typically: particles are pushed in the direction of the negative gradient of the deposition height.

It is natural to assume left-right symmetry of the models. This is realized in the following way. There is an involution

$$R : S \to S, \qquad R \circ R = Id$$

which acts on the conserved quantities and the jump rates as follows:

$$\begin{aligned} \eta(R\omega) = \eta(\omega), \qquad \xi(R\omega) = -\xi(\omega), \\ r(R\omega_2, R\omega_1; R\omega_2', R\omega_1') = r(\omega_1, \omega_2; \omega_1', \omega_2'). \end{aligned} \tag{14}$$

Correspondingly on the macroscopic level we shall use the traditional notation $\rho(t,x)$ (instead of $v(t,x)$ of the general formulation) and $u(t,x)$. The limiting partial differential equations will be also invariant under the left-right reflection symmetry: $\big(\rho(t,x), u(t,x)\big) \mapsto \big(\rho(t,-x), -u(t,-x)\big)$.

4.1 A model with $|S| = 3$

The state space is $S = \{-1, 0, 1\}$. The left-right reflection symmetry is implemented by $R : S \to S$, $R\omega = -\omega$. The two conserved quantities are $\xi(\omega) = \omega$ (the spin itself) and $\eta(\omega) = 1 - |\omega|$ (the number of zeros). (It is easy to see that up to linear combinations these are the only two conserved quantities we can define on S.) From condition (A) it follows that $r(\omega_1, \omega_2; \omega_1', \omega_2') > 0$ only if $\omega_1' = \omega_2$ and $\omega_2' = \omega_1$. I.e. the dynamics consists of exchanges of nearest neighbour spins. It follows that, if condition (C) is satisfied with a probability measure π on S then

$$\begin{aligned} &r(1,-1;-1,1) - r(-1,1;1,-1) \\ &\qquad = r(1,0;0,1) - r(0,1;1,0) + r(0,-1;-1,0) - r(-1,0;0,-1) \end{aligned}$$

holds. In that case condition (C) is satisfied with any probability measure π on S. Our natural parametrization is

$$\pi_{\rho,u}(0) = \rho, \quad \pi_{\rho,u}(\pm 1) = \frac{1 - \rho \pm u}{2},$$

with the parameter range $\{(\rho, u) : \rho \in [0,1],\ u \in [-1,1],\ \rho + |u| \le 1\}$.

The reflection symmetry condition (14) reads

$$r(1,0;0,1) = r(0,-1;-1,0), \qquad r(0,1;1,0) = r(-1,0;0,-1).$$

These conditions leave us with

$$\begin{array}{rclcrcl} r(1,-1;-1,1) & = & a, & \qquad & r(-1,1;1,-1) & = & 2c+a, \\ r(0,-1;-1,0) & = & b, & & r(-1,0;0,-1) & = & c+b, \\ r(1,0;0,1) & = & b, & & r(0,1;1,0) & = & c+b, \end{array}$$

where $a, b \geq 0$ and $c \geq \max\{-b, -a/2\}$ are free parameters. Without loss of generality we may choose $c \geq 0$ (otherwise, rename $\tilde{\omega} := -\omega$). It is easy to check that condition (B) is satisfied if and only if $(a+2c)(b+c) > 0$. We are not interested in the $c = 0$ case, since that defines the reversible process which would imply diffusive rather than hyperbolic (Eulerian) scaling. By fixing an appropriate time scale we choose $c = 1$. It is easy to compute the microscopic fluxes ϕ_j and ψ_j given by formula (5):

$$\begin{aligned} \phi_j &= \frac{1}{2}(\omega_j - \omega_{j+1})\left((\omega_j - 1)(1+\omega_{j+1}) - 2a\omega_j\omega_{j+1} + 2b\,(1+\omega_j\omega_{j+1})\right), \\ \psi_j &= b(\omega_{j+1}^2 - \omega_j^2) + \frac{1}{2}(1-\omega_j)(1+\omega_{j+1})(\omega_j + \omega_{j+1}) \end{aligned}$$

The macroscopic fluxes are computed with formula (6). Inserted in (8) this leads to the hydrodynamic equation:

$$\left\{ \begin{array}{rcl} \partial_t \rho + \partial_x(\rho u) & = & 0 \\ \partial_t u + \partial_x(\rho + u^2) & = & 0. \end{array} \right. \tag{15}$$

This system is known in the pde community as *Leroux's equation.* The system has some very special features: it belongs to the so-called Temple class and it was much investigated. For details see [11]. Validity of this pde in the hydrodynamic limit *up to the occurrence of shocks* follows from our general Theorem.

Remark: It is an easy exercise to see that all models with $|S| = 3$ satisfying the general conditions (A, B, C), without the extra assumption of left-right reflection symmetry, are essentially equivalent, in the sense that in the hydrodynamic limit they lead to pde-s which can be transformed to (15) by linear combinations of the functions involved.

4.2 A finite bricklayer model

In the following example we give a finite version of the infinite bricklayers model introduced in [13]. Let $S = \{0,1\} \times \{-1,1\}$. The elements of S will be denoted $\omega =: (n, z)$. Left-right reflection symmetry is implemented as

$R : S \to S$, $R(n, z) = (n, -z)$. The conserved quantities are $\xi(\omega) = z$ and $\eta(\omega) = n$. Condition (A) leaves twenty (possibly) non-zero rates. Due to the left-right reflection symmetry conditions eight pairs of rates are equal. Using the notation

$$r(\omega_1, \omega_2; \omega_1', \omega_2') = r\left(\genfrac{}{}{0pt}{}{n_1}{z_1}, \genfrac{}{}{0pt}{}{n_2}{z_2}; \genfrac{}{}{0pt}{}{n_1'}{z_1'}, \genfrac{}{}{0pt}{}{n_2'}{z_2'}\right)$$

in the following table we list the (possibly) non-zero rates, parameterized by twelve nonnegative parameters.

$$\begin{array}{ll}
r\left(\genfrac{}{}{0pt}{}{0}{-}, \genfrac{}{}{0pt}{}{0}{+}; \genfrac{}{}{0pt}{}{0}{+}, \genfrac{}{}{0pt}{}{0}{-}\right) = a, & r\left(\genfrac{}{}{0pt}{}{0}{+}, \genfrac{}{}{0pt}{}{0}{-}; \genfrac{}{}{0pt}{}{0}{-}, \genfrac{}{}{0pt}{}{0}{+}\right) = b, \\
r\left(\genfrac{}{}{0pt}{}{1}{-}, \genfrac{}{}{0pt}{}{1}{+}; \genfrac{}{}{0pt}{}{1}{+}, \genfrac{}{}{0pt}{}{1}{-}\right) = c, & r\left(\genfrac{}{}{0pt}{}{1}{+}, \genfrac{}{}{0pt}{}{1}{-}; \genfrac{}{}{0pt}{}{1}{-}, \genfrac{}{}{0pt}{}{1}{+}\right) = d, \\
r\left(\genfrac{}{}{0pt}{}{0}{-}, \genfrac{}{}{0pt}{}{1}{+}; \genfrac{}{}{0pt}{}{0}{+}, \genfrac{}{}{0pt}{}{1}{-}\right) = e, & r\left(\genfrac{}{}{0pt}{}{1}{-}, \genfrac{}{}{0pt}{}{0}{+}; \genfrac{}{}{0pt}{}{1}{+}, \genfrac{}{}{0pt}{}{0}{-}\right) = e, \\
r\left(\genfrac{}{}{0pt}{}{0}{+}, \genfrac{}{}{0pt}{}{1}{-}; \genfrac{}{}{0pt}{}{0}{-}, \genfrac{}{}{0pt}{}{1}{+}\right) = f, & r\left(\genfrac{}{}{0pt}{}{1}{+}, \genfrac{}{}{0pt}{}{0}{-}; \genfrac{}{}{0pt}{}{1}{-}, \genfrac{}{}{0pt}{}{0}{+}\right) = f, \\
r\left(\genfrac{}{}{0pt}{}{0}{-}, \genfrac{}{}{0pt}{}{1}{-}; \genfrac{}{}{0pt}{}{1}{-}, \genfrac{}{}{0pt}{}{0}{-}\right) = p, & r\left(\genfrac{}{}{0pt}{}{1}{+}, \genfrac{}{}{0pt}{}{0}{+}; \genfrac{}{}{0pt}{}{0}{+}, \genfrac{}{}{0pt}{}{1}{+}\right) = p, \\
r\left(\genfrac{}{}{0pt}{}{0}{+}, \genfrac{}{}{0pt}{}{1}{+}; \genfrac{}{}{0pt}{}{1}{+}, \genfrac{}{}{0pt}{}{0}{+}\right) = q, & r\left(\genfrac{}{}{0pt}{}{1}{-}, \genfrac{}{}{0pt}{}{0}{-}; \genfrac{}{}{0pt}{}{0}{-}, \genfrac{}{}{0pt}{}{1}{-}\right) = q, \\
r\left(\genfrac{}{}{0pt}{}{0}{+}, \genfrac{}{}{0pt}{}{1}{-}; \genfrac{}{}{0pt}{}{1}{+}, \genfrac{}{}{0pt}{}{0}{-}\right) = r, & r\left(\genfrac{}{}{0pt}{}{1}{+}, \genfrac{}{}{0pt}{}{0}{-}; \genfrac{}{}{0pt}{}{0}{+}, \genfrac{}{}{0pt}{}{1}{-}\right) = r, \\
r\left(\genfrac{}{}{0pt}{}{0}{-}, \genfrac{}{}{0pt}{}{1}{+}; \genfrac{}{}{0pt}{}{1}{-}, \genfrac{}{}{0pt}{}{0}{+}\right) = s, & r\left(\genfrac{}{}{0pt}{}{1}{-}, \genfrac{}{}{0pt}{}{0}{+}; \genfrac{}{}{0pt}{}{0}{-}, \genfrac{}{}{0pt}{}{1}{+}\right) = s, \\
r\left(\genfrac{}{}{0pt}{}{0}{-}, \genfrac{}{}{0pt}{}{1}{+}; \genfrac{}{}{0pt}{}{1}{+}, \genfrac{}{}{0pt}{}{0}{-}\right) = x, & r\left(\genfrac{}{}{0pt}{}{1}{-}, \genfrac{}{}{0pt}{}{0}{+}; \genfrac{}{}{0pt}{}{0}{+}, \genfrac{}{}{0pt}{}{1}{-}\right) = x, \\
r\left(\genfrac{}{}{0pt}{}{0}{+}, \genfrac{}{}{0pt}{}{1}{-}; \genfrac{}{}{0pt}{}{1}{-}, \genfrac{}{}{0pt}{}{0}{+}\right) = y, & r\left(\genfrac{}{}{0pt}{}{1}{+}, \genfrac{}{}{0pt}{}{0}{-}; \genfrac{}{}{0pt}{}{0}{-}, \genfrac{}{}{0pt}{}{1}{+}\right) = y.
\end{array}$$

All the other jump rates are zero.

It can be shown that condition (C) is satisfied with the measures $\pi_{\rho,u}$

$$\pi_{\rho,u}(n, z) = \big(n\rho + (1-n)(1-\rho)\big)\frac{1+zu}{2}, \quad n = 0, 1, \quad z = +, -, \tag{16}$$

with the parameters $\rho \in (0, 1)$, $u \in (-1, +1)$, if

$$\begin{aligned} c + f + p + y &= d + e + q + x \\ a + f + q + y &= b + e + p + x. \end{aligned} \tag{17}$$

So, we we are left with a nine-parameter family of models. Given the formulas (5) we compute the fluxes ϕ_j and ψ_j. Using the condition (17) eventually we get

$$
\begin{aligned}
2\phi_j \;=\;\; & (b-a)+(p-q)(n_j+n_{j+1})-(b+a)(z_{j+1}-z_j) \\
& +(a+b-e-f-x-y)(n_j+n_{j+1})(z_{j+1}-z_j)+(a-b)z_{j+1}z_j \\
& -(a+b+c+d-2e-2f-2x-2y)n_jn_{j+1}(z_{j+1}-z_j) \\
& -(p-q)(n_j+n_{j+1})z_{j+1}z_j \\
4\psi_j \;=\;\; & -(p+q+r+s+x+y)(n_{j+1}-n_j) \\
& +(p-q)(n_j+n_{j+1})(z_j+z_{j+1})+(y-x)(n_{j+1}-n_j)(z_{j+1}-z_j) \\
& -2(p-q)n_jn_{j+1}(z_j+z_{j+1}) \\
& -(p+q-r-s-x-y)(n_{j+1}-n_j)z_{j+1}z_j
\end{aligned}
$$

The macroscopic fluxes are again explicitly computable. From (6) and (16) we get

$$
\begin{aligned}
\Phi(\rho,u) &= \big((p-q)\rho-(a-b)/2\big)\big(1-u^2\big), \\
\Psi(\rho,u) &= (p-q)\rho(1-\rho)u.
\end{aligned}
$$

Without loss of generality we may assume $p-q\geq 0$ (otherwise rename the microscopic variables $\tilde{n}_j := n_j$, $\tilde{z}_j := -z_j$). Further on, $p=q$ leads to diffusive rather then hyperbolic (Eulerian) scaling of the particle density, so we are interested in the $p>q$ cases only. By setting the appropriate time scale we can choose $p-q=1$ and denote $\gamma := \frac{a-b}{2(p-q)}$. So, eventually we get the system of pdes

$$
\left\{
\begin{array}{l}
\partial_t\rho+\partial_x\big(\rho(1-\rho)u\big)=0 \\
\partial_t u+\partial_x\big((\rho-\gamma)(1-u^2)\big)=0
\end{array}
\right.
\tag{18}
$$

In [8] another family of four-state models with two conserved quantities, the so-called *two channel traffic models* are analyzed. These models also satisfy conditions (A), (B) and (C). As a consequence our general Theorem is applicable to the two channel traffic models, too.

About the relation of our four state deposition models (treated in this subsection) and the two channel traffic models treated in [8]: Due to the different symmetry conditions imposed — we impose the left-right reflection symmetry described in the first paragraph of this section, while in [8] symmetry between the two traffic channels is imposed — the two families of models do not intersect. The one parameter family of partial differential equations derived in [8] essentially differs form our partial differential equations (18). (Actually there is no parameter value for which the two pde-s are equivalent.) However, the two families of models show many similarities and do have common generalizations.

5 Sketch of proof

The present section is divided into four subsections. In subsection 5.1 we present the first steps of the 'relative entropy method' applied. As there is no real novelty in this part, we only list the main steps *without the computational details* which are essentially the same as in the original work [15] of Yau or in Chapter 6. of [4] or in [3].

It turns out that for a general two (or more) component system some identity relating the macroscopic fluxes Φ and Ψ is essentially needed for completing the proof. These relations are reminiscent of Onsager's reciprocity relations of nonequilibrium thermodynamics, see e.g. Chapter 10.D of [9]. However an essential difference is worth noting: while the traditional Onsager relations are derived under the condition of reversibility of the microscopic dynamics, in our case conditions (A) and (C) are involved which do not imply reversibility by any means. Seemingly, these relations were not explicitly noted so far in the context of mathematically rigorous hydrodynamic limits. This omission is probably explained by the fact that in the concrete models investigated so far these identities just droped out without any computations.

In subsection 5.2 we prove that under the conditions (A) and (C) these identities hold in general. We shall refer to these identities as *Onsager relations*. It also follows from these identities that the systems of partial differential equations arising as hydrodynamic limits under Eulerian scaling are indeed of hyperbolic type and the thermodynamic equilibrium entropy $S(u,v)$ is globally convex Lax entropy of the hydrodynamic equations, as it is commonly assumed. In subsection 5.3 we formulate the consequences of the Onsager relations which are crucial for the further steps of the proof of the hydrodynamic limit.

Finally, in subsection 5.4 we sketch the last steps of the proof. Here again we follow the standard steps of the relative entropy method, so we omit all computational details, referring only to the main stations of the proof. For the computational details of subsections 5.1 and 5.4 we refer the reader to Chapter 6. of [4] or to [3]. However, we warn the reader that the omitted details (in particular the last two steps: the control of the block replacement and the one-block estimate) are rather sophisticated and mathematically deep.

5.1 First transformations

In order to obtain the main estimate (12) our aim is to get a Grönwall type inequality: we will prove that for every $t \in [0,T]$

$$H^N(t) - H^N(0) \leq C \int_0^t H^N(s)ds + o(N), \tag{19}$$

where the error term is uniform in $t \in [0,T]$. Because it is assumed that $H^N(0) = o(N)$, the Theorem follows.

For proving (19) we try to bound (from above) $\partial_t H^N(s)$ by const $\cdot$ $H^N(s) + o(N)$, uniformly for $s \in [0,T]$. We start form the inequality (20) which is derived in [4] under very general conditions, valid in our case.

$$\partial_t H^N(t) \leq N \int_{\Omega^N} \frac{L^{*N} f_t^N}{f_t^N} d\mu_t^N - \int_{\Omega^N} \frac{\partial_t f_t^N}{f_t^N} d\mu_t^N. \tag{20}$$

Next we transform the two terms appearing on the right hand side of (20). Equation (21) follows form the smoothness of the functions $\theta(t,x)$ and $\tau(t,x)$ and from the entropy inequality applied to the measures μ_t^N compared with the absolute reference measure π^N.

$$\begin{aligned} N \int_{\Omega^N} \frac{L^{*N} f_t^N}{f_t^N} d\mu_t^N &= -\sum_{j\in\mathbb{T}^N} \partial_x \theta(t,j/N) \int_{\Omega^N} \phi_j d\mu_t^N \\ &\quad - \sum_{j\in\mathbb{T}^N} \partial_x \tau(t,j/N) \int_{\Omega^N} \psi_j d\mu_t^N \\ &\quad + \mathcal{O}(1) \end{aligned} \tag{21}$$

Equation (22) follows from direct computation of the time derivative of the function f_t^N.

$$\begin{aligned} \int_{\Omega^N} \frac{\partial_t f_t^N}{f_t^N} d\mu_t^N &= \sum_{j\in\mathbb{T}^N} \partial_t \theta(t,j/N) \int_{\Omega^N} (\xi_j - u(t,j/N)) d\mu_t^N \\ &\quad + \sum_{j\in\mathbb{T}^N} \partial_t \tau(t,j/N) \int_{\Omega^N} (\eta_j - v(t,j/N)) d\mu_t^N \end{aligned} \tag{22}$$

Next we replace the local variables ϕ_j and ψ_j in (21), respectively, ξ_j and η_j in (22) by their block averages defined as follows: if δ_j is a local microscopic variable its block average is defined as

$$\delta_j^l := \frac{\delta_j + \cdots + \delta_{j+l-1}}{l}.$$

In the following two block-replacements we use again the smoothness of the functions $\theta(t,x)$ and $\tau(t,x)$ and the entropy inequality applied to the measures μ_t^N compared with the absolute reference measure π^N.

$$\begin{aligned} N\int_{\Omega^N} \frac{L^{*N} f_t^N}{f_t^N} d\mu_t^N &= -\sum_{j\in\mathbb{T}^N} \partial_x\theta(t,j/n)\int_{\Omega^N} \phi_j^l d\mu_t^N \\ &\quad -\sum_{j\in\mathbb{T}^N} \partial_x\tau(t,j/n)\int_{\Omega^N} \psi_j^l d\mu_t^N \\ &\quad +\mathcal{O}\left(l\right) \end{aligned} \tag{23}$$

$$\begin{aligned} \int_{\Omega^N} \frac{\partial_t f_t^N}{f_t^N} d\mu_t^N &= \sum_{j\in\mathbb{T}^N} \partial_t\theta(t,j/N)\int_{\Omega^N} (\xi_j^l - u(t,j/N)) d\mu_t^N \\ &\quad +\sum_{j\in\mathbb{T}^N} \partial_t\tau(t,j/N)\int_{\Omega^N} (\eta_j^l - v(t,j/N)) d\mu_t^N \\ &\quad +\mathcal{O}\left(l\right) \end{aligned} \tag{24}$$

The last transformation of this first, preparatory part is replacing in (23) the block averages ϕ_j^l, respectively, ψ_j^l by their equilibrium averages computed at the empirical densities, $\Phi(\xi_j^l,\eta_j^l)$, respectively, $\Psi(\xi_j^l,\eta_j^l)$. The error terms appearing in the third and fourth lines of the right hand side of (25) are the most important error terms to be controlled by the so called *one block estimate* towards the end of the proof.

$$\begin{aligned} N\int_{\Omega^N} \frac{L^{*N} f_t^N}{f_t^N} d\mu_t^N &= -\sum_{j\in\mathbb{T}^N} \partial_x\theta(t,j/n)\int_{\Omega^N} \Phi(\xi_j^l,\eta_j^l) d\mu_t^N \\ &\quad -\sum_{j\in\mathbb{T}^N} \partial_x\tau(t,j/n)\int_{\Omega^N} \Psi(\xi_j^l,\eta_j^l) d\mu_t^N \\ &\quad -\sum_{j\in\mathbb{T}^N} \partial_x\theta(t,j/n)\int_{\Omega^N} \left(\phi_j^l - \Phi(\xi_j^l,\eta_j^l)\right) d\mu_t^N \\ &\quad -\sum_{j\in\mathbb{T}^N} \partial_x\tau(t,j/n)\int_{\Omega^N} \left(\psi_j^l - \Psi(\xi_j^l,\eta_j^l)\right) d\mu_t^N \\ &\quad +\mathcal{O}\left(l\right) \end{aligned} \tag{25}$$

Before going on with the standard steps of the relative entropy proof we need to make a detour.

5.2 An Onsager type identity

Lemma 1. *Suppose we have a particle system with two conserved quantities and rates satisfying conditions (A) and (C). Then there exists a potential function* $(\theta, \tau) \mapsto U(\theta, \tau)$ *such that*

$$\begin{aligned} \Phi(\theta, \tau) &:= \Phi(u(\theta, \tau), v(\theta, \tau)) = U'_\theta, \\ \Psi(\theta, \tau) &:= \Psi(u(\theta, \tau), v(\theta, \tau)) = U'_\tau, \end{aligned} \tag{26}$$

or, equivalently

$$\Phi'_\tau = \Psi'_\theta. \tag{27}$$

Proof. We prove (27). Throughout the forthcoming proof we adopt the notation $\xi_j := \xi(\omega_j)$, $\xi'_j := \xi(\omega'_j)$, etc.

From the definitions

$$\pi_{\theta,\tau}(\omega_1)\pi_{\theta,\tau}(\omega_2) = \exp\{\theta(\xi_1 + \xi_2) + \tau(\eta_1 + \eta_2) - 2G(\theta, \tau)\}\pi(\omega_1)\pi(\omega_2),$$

and

$$\begin{aligned} &\big(\pi_{\theta,\tau}(\omega_1)\pi_{\theta,\tau}(\omega_2)\big)'_\theta = \\ &\qquad \pi(\omega_1)\pi(\omega_2)e^{\theta(\xi_1+\xi_2)+\tau(\eta_1+\eta_2)-2G(\theta,\tau)}\left\{(\xi_1 + \xi_2) - 2u(\theta, \tau)\right\} = \\ &\qquad \frac{\pi(\omega_1)\pi(\omega_2)}{Z(\theta, \tau)^3} \sum_{\omega_3 \in S} \pi(\omega_3)(\xi_1 + \xi_2 - 2\xi_3)e^{\theta(\xi_1+\xi_2+\xi_3)+\tau(\eta_1+\eta_2+\eta_3)}, \end{aligned}$$

where $Z(\theta, \tau) = \exp\{G(\theta, \tau)\}$. Similarly,

$$\begin{aligned} &\big(\pi_{\theta,\tau}(\omega_1)\pi_{\theta,\tau}(\omega_2)\big)'_\tau = \\ &\qquad \frac{\pi(\omega_1)\pi(\omega_2)}{Z(\theta, \tau)^3} \sum_{\omega_3 \in S} \pi(\omega_3)(\eta_1 + \eta_2 - 2\eta_3)e^{\theta(\xi_1+\xi_2+\xi_3)+\tau(\eta_1+\eta_2+\eta_3)}. \end{aligned}$$

Hence

$$\begin{aligned}
\Phi'_\tau(\theta,\tau) &= \frac{1}{Z(\theta,\tau)^3}\sum_{\omega_1,\omega_2,\omega_3\in S}\pi(\omega_1)\pi(\omega_2)\pi(\omega_3)e^{\theta(\xi_1+\xi_2+\xi_3)+\tau(\eta_1+\eta_2+\eta_3)}\\
&\qquad\times\ (\eta_1+\eta_2-2\eta_3)\sum_{\omega'_1,\omega'_2\in S} r(\omega_1,\omega_2,\omega'_1,\omega'_2)(\xi'_2-\xi_2),\\
\Psi'_\theta(\theta,\tau) &= \frac{1}{Z(\theta,\tau)^3}\sum_{\omega_1,\omega_2,\omega_3\in S}\pi(\omega_1)\pi(\omega_2)\pi(\omega_3)e^{\theta(\xi_1+\xi_2+\xi_3)+\tau(\eta_1+\eta_2+\eta_3)}\\
&\qquad\times\ (\xi_1+\xi_2-2\xi_3)\sum_{\omega'_1,\omega'_2\in S} r(\omega_1,\omega_2,\omega'_1,\omega'_2)(\eta'_2-\eta_2),
\end{aligned}$$

For the proof of the lemma it is enough to prove for any $K\in[3\min\xi,3\max\xi]$ and $L\in[3\min\eta,3\max\eta]$ the following expression equals 0:

$$\sum_{\substack{\omega_1,\omega_2,\omega_3,\omega'_1,\omega'_2\in S:\\ \xi_1+\xi_2+\xi_3=K\\ \eta_1+\eta_2+\eta_3=L}}\pi(\omega_1)\pi(\omega_2)\pi(\omega_3)r(\omega_1,\omega_2;\omega'_1,\omega'_2)\times\big((\eta_1+\eta_2-2\eta_3)(\xi'_2-\xi_2)-(\xi_1+\xi_2-2\xi_3)(\eta'_2-\eta_2)\big) \tag{28}$$

From condition (A) imposed on the rate functions it follows that in all nonzero terms of the above sum one can replace $\eta_1+\eta_2$ by $\eta'_1+\eta'_2$ and $\eta'_2-\eta_2$ by $\eta_1-\eta'_1$, and similarly for the ξ-s. After straightforward computations (28) becomes

$$\sum_{\substack{\omega_1,\omega_2,\omega_3,\omega'_1,\omega'_2\in S:\\ \xi_1+\xi_2+\xi_3=K\\ \eta_1+\eta_2+\eta_3=L}}\pi(\omega_1)\pi(\omega_2)\pi(\omega_3)r(\omega_1,\omega_2;\omega'_1,\omega'_2)\times\big(\Delta(\omega_1,\omega_2,\omega_3)-\Delta(\omega'_1,\omega'_2,\omega_3)\big) \tag{29}$$

where $\Delta : S\times S\times S\to Z$ is defined as follows

$$\Delta(\omega_1,\omega_2,\omega_3):=\xi_1(\eta_2-\eta_3)+\xi_2(\eta_3-\eta_1)+\xi_3(\eta_1-\eta_2).$$

Note that Δ is *antisymmetric* regarding permutation of its variables. Next, after rearranging the sum, from the definition of the function Q in condition (C) expression (29) becomes

$$-\sum_{\substack{\omega_1,\omega_2,\omega_3\in S:\\ \xi_1+\xi_2+\xi_3=K\\ \eta_1+\eta_2+\eta_3=L}}\pi(\omega_3)\pi(\omega_1)\pi(\omega_2)Q(\omega_1,\omega_2)\Delta(\omega_1,\omega_2,\omega_3).$$

Finally, from the antisymmetry of the function Δ and condition (C) imposed on the function Q it follows indeed that this last sum equals zero. $\square$

5.3 Consequences of the Onsager relations

Relation (27) is the same as saying that the matrix $D(u(\theta,\tau),v(\theta,\tau))\cdot G''(\theta,\tau)$ is symmetric. Using (4) this also reads as

$$S''(u,v)\cdot D(u,v)=\big(S''(u,v)\cdot D(u,v)\big)^{\dagger}. \tag{30}$$

This relation implies that only *hyperbolic* two-by-two systems of conservation laws (8) can arise as hydrodynamic limits. Indeed, as the following elementary argument shows relation (30) can hold with a positive definite matrix S'' only if $D(u,v)$ can be diagonalized (in the real sense), which is exactly the condition of hyperbolicity of the system (8). Indeed, since S'' is positive definite, we can write

$$D=(S'')^{-1/2}\left((S'')^{-1/2}(S''D)(S'')^{-1/2}\right)(S'')^{1/2}, \tag{31}$$

which means that D is similar to the real symmetric matrix $(S'')^{-1/2}(S''D)(S'')^{-1/2}$, and from this the (real) diagonalizability of D follows. Furthermore, (30) is spelled out as

$$S''_{uu}\Phi'_v+S''_{uv}\Psi'_v=S''_{vu}\Phi'_u+S''_{vv}\Psi'_u, \tag{32}$$

which is readily recognized as the the partial differential equation defining the *Lax entropies* of the system (8). The function $F(u,v):=U(\theta(u,v),\tau(u,v))$ is the corresponding (macroscopic) entropy-flux. See [11] or [12] for the pde notions involved. Thus, part (i) of the Theorem is proved.

Now we turn to two further consequences of Lemma 1 which turn out to be of crucial importance in the hydrodynamic behavior.

First, the time derivatives of θ and τ are expressed. From the pde (8) it follows that

$$\partial_t\theta=-\theta'_u\Phi'_u\partial_x u-\theta'_u\Phi'_v\partial_x v-\theta'_v\Psi'_u\partial_x u-\theta'_v\Psi'_v\partial_x v.$$

Using the identity (32) we replace

$$\theta'_u\Phi'_v=\theta'_v\Phi'_u+\tau'_v\Psi'_u-\tau'_u\Psi'_v$$

in the second term of the right hand side. Using also the straightforward identities $u'_\tau= v'_\theta$ and $\theta'_v=\tau'_u$ (see subsection 2.4) we finally get

$$\partial_t\theta=\Phi'_u\partial_x\theta+\Psi'_u\partial_x\tau, \tag{33}$$

and by identical considerations

$$\partial_t \tau = \Phi'_v \partial_x \theta + \Psi'_v \partial_x \tau. \tag{34}$$

Second, due to identity (26),

$$\sum_{j\in\mathbb{T}^N} \Big(\partial_x \theta(j/N) \Phi(u(j/N), v(j/N)) + \partial_x \tau(j/N) \Psi(u(j/N), v(j/N)) \Big)$$
$$= \sum_{j\in\mathbb{T}^N} \partial_x U(u(j/N), v(j/N)) = \mathcal{O}(1). \tag{35}$$

5.4 End of proof

Now we return to proving (19). Denote

$$\mathcal{D}\Phi(u,v;\tilde{u},\tilde{v}) := \Phi(\tilde{u},\tilde{v}) - \Phi(u,v) - \Phi'_u(u,v)(\tilde{u}-u) - \Phi'_v(u,v)(\tilde{v}-v)$$

and similarly for $\mathcal{D}\Psi(u,v;\tilde{u},\tilde{v})$. Applying (33), (34) and (35), from (24) and (25) we obtain

$$\begin{aligned}
\int_{\Omega^N} \frac{\partial_t f_t^N - N L^{*N} f_t^N}{f_t^N} d\mu_t^N = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (36)\\
& \sum_{j\in\mathbb{T}^N} \partial_x \theta(t, j/n) \int_{\Omega^N} \mathcal{D}\Phi(u(t,j/N), v(t,j/N); \xi_j^l, \eta_j^l) d\mu_t^N \\
& + \sum_{j\in\mathbb{T}^N} \partial_x \tau(t, j/n) \int_{\Omega^N} \mathcal{D}\Psi(u(t,j/N), v(t,j/N); \xi_j^l, \eta_j^l) d\mu_t^N \\
& + \sum_{j\in\mathbb{T}^N} \partial_x \theta(t, j/n) \int_{\Omega^N} \Big(\phi_j^l - \Phi(\xi_j^l, \eta_j^l) \Big) d\mu_t^N \\
& + \sum_{j\in\mathbb{T}^N} \partial_x \tau(t, j/n) \int_{\Omega^N} \Big(\psi_j^l - \Psi(\xi_j^l, \eta_j^l) \Big) d\mu_t^N \\
& + \mathcal{O}(l)
\end{aligned}$$

The first two terms on the right hand side of (36) are estimated by the entropy inequality, comparing the measure μ_t^N with the *local equilibrium measure* ν_t^N:

$$\begin{aligned}
\sum_{j\in\mathbb{T}^N} \int_{\Omega^N} \Big(\Big| \mathcal{D}\Phi(u(t,j/N), v(t,j/N); \xi_j^l, \eta_j^l) \Big| & \qquad\qquad\qquad\qquad\qquad (37)\\
+ \Big| \mathcal{D}\Psi(u(t,j/N), v(t,j/N); \xi_j^l, \eta_j^l) \Big| \Big) d\mu_t^N & \\
\le C H^N(t) + \mathcal{O}(N l^{-1}). &
\end{aligned}$$

The last two terms in (36) are estimated only *integrated against time.* Applying the so-called one block estimate (see e.g. Chapter 5 of [4]) one gets

$$\begin{aligned}
&\lim_{l\to\infty}\lim_{N\to\infty} N^{-1}\sum_{j\in\mathbb{T}^N}\int_0^t ds\int_{\Omega^N}\left|\phi_j^l-\Phi(\xi_j^l,\eta_j^l)\right| d\mu_t^N = 0,\\
&\lim_{l\to\infty}\lim_{N\to\infty} N^{-1}\sum_{j\in\mathbb{T}^N}\int_0^t ds\int_{\Omega^N}\left|\psi_j^l-\Psi(\xi_j^l,\eta_j^l)\right| d\mu_t^N = 0.
\end{aligned} \tag{38}$$

This is the only part of the proof where condition (B) is used, which ensures ergodicity of the Markov process X_t^N on the 'hyperplanes' $\Omega_{K,L}^N$.

Finally, inserting (37) and (38) in (36), via (20) we obtain (19) and thus the part (ii) of the Theorem is also proved.

6 Particle systems with several conserved variables

As noted in the introduction, the results described in the previous sections are also valid for particle systems with more than 2 conserved quantities.

Before we formulate the general results we have to summarize some notations . Let $n \geq 2$ be fixed integer, and $\boldsymbol{\xi} = (\xi^1, \xi^2, \dots, \xi^n) : S \to \mathbb{R}^n$ the vector of conserved quantities. Throughout the present section bold face symbols will denote n-vectors.

We require the rate function to satisfy similar conditions as listed in subsection 2.2 (in place of conditions (A) and (B) we need the suitable generalizations). For every $\boldsymbol{\theta} \in \mathbb{R}^n$ we can define momentum generating function $G(\boldsymbol{\theta})$ as

$$G(\boldsymbol{\theta}) := \log \sum_{\omega\in S} e^{\boldsymbol{\theta}\cdot\boldsymbol{\xi}(\omega)}\pi(\omega),$$

and the probability measures

$$\begin{aligned}
\pi_{\boldsymbol{\theta}}(\omega) &:= \pi(\omega)\exp(\boldsymbol{\theta}\cdot\boldsymbol{\xi}(\omega) - G(\boldsymbol{\theta}))\\
\pi_{\boldsymbol{\theta}}^N &:= \prod_{j\in\mathbb{T}^N}\pi_{\boldsymbol{\theta}}
\end{aligned}$$

on S, respectively, on Ω^N. We define the expectation of the conserved quantities with respect to the measure $\pi_{\boldsymbol{\theta}}^N$:

$$\boldsymbol{u}(\boldsymbol{\theta}) := \mathbf{E}_{\boldsymbol{\theta}}(\boldsymbol{\xi}) = \nabla_{\boldsymbol{\theta}} G(\boldsymbol{\theta}).$$

One can easily show, that $\nabla^2_{\boldsymbol{\theta}} G(\boldsymbol{\theta}) = \mathbf{Cov}_{\boldsymbol{\theta}}(\boldsymbol{u}, \boldsymbol{u})$ is positive definite. As a consequence, the function $\boldsymbol{\theta} \mapsto \boldsymbol{u}$ is invertible, and

$$\boldsymbol{\theta}(\boldsymbol{u}) = \nabla_{\boldsymbol{u}} S(\boldsymbol{u}),$$

where $S(\boldsymbol{u})$ is the convex conjugate of $G(\boldsymbol{\theta})$:

$$S(\boldsymbol{u}) := \sup_{\boldsymbol{u} \in \mathbb{R}^n} (\boldsymbol{u} \cdot \boldsymbol{\theta} - G(\boldsymbol{\theta})).$$

We introduce the flux of the vector of the conserved quantities and its expectation:

$$\begin{aligned}
\boldsymbol{\phi}(\omega_1, \omega_2) &:= \sum_{\omega_1', \omega_2' \in S} r(\omega_1, \omega_2; \omega_1', \omega_2')(\boldsymbol{\xi}(\omega_2') - \boldsymbol{\xi}(\omega_2)) \\
\boldsymbol{\Phi}(\boldsymbol{u}) &:= \mathbf{E}_{\boldsymbol{\theta}(\boldsymbol{u})} \boldsymbol{\phi} \\
&= \sum_{\substack{\omega_1, \omega_2, \\ \omega_1', \omega_2' \in S}} r(\omega_1, \omega_2; \omega_1', \omega_2')(\boldsymbol{\xi}(\omega_2') - \boldsymbol{\xi}(\omega_2)) \pi_{\boldsymbol{u}}(\omega_1) \pi_{\boldsymbol{u}}(\omega_2).
\end{aligned}$$

Now we are able to formulate results of the previous sections in the more general setting.

Using the arguments presented in section 5 one can show that under Eulerian scaling the vector of the local densities of the conserved quantities $\boldsymbol{u}(t, x)$ evolve according to the following n-component partial differential equation:

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{\Phi}(\boldsymbol{u}) = 0. \tag{39}$$

Lemma 1 applies for any two conserved quantities ξ^i, ξ^j $(i \neq j)$, thus if we denote the derivative matrix of the flux vector $\boldsymbol{\Phi}(\boldsymbol{u})$ by $D(\boldsymbol{u}) := \nabla_{\boldsymbol{u}} \boldsymbol{\Phi}(\boldsymbol{u})$ and the second derivative matrix of the thermodynamic entropy $S''(\boldsymbol{u}) := \nabla^2_{\boldsymbol{u}} S(\boldsymbol{u})$, we get

$$S''(\boldsymbol{u}) \cdot D(\boldsymbol{u}) = \left(S''(\boldsymbol{u}) \cdot D(\boldsymbol{u})\right)^\dagger \tag{40}$$

Since $S''(\boldsymbol{u})$ is positive definite (31) implies that $D(\boldsymbol{u})$ can be diagonalized which means that the arising system of partial differential equations is *hyperbolic*. Moreover, (40) spelled out is

$$\frac{\partial^2 S}{\partial u_i \partial u_i} \frac{\Phi_i}{\partial u_j} + \frac{\partial^2 S}{\partial u_i \partial u_j} \frac{\Phi_j}{\partial u_j} = \frac{\partial^2 S}{\partial u_j \partial u_i} \frac{\Phi_i}{\partial u_i} + \frac{\partial^2 S}{\partial u_j \partial u_j} \frac{\Phi_j}{\partial u_i}, \tag{41}$$

with $1 \leq i < j \leq n$. These are exactly the $n(n-1)/2$ equations defining the Lax entropies of the hyperbolic system (39). It is well-known that in the case of $n \geq 3$ only very special n-component hyperbolic conservation laws possess Lax entropies. In general, the defining equations (41) are overdetermined. In [11] these paricular systems of hyperbolic coservation laws are called of *physical* type. From the previous arguments it follows that only physical hyperbolic equations can arise as the hydrodynamic limit of an interacting particle system satisfying our conditions.

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Bálint Tóth
Institute of Mathematics
Technical University Budapest
Egry József u. 1.
H-1111 Budapest, Hungary
`balint@math.bme.hu`

Benedek Valkó
Institute of Mathematics
Technical University Budapest
Egry József u. 1.
H-1111 Budapest, Hungary
`valko@math.bme.hu`