# PERTURBATION OF SINGULAR EQUILIBRIA OF HYPERBOLIC TWO-COMPONENT SYSTEMS: A UNIVERSAL HYDRODYNAMIC LIMIT

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**Abstract**

We consider one-dimensional, locally finite interacting particle systems with two conservation laws which under Eulerian hydrodynamic limit lead to two-by-two systems of conservation laws:

$$\begin{cases} \partial_t \rho + \partial_x \Psi(\rho, u) = 0 \\ \partial_t u + \partial_x \Phi(\rho, u) = 0, \end{cases}$$

with $(\rho, u) \in \mathcal{D} \subset \mathbb{R}^2$, where $\mathcal{D}$ is a convex compact polygon in $\mathbb{R}^2$. The system is *typically* strictly hyperbolic in the interior of $\mathcal{D}$ with possible non-hyperbolic degeneracies on the boundary $\partial\mathcal{D}$. We consider the case of isolated singular (i.e. non hyperbolic) point on the interior of one of the edges of $\mathcal{D}$, call it $(\rho_0, u_0) = (0, 0)$ and assume $\mathcal{D} \subset \{\rho \geq 0\}$. This can be achieved by a linear transformation of the conserved quantities. We investigate the propagation of *small nonequilibrium perturbations* of the steady state of the microscopic interacting particle system, corresponding to the densities $(\rho_0, u_0)$ of the conserved quantities. We prove that for a very rich class of systems, under proper hydrodynamic limit the propagation of these small perturbations are *universally* driven by the two-by-two system

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0 \\ \partial_t u + \partial_x (\rho + \gamma u^2) = 0 \end{cases}$$

where the parameter $\gamma := \frac{1}{2}\Phi_{uu}(\rho_0, u_0)$ (with a proper choice of space and time scale) is the only trace of the microscopic structure. The proof is valid for the cases with $\gamma > 1$.

The proof relies on the relative entropy method and thus, it is valid only in the regime of smooth solutions of the pde. But there are essentially new elements: in order to control the fluctuations of the terms with Poissonian (rather than Gaussian) decay coming from the low density approximations we have to apply refined pde estimates. In particular Lax entropies of these pde systems play a *not merely technical* key role in the main part of the proof.

## Contents



# 1 Introduction

## 1.1 The PDE to be derived and some facts about it

We consider the pde

$$\begin{cases} \partial_t \rho + \partial_x \big(\rho u\big) = 0 \\ \partial_t u + \partial_x \big(\rho + \gamma u^2\big) = 0 \end{cases} \tag{1.1}$$

where $\rho = \rho(t, x) \in [0, \infty)$, $u = u(t, x) \in (-\infty, \infty)$ are density, respectively, velocity field and $\gamma \in \mathbb{R}$ is a fixed parameter. For any fixed $\gamma$ this is a *hyperbolic system of conservation laws* in the domain $(\rho, u) \in \mathbb{R}_+ \times \mathbb{R}$.

Phenomenologically, the pde describes a deposition/domain growth – or, in biological term: chemotaxis – mechanism: $\rho(t, x)$ is the density of population performing the deposition and $h(t, x)$ is the height of the deposition. Let

$$u(t, x) := -\partial_x h(t, x).$$

The physics of the phenomenon is contained in the following two rules:

(a) The velocity field of the population is proportional to the *negative gradient of the height* of the deposition. That is, the population is pushed towards the local decrease of the deposition height. This rule, together with the conservation of total mass of the population leads to the continuity equation (the first equation in our system).

(b) The deposition rate is

$$\partial_t h = \rho + \gamma\big(\partial_x h\big)^2.$$

The first term on the right hand side is just saying that deposition is done additively by the population. The second term is a self-generating deposition, introduced and phenomenologically motivated by Kardar-Parisi-Zhang [9] and commonly accepted in the literature. Differentiating this last equation with respect to the space variable $x$ results in the second equation of our system.

The pde (1.1) is invariant under the scaling:

$$\widetilde{\rho}(t,x) := A^{2\beta}\rho(A^{1+\beta}t, Ax), \qquad \widetilde{u}(t,x) := A^{\beta}u(A^{1+\beta}t, Ax),$$

where $A > 0$ and $\beta \in \mathbb{R}$ are arbitrarily fixed. The choice $\beta = 0$ gives the straightforward hyperbolic scale invariance, valid for any system of conservation laws. More interesting is the $\beta = 1/2$ case. This is the natural scale invariance of the system, since the physical variables (density and velocity fields) change *covariantly* under this scaling. This is the (presumed, but never rigorously proved) asymptotic scale invariance of the Kardar-Parisi-Zhang deposition phenomena. The nontrivial scale invariance of the pde (1.1) suggests its *universality* in some sense. Our main result indeed states its validity in a very wide context.

It is also clear that the pde is invariant under the left-right reflection symmetry $x \mapsto -x$:

The parameter $\gamma$ of the pde (1.1) is of crucial importance: different values of $\gamma$ lead to completely different behavior. Here are listed some particular cases which arose in the past in various contexts:

— The pde (1.1) with $\gamma = 0$ arose in the context of the 'true self-repelling motion' constructed by Tóth and Werner in [23]. For a survey of this case see also [24]. The same equation, with viscosity terms added, appear in mathematical biology under the name of (negative) chemotaxis equations (see e.g. [17], [15], [14]) ).

— Taking $\gamma = 1/2$ we get the 'shallow water equation'. See [3], [13]. This is the only value of the parameter $\gamma$ when $m = \rho u$ is conserved and as a consequence the pde (1.1) can be interpreted as gas dynamics equation.

— With $\gamma = 1$ the pde is called 'Leroux's equation' which is of Temple class and for this reason much investigated. For many details about this equation see [19]. In the recent paper [6] Leroux's system has been derived as hydrodynamic limit under Eulerian scaling for a two-component lattice gas, going even beyond the appearance of shocks.

The main facts about the pde (1.1) are presented in Section 11. Here we only mention that

1. For any $\gamma \in \mathbb{R}$ the system (1.1) is strictly *hyperbolic* in $(\rho, u) \in (0, \infty) \times \mathbb{R}$, with hyperbolicity marginally lost at $(\rho, u) = (0, 0)$ for $\gamma \neq 1/2$ and at $\rho = 0$ for $\gamma = 1/2$. This follows from straightforward computations.

2. The *Riemann invariants* (or characteristic coordinates) are explicitly computed in section 11, for a first impression see Figure 3 of the Appendix where the level lines of the Riemann invariants are shown. It turns out that the picture changes qualitatively at the critical values $\gamma = 1/2$, $\gamma = 3/4$ and $\gamma = 1$. It is of crucial importance for our later problem that the level curves, expressed as $u \mapsto \rho(u)$ are convex for $\gamma < 1$, linear for $\gamma = 1$ and concave for $\gamma > 1$.

3. For any $\gamma \geq 0$ the system (1.1) is *genuinely nonlinear* in $(\rho, u) \in (0, \infty) \times \mathbb{R}$, with genuine nonlinearity marginally lost at $(\rho, u) = (0, 0)$ for $\gamma \neq 0, 1/2$ and at $\rho = 0$ for $\gamma = 0, 1/2$. (For $\gamma < 0$ genuine nonlinearity is lost on the parabola $\rho = -4\gamma(2\gamma - 1)^2(\gamma + 1)^{-2}u^2$.)

4. The system is sufficiently rich in *Lax entropies.*

5. For $\gamma \geq 0$ the system (1.1) satisfies the conditions of the Lax-Chuey-Conley-Smoller *Maximum Principle* (see [11], [12], [19]).

From the Maximum Principle a very essential difference between the cases $\gamma < 1$, $\gamma = 1$ and $\gamma > 1$ follows, which is of crucial importance for our further work. In the case $\gamma < 1$ all convex domains bounded by level curves of the Riemann invariants are *unbounded (non-compact)* and thus there is no a priori bound on the entropy solutions. Even starting with smooth initial data with compact support nothing prevents the solutions to blow up indefinitely. On the other hand, if $\gamma \geq 1$ any bounded subset of $\mathbb{R}_+ \times \mathbb{R}$ is contained in a compact convex domain bounded by level sets of the Riemann invariants, which fact yields a priori bounds on the entropy solutions, given bounded initial data.

The goal of the present paper is to derive the two-by-two hyperbolic system of conservation laws (1.1) as decent hydrodynamic limit of some systems of interacting particles with two conserved quantities.

We consider one-dimensional, locally finite interacting particle systems with two conservation laws which under *Eulerian* hydrodynamic limit lead to two-by-two systems of conservation laws

$$\begin{cases} \partial_t \rho + \partial_x \Psi(\rho, u) = 0 \\ \partial_t u + \partial_x \Phi(\rho, u) = 0, \end{cases}$$

with $(\rho, u) \in \mathcal{D} \subset \mathbb{R}^2$, where $\mathcal{D}$ is a convex compact polygon in $\mathbb{R}^2$. The system is *typically* strictly hyperbolic in the interior of $\mathcal{D}$ with possible non-hyperbolic degeneracies on the boundary $\partial\mathcal{D}$. We consider the case of isolated singular (i.e. non hyperbolic) point on the interior of one of the edges of $\mathcal{D}$, call it $(\rho_0, u_0) = (0, 0)$ and assume $\mathcal{D} \subset \{\rho \geq 0\}$ (otherwise we apply an appropriate linear transformation on the conserved quantities) We investigate the propagation of *small nonequilibrium perturbations* of the steady state of the microscopic interacting particle system, corresponding to the densities $(\rho_0, u_0)$ of the conserved quantities. We prove that for a very rich class of systems, under proper hydrodynamic limit the propagation of these small perturbations are *universally* driven by the system (1.1), where the parameter $\gamma := \frac{1}{2}\Phi_{uu}(\rho_0, u_0)$ (with a proper choice of space and time scale) is the only trace of the microscopic structure. The proof is valid for the cases with $\gamma > 1$.

Actually, in order to simplify some of the arguments, we impose the left-right reflection symmetry of the pde (1.1) on the systems of interacting particles on microscopic level, see condition (C) in subsection 2.2. But we note that the whole proof can be extended without this condition, just some arguments would be longer.

The proof essentially relies on H-T. Yau's relative entropy method and thus, it is valid only in the regime of smooth solutions of the pde (1.1).

We should emphasize here the essential new ideas of the proof. Since we consider a *low density* limit, the distribution of particle numbers in blocks of mesoscopic size will have a *Poissonian* tail. The fluctuations of the other conserved quantity will be Gaussian, as usual. It follows that when controlling the fluctuations of the empirical block averages the usual large deviation approach would lead us to the disastrous estimate $\mathbf{E}\big(\exp\{\varepsilon\ GAU \cdot POI\}\big) = \infty$. It turns out that some very special cutoff must be applied. Since the large fluctuations which are cut off can not be estimated by robust methods (i.e. by applying entropy inequality), only some cancellation due to martingales can help. This is the reason why the cutoff function must be chosen in a very special way, in terms of a particular Lax entropy of the Euler equation (2.15). In this way the proof becomes a mixture (in our opinion rather interesting mixture) of probabilistic and pde arguments. The fine properties of the limiting pde, in particular the global behavior of Riemann invariants and some particular Lax entropies, play an essential role in the proof. The radical difference between the $\gamma \geq 1$ vs. $\gamma < 1$ cases, in particular applicability vs. non-applicability of the Lax-Chuey-Conley-Smoller maximum principle, manifests itself on the microscopic, probabilistic level.

## 1.2 The structure of the paper

In Section 2 we define the class of models to which our main theorem applies: we formulate the conditions to be satisfied by the interacting particle systems to be considered, we compute the steady state measures and the fluxes corresponding to the conserved quantities. At the end of this section we formulate the Eulerian hydrodynamic limit, for later reference.

In Section 3 first we perform asymptotic analysis of the Euler equations close to the singular point considered, then we formulate our main result, Theorem 1, and its immediate consequences.

Sections 4 to 10 are devoted to the proof of Theorem 1.

In Section 4 we perform the necessary preliminary computations for the proof. After introducing the minimum necessary notation we apply some standard procedures in the context of relative entropy method. Empirical block averages are introduced, numerical error terms are separated and estimated. In this first estimates only straightforward numerical approximations (Taylor expansion bounds) and the most direct entropy inequality is applied.

Section 5 is of crucial importance: here it is shown why the traditional approach of the relative entropy method fails to apply. Here it becomes apparent that in the fluctuation bound (usually referred to as *large deviation estimate*) instead of the tame $\mathbf{E}\big(\exp\{\varepsilon\ GAU^2\}\big)$ we would run into the wild $\mathbf{E}\big(\exp\{\varepsilon\ GAU \cdot POI\}\big)$ which is, of course, infinite. It is explained here what kind of cutoff is applied: the large fluctuations cut off can not be estimated by robust methods (i.e. by applying entropy inequality). Only some cancellation due to martingales can help. This is the reason why the cutoff function must be chosen in a very particular way, in terms of a particular Lax entropy of the Euler equation. The cutoff function is constructed and its key estimates are stated. Proofs of the lemmas formulated in this section are postponed to Section 9. At the end of this section the outline of the further steps is presented.

In Section 6 all the necessary probabilistic ingredients of the forthcoming steps are gathered. These are: fixed time large deviation bounds and fixed time fluctuation bounds, the time averaged block replacement bounds (one block estimates) and the time averaged gradient bounds (two block estimates). The proof of these last two rely on Varadhan's large deviation bound cited in that section and on some probability lemmas stated and proved in section 10. We should mention here that these proofs of the one- and two block estimates, in particular the probability lemmas involved also contain some new (and, we hope, instructive) elements.

Sections 7 and 8 conclude the proof: the various terms arising in section 5 are estimated using all the tools (probabilistic and pde) developed in earlier sections. One can see that these estimates rely heavily on the fine properties of the Lax entropy used in the cutoff procedure.

As we already mentioned sections 9 and 10 are devoted to proofs of various lemmas stated in earlier parts. Section 9 deals with the pde estimates while Section 10 is probabilistic.

Finally in the Appendix (Section 11) we give some details about the pde (1.1). This is included for sake of completeness and in order to let the reader have some more information about these, certainly interesting, pde-s. Strictly technically speaking this Appendix is not used in the proof.

## 2 Microscopic models

Our interacting particle systems to be defined in the present section model on a microscopic level the same deposition phenomena as the pde (1.1). There will be two conserved physical quantities: the particle number $\eta_j \in \mathbb{N}$ and the (discrete) negative gradient of the deposition height $\zeta_j \in \mathbb{Z}$.
The dynamical driving mechanism is of such nature that

(i) The deposition height growth is influenced by the local particle density. Typically: growth is enhanced by higher particle densities.

(ii) The particle motion is itself influenced by the deposition profile. Typically: particles are pushed in the direction of the negative gradient of the deposition height.

The left-right reflection symmetry of the pde will be also implemented on the microscopic level. Actually, this is not really necessary in order to prove our main result, but without this assumption some of the arguments would be somewhat longer.

### 2.1 State space, conserved quantities

Throughout this paper we denote by $\mathbb{T}^n$ the discrete tori $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, and by $\mathbb{T}$ the continuous torus $\mathbb{R}/\mathbb{Z}$. We will denote the local spin state by $\Omega$, we only consider the case when $\Omega$ is finite. The state space of the interacting particle system of size $n$ is

$$\Omega^n := \Omega^{\mathbb{T}^n}.$$

Configurations will be denoted

$$\underline{\omega} := (\omega_j)_{j\in\mathbb{T}^n} \in \Omega^n,$$

For sake of simplicity we consider discrete (integer valued) conserved quantities only. The two conserved quantities are

$$\begin{aligned} &\eta : \Omega \to \mathbb{N}, \\ &\zeta : \Omega \to v_0\mathbb{Z}, \text{ or } \zeta : \Omega \to v_0\big(\mathbb{Z} + 1/2\big). \end{aligned} \tag{2.1}$$

The trivial scaling factor $v_0$ will be conveniently chosen later (see (2.4)). We also use the notations $\eta_j = \eta(\omega_j)$, $\zeta_j = \zeta(\omega_j)$. This means that the sums $\sum_j \eta_j$ and $\sum_j \zeta_j$ are conserved by the dynamics. We assume that the conserved quantities are different and non-trivial, i.e. the functions $\zeta, \eta$ and the constant function 1 on $\Omega$ are linearly independent.

The left-right reflection symmetry of the model is implemented by an involution

$$R : \Omega \to \Omega, \qquad R \circ R = Id$$

which acts on the conserved quantities as follows:

$$\eta(R\omega) = \eta(\omega), \qquad \zeta(R\omega) = -\zeta(\omega). \tag{2.2}$$

### 2.2 Rate functions, infinitesimal generators, stationary measures

Consider a (fixed) probability measure $\pi$ on $\Omega$, which is invariant under the action of the involution $R$, i.e. $\pi(R\omega) = \pi(\omega)$. Since eventually we consider *low densities* of $\eta$, in order to exclude trivial cases we assume that

$$\pi\big(\, \zeta = 0 \,\big|\, \eta = 0 \,\big) < 1. \tag{2.3}$$

The scaling factor $v_0$ in (2.1) is chosen so that

$$\mathbf{Var}\big(\zeta \,|\, \eta = 0\big) = 1. \tag{2.4}$$

This choice simplifies some formulas (fixing a recurring constant to be equal to 1, see (3.4)) but does not restrict generality.

For later use we introduce the notation

$$\begin{aligned} \rho^* &:= \max\{\eta(\omega) : \pi(\omega) > 0\}, \\ u^* &:= \max\{\zeta(\omega) : \pi(\omega) > 0\}, \\ u_* &:= \max\{\zeta(\omega) : \eta(\omega) = 0,\, \pi(\omega) > 0\}, \end{aligned}$$

For $\tau, \theta \in \mathbb{R}$ let $G(\tau, \theta)$ be the moment generating function defined below:

$$G(\tau, \theta) := \log \sum_{\omega \in \Omega} e^{\tau\eta(\omega) + \theta\zeta(\omega)} \pi(\omega).$$

In thermodynamic terms $G(\tau, \theta)$ corresponds to the Gibbs free energy. We define the probability measures

$$\pi_{\tau,\theta}(\omega) := \pi(\omega) \exp(\tau\eta(\omega) + \theta\zeta(\omega) - G(\tau, \theta)) \tag{2.5}$$

on $\Omega$. We are going to define dynamics which conserve the quantities $\sum_j \eta_j$ and $\sum_j \zeta_j$, posses no other (hidden) conserved quantities and for which the product measures

$$\pi^n_{\tau,\theta} := \prod_{j \in \mathbb{T}^n} \pi_{\tau,\theta}$$

are stationary.

We need to separate a symmetric (reversible) part of the dynamics which will be speeded up sufficiently in order to enhance convergence to local equilibrium and thus help estimating some error term in the hydrodynamic limiting procedure. So we consider two *rate functions* $r : \Omega \times \Omega \times \Omega \times \Omega \to \mathbb{R}_+$ and $s : \Omega \times \Omega \times \Omega \times \Omega \to \mathbb{R}_+$, $r$ will define the *asymmetric* component of the dynamics, while $s$ will define the *reversible* component. The dynamics of the system consists of elementary jumps affecting nearest neighbor spins, $(\omega_j, \omega_{j+1}) \longrightarrow (\omega'_j, \omega'_{j+1})$, performed with rate $\lambda r(\omega_j, \omega_{j+1}; \omega'_j, \omega'_{j+1}) + \kappa s(\omega_j, \omega_{j+1}; \omega'_j, \omega'_{j+1})$, where $\lambda, \kappa > 0$ are speed-up factors, depending on the size of the system in the limiting procedure.

We require that the rate functions $r$ and $s$ satisfy the following conditions.

(A) *Conservation laws:* If $r(\omega_1, \omega_2; \omega'_1, \omega'_2) > 0$ *or* $s(\omega_1, \omega_2; \omega'_1, \omega'_2) > 0$ then

$$\begin{aligned} \eta(\omega_1) + \eta(\omega_2) &= \eta(\omega'_1) + \eta(\omega'_2). \\ \zeta(\omega_1) + \zeta(\omega_2) &= \zeta(\omega'_1) + \zeta(\omega'_2), \end{aligned}$$

(B) *Irreducibility:* For every $N \in [0, n\rho^*]$, $Z \in [-nu^*, nu^*]$ the set

$$\Omega^n_{N,Z} := \left\{ \underline{\omega} \in \Omega^n : \sum_{j \in \mathbb{T}^n} \eta_j = N, \sum_{j \in \mathbb{T}^n} \zeta_j = Z \right\}$$

is an irreducible component of $\Omega^n$, i.e. if $\underline{\omega}, \underline{\omega}' \in \Omega^n_{N,Z}$ then there exists a series of elementary jumps with positive rates transforming $\underline{\omega}$ into $\underline{\omega}'$.

(C) *Left-right symmetry:* The jump rates are invariant under left-right reflection *and* the action of the involution $R$ (jointly):

$$r(R\omega_2, R\omega_1; R\omega_2', R\omega_1') = r(\omega_1, \omega_2; \omega_1', \omega_2').$$
$$s(R\omega_2, R\omega_1; R\omega_2', R\omega_1') = s(\omega_1, \omega_2; \omega_1', \omega_2').$$

(D) *Stationarity of the asymmetric part:* For any $\omega_1, \omega_2, \omega_3 \in \Omega$

$$Q(\omega_1, \omega_2) + Q(\omega_2, \omega_3) + Q(\omega_3, \omega_1) = 0,$$

where

$$Q(\omega_1, \omega_2) := \sum_{\omega_1', \omega_2' \in \Omega} \left\{ \frac{\pi(\omega_1')\pi(\omega_2')}{\pi(\omega_1)\pi(\omega_2)} r(\omega_1', \omega_2'; \omega_1, \omega_2) - r(\omega_1, \omega_2; \omega_1', \omega_2') \right\}.$$

(E) *Reversibility of the symmetric part:* For any $\omega_1, \omega_2, \omega_1', \omega_2' \in \Omega$

$$\pi(\omega_1)\pi(\omega_2)s(\omega_1, \omega_2; \omega_1', \omega_2') = \pi(\omega_1')\pi(\omega_2')s(\omega_1', \omega_2'; \omega_1, \omega_2).$$

For a precise formulation of the infinitesimal generator on $\Omega^n$ we first define the map $\Theta_j^{\omega'\omega''} : \Omega^n \to \Omega^n$ for every $\omega', \omega'' \in \Omega$, $j \in \mathbb{T}^n$:

$$\left(\Theta_j^{\omega'\omega''} \underline{\omega}\right)_i = \begin{cases} \omega' & \text{if} \quad i = j \\ \omega'' & \text{if} \quad i = j+1 \\ \omega_i & \text{if} \quad i \neq j, j+1. \end{cases}$$

The infinitesimal generators defined by these rates will be denoted:

$$L^n f(\underline{\omega}) = \sum_{j \in \mathbb{T}^n} \sum_{\omega', \omega'' \in \Omega} r(\omega_j, \omega_{j+1}; \omega', \omega'') \big( f(\Theta_j^{\omega'\omega''} \underline{\omega}) - f(\underline{\omega}) \big).$$

$$K^n f(\underline{\omega}) = \sum_{j \in \mathbb{T}^n} \sum_{\omega', \omega'' \in \Omega} s(\omega_j, \omega_{j+1}; \omega', \omega'') \big( f(\Theta_j^{\omega'\omega''} \underline{\omega}) - f(\underline{\omega}) \big).$$

We denote by $\mathcal{X}_t^n$ the Markov process on the state space $\Omega^n$ with infinitesimal generator $G^n := \lambda(n)L^n + \kappa(n)K^n$. with speed-up factors $\lambda(n)$ and $\kappa(n)$ to be specified later

**Remarks:**

(1) Conditions (A) and (B) together imply that $\sum_j \eta_j$ and $\sum_j \zeta_j$ are indeed the only conserved quantities of the dynamics.

(2) Condition (C) together with (2.2) is implementation on a microscopic level of the left-right symmetry of the pde (1.1). Actually, our main result, Theorem 1, is valid without this assumption but some of the arguments would be more technical.

(3) Condition (D) implies that the product measures $\pi^n_{\tau,\theta}$ are indeed stationary for the dynamics defined by the asymmetric rates $r$. This is seen by applying similar computations to those of [1], [2], [18] or [22]. Mind that this is *not* a detailed balance condition for the rates $r$.

(4) Condition (E) is a straightforward detailed balance condition. It implies that the product measures $\pi^n_{\tau,\theta}$ are reversible for the dynamics defined by the symmetric rates $s$.

We will refer to the measures $\pi^n_{\tau,\theta}$ as the *canonical* measures. Since $\sum_j \zeta_j$ and $\sum_j \eta_j$ are conserved the canonical measures on $\Omega^n$ are *not* ergodic. The conditioned measures defined on $\Omega^n_{N,Z}$ by:

$$\pi^n_{N,Z}(\underline{\omega}) := \pi^n_{\tau,\theta}\big(\underline{\omega}\big| \sum_{j\in\mathbb{T}^n} \eta_j = N, \sum_{j\in\mathbb{T}^n} \zeta_j = Z, \big) = \frac{\pi^n_{\tau,\theta}(\underline{\omega})\,1\!\!1\{\underline{\omega} \in \Omega^n_{N,Z}\}}{\pi^n_{\tau,\theta}(\Omega^n_{N,Z})}$$

are also stationary and due to condition (B) satisfied by the rate functions they are ergodic. We shall call these measures the *microcanonical measures* of our system. (It is easy to see that the measure $\pi^n_{N,Z}$ does not depend on the choice of thew values of $\tau$ and $\theta$ in the previous definition.)

The assumptions are by no means excessively restrictive. Here follow some concrete examples of interacting particle systems which belong to the class specified by conditions (A)-(E) and also satisfy the further conditions (F), (G), (H), (I) to be formulated later.

$\underline{\{-1,0,+1\}\text{-}model}$ The model is described and analyzed in full detail in [22] and [6]. The one spin state space is $\Omega = \{-1,0,+1\}$ . The left-right reflection symmetry is implemented by $R : \Omega \to \Omega$, $R\omega = -\omega$. The dynamics consists of nearest neighbor spin exchanges and the two conserved quantities are $\eta(\omega) = 1 - |\omega|$ and $\zeta(\omega) = \omega$. The jump rates are

$$\begin{aligned} r(1,-1;-1,1) &= 0, & r(-1,1;1,-1) &= 2,\\ r(0,-1;-1,0) &= 0, & r(-1,0;0,-1) &= 1,\\ r(1,0,0,1) &= 0, & r(0,1,1,0) &= 1. \end{aligned}$$

and

$$s(\omega_1,\omega_2;\omega_1',\omega_2') = \begin{cases} 1 & \text{if } (\omega_1,\omega_2) = (\omega_2',\omega_1') \text{ and } \omega_1 \neq \omega_2 \\ 0 & \text{otherwise.} \end{cases}$$

The one dimensional marginals of the stationary measures are

$$\pi_{\rho,u}(0) = \rho, \quad \pi_{\rho,u}(\pm 1) = \frac{1-\rho \pm u}{2}$$

with the domain of variables $\mathcal{D} = \{(\rho,u) \in \mathbb{R}_+ \times \mathbb{R} : \rho + |u| \leq 1\}$.

_Two-lane models_ The following family of examples are finite state space versions of the brick-layers models introduced in [24]. Let $\Omega = \{0, 1, \dots, \bar{n}\} \times \{-\bar{z}, -\bar{z}+1, \dots, \bar{z}-1, \bar{z}\}$, where $\bar{n} \in \mathbb{N}$ and $\bar{z} \in \{\frac{1}{2}, 1, \frac{3}{2}, 2 \dots\}$. The elements of $\Omega$ will be denoted $\omega := \binom{\eta}{\zeta}$. Naturally enough, $\sum_j \eta_j$ and $\sum_j \zeta_j$ will be the conserved quantities of the dynamics. Left-right reflection symmetry is implemented as $R : \Omega \to \Omega$, $R\binom{\eta}{\zeta} = \binom{\eta}{-\zeta}$. We allow only the following elementary changes to occur at neighboring sites $j, j+1$:

$$\binom{\eta_j}{\zeta_j}, \binom{\eta_{j+1}}{\zeta_{j+1}} \to \binom{\eta_j}{\zeta_j \mp 1}, \binom{\eta_{j+1}}{\zeta_{j+1} \pm 1}, \qquad \binom{\eta_j}{\zeta_j}, \binom{\eta_{j+1}}{\zeta_{j+1}} \to \binom{\eta_j \mp 1}{\zeta_j}, \binom{\eta_{j+1} \pm 1}{\zeta_{j+1}}$$

with appropriate rates. Beside the conditions already imposed we also assume that the one dimensional marginals of the steady state measures factorize as follows:

$$\pi(\omega) = \pi\binom{\eta}{\zeta} = p(\eta) q(\zeta).$$

The simplest case, with $\bar{n} = 1$ and $\bar{z} = 1/2$, that is with $\Omega = \{0, 1\} \times \{-1/2, +1/2\}$, was introduced and fully analyzed in [22] and [16]. For a full description (i.e. identification of the rates which satisfy the imposed conditions, Eulerian hydrodynamic limit, etc. see those papers.) It turns out that conditions (A)-(E) impose some nontrivial combinatorial constraints on the rates which are satisfied by a finite parameter family of models. The number of free parameters increases with $\bar{n}$ and $\bar{z}$. Since the concrete expressions of the rates are not relevant for our further presentation we omit the lengthy computations.

## 2.3 Expectations

Expectation, variance, covariance with respect to the measures $\pi^n_{\tau,\theta}$ will be denoted by $\mathbf{E}_{\tau,\theta}(.)$, $\mathbf{Var}_{\tau,\theta}(.)$, $\mathbf{Cov}_{\tau,\theta}(.)$.

We compute the expectations of the conserved quantities with respect to the canonical measures, as functions of the parameters $\tau$ and $\theta$:

$$\rho(\tau, \theta) := \mathbf{E}_{\tau,\theta}(\eta) = \sum_{\omega \in \Omega} \eta(\omega) \pi_{\tau,\theta}(\omega) = G_\tau(\tau, \theta).$$

$$u(\tau, \theta) := \mathbf{E}_{\tau,\theta}(\zeta) = \sum_{\omega \in \Omega} \zeta(\omega) \pi_{\tau,\theta}(\omega) = G_\theta(\tau, \theta),$$

Elementary calculations show, that the matrix-valued function

$$\begin{pmatrix} \rho_\tau & \rho_\theta \\ u_\tau & u_\theta \end{pmatrix} = \begin{pmatrix} G_{\tau\tau} & G_{\tau\theta} \\ G_{\theta\tau} & G_{\theta\theta} \end{pmatrix} =: G''(\tau, \theta)$$

is equal to the covariance matrix $\mathbf{Cov}_{\tau,\theta}(\eta, \zeta)$ and therefore it is strictly positive definite. It follows that the function $(\tau, \theta) \mapsto (\rho(\tau, \theta), u(\tau, \theta))$ is invertible. We denote the inverse function by $(\rho, u) \mapsto (\tau(\rho, u), \theta(\rho, u))$. Denote by $(\rho, u) \mapsto S(\rho, u)$ the convex conjugate (Legendre transform) of the strictly convex function $(\tau, \theta) \mapsto G(\tau, \theta)$:

$$S(\rho, u) := \sup_{\tau,\theta} \big(\rho\tau + u\theta - G(\tau, \theta)\big), \tag{2.6}$$

and

$$\begin{aligned}\mathcal{D} &:= \{(\rho,u)\in\mathbb{R}_+\times\mathbb{R} : S(\rho,u)<\infty\} \\ &= \operatorname{co}\{(\eta,\zeta):\pi(\omega)>0\},\end{aligned} \tag{2.7}$$

where co stands for convex hull. The nondegeneracy condition (2.3) implies that $\partial\mathcal{D}\cap\{\rho=0\}=\{(0,u):|u|\le u_*\}$. For $(\rho,u)\in\mathcal{D}$ we have

$$\tau(\rho,u)=S_\rho(\rho,u), \qquad \theta(\rho,u)=S_u(\rho,u).$$

In probabilistic terms: $S(\rho,u)$ is the rate function of joint large deviations of $(\sum_j\eta_j,\sum_j\zeta_j)$. In thermodynamic terms: $S(\rho,u)$ corresponds to the equilibrium thermodynamic entropy. Let

$$\begin{pmatrix}\tau_\rho & \tau_u\\ \theta_\rho & \theta_u\end{pmatrix}=\begin{pmatrix}S_{\rho\rho} & S_{\rho u}\\ S_{u\rho} & S_{uu}\end{pmatrix}=: S''(\rho,u).$$

It is obvious that the matrices $G''(\tau,\theta)$ and $S''(\rho,u)$ are strictly positive definite and are inverse of each other:

$$G''(\tau,\theta)S''(\rho,u)=I=S''(\rho,u)G''(\tau,\theta), \tag{2.8}$$

where either $(\tau,\theta)=(\tau(\rho,u),\theta(\rho,u))$ or $(\rho,u)=(\rho(\tau,\theta),u(\tau,\theta))$. With slight abuse of notation we shall denote: $\pi_{\tau(\rho,u),\theta(\rho,u)}=:\pi_{\rho,u}$, $\pi^n_{\tau(\rho,u),\theta(\rho,u)}=:\pi^n_{\rho,u}$, $\mathbf{E}_{\tau(\rho,u),\theta(\rho,u)}=:\mathbf{E}_{\rho,u}$, etc.

As a general convention, if $\xi:\Omega^m\to\mathbb{R}$ is a local function then its expectation with respect to the canonical measure $\pi^m_{\rho,u}$ is denoted by

$$\Xi(\rho,u):=\mathbf{E}_{\rho,u}(\xi)=\sum_{\omega_1,\dots,\omega_m\in\Omega^m}\xi(\omega_1,\dots,\omega_m)\pi_{\rho,u}(\omega_1)\cdots\pi_{\rho,u}(\omega_m).$$

## 2.4 Fluxes

We introduce the fluxes of the conserved quantities. The infinitesimal generators $L^n$ and $K^n$ act on the conserved quantities as follows:

$$\begin{aligned}L^n\eta_i &= -\psi(\omega_i,\omega_{i+1})+\psi(\omega_{i-1},\omega_i) &&=: -\psi_i+\psi_{i-1},\\ L^n\zeta_i &= -\phi(\omega_i,\omega_{i+1})+\phi(\omega_{i-1},\omega_i) &&=: -\phi_i+\phi_{i-1},\\ K^n\eta_i &= -\psi^s(\omega_i,\omega_{i+1})+\psi^s(\omega_{i-1},\omega_i) &&=: -\psi^s_i+\psi^s_{i-1},\\ K^n\zeta_i &= -\phi^s(\omega_i,\omega_{i+1})+\phi^s(\omega_{i-1},\omega_i) &&=: -\phi^s_i+\phi^s_{i-1},\end{aligned}$$

where

$$\begin{aligned}\psi(\omega_1,\omega_2) &:= \sum_{\omega_1',\omega_2'\in\Omega} r(\omega_1,\omega_2;\omega_1',\omega_2')\big(\eta(\omega_2')-\eta(\omega_2)\big)\\ \phi(\omega_1,\omega_2) &:= \sum_{\omega_1',\omega_2'\in\Omega} r(\omega_1,\omega_2;\omega_1',\omega_2')\big(\zeta(\omega_2')-\zeta(\omega_2)\big)\end{aligned} \tag{2.9}$$

$$\begin{aligned}\psi^s(\omega_1,\omega_2) &:= \sum_{\omega_1',\omega_2'\in\Omega} s(\omega_1,\omega_2;\omega_1',\omega_2')\big(\eta(\omega_2')-\eta(\omega_2)\big)\\ \phi^s(\omega_1,\omega_2) &:= \sum_{\omega_1',\omega_2'\in\Omega} s(\omega_1,\omega_2;\omega_1',\omega_2')\big(\zeta(\omega_2')-\zeta(\omega_2)\big)\end{aligned} \tag{2.10}$$

Note that due to the left-right symmetry and conservations, i.e. (2.2) and conditions (A) and (C), the microscopic fluxes have the following symmetries:

$$
\begin{aligned}
\phi(\omega_1,\omega_2) &= \quad \phi(R\omega_2,R\omega_1),\\
\psi(\omega_1,\omega_2) &= -\psi(R\omega_2,R\omega_1).
\end{aligned}
$$

In order to simplify some of our further arguments (in particular, see (7.7) in subsection 7.3) we impose one more microscopic condition

(F) *Gradient condition on symmetric fluxes:* The microscopic fluxes of the symmetric part, defined in (2.10) satisfy the following gradient conditions

$$
\begin{aligned}
\psi^s(\omega_1,\omega_2) &= \kappa(\omega_1)-\kappa(\omega_2) =: \kappa_1-\kappa_2 \qquad (2.11)\\
\phi^s(\omega_1,\omega_2) &= \chi(\omega_1)-\chi(\omega_2) =: \chi_1-\chi_2.
\end{aligned}
$$

**Remark:** (1) This is a technical assumption (referring actually to the measure $\pi$) which simplifies considerably the arguments of subsection 9.2. The symmetric part $K^n$ has the role of enhancing convergence to local equilibrium. Its effect is *not seen* in the limit, so in principle we can choose it conveniently. Without this assumption we would be forced to use all the non-gradient technology developed in [25] (see also [10]), which would make the paper even longer. (2) It is easy to see that $\eta(\omega_1)=\eta(\omega_2)=0$ implies $\psi^s(\omega_1,\omega_2)=0$ and thus (by choosing a suitable additive constant) $\omega\mapsto\kappa(\omega)$ can be chosen so that

$$
\eta(\omega)=0 \ \Rightarrow \ \kappa(\omega)=0. \qquad (2.12)
$$

The *macroscopic fluxes* are:

$$
\begin{aligned}
\Psi(\rho,u) &:= \mathbf{E}_{\rho,u}(\psi)\\
&= \sum_{\substack{\omega_1,\omega_2,\\ \omega_1',\omega_2'\in\Omega}} r(\omega_1,\omega_2;\omega_1',\omega_2')\big(\eta(\omega_2')-\eta(\omega_2)\big)\pi_{\rho,u}(\omega_1)\pi_{\rho,u}(\omega_2),\\
\Phi(\rho,u) &:= \mathbf{E}_{\rho,u}(\phi) \qquad (2.13)\\
&= \sum_{\substack{\omega_1,\omega_2,\\ \omega_1',\omega_2'\in\Omega}} r(\omega_1,\omega_2;\omega_1',\omega_2')\big(\zeta(\omega_2')-\zeta(\omega_2)\big)\pi_{\rho,u}(\omega_1)\pi_{\rho,u}(\omega_2).
\end{aligned}
$$

These are smooth regular functions of the variables $(\rho,u)\in\mathcal{D}$. Note that due to reversibility of $K^n$, for any value of $\rho$ and $u$

$$
\mathbf{E}_{\rho,u}(\psi^s)=0=\mathbf{E}_{\rho,u}(\phi^s).
$$

(These identities hold true without assuming condition (F).)

For later use we mention here that according to [22], the macroscopic fluxes $\Psi(\rho, u)$ and $\Phi(\rho, u)$ satisfy the following *Onsager reciprocity relation*

$$\Psi_u(\rho, u)\mathbf{Var}_{\rho,u}(\zeta) - \Phi_u(\rho, u)\mathbf{Cov}_{\rho,u}(\eta, \zeta) = \qquad (2.14)$$

$$\Phi_\rho(\rho, u)\mathbf{Var}_{\rho,u}(\eta) - \Psi_\rho(\rho, u)\mathbf{Cov}_{\rho,u}(\eta, \zeta).$$

For the concrete examples presented at the end of subsection 2.2 the following domains $\mathcal{D}$ and macroscopic rates are gotten:

<u>$\{-1, 0, +1\}$-*model*</u>:

$$\begin{aligned}
&\mathcal{D} = \{(\rho, u) \in \mathbb{R}_+ \times \mathbb{R} \, : \, \rho + |u| \leq 1\} \\
&\Psi(\rho, u) = \rho u \\
&\Phi(\rho, u) = \rho + u^2.
\end{aligned}$$

<u>*Two lane models*</u> with $\bar{n} = 1$:

$$\begin{aligned}
&\mathcal{D} = \{(\rho, u) \in \mathbb{R}_+ \times \mathbb{R} \, : \, \rho \leq 1, \;\; |u| \leq \bar{z}\} \\
&\Psi(\rho, u) = \rho(1 - \rho)\psi(u) \\
&\Phi(\rho, u) = \varphi_0(u) + \rho\varphi_1(u),
\end{aligned}$$

where $\psi(u)$ is odd, while $\varphi_0(u)$ and $\varphi_1(u)$ are even functions of $u$, determined by the jump rates of the model. In the simplest particular case with $\bar{z} = 1/2$

$$\begin{aligned}
&\Psi(\rho, u) = \rho(1 - \rho)u \\
&\Phi(\rho, u) = (\rho - \gamma)(1 - u^2),
\end{aligned}$$

where $\gamma \in \mathbb{R}$ is the only model dependent parameter which appears in the macroscopic fluxes. For details see [22].

### 2.5 The hdl under Eulerian scaling

Given a system of interacting particles as defined in the previous subsections, by applying Yau's relative entropy method (see [26] or the monograph [10]), one shows that under Eulerian scaling the local densities of the conserved quantities $\rho(t, x)$, $u(t, x)$ evolve according to the system of partial differential equations:

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x \Psi(\rho, u) = 0 \\ \partial_t u + \partial_x \Phi(\rho, u) = 0 \end{array} \right. \qquad (2.15)$$

where $\Psi(\rho, u)$ and $\Phi(\rho, u)$ are the macroscopic fluxes defined in (2.13).

The precise statement of the hydrodynamical limit is as follows: Consider a microscopic system which satisfies conditions (A)-(E) of subsection 2.2. Note that condition (F) of subsection

2.4 is not assumed. Let $\Psi(\rho, u)$ and $\Phi(\rho, u)$ be the macroscopic fluxes computed for this system and $\rho(t,x), u(t,x)$ $x \in \mathbb{T}$, $t \in [0,T]$ be *smooth* solution of the pde (2.15). Let the microscopic system of size $n$ be driven by the infinitesimal generator

$$G^n = nL^n + n^{1+\delta} K^n,$$

where $\delta \in [0,1)$ is fixed. This means that the main, asymmetric part of the generator is speeded up by $n$ and the additional symmetric part by $n^{1+\delta}$. Let $\mu_0^n$ be a probability distribution on $\Omega^n$ which is the initial distribution of the microscopic system of size $n$, and

$$\mu_t^n := \mu_0^n e^{tG^n}$$

the distribution of the system at (macroscopic) time $t$. The *local equilibrium* measure $\nu_t^n$ (itself a probability measure on $\Omega^n$) is defined by

$$\nu_t^n := \prod_{j \in \mathbb{T}^n} \pi_{\rho(t,\frac{j}{n}),u(t,\frac{j}{n})}.$$

This measure *mimics on a microscopic scale* the macroscopic evolution driven by the pde (2.15).

We denote by $H(\mu_t^n|\pi^n)$, respectively, by $H(\mu_t^n|\nu_t^n)$ the relative entropy of the measure $\mu_t^n$ with respect to the absolute reference measure $\pi^n$, respectively, with respect to the local equilibrium measure $\nu_t^n$.

The precise statement of the Eulerian hydrodynamic limit is the following

**Theorem.** *Assume conditions (A)-(E) and let $\delta \in [0,1)$ be fixed. If*

$$H\left(\mu_0^n \mid \nu_0^n\right) = o(n)$$

*then*

$$H\left(\mu_t^n \mid \nu_t^n\right) = o(n)$$

*uniformly for $t \in [0,T]$.*

**Remark:** Note that due to finiteness of the state space $\Omega$ the condition

$$H\left(\mu_0^n \mid \pi_{1,0}^n\right) = \mathcal{O}(n)$$

holds automatically.

The Theorem follows from direct application of Yau's relative entropy method. For the proof and its direct consequences see [26], [10] or [22]. For the main consequences of this Theorem see e.g. Corollary 1 of [22].

# 3 Low density asymptotics and the main result: hydrodynamic limit under intermediate scaling

## 3.1 General properties and low density asymptotics of the macroscopic fluxes

The fluxes in the Euler equation (2.15) are regular smooth functions of in $(\rho, u) \in \overline{\mathcal{D}}$.

From the left-right symmetry of the microscopic models it follows that

$$\Phi(\rho, -u) = \Phi(\rho, u), \qquad \Psi(\rho, -u) = -\Psi(\rho, u). \tag{3.1}$$

It is also obvious that for $u \in [-u_*, u_*]$

$$\Psi(0, u) = 0. \tag{3.2}$$

We make two assumptions about the low density asymptotics of the macroscopic fluxes. Here is the first one:

(G) We assume that $\Psi_{\rho u}(0,0) \neq 0$. Actually, by possibly redefining the time scale and orientation of space, without loss of generality we assume

$$\Psi_{\rho u}(0,0) = 1. \tag{3.3}$$

From the Onsager relation (2.14) and obvious parity considerations it also follows that

$$\Phi_{\rho}(0,0) = \Psi_{\rho u}(0,0)\mathbf{Var}_{0,0}(\zeta) = 1. \tag{3.4}$$

Note, that here we rely on the choice (2.4) of the scaling factor $v_0$ in (2.1).

We denote

$$\gamma := \frac{1}{2}\Phi_{uu}(0,0). \tag{3.5}$$

Our results will hold for $\gamma > 1$ only.

From (3.1) and (3.3) it follows that

$$\Phi_u(0,u) - \Psi_{\rho}(0,u) = (2\gamma - 1)u + \mathcal{O}(|u|^3). \tag{3.6}$$

The second condition imposed on the low density asymptotics of the macroscopic fluxes is:

(H) For $u \in [-u_*, u_*]$, $u \neq 0$

$$\Phi_u(0,u) - \Psi_{\rho}(0,u) \neq 0, \tag{3.7}$$

$$\Phi_{\rho}(0,u) \neq 0, \quad \Psi_{\rho u}(0,u) \neq 0 \tag{3.8}$$

**Remarks:** (1) (G) is a very natural nondegeneracy condition: if $\Psi_{\rho u}(0,0)$ vanished then in the perturbation calculus to be performed, higher order terms would be dominant and a different scaling limit should be taken.
(2) Due to (3.1), (3.3) and (3.6) conditions (3.7), (3.8) hold anyway in a neighborhood of $u=0$, and this would suffice, but the forthcoming arguments, in particular the proof of Lemmas 2 and 3 would be less transparent. We assume condition (H) for technical convenience only. Condition (3.7) amounts to forbidding other non-hyperbolic pointson $\overline{\partial\mathcal{D}\cap\{\rho=0\}}$, beside the point $(\rho,u)=(0,0)$. Condition (3.8) reflects the natural monotonicity requirements (i) and (ii) formulated about the microscopic models at the beginning of Section 2.

We are interested in the behavior of the pde near the isolated non-hyperbolic point $(\rho,u)=(0,0)$. The asymptotic expansion for $\rho+u^2\ll 1$ of the macroscopic fluxes and their first partial derivatives is

$$\begin{aligned}
&\Psi(\rho,u)=\rho u\big(1+\mathcal{O}(\rho+u^2)\big), \quad \Phi(\rho,u)=(\rho+\gamma u^2)\big(1+\mathcal{O}(\rho+u^2)\big),\\
&\Psi_\rho(\rho,u)=u\big(1+\mathcal{O}(\rho+u^2)\big), \quad \Phi_\rho(\rho,u)=1+\mathcal{O}(\rho+u^2),\\
&\Psi_u(\rho,u)=\rho\big(1+\mathcal{O}(\rho+u^2)\big), \quad \Phi_u(\rho,u)=2\gamma u\big(1+\mathcal{O}(\rho+u^2)\big).
\end{aligned}\tag{3.9}$$

We are looking for "small solutions" of the pde (2.15): Let $\rho_0(x)$ and $u_0(x)$ be given profiles and assume that $\rho^\varepsilon(t,x)$, $u^\varepsilon(t,x)$ is solution of the pde (2.15) with initial condition

$$\rho^\varepsilon(0,x)=\varepsilon^2\rho_0(x), \quad u^\varepsilon(0,x)=\varepsilon\, u_0(x).$$

Then, at least formally,

$$\varepsilon^{-2}\rho^\varepsilon(\varepsilon^{-1}t,x)\to\rho(t,x), \quad \varepsilon^{-1}u^\varepsilon(\varepsilon^{-1}t,x)\to u(t,x),$$

where $\rho(t,x)$, $u(t,x)$ is solution of the pde (1.1) with initial condition

$$\rho(0,x)=\rho_0(x), \quad u(0,x)=u_0(x).$$

### 3.2 The main result

The asymptotic computations of subsection 3.1 suggest the scaling under which we should derive the pde (1.1) as hydrodynamic limit: fix a (small) positive $\beta$ and choose the scaling

| | MICRO | MACRO |
|---|---|---|
| space | $nx$ | $x$ |
| time | $n^{1+\beta}t$ | $t$ |
| particle density | $n^{-2\beta}\rho$ | $\rho$ |
| 'slope of the wall' | $n^{-\beta}u$ | $u$ |

Ideally the result should be valid for $0<\beta<1/2$ but we are able to prove much less than that.

Choose a model satisfying the conditions (A)-(F) of section 2 and conditions (G-H) of subsection 3.1, and let $\gamma$ be given by (3.5), corresponding to the microscopic system chosen.

Let the microscopic system of size $n$ (defined on the discrete torus $\mathbb{T}^n$) evolve (on macroscopic time scale) according to the infinitesimal generator

$$G^n = n^{1+\beta} L^n + n^{1+\beta+\delta} K^n.$$

with $\beta > 0$ and some further conditions to be imposed on $\beta$ and $\delta$ (see Theorem 1). Denote by $\mu_t^n$ the true distribution of the microscopic system at macroscopic time $t$:

$$\mu_t^n := \mu_0^n e^{tG^n},$$

where $\mu_0^n$ is the initial distribution.

We use the translation invariant product measure

$$\pi^n := \pi^n_{n^{-2\beta},0}$$

as *absolute reference measure.* Global entropy will be considered relative to this measure, Radon-Nikodym derivatives of $\mu_t^n$ and of the local equilibrium measure $\nu_t^n$ to be defined below, with respect to $\pi^n$ will be used.

Given a smooth solution $\big(\rho(t,x), u(t,x)\big)$, $(t,x) \in [0,T] \times \mathbb{T}$, of the pde (1.1) define the *local equilibrium measure* $\nu_t^n$ on $\Omega^n$ as follows

$$\nu_t^n := \prod_{j \in \mathbb{T}^n} \pi_{n^{-2\beta}\rho(t,\frac{j}{n}),\, n^{-\beta}u(t,\frac{j}{n})}. \tag{3.10}$$

This time-dependent measure *mimics on a microscopic level* the macroscopic evolution governed by the pde (1.1).

Our main result is the following

**Theorem 1.** *Assume that the microscopic system of interacting particles satisfies conditions (A)-(F) of subsubsections 2.2, 2.4 and the uniform log-Sobolev condition (I) of subsection 6.2. Additionally, assume that the macroscopic fluxes satisfy conditions (G), (H) of subsection 3.1 and $\gamma > 1$. Choose $\beta \in (0,1/2)$ and $\delta \in (1/2,1)$ so that*

$$2\delta - 8\beta > 1 \quad \textit{and} \quad \delta + 3\beta < 1. \tag{3.11}$$

*Let $\big(\rho(t,x), u(t,x)\big)$, $(t,x) \in [0,T] \times \mathbb{T}$, be* smooth *solution of the pde (1.1), such that $\inf_{x\in\mathbb{T}} \rho(0,x) > 0$ and let $\nu_t^n$, $t \in [0,T]$ be the corresponding local equilibrium measure defined in (3.10).*

*Under these conditions, if*

$$H\big(\mu_0^n \,\big|\, \nu_0^n\big) = o(n^{1-2\beta}) \tag{3.12}$$

*then*

$$H\big(\mu_t^n \,\big|\, \nu_t^n\big) = o(n^{1-2\beta}) \tag{3.13}$$

*uniformly for $t \in [0,T]$.*

**Remarks:**
(i) From (3.12) via the identity (4.5) and the entropy inequality it also follows that

$$H\big(\mu_0^n \,\big|\, \pi^n\big) = \mathcal{O}(n^{1-2\beta}). \tag{3.14}$$

See the beginning of subsection 4.2
(i) If $\gamma > 3/4$, in smooth solutions vacuum does not appear. That is $\inf_{x\in\mathbb{T}} \rho(0,x) > 0$ implies $\inf_{(t,x)\in[0,T]\times\mathbb{T}} \rho(t,x) > 0$.
(ii) Although for the $\{-1,0,+1\}$*-model* we have $\gamma = 1$, our proof can also be extended to cover this model. Actually, in that case the proof is much simpler, since the Eulerian pde is equal to the limit pde (1.1) and thus the cutoff function (see Section 5) can be determined explicitly.

**Corollary 1.** *Assume the conditions of Theorem 1. Let $g : \mathbb{T} \to \mathbb{R}$ be a test function. Then for any $t \in [0,T]$*
*(i)*

$$\begin{aligned} n^{2\beta-1} \sum_{j\in\mathbb{T}^n} g\big(\frac{j}{n}\big)\eta_j(t) \ &\overset{\mathbf{P}}{\to} \ \int_{\mathbb{T}} g(x)\rho(t,x)\,dx, \\ n^{\beta-1} \sum_{j\in\mathbb{T}^n} g\big(\frac{j}{n}\big)\zeta_j(t) \ &\overset{\mathbf{P}}{\to} \ \int_{\mathbb{T}} g(x)u(t,x)\,dx. \end{aligned}$$

*(ii)*

$$H\big(\mu_0^n \,\big|\, \pi^n\big) - H\big(\mu_t^n \,\big|\, \pi^n\big) = o(n^{1-2\beta}).$$

See the proof of Corollary 1 in [22].

# 4 Notations and general preparatory computations

This section completely standard in the context of the relative entropy method. So we shall be sketchy.

## 4.1 Notation

We denote

$$\begin{aligned} h^n(t) \ &:= \ n^{-(1-2\beta)} H\big(\mu_t^n \,\big|\, \nu_t^n\big). \\ s^n(t) \ &:= \ n^{-(1-2\beta)} \left(H\big(\mu_0^n \,\big|\, \pi^n\big) - H\big(\mu_t^n \,\big|\, \pi^n\big)\right). \end{aligned}$$

We know *a priori* that $t \mapsto s^n(t)$ is monotone increasing and due to (3.14)

$$s^n(t) = \mathcal{O}(1), \qquad \text{uniformly for } t \in [0,\infty). \tag{4.1}$$

In fact, from Theorem 1 it follows (see Corollary 1) that as long as the solution $\rho(t,x), u(t,x)$ of the pde (1.1) is smooth

$$s^n(t) = o(1), \qquad \text{uniformly for } t \in [0,T].$$

For $(\rho,u) \in (0,\infty) \times (-\infty,\infty)$ denote

$$\begin{aligned}
\tau^n(\rho,u) &:= \tau(n^{-2\beta}\rho, n^{-\beta}u) - \tau(n^{-2\beta},0) \\
\theta^n(\rho,u) &:= n^{\beta}\theta(n^{-2\beta}\rho, n^{-\beta}u).
\end{aligned}$$

Note that, for symmetry reasons $\theta(n^{-2\beta},0) = 0$. Mind that $\tau$ is chemical potential rather than fugacity and for small densities the fugacity $\lambda := e^{\tau}$ scales like $\rho$, i.e. $\tau(n^{-2\beta},0) \sim -2\beta \log n$. If $\rho > 0$ and $u \in \mathbb{R}$ are fixed then $\tau^n(\rho,u)$ and $\theta^n(\rho,u)$ stay of order 1, as $n \to \infty$.

Given the smooth solution $\rho(t,x), u(t,x)$, with $\rho(t,x) > 0$ we shall use the notation

$$\begin{aligned}
\tau^n(t,x) &:= \tau^n(\rho(t,x), u(t,x)), \\
\theta^n(t,x) &:= \theta^n(\rho(t,x), u(t,x)), \\
v(t,x) &:= \log \rho(t,x).
\end{aligned}$$

The following asymptotics hold uniformly in $(t,x) \in [0,T] \times \mathbb{T}$:

$$\begin{array}{rlrl}
\tau^n(t,x) = & v(t,x) + \mathcal{O}(n^{-2\beta}), & \theta^n(t,x) = & u(t,x) + \mathcal{O}(n^{-2\beta}) \\
\partial_x\tau^n(t,x) = & \partial_x v(t,x) + \mathcal{O}(n^{-2\beta}), & \partial_x\theta^n(t,x) = & \partial_x u(t,x) + \mathcal{O}(n^{-2\beta}) \\
\partial_t\,\tau^n(t,x) = & \partial_t\, v(t,x) + \mathcal{O}(n^{-2\beta}), & \partial_t\,\theta^n(t,x) = & \partial_t\, u(t,x) + \mathcal{O}(n^{-2\beta})
\end{array} \tag{4.2}$$

The logarithm of the Radom-Nikodym derivative of the time dependent reference measure $\nu_t^n$ with respect to the absolute referencee measure $\pi^n$ is denoted by $f_t^n$:

$$\begin{aligned}
f_t^n(\underline{\omega}) \;&:=\; \log \frac{d\nu_t^n}{d\pi^n}(\underline{\omega}) \\
&= \sum_{j\in\mathbb{T}^n} \Big\{ \tau^n(t,\frac{j}{n})\eta_j + n^{-\beta}\theta^n(t,\frac{j}{n})\zeta_j \\
&\qquad -G(\tau^n(t,\frac{j}{n}) + \tau(n^{-2\beta},0), n^{-\beta}\theta^n(t,\frac{j}{n})) + G(\tau(n^{-2\beta},0),0) \Big\}
\end{aligned} \tag{4.3}$$

### 4.2 Preparatory computations

In order to obtain the main estimate (3.13) our aim is to get a Grönwall type inequality: we will prove that for every $t \in [0,T]$

$$h^n(t) - h^n(0) = \int_0^t \partial_t h^n(s)ds \le C \int_0^t h^n(s)ds + o(1), \tag{4.4}$$

where the error term is uniform in $t \in [0,T]$. Because it is assumed that $h^n(0) = o(1)$, the Theorem follows.

We start with the identity

$$H(\mu_t^n|\nu_t^n) - H(\mu_t^n|\pi^n) = -\int_{\Omega^n} f_t^n \, d\mu_t^n. \tag{4.5}$$

From this identity, the explicit form of the Radon-Nikodym derivative (4.3), the asymptotics (4.2), via the entropy inequality and (3.12) the a priori entropy bound (3.14) follows indeed, as remarked after the formulation of Theorem 1.

Next we differentiate (4.5) to obtain

$$\partial_t h^n(t) = -\int_{\Omega^n} \left( n^{3\beta} L^n f_t^n + n^{3\beta+\delta} K^n f_t^n + n^{-1+2\beta} \partial_t f_t^n \right) d\mu_t^n - \partial_t s^n(t). \tag{4.6}$$

Usually, an adjoint version of (4.6) is being used in form of an inequality. In our case this form is needed. We emphasize that the term $-\partial_t s^n(t)$ on the right hand side will be of crucial importance.

We compute the three terms under the integral.

$$\begin{aligned} n^{3\beta} L^n f_t^n(\underline{\omega}) \quad &= \quad \frac{1}{n} \sum_{j \in \mathbb{T}^n} \partial_x v(t, \frac{j}{n}) n^{3\beta} \psi_j + \frac{1}{n} \sum_{j \in \mathbb{T}^n} \partial_x u(t, \frac{j}{n}) n^{2\beta} \phi_j \\ &\quad + A_1^n(t,\underline{\omega}) + A_2^n(t,\underline{\omega}) + A_3^n(t,\underline{\omega}) + A_4^n(t,\underline{\omega}), \end{aligned} \tag{4.7}$$

where $A_i^n(t,\underline{\omega})$, $i = 1, \ldots, 4$ are error terms which will be easy to estimate:

$$\begin{aligned} A_1^n(t,\underline{\omega}) \quad &:= \quad \frac{1}{n} \sum_{j \in \mathbb{T}^n} \big( \partial_x \tau^n(t, \frac{j}{n}) - \partial_x v(t, \frac{j}{n}) \big) n^{3\beta} \psi_j, \\ A_2^n(t,\underline{\omega}) \quad &:= \quad \frac{1}{n} \sum_{j \in \mathbb{T}^n} \big( \partial_x \theta^n(t, \frac{j}{n}) - \partial_x u(t, \frac{j}{n}) \big) n^{2\beta} \phi_j, \\ A_3^n(t,\underline{\omega}) \quad &:= \quad \frac{1}{n} \sum_{j \in \mathbb{T}^n} \big( \nabla^n \tau^n(t, \frac{j}{n}) - \partial_x \tau^n(t, \frac{j}{n}) \big) n^{3\beta} \psi_j \\ A_4^n(t,\underline{\omega}) \quad &:= \quad \frac{1}{n} \sum_{j \in \mathbb{T}^n} \big( \nabla^n \theta^n(t, \frac{j}{n}) - \partial_x \theta^n(t, \frac{j}{n}) \big) n^{2\beta} \phi_j. \end{aligned}$$

Here and in the sequel $\nabla^n$ denotes the discrete gradient:

$$\nabla^n f(x) := n\big(f(x + 1/n) - f(x)\big).$$

See subsection 4.4 for the estimate of the error terms $A_j^n(t,\underline{\omega})$, $j = 1, \ldots, 12$.

Next,

$$\begin{aligned} n^{3\beta+\delta} K^n f_t^n(\underline{\omega}) \quad &= \quad n^{-1+3\beta+\delta} \frac{1}{n} \sum_{j \in \mathbb{T}^n} \big( (\nabla^n)^2 \, \tau^n(t, \frac{j}{n}) \kappa_j + (\nabla^n)^2 \, \theta^n(t, \frac{j}{n}) \chi_j \big) \\ &=: \quad A_5^n(t,\underline{\omega}) \end{aligned} \tag{4.8}$$

is itself a numerical error term. Finally

$$
\begin{aligned}
n^{-1+2\beta}\partial_t f_t^n(\underline{\omega}) \quad = \quad & \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t v(t,\frac{j}{n})\big(n^{2\beta}\eta_j-\rho(t,\frac{j}{n})\big) \\
& +\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t u(t,\frac{j}{n})\big(n^{\beta}\zeta_j-u(t,\frac{j}{n})\big) \\
& +A_6^n(t,\underline{\omega})+A_7^n(t,\underline{\omega}),
\end{aligned}
\tag{4.9}
$$

where

$$
\begin{aligned}
A_6^n(t,\underline{\omega}) \quad &:= \quad \frac{1}{n}\sum_{j\in\mathbb{T}^n}\big(\partial_t\tau^n(t,\frac{j}{n})-\partial_t v(t,\frac{j}{n})\big)\big(n^{2\beta}\eta_j-\rho(t,\frac{j}{n})\big), \\
A_7^n(t,\underline{\omega}) \quad &:= \quad \frac{1}{n}\sum_{j\in\mathbb{T}^n}\big(\partial_t\theta^n(t,\frac{j}{n})-\partial_t u(t,\frac{j}{n})\big)\big(n^{\beta}\zeta_j-u(t,\frac{j}{n})\big).
\end{aligned}
$$

are again easy-to-estimate error terms.

## 4.3 Blocks

We fix once and for all a weight function $a:\mathbb{R}\to\mathbb{R}$. It is assumed that:
(1) $a(x)>0$ for $x\in(-1,1)$ and $a(x)=0$ otherwise,
(2) it has total weight $\int a(x)\,dx=1$,
(3) it is even: $a(-x)=a(x)$, and
(4) it is twice continuously differentiable.

We choose a *mesoscopic* block size $l=l(n)$ such that

$$
1\ll n^{(1+\delta+5\beta)/3}\ll l(n)\ll n^{\delta-\beta}\ll n.
\tag{4.10}
$$

This can be done due to condition (3.11) imposed on $\beta$ and $\delta$.

Given a local variable (depending on $m$ consecutive spins)

$$
\xi_i=\xi_i(\underline{\omega})=\xi(\omega_i,\dots,\omega_{i+m-1}),
$$

its *block average at macroscopic space coordinate* $x$ is defined as

$$
\widehat{\xi}^n(x)=\widehat{\xi}^n(\underline{\omega},x):=\frac{1}{l}\sum_j a\left(\frac{nx-j}{l}\right)\xi_j.
\tag{4.11}
$$

Since $l=l(n)$, we do not denote explicitly dependence of the block average on the mesoscopic block size $l$.

Note that $x\mapsto\widehat{\xi}^n(x)$ is smooth

$$
\partial_x\widehat{\xi}^n(x)=\partial_x\widehat{\xi}^n(\underline{\omega},x)=\frac{n}{l}\frac{1}{l}\sum_j a'\left(\frac{nx-j}{l}\right)\xi_j,
$$

and it is straightforward that

$$\sup_{\underline{\omega}\in\Omega^n}\sup_{x\in\mathbb{T}}\left|\partial_x\widehat{\xi}^n(\underline{\omega},x)\right| \leq C\left(\max_{\omega_1,\dots\omega_m}\xi(\omega_1,\dots,\omega_m)\right)\frac{n}{l}. \tag{4.12}$$

For a more sophisticated bound on $\left|\partial_x\widehat{\xi}^n(\underline{\omega},x)\right|$ see (6.10).

We shall use the handy (but slightly abused) notation

$$\widehat{\xi}^n(t,x) := \widehat{\xi}^n(\mathcal{X}^n_t,x).$$

This is the empirical block average process of the local observable $\xi_i$.

For the scaled block average of the two conserved quantities we shall also use the notation

$$\widehat{\rho}^n(t,x) := n^{2\beta}\widehat{\eta}^n(t,x), \qquad \widehat{u}^n(t,x) := n^{\beta}\widehat{\zeta}^n(t,x).$$

Introducing block averages the main terms on the right hand side of (4.7) and (4.9) become:

$$\begin{aligned}\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x v(t,\frac{j}{n})n^{3\beta}\psi_j + \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x u(t,\frac{j}{n})n^{2\beta}\phi_j = \qquad\qquad&\\ \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x v(t,\frac{j}{n})n^{3\beta}\widehat{\psi}^n(\frac{j}{n}) + \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x u(t,\frac{j}{n})n^{2\beta}\widehat{\phi}^n(\frac{j}{n})&\\ +A^n_8(t,\underline{\omega}) + A^n_9(t,\underline{\omega}),&\end{aligned} \tag{4.13}$$

respectively

$$\begin{aligned}\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t v(t,\frac{j}{n})\big(n^{2\beta}\eta_j - \rho(t,\frac{j}{n})\big) + \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t u(t,\frac{j}{n})\big(n^{\beta}\zeta_j - u(t,\frac{j}{n})\big) =&\\ \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t v(t,\frac{j}{n})\big(n^{2\beta}\widehat{\eta}^n(\frac{j}{n}) - \rho(t,\frac{j}{n})\big)&\\ +\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t u(t,\frac{j}{n})\big(n^{\beta}\widehat{\zeta}^n(\frac{j}{n}) - u(t,\frac{j}{n})\big)&\\ +A^n_{10}(t,\underline{\omega}) + A^n_{11}(t,\underline{\omega}),&\end{aligned} \tag{4.14}$$

where the error terms are

$$A^n_8(t,\underline{\omega}) := \frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big(\partial_x v\big(t,\frac{j}{n}\big) - \frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\partial_x v\big(t,\frac{k}{n}\big)\Big)n^{3\beta}\psi_j,$$

$$A^n_9(t,\underline{\omega}) := \frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big(\partial_x u\big(t,\frac{j}{n}\big) - \frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\partial_x u\big(t,\frac{k}{n}\big)\Big)n^{2\beta}\phi_j,$$

$$A^n_{10}(t,\underline{\omega}) := \frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big(\partial_t v\big(t,\frac{j}{n}\big) - \frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\partial_t v\big(t,\frac{k}{n}\big)\Big)n^{2\beta}\eta_j,$$

$$A^n_{11}(t,\underline{\omega}) := \frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big(\partial_t u\big(t,\frac{j}{n}\big) - \frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\partial_t u\big(t,\frac{k}{n}\big)\Big)n^{\beta}\zeta_j.$$

These error terms will be estimated in subsection 4.4.

Since $[0,T]\times\mathbb{T}\ni(t,x)\mapsto(\rho(t,x),u(t,x))$, is a *smooth* solution of the pde (1.1), we have

$$\partial_t v=-u\partial_x v-\partial_x u,\qquad \partial_t u=-\rho\partial_x v-\gamma u\partial_x u.$$

Inserting these expressions into the main terms of (4.14) eventually we obtain for the integrand in (4.6)

$$\begin{aligned}
&n^{3\beta}L^n f_t^n(\underline{\omega})+n^{3\beta+\delta}K^n f_t^n(\underline{\omega})+n^{-1+2\beta}\partial_t f_t^n(\underline{\omega})= \qquad (4.15)\\
&\qquad \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x v(t,\frac{j}{n})\Big\{n^{3\beta}\widehat{\psi}^n(\frac{j}{n})-\rho(t,\frac{j}{n})u(t,\frac{j}{n})\\
&\qquad\qquad -u(t,\frac{j}{n})\big(n^{2\beta}\widehat{\eta}^n(\frac{j}{n})-\rho(t,\frac{j}{n})\big)-\rho(t,\frac{j}{n})\big(n^{\beta}\widehat{\zeta}^n(\frac{j}{n})-u(t,\frac{j}{n})\big)\Big\}\\
&\qquad +\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x u(t,\frac{j}{n})\Big\{n^{2\beta}\widehat{\phi}^n(\frac{j}{n})-\big(\rho(t,\frac{j}{n})+\gamma u(t,\frac{j}{n})^2\big)\\
&\qquad\qquad -\big(n^{2\beta}\widehat{\eta}^n(\frac{j}{n})-\rho(t,\frac{j}{n})\big)-2\gamma u(t,\frac{j}{n})\big(n^{\beta}\widehat{\zeta}^n(\frac{j}{n})-u(t,\frac{j}{n})\big)\Big\}\\
&\qquad +\sum_{k=1}^{12}A_k^n(t,\underline{\omega}),
\end{aligned}$$

where

$$\begin{aligned}
A_{12}^n(t) &:= \frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big((\partial_x v)\rho u+(\partial_x u)\big(\rho+\gamma u^2\big)\Big)(t,\frac{j}{n})\\
&= \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x\big(\rho u+\frac{\gamma}{3}u^3\big)(t,\frac{j}{n})
\end{aligned}$$

## 4.4 The error terms $A_k^n$, $k=1,\dots,12$

**Lemma 1.** *There exists a finite constant $C$, such that for any $n\in\mathbb{N}$, $t\in[0,T]$ and for any sequence of real numbers $b_j$, $j=1,\dots,n$ the following bounds hold:*

$$\mathbf{E}_{\mu_t^n}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}b_j\eta_j\Big)\le Cn^{-2\beta}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}b_j+\sqrt{\frac{1}{n}\sum_{j\in\mathbb{T}^n}b_j^2}\right) \tag{4.16}$$

$$\mathbf{E}_{\mu_t^n}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}b_j\zeta_j\Big)\le Cn^{-\beta}\sqrt{\frac{1}{n}\sum_{j\in\mathbb{T}^n}b_j^2} \tag{4.17}$$

$$\mathbf{E}_{\mu_t^n}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}b_j\psi_j\Big)\le Cn^{-2\beta}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}b_j+\sqrt{\frac{1}{n}\sum_{j\in\mathbb{T}^n}b_j^2}\right) \tag{4.18}$$

$$\mathbf{E}_{\mu_t^n}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}b_j\phi_j\Big)\le C\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}b_j+n^{-\beta}\sqrt{\frac{1}{n}\sum_{j\in\mathbb{T}^n}b_j^2}\right) \tag{4.19}$$

*Proof.* The proof relies on the entropy inequality

$$\mathbf{E}_{\mu_t^n}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n} b_j\big(\xi_j-\mathbf{E}_{\pi^n}(\xi_j)\big)\Big) \le \tag{4.20}$$
$$\frac{1}{\gamma n}H(\mu_t^n|\pi^n)+\frac{1}{\gamma n}\log \mathbf{E}_{\pi^n}\Big(\exp\Big\{\gamma\sum_{j\in\mathbb{T}^n} b_j\big(\xi_j-\mathbf{E}_{\pi^n}(\xi_j)\big)\Big\}\Big),$$

where $\xi_j$ stands for either of $\eta_j$, $\zeta_j$, $\psi_j$ or $\phi_j$. We note that all these variables are bounded and

$$\begin{aligned}
&\big|\,\mathbf{E}_{\pi^n}\big(\eta_j\big)\,\big| \le Cn^{-2\beta}, && \mathbf{Var}_{\pi^n}\big(\eta_j\big) \le Cn^{-2\beta},\\
&\big|\,\mathbf{E}_{\pi^n}\big(\zeta_j\big)\,\big| = 0, && \mathbf{Var}_{\pi^n}\big(\zeta_j\big) \le C,\\
&\big|\,\mathbf{E}_{\pi^n}\big(\psi_j\big)\,\big| = 0, && \mathbf{Var}_{\pi^n}\big(\psi_j\big) \le Cn^{-2\beta},\\
&\big|\,\mathbf{E}_{\pi^n}\big(\phi_j\big)\,\big| \le C, && \mathbf{Var}_{\pi^n}\big(\phi_j\big) \le C.
\end{aligned}$$

From these bounds and the entropy inequality (4.20) the statement of the lemma follows directly. □

Now we turn to the estimates on the error terms. We use the bounds (4.16), (4.17), (4.18) and (4.19) of Lemma 1, the asymptotics (4.2) and uniform approximation of $\partial_x$ of smooth functions by their discrete derivative $\nabla^n$. Straightforward computations yield

$$\begin{aligned}
&\mathbf{E}_{\mu_t^n}\big(A_1^n(t)\big) \le C\big(n^{-1+\beta}+n^{-\beta}\big) = o(1),\\
&\mathbf{E}_{\mu_t^n}\big(A_2^n(t)\big) \le C\big(n^{-1+2\beta}+n^{-\beta}\big) = o(1),\\
&\mathbf{E}_{\mu_t^n}\big(A_3^n(t)\big) \le Cn^{-1+\beta} = o(1),\\
&\mathbf{E}_{\mu_t^n}\big(A_4^n(t)\big) \le C\big(n^{-1+2\beta}+n^{-1+\beta}\big) = o(1),\\
&\mathbf{E}_{\mu_t^n}\big(A_5^n(t)\big) \le C\big(n^{-1+\beta+\delta}+n^{-1+2\beta+\delta}\big) = o(1),\\
&\mathbf{E}_{\mu_t^n}\big(A_6^n(t)\big) \le Cn^{-2\beta} = o(1),\\
&\mathbf{E}_{\mu_t^n}\big(A_7^n(t)\big) \le Cn^{-2\beta} = o(1),\\
&\mathbf{E}_{\mu_t^n}\big(A_8^n(t)\big) \le C\big(n^{-1+\beta}+n^{\beta}l^{-1}+n^{-1+\beta}l\big) = o(1),\\
&\mathbf{E}_{\mu_t^n}\big(A_9^n(t)\big) \le C\big(n^{-1+2\beta}+n^{\beta}l^{-1}+n^{-1+\beta}l\big) = o(1),\\
&\mathbf{E}_{\mu_t^n}\big(A_{10}^n(t)\big) \le C\big(n^{-1}+l^{-1}+n^{-1}l\big) = o(1),\\
&\mathbf{E}_{\mu_t^n}\big(A_{11}^n(t)\big) \le C\big(l^{-1}+n^{-1}l\big) = o(1).
\end{aligned}$$

Finally, $A_{12}^n(t)$ is a simple numerical error term (no probability involved):

$$A_{12}^n(t) \le Cn^{-1} = o(1).$$

### 4.5 Sumup

Thus, integrating (4.6), using (4.15) and the bounds of subsection 4.4 we obtain

$$h^n(t) = \int_0^t \mathcal{A}^n(s)\,ds + \int_0^t \mathcal{B}^n(s)\,ds - s^n(t) + o(1), \tag{4.21}$$

where

$$\mathcal{A}^n(t) := \mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big\{(\partial_x v)\big\{n^{3\beta}\widehat{\psi}^n - \rho u - u(\widehat{\rho}^n - \rho) - \rho(\widehat{u}^n - u)\big\}\Big\}(t,\frac{j}{n})\Big) \tag{4.22}$$

and

$$\mathcal{B}^n(t) := \mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big\{(\partial_x u)\big\{n^{2\beta}\widehat{\phi}^n - (\rho+\gamma u^2) - (\widehat{\rho}^n - \rho) - 2\gamma u(\widehat{u}^n - u)\big\}\Big\}(t,\frac{j}{n})\Big) \tag{4.23}$$

The main difficulty is caused by $\mathcal{A}^n(t)$. The term $\mathcal{B}^n(t)$ is estimated exactly as it is done in [21] for the one-component systems: since $\Phi(\rho,u) = \rho + \gamma u^2$ is linear in $\rho$ and quadratic in $u$ no problem is caused by the low particle density. By repeating the arguments of [21] we obtain

$$\int_0^t \mathcal{B}^n(s)ds \le C\int_0^t h^n(s)ds + o(1). \tag{4.24}$$

In the rest of the proof we concentrate on the essentially difficult term $\mathcal{A}^n(t)$.

# 5 Cutoff

We define the *rescaled macroscopic fluxes*

$$\Psi^n(\rho,u) := n^{3\beta}\Psi(n^{-2\beta}\rho, n^{-\beta}u), \qquad \Phi^n(\rho,u) := n^{2\beta}\Phi(n^{-2\beta}\rho, n^{-\beta}u). \tag{5.1}$$

defined on the scaled domain

$$\mathcal{D}^n := \{(\rho,u) : (n^{-2\beta}\rho, n^{-\beta}u) \in \mathcal{D}\}. \tag{5.2}$$

The first partial derivatives of the scaled fluxes are

$$\begin{aligned} \Psi^n_\rho(\rho,u) &= n^{\beta}\Psi_\rho(n^{-2\beta}\rho, n^{-\beta}u), & \Phi^n_\rho(\rho,u) &= \Phi_\rho(n^{-2\beta}\rho, n^{-\beta}u),\\ \Psi^n_u(\rho,u) &= n^{2\beta}\Psi_u(n^{-2\beta}\rho, n^{-\beta}u), & \Phi^n_u(\rho,u) &= n^{\beta}\Phi_u(n^{-2\beta}\rho, n^{-\beta}u). \end{aligned} \tag{5.3}$$

For any $(\rho,u) \in \mathbb{R}_+ \times \mathbb{R}$

$$\begin{aligned} \lim_{n\to\infty}\Psi^n(\rho,u) &= \rho u, & \lim_{n\to\infty}\Phi^n(\rho,u) &= \rho + \gamma u^2,\\ \lim_{n\to\infty}\Psi^n_\rho(\rho,u) &= u, & \lim_{n\to\infty}\Phi^n_\rho(\rho,u) &= 1,\\ \lim_{n\to\infty}\Psi^n_u(\rho,u) &= \rho, & \lim_{n\to\infty}\Phi^n_u(\rho,u) &= 2\gamma u. \end{aligned} \tag{5.4}$$

The convergence is uniform in compact subsets of $\mathbb{R}_+ \times \mathbb{R}$

Note that

$$\Psi^n(\widehat{\rho}^n(t,x), \widehat{u}^n(t,x)) = n^{3\beta}\Psi(\widehat{\eta}^n(t,x), \widehat{\zeta}^n(t,x)),$$

$$\Phi^n(\widehat{\rho}^n(t,x), \widehat{u}^n(t,x)) = n^{2\beta}\Phi(\widehat{\eta}^n(t,x), \widehat{\zeta}^n(t,x)).$$

### 5.1 The direct approach — why it fails?

The most natural thing is to write the summand in $\mathcal{A}^n(t)$ as

$$n^{3\beta}\widehat{\psi}^n - \rho u - u(\widehat{\rho}^n - \rho) - \rho(\widehat{u}^n - u) = \tag{5.5}$$
$$n^{3\beta}(\widehat{\psi}^n - \Psi(\widehat{\eta}^n, \widehat{\zeta}^n)) + (\Psi^n(\widehat{\rho}^n, \widehat{u}^n) - \widehat{\rho}^n\widehat{u}^n) + (\widehat{\rho}^n - \rho)(\widehat{u}^n - u)$$

By applying Varadhan's "one block estimate" and controlling the error terms in the Taylor expansion of $\Psi$, the first two terms on the right hand side can be dealt with. However, the last term causes serious problems: with proper normalization, it is distributed with respect to the local equilibrium measure $\nu_t^n$, like a product of independent Poisson and Gaussian random variables, and thus it does *not* have a finite exponential moment. Since the robust estimates heavily rely on the entropy inequality where the finite exponential moment is needed, we have to choose another approach for estimating $\mathcal{A}^n(t)$.

Instead of writing plainly (5.5), we introduce a cutoff. We let

$$M > \sup\{\rho(t,x) \vee |u(t,x)| : (t,x) \in [0,T] \times \mathbb{T}\}.$$

The value of $M$ will be specified by the large deviation bounds given in Proposition 2 (via Lemma 9).

Let $I^n, J^n : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be bounded functions so that

$$\begin{aligned} &I^n + J^n = 1, \\ &I^n(\rho,u) = 1 \quad \text{for} \quad \rho \vee |u| \le M, \\ &I^n(\rho,u) = 0 \quad \text{for 'large' } (\rho,u). \end{aligned}$$

The last property will be specified later.

We split the right hand side of (5.5) in a most natural way, according to this cutoff:

$$n^{3\beta}\widehat{\psi}^n - \rho u - u(\widehat{\rho}^n - \rho) - \rho(\widehat{u}^n - u) = \tag{5.6}$$
$$\begin{aligned} &n^{3\beta}\widehat{\psi}^n J^n(\widehat{\rho}^n, \widehat{u}^n) - (\widehat{\rho}^n u + \rho\widehat{u}^n - \rho u) J^n(\widehat{\rho}^n, \widehat{u}^n) \\ &+ n^{3\beta}(\widehat{\psi}^n - \Psi(\widehat{\eta}^n, \widehat{\zeta}^n)) I^n(\widehat{\rho}^n, \widehat{u}^n) \\ &+ (\Psi^n(\widehat{\rho}^n, \widehat{u}^n) - \widehat{\rho}^n\widehat{u}^n) I^n(\widehat{\rho}^n, \widehat{u}^n) + (\widehat{\rho}^n - \rho)(\widehat{u}^n - u) I^n(\widehat{\rho}^n, \widehat{u}^n) \end{aligned}$$

The second term on the right hand side is linear in the block averages, so it does not cause any problem. The third term is estimated by use of Varadhan's one block estimate. The fourth term is Taylor approximation. Finally, the last term can be handled with the entropy inequality *if the cutoff $I^n(\rho,u)$ is strong enough* to tame the tail of the Gaussian×Poisson random variable.

The main difficulty is caused by the first term on the right hand side. This term certainly can not be estimated with the robust method, i.e. with entropy inequality: we would run into the same problem we wanted to overcome by introducing the cutoff. The only way this term may be small is by some cancellation. It turns out that the desired cancellations indeed occur (in form of a martingale appearing in the space-time average) if and only if

$$J^n(\rho,u) = S^n_\rho(\rho,u), \tag{5.7}$$

where $S^n(\rho,u)$ is a particular *Lax entropy of the scaled Euler equation*

$$\begin{cases} \partial_t\rho + \partial_x\Psi^n(\rho,u) = 0 \\ \partial_t u + \partial_x\Phi^n(\rho,u) = 0, \end{cases} \tag{5.8}$$

with $\Psi^n(\rho,u)$ and $\Phi^n(\rho,u)$ defined in (5.1). That is $S^n$ is solution of the pde

$$\Psi^n_u S^n_{\rho\rho} + \big(\Phi^n_u - \Psi^n_\rho\big)S^n_{\rho u} - \Phi^n_\rho S^n_{uu} = 0. \tag{5.9}$$

## 5.2 The cutoff function

In the present subsection we construct the cutoff function (5.7) and we state some estimates related to it. These bounds will be of paramount importance in our further proof. They are stated in the technical Lemmas 2, 3 and 4. The proof of these lemmas is pure classical pde theory and it is postponed to section 9.

First, in subsubsection 5.2.1, we formulate our construction and estimates in terms of Lax entropies of the *unscaled* Euler equation (2.15). Then in subsubsection 5.2.2 we rescale these estimates in order to get the necessary bounds on $S^n$ and its derivatives.

### 5.2.1

A Lax entropy/flux pair $S(\rho,u)$, $F(\rho,u)$ of the system (2.15) is solution of the system of pdes

$$F_\rho = \Psi_\rho S_\rho + \Phi_\rho S_u, \qquad F_u = \Psi_u S_\rho + \Phi_u S_u, \tag{5.10}$$

defined on $\mathcal{D}$. In particular the Lax entropy $S(\rho,u)$ solves the pde:

$$\Psi_u S_{\rho\rho} + \big(\Phi_u - \Psi_\rho\big)S_{\rho u} - \Phi_\rho S_{uu} = 0, \tag{5.11}$$

The linear pde (5.11) is hyperbolic in $\mathcal{D}$. One family of its characteristic curves are solutions of the following ODE, meant in the domain $\mathcal{D}$:

$$\frac{d\rho}{du} = \frac{\sqrt{\big(\Phi_u - \Psi_\rho\big)^2 + 4\Phi_\rho\Psi_u} - \big(\Phi_u - \Psi_\rho\big)}{2\Phi_\rho}, \tag{5.12}$$

The other family is obtained by reflecting $u$ to $-u$.

First we conclude that the line segment $\mathcal{D}\cap\{u=0\}$ is *not* characteristic for the hyperbolic pde (5.11). That is: is intersects transversally the characteristic lines defined by the differential equation (5.12). Indeed, from the Onsager relation (2.14) and obvious parity considerations it follows, that the right hand side of (5.12) restricted to $\{u = 0\}$ becomes $\big(\mathbf{Var}_{r,0}(\eta)/\mathbf{Var}_{r,0}(\zeta)\big)^{1/2}$ and this expression is obviously finite for $r \in [0, \rho^*)$. It follows that the Cauchy problem (5.11), with the following initial condition:

$$S(r, 0) = s(r), \qquad S_u(r, 0) = 0, \qquad r \in [0, \rho^*) \tag{5.13}$$

is *well posed.*

In our concrete problem the function $s(r)$ will be chosen as follows: we fix $0 < \underline{r} < \overline{r}$, and define

$$s(r) = \begin{cases} 0 & \text{if } r \in [0, \underline{r}), \\ \dfrac{r \log(r/\underline{r}) - (r - \underline{r})}{\log(\overline{r}/\underline{r})} & \text{if } r \in [\underline{r}, \overline{r}), \\ r - \dfrac{\overline{r} - \underline{r}}{\log(\overline{r}/\underline{r})} & \text{if } r \in [\overline{r}, \infty). \end{cases} \tag{5.14}$$

Note that $s(r)$ and $s'(r)$ are continuous.

We first analyze the global structure of the characteristic curves. Due to the assumption (H) imposed on-, and regularity of the flux functions $\Phi$ and $\Psi$, there exists some $\rho_0 > 0$ such that the ODE (5.12) is regular in $\{(\rho, u) \in \mathcal{D} : \rho < \rho_0 \text{ and } (\rho, u) \neq (0, 0)\}$. We shall not be concerned about what happens outside this strip. Denote by $\sigma(u; r)$ the solution of the ODE (5.12) with initial condition $\sigma(0; r) = r$.

**Lemma 2.** *There exist constants $0 < C_2 < C_1 < \infty$ and $r_0 > 0$ such that for any $r \in [0, r_0]$*

$$\begin{aligned} r + C_1\sqrt{r}u \le \sigma(u; r) \le r + C_2\sqrt{r}u \qquad & \text{if } u \le 0, \\ r \le \sigma(u; r) \le r + C_1\left(\sqrt{r}u \wedge r^{\frac{4\gamma-3}{4\gamma-2}} u^{\frac{1}{2\gamma-1}}\right) \qquad & \text{if } u \ge 0. \end{aligned} \tag{5.15}$$

*The inequalities are valid as long as $(\sigma(u; r), u) \in \mathcal{D}$. The map $u \mapsto \sigma(u; r)$ is regular and monotone increasing.*

See subsection 9.1 for the proof of this lemma.

For $r < r_0$ we partition the domain $\mathcal{D}$ in three parts as follows

$$\begin{aligned} \mathcal{D}_1(r) \quad &:= \quad \{(\rho, u) \in \mathcal{D} : \rho < \sigma(-|u|; r)\} \\ \mathcal{D}_2(r) \quad &:= \quad \{(\rho, u) \in \mathcal{D} : \rho > \sigma(|u|; r)\} \\ \mathcal{D}_3(r) \quad &:= \quad \{(\rho, u) \in \mathcal{D} : \sigma(-|u|; r) \le \rho \le \sigma(|u|; r)\} \\ &= \quad \mathcal{D} \setminus \big(\mathcal{D}_1(r) \cup \mathcal{D}_2(r)\big). \end{aligned}$$

See Figure 1 for a sketch of the domains $\mathcal{D}_1(r), \mathcal{D}_2(r), \mathcal{D}_3(r)$.

From Lemma 2 it follows that

$$\begin{aligned} &\{(\rho, u) : 0 \le \rho < r - C_1\sqrt{r}|u|\} \;\subset\; \mathcal{D}_1(r) \;\subset\; \{(\rho, u) : 0 \le \rho < r - C_2\sqrt{r}|u|\}, \\ &\{(\rho, u) : r + C_1\sqrt{r}|u| < \rho\} \;\subset\; \mathcal{D}_2(r) \;\subset\; \{(\rho, u) : r < \rho\}, \end{aligned} \tag{5.16}$$

and for $0 \le r \le r' \le r_0$

$$\mathcal{D}_1(r) \subset \mathcal{D}_1(r'), \qquad \mathcal{D}_2(r') \subset \mathcal{D}_2(r). \tag{5.17}$$

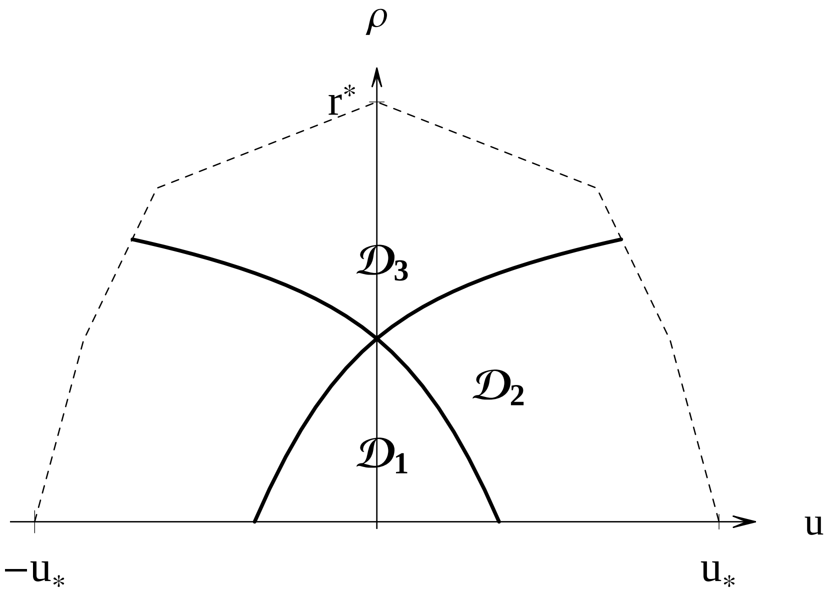



Figure 1: $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$

From now $r_0$ is *fixed for ever* and we denote

$$\widetilde{\mathcal{D}} := \mathcal{D}_1(r_0).$$

This domain is a *rectangle* in characteristic coordinates with diagonal $\widetilde{\mathcal{D}} \cap \{u = 0\}$, as opposed to $\mathcal{D}$ which may not be a full characteristic rectangle. (Actually, choosing the characteristic coordinates in a natural symmetric way, $z(\rho, u) = w(\rho, -u)$, the domain $\widetilde{\mathcal{D}}$ is a square in characteristic coordinates.) Note that $\mathcal{D}_3(r_0) \cap \{u \ge 0\}$ and $\mathcal{D}_3(r_0) \cap \{u \le 0\}$ are also characteristic rectangles.

Next we turn to the construction of a particular family of Lax entropies which will serve for obtaining the cutoff functions needed. We fix $0 < \underline{r} < \overline{r} < r_0$. and define $S : \mathcal{D} \to \mathbb{R}$ as follows:

(i) In $\widetilde{\mathcal{D}}$: $S(\rho, u)$ is solution of the Cauchy problem (5.11)+(5.13) with $s(r)$ given in (5.14). Note that

$$S(\rho, u) = \begin{cases} 0 & \text{if } (\rho, u) \in \mathcal{D}_1(\underline{r}) \subset \widetilde{\mathcal{D}}, \\ \rho - \dfrac{\overline{r} - \underline{r}}{\log(\overline{r}/\underline{r})} & \text{if } (\rho, u) \in \mathcal{D}_2(\overline{r}) \cap \widetilde{\mathcal{D}}. \end{cases} \tag{5.18}$$

(ii) In $\mathcal{D}_2(\overline{r})$:

$$S(\rho, u) := \rho - \frac{\overline{r} - \underline{r}}{\log(\overline{r}/\underline{r})}, \qquad \text{if } (\rho, u) \in \mathcal{D}_2(\overline{r}) \tag{5.19}$$

Note that there is no contradiction: in $\widetilde{\mathcal{D}} \cap \mathcal{D}_2(\underline{r})$, (i) yields the same expression.

(iii) In $\mathcal{D}_3(\overline{r}) \setminus \widetilde{\mathcal{D}}$: $S(\rho, u)$ is defined as solution of the Goursat problem (5.11) with boundary conditions on the characteristic lines $\partial\widetilde{\mathcal{D}} \cap \mathcal{D}_3(\overline{r})$, respectively, $\partial\mathcal{D}_2(\overline{r}) \setminus \widetilde{\mathcal{D}}$ provided by (i), respectively, (ii).

Note that $S(\rho, u)$ is solution of the pde (5.11), *globally* in $\widetilde{\mathcal{D}}$.

We denote

$$\begin{aligned}\mathcal{D}_3(\underline{r}, \overline{r}) \quad &:= \quad \mathcal{D} \setminus \big(\mathcal{D}_1(\underline{r}) \cup \mathcal{D}_2(\overline{r})\big) \\ &= \quad \big(\mathcal{D}_1(\overline{r}) \cap \mathcal{D}_2(\underline{r})\big) \cup \big(\mathcal{D}_1(\overline{r}) \cap \mathcal{D}_3(\underline{r})\big) \\ &\quad\ \cup \big(\mathcal{D}_3(\overline{r}) \cap \mathcal{D}_2(\underline{r})\big) \cup \big(\mathcal{D}_3(\overline{r}) \cap \mathcal{D}_3(\underline{r})\big).\end{aligned}$$

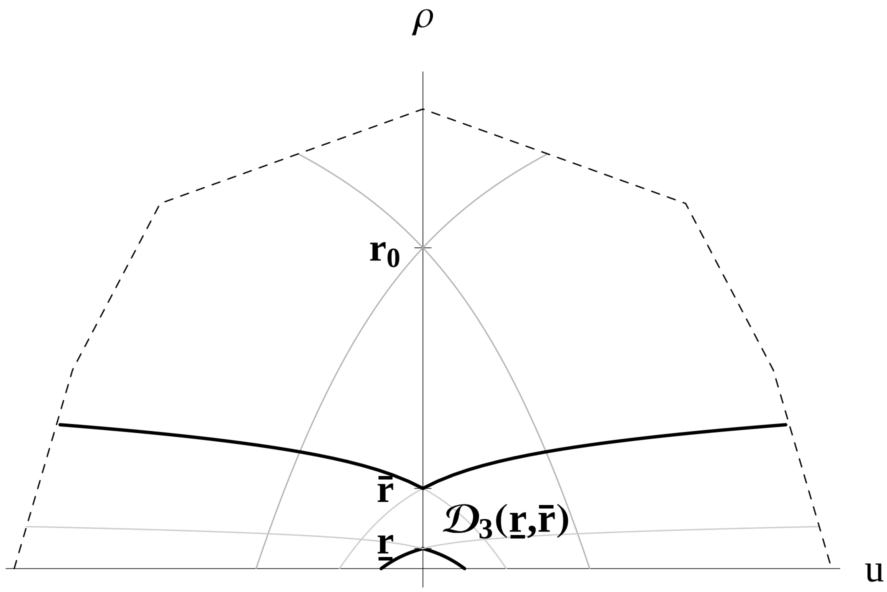



Figure 2: $\mathcal{D}_3(\underline{r}, \overline{r})$

The following lemma provides the necessary bounds on the partial derivatives of $S(\rho, u)$ (up to second order) in the domain $\mathcal{D}_3(\underline{r}, \overline{r})$.

**Lemma 3.** *There exists a constant* $C < \infty$ *such that for any* $0 < \underline{r} < \overline{r} < r_0$ *and* $S(\rho, u)$

*defined as above the following global bounds hold:*

$$|S_\rho(\rho,u) - 1\!\!1_{\mathcal{D}_2(\overline{r})}(\rho,u)| \le C\, 1\!\!1_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u), \tag{5.20}$$

$$|S_u(\rho,u)| \le \frac{C(\sqrt{\overline{r}}-\sqrt{\underline{r}})}{\log(\overline{r}/\underline{r})}\, 1\!\!1_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u), \tag{5.21}$$

$$|S_{\rho\rho}(\rho,u)| \le \frac{C}{\log(\overline{r}/\underline{r})}\frac{1}{\underline{r}+\rho}\, 1\!\!1_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u), \tag{5.22}$$

$$|S_{\rho u}(\rho,u)| \le \frac{C}{\log(\overline{r}/\underline{r})}\frac{1}{\sqrt{\underline{r}}+\sqrt{\rho}+|u|}\, 1\!\!1_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u), \tag{5.23}$$

$$|S_{uu}(\rho,u)| \le \frac{C}{\log(\overline{r}/\underline{r})}\, 1\!\!1_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u), \tag{5.24}$$

This lemma is proved in subsection 9.2.

Beside the bounds on the partial derivatives of $S(\rho,u)$ we shall also need a bound on the function $F(\rho,u) - \Psi(\rho,u)S_\rho(\rho,u)$. From (5.10) and (5.18) it follows that

$$F(\rho,u) = \begin{cases} 0 & \text{if } (\rho,u) \in \mathcal{D}_1(\underline{r}), \\ \Psi(\rho,u) & \text{if } (\rho,u) \in \mathcal{D}_2(\overline{r}). \end{cases}$$

**Lemma 4.** *With the assumptions and notations of Lemma 3*

$$|F(\rho,u) - \Psi(\rho,u)S_\rho(\rho,u)| \le \frac{C\sqrt{\overline{r}}}{\log(\overline{r}/\underline{r})}(\overline{r}+u^2)\, 1\!\!1_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u). \tag{5.25}$$

See subsection 9.3 for the proof of this lemma.

### 5.2.2

The *scaled* functions $S^n(\rho,u)$, $F^n(\rho,u)$ are defined on the scaled domain $\mathcal{D}^n$ given in (5.2), as follows: fix $0 < \underline{r} < \overline{r} < \infty$ and define the *unscaled* Lax entropy/flux pair as in the previous section but with *downscaled* initial conditions

$$S(r,0) = n^{-2\beta}s(n^{2\beta}r), \qquad S_u(r,0) = 0. \qquad r \in [0,\rho^*), \tag{5.26}$$

with the function $r \mapsto s(r)$ given in (5.14). Now, define the pair of scaled functions $S^n, F^n : \mathcal{D}^n \to \mathbb{R}$ as

$$S^n(\rho,u) := n^{2\beta}S(n^{-2\beta}\rho, n^{-\beta}u), \qquad F^n(\rho,u) := n^{3\beta}F(n^{-2\beta}\rho, n^{-\beta}u). \tag{5.27}$$

It is straightforward to check that $S^n$, $F^n$ form a Lax entropy/flux pair of the pde (5.8):

$$F^n_\rho = \Psi^n_\rho S^n_\rho + \Phi^n_\rho S^n_u, \qquad F^n_u = \Psi^n_u S^n_\rho + \Phi^n_u S^n_u, \tag{5.28}$$

in particular $S^n$ solves the pde (5.9).

We partition the scaled domain

$$\mathcal{D}^n = \mathcal{D}_1^n(\underline{r}) \cup \mathcal{D}_2^n(\overline{r}) \cup \mathcal{D}_3^n(\underline{r}, \overline{r})$$

with the partition elements

$$\begin{aligned}
\mathcal{D}_1^n(\underline{r}) &:= \{(\rho, u) \in \mathcal{D}_n : (n^{-2\beta}\rho, n^{-\beta}u) \in \mathcal{D}_1(n^{-2\beta}\underline{r})\} \\
\mathcal{D}_2^n(\overline{r}) &:= \{(\rho, u) \in \mathcal{D}_n : (n^{-2\beta}\rho, n^{-\beta}u) \in \mathcal{D}_2(n^{-2\beta}\overline{r})\} \\
\mathcal{D}_3^n(\underline{r}, \overline{r}) &:= \{(\rho, u) \in \mathcal{D}_n : (n^{-2\beta}\rho, n^{-\beta}u) \in \mathcal{D}_3(n^{-2\beta}\underline{r}, n^{-2\beta}\overline{r})\}.
\end{aligned}$$

In the following Proposition we summarize our main estimates formulated in terms of the scaled objects. The statement is a mere corollary of the previous lemmas. It follows by simple scaling from (5.16) and (5.20)-(5.25).

**Proposition 1.** *There exists a constant $C < \infty$, such that given any $0 < \underline{r} < \overline{r} < \infty$ and the scaled Lax entropy/flux pair defined as prescribed above, the following bounds hold uniformly in $n$:*

$$\mathcal{D}_1^n(\underline{r}) \supset \{(\rho, u) : 0 \le \rho \le \underline{r} - C^{-1}\sqrt{\underline{r}}|u|\} \tag{5.29}$$

$$\mathcal{D}_3^n(\underline{r}, \overline{r}) \subset \{(\rho, u) : \rho \le \overline{r} + C\sqrt{\overline{r}}|u|\} \tag{5.30}$$

$$|S_\rho^n(\rho, u) - 1\!\!1_{\mathcal{D}_2^n(\overline{r})}(\rho, u)| \le C\, 1\!\!1_{\mathcal{D}_3^n(\underline{r}, \overline{r})}(\rho, u), \tag{5.31}$$

$$|S_u^n(\rho, u)| \le \frac{C(\sqrt{\overline{r}} - \sqrt{\underline{r}})}{\log(\overline{r}/\underline{r})} 1\!\!1_{\mathcal{D}_3^n(\underline{r}, \overline{r})}(\rho, u), \tag{5.32}$$

$$|S_{\rho\rho}^n(\rho, u)| \le \frac{C}{\log(\overline{r}/\underline{r})} \frac{1}{\underline{r} + \rho} 1\!\!1_{\mathcal{D}_3^n(\underline{r}, \overline{r})}(\rho, u), \tag{5.33}$$

$$|S_{\rho u}^n(\rho, u)| \le \frac{C}{\log(\overline{r}/\underline{r})} \frac{1}{\sqrt{\underline{r}} + \sqrt{\rho} + |u|} 1\!\!1_{\mathcal{D}_3^n(\underline{r}, \overline{r})}(\rho, u), \tag{5.34}$$

$$|S_{uu}^n(\rho, u)| \le \frac{C}{\log(\overline{r}/\underline{r})} 1\!\!1_{\mathcal{D}_3^n(\underline{r}, \overline{r})}(\rho, u), \tag{5.35}$$

$$|F^n(\rho, u) - \Psi^n(\rho, u) S_\rho^n(\rho, u)| \le \frac{C\sqrt{\overline{r}}}{\log(\overline{r}/\underline{r})}(\overline{r} + u^2) 1\!\!1_{\mathcal{D}_3^n(\underline{r}, \overline{r})}(\rho, u). \tag{5.36}$$

Due to (5.29) we can choose $\underline{r}$ large enough to ensure that for all $n$

$$\{(\rho, u) : \rho \vee |u| \le M\} \subset \mathcal{D}_1^n(\underline{r}), \tag{5.37}$$

where $M$ is specified by the large deviation bounds given in Proposition 2 (via Lemma 9).

Further on, we choose $\overline{r}$ so large that

$$\frac{C}{\log(\overline{r}/\underline{r})} < \left(100 \sup_{(t,x) \in [0,T] \times \mathbb{T}} |\log \rho(t, x)|\right)^{-1}, \tag{5.38}$$

and thus the bounds (5.33)-(5.35) of Proposition 1 are sufficient for our further purposes.

### 5.3 Outline of the further steps of proof

In section 6 we present the main probabilistic technical ingredients of the forthcoming proof. These are variants of entropy inequalities and of the celebrated one and two block estimates. In Section 7 we give an estimate for the terms with 'large' values of $(\rho, u)$, we prove that

$$\left|\mathbf{E}\left(\int_0^t \frac{1}{n}\sum_{j\in\mathbb{T}^n}\left\{(\partial_x v)\left(n^{3\beta}\widehat{\psi}^n\right)J^n(\widehat{\rho}^n,\widehat{u}^n)\right\}(s,\frac{j}{n})\,ds\right)\right| \tag{5.39}$$
$$\leq \frac{1}{2}\,h^n(t)+\frac{1}{2}\,s^n(t)+C\int_0^t h^n(s)\,ds+o(1).$$

In Section 8 we estimate the terms with 'small' values of $(\rho, u)$, the section is divided into four subsections.

In subsection 8.1 we prove

$$\left|\mathbf{E}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\left\{(\partial_x v)\,\left(\widehat{\rho}^n u+\rho\widehat{u}^n-\rho u\right)\,J^n(\widehat{\rho}^n,\widehat{u}^n)\right\}(s,\frac{j}{n})\right)\right| \tag{5.40}$$
$$\leq C\,h^n(s)+o(1).$$

In subsection 8.2 we prove the one block estimate

$$\left|\mathbf{E}\left(\int_0^t \frac{1}{n}\sum_{j\in\mathbb{T}^n}\left\{(\partial_x v)\,n^{3\beta}\left(\widehat{\psi}^n-\Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\right)\,I^n(\widehat{\rho}^n,\widehat{u}^n)\right\}(s,\frac{j}{n})\,ds\right)\right| \tag{5.41}$$
$$= o(1).$$

In subsection 8.3 we control the Taylor approximation

$$\left|\mathbf{E}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\left\{(\partial_x v)\,\left(\Psi^n(\widehat{\rho}^n,\widehat{u}^n)-\widehat{\rho}^n\widehat{u}^n\right)\,I^n(\widehat{\rho}^n,\widehat{u}^n)\right\}(s,\frac{j}{n})\right)\right| \tag{5.42}$$
$$\leq C\,h^n(s)+o(1).$$

Finally, in subsection 8.4 we control the fluctuations

$$\left|\mathbf{E}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\left\{(\partial_x v)\,\left(\widehat{\rho}^n-\rho\right)\left(\widehat{u}^n-u\right)\,I^n(\widehat{\rho}^n,\widehat{u}^n)\right\}(s,\frac{j}{n})\right)\right| \tag{5.43}$$
$$\leq C\,h^n(s)+o(1).$$

Having all these done, from (4.22), (5.6) and the bounds (5.39), (5.40), (5.41), (5.42), (5.43) it follows that

$$\int_0^t \mathcal{A}^n(s)ds \leq \frac{1}{2}\,h^n(t)+\frac{1}{2}\,s^n(t)+C\int_0^t h^n(s)ds+o(1). \tag{5.44}$$

Finally, from (4.21), (4.24), (5.44) and noting that $s^n(t)\geq 0$ we get the desired Grönwall inequality (4.4) and the Theorem follows. Note the importance of the term $-\partial_t s^n(t)$ on the right hand side of (4.6).

# 6 Tools

## 6.1 Fixed time estimates

In the estimates with fixed time $s \in [0,T]$ we shall use the notation

$$L = L(n) := n^{-2\beta} l. \tag{6.1}$$

Note that $L \gg 1$ as $n \to \infty$.

The following general entropy estimate will be exploited all over:

**Lemma 5.** (Fixed time entropy inequality)
*Let $l \le n$, $\mathcal{V} : \Omega^l \to \mathbb{R}$ and denote $\mathcal{V}_j(\underline{\omega}) := \mathcal{V}(\omega_j, \dots, \omega_{j+l-1})$. Then for any $\gamma > 0$*

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n} \mathcal{V}_j(\mathcal{X}^n_s)\Big) \le \frac{1}{\gamma} h^n(s) + \frac{1}{\gamma L}\frac{1}{n}\sum_{j\in\mathbb{T}^n} \log \mathbf{E}_{\nu^n_s}\left(\exp\left\{\gamma L \mathcal{V}_j\right\}\right). \tag{6.2}$$

This lemma is standard tool in the context of relative entropy method. For its proof we refer the reader to the original paper [26] or the monograph [10].

**Proposition 2.** (Fixed time large deviation bounds)
*(i) For any $\varepsilon > 0$ there exists $M < \infty$ such that for any $s \in [0,T]$*

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n} \left\{\left(1 + \widehat{\rho}^n + |\widehat{u}^n|\right) 1\!\!1_{\{\widehat{\rho}^n \vee |\widehat{u}^n| > M\}}\right\}(s, \frac{j}{n})\Big) \le \varepsilon\, h^n(s) + o(1). \tag{6.3}$$

*(ii) There exist $C < \infty$ and $M < \infty$ such that for any $s \in [0,T]$*

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n} \left\{|\widehat{u}^n|^2\, 1\!\!1_{\{\widehat{\rho}^n \vee |\widehat{u}^n| > M\}}\right\}(s, \frac{j}{n})\Big) \le C\, h^n(s) + o(1). \tag{6.4}$$

The proof of Proposition 2 is postponed to subsection 10.1. It relies on the entropy inequality (6.2) of Lemma 5, the stochastic dominations formulated in Lemma 8 (see subsection 10.1) and standard large deviation bounds.

We shall refer to (6.3) and (6.4) as *large deviation bounds.*

**Proposition 3.** (Fixed time fluctuation bounds)
*For any $M < \infty$ there exists a $C < \infty$ such that the following bounds hold:*

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n} |\widehat{u}^n - u|^2\,(s, \frac{j}{n})\Big) \le C\, h^n(s) + o(1), \tag{6.5}$$

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n} \left\{|\widehat{\rho}^n - \rho|^2\, 1\!\!1_{\{\widehat{\rho}^n \le M\}}\right\}(s, \frac{j}{n})\Big) \le C\, h^n(s) + o(1). \tag{6.6}$$

The proof of Proposition 3 is postponed to subsection 10.2. It relies on the entropy inequality (6.2) of Lemma 5, and Gaussian fluctuation estimates.

We shall refer to (6.5) and (6.6) as *fluctuation bounds.*

## 6.2 Convergence to local equilibrium and a priori bounds

The hydrodynamic limit relies on macroscopically fast convergence to (local) equilibrium in blocks of mesoscopic size $l$. Fix the block size $l$ and $(N, Z) \in \mathbb{N} \times (w_0/2)\mathbb{Z}$ with the restriction $N \in [0, l \max \eta]$, $Z \in [l \min \zeta, l \max \zeta]$ and denote

$$\begin{aligned}
\Omega^l_{N,Z} &:= \{\underline{\omega} \in \Omega^l : \sum_{j=1}^{l} \eta_j = N, \sum_{j=1}^{l} \zeta_j = Z\}, \\
\pi^l_{N,Z}(\underline{\omega}) &:= \pi^l_{\lambda,\theta}(\underline{\omega} \,|\, \sum_{j=1}^{l} \eta_j = N. \sum_{j=1}^{l} \zeta_j = Z),
\end{aligned}$$

Expectation with respect to the measure $\pi^l_{N,Z}$ is denoted by $\mathbf{E}^l_{N,Z}(\,\cdot\,)$. For $f : \Omega^l_{N,Z} \to \mathbb{R}$ let

$$K^l_{N,Z} f(\underline{\omega}) := \sum_{j=1}^{l-1} \sum_{\omega', \omega''} s(\omega_j, \omega_{j+1}; \omega', \omega'') \big(f(\Theta^{\omega',\omega''}_{j,j+1}\underline{\omega}) - f(\underline{\omega})\big),$$

$$D^l_{N,Z}(f) := \frac{1}{2} \sum_{j=1}^{l-1} \mathbf{E}^l_{N,Z} \left( \sum_{\omega', \omega''} s(\omega_j, \omega_{j+1}; \omega', \omega'') \big(f(\Theta^{\omega',\omega''}_{j,j+1}\underline{\omega}) - f(\underline{\omega})\big)^2 \right).$$

In plain words: $\Omega^l_{N,Z}$ is the hyperplane of configurations $\underline{\omega} \in \Omega^l$ with fixed values of the conserved quantities, $\pi^l_{N,Z}$ is the *microcanonical distribution* on this hyperplane, $K^l_{N,Z}$ is the symmetric infinitesimal generator restricted to the hyperplane $\Omega^l_{N,Z}$, and finally $D^l_{N,Z}$ is the Dirichlet form associated to $K^l_{N,Z}$. Note, that $K^l_{N,Z}$ is defined with *free boundary conditions.*

The convergence to local equilibrium is *quantitatively controlled* by the following uniform logarithmic Sobolev estimate, assumed to hold:

(I) *Logarithmic Sobolev inequality:* There exists a finite constant $\aleph$ such that for any $l \in \mathbb{N}$, $(N, Z) \in \mathbb{N} \times (w_0/2)\mathbb{Z}$ with the restriction $N \in [0, l \max \eta]$, $Z \in [l \min \zeta, l \max \zeta]$, and any $h : \Omega^l_{N,Z} \to \mathbb{R}_+$ with $\mathbf{E}^l_{N,Z}(h) = 1$ the following bound holds:

$$\mathbf{E}^l_{N,Z}\big(h \log h\big) \leq \aleph\, l^2 D^l_{N,Z}\left(\sqrt{h}\right). \tag{6.7}$$

**Remark:** The uniform logarithmic Sobolev inequality (6.7) is expected to hold for a very wide range of locally finite interacting particle systems, though we do not know about a fully general proof. In [27] the logarithmic Sobolev inequality is proved for symmetric $K$-exclusion processes. This implies that (6.7) holds for the two lane models defined in subsection 2. In [6] Yau's method of proving logarithmic Sobolev inequality is applied and the logarithmic Sobolev inequality is stated for random stirring models with arbitrary number of colors. In particular, (6.7) follows for the $\{-1, 0, +1\}$-model defined in subsection 2.

The following large deviation bound goes back to Varadhan [25]. See also the monographs [10] and [4].

**Lemma 6.** (Time-averaged entropy inequality, local equilibrium)
*Let $l \le n$, $\mathcal{V} : \Omega^l \to \mathbb{R}_+$ and denote $\mathcal{V}_j(\underline{\omega}) := \mathcal{V}(\omega_j, \dots, \omega_{j+l-1})$. Then for any $\gamma > 0$*

$$\mathbf{E}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\int_0^t \mathcal{V}_j(\mathcal{X}^n_s)\,ds\right) \le \tag{6.8}$$

$$\frac{\aleph\, l^3}{2\,\gamma\, n^{1+3\beta+\delta}}\Big(s^n(t) + \frac{2\,n^{1+3\beta+\delta}\,t}{\aleph l^3}\max_{N,Z}\log \mathbf{E}^l_{N,Z}\big(\exp\{\gamma\mathcal{V}\}\big)\Big).$$

**Remarks:** (1) Since

$$\frac{n^{1+3\beta+\delta}}{l^3} = o(1),$$

in order to apply efficiently Lemma 6 one has to chose $\gamma = \gamma(n)$ so that

$$\mathbf{E}^l_{N,Z}\big(\exp\{\gamma\mathcal{V}\}\big) = \mathcal{O}(1),$$

*uniformly* in the block size $l = l(n) \in \mathbb{N}$, and in $N \in [0, l\max\eta]$ and $Z \in [l\min\zeta, l\max\zeta]$.
(2) Assuming only uniform bound of size $\sim \big(\aleph l^2\big)^{-1}$ on the spectral gap of $K^l_{N,Z}$ (rather than the stronger logarithmic Sobolev inequality (6.7)) and using Rayleigh-Schrödinger perturbation (see Appendix 3 of [10]) we would get

$$\mathbf{E}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\int_0^t \mathcal{V}_j(\mathcal{X}^n_s)\,ds\right) \le$$

$$\frac{\aleph\, l^3\,\|\mathcal{V}\|_\infty}{2\,n^{1+3\beta+\delta}}s^n(t) + t\,\|\mathcal{V}\|_\infty\Big(\frac{\max_{N,Z}\mathbf{E}^l_{N,Z}\big(\mathcal{V}\big)}{\|\mathcal{V}\|_\infty} + \frac{\max_{N,Z}\mathbf{Var}^l_{N,Z}\big(\mathcal{V}\big)}{\|\mathcal{V}\|^2_\infty}\Big),$$

which would not be sufficient for our needs.
(3) The proof of the bound (6.8) explicitly relies on the logarithmic Sobolev inequality (6.7). It appears in [28] and it is reproduced in several places, see e.g. [4], [5]. We do not repeat it here.

The main probabilistic ingredients of our proof are summarized in Proposition 4 which is consequence of Lemma 6. These are variants of the celebrated *one block estimate*, respectively, *two blocks estimate* of Varadhan and co-authors.

**Proposition 4.** (Time-averaged block replacement and gradient bounds)
*Given a local variable $\xi : \Omega^m \to \mathbb{R}$ there exists a constant $C$ such that the following bounds hold:*
*(i)*

$$\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\big|\{\widehat{\xi}^n - \Xi(\widehat{\eta}^n, \widehat{\zeta}^n)\}(s,x)\big|^2\,dx\,ds\right) \le C\,\frac{l^2}{n^{1+3\beta+\delta}}\big(s^n(t) + o(1)\big). \tag{6.9}$$

*(ii)*

$$\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\big|\partial_x\widehat{\xi}^n(s,x)\big|^2\,dx\,ds\right) \le Cn^{1-3\beta-\delta}\big(s^n(t) + o(1)\big). \tag{6.10}$$

*(iii) Further on, if $\xi : \Omega \to \mathbb{R}$ (that is: it depends on a single spin) and $\xi(\omega) = 0$ whenever $\eta(\omega) = 0$ then the following stronger version of the gradient bound holds:*

$$\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\frac{\big|\partial_x\widehat{\xi}^n(s,x)\big|^2}{\widehat{\eta}^n(s,x)}\,dx\,ds\right) \le Cn^{1-3\beta-\delta}\big(s^n(t) + o(1)\big). \tag{6.11}$$

The proof of Proposition 4 is postponed to subsection 10.3. It relies on the large deviation bound (6.8) and some elementary probability estimates stated in Lemma 12 (see subsection 10.3).

We shall refer to (6.9), respectively, (6.10) and (6.11) as the *block replacement bounds*, respectively, the *gradient bounds.*

We shall apply (6.9) to $\xi = \phi$ and $\xi = \psi$. From (6.10) it follows that

$$\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\partial_x \widehat{u}^n(s,x)\right|^2 dx\, ds\right) \le Cn^{1-\beta-\delta}\big(s^n(t)+o(1)\big), \tag{6.12}$$

$$\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\partial_x \widehat{\rho}^n(s,x)\right|^2 dx\, ds\right) \le Cn^{1+\beta-\delta}\big(s^n(t)+o(1)\big). \tag{6.13}$$

Using (6.11) the last bound is improved to

$$\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \frac{\left|\partial_x \widehat{\rho}^n(s,x)\right|^2}{\widehat{\rho}^n(s,x)} dx\, ds\right) \le Cn^{1-\beta-\delta}\big(s^n(t)+o(1)\big). \tag{6.14}$$

The bound (6.11) will also be applied to $\xi = \kappa$ (see (2.11) and (2.12)) to get

$$\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \frac{\left|n^{2\beta}\partial_x \widehat{\kappa}^n(s,x)\right|^2}{\widehat{\rho}^n(s,x)} dx\, ds\right) \le Cn^{1-\beta-\delta}\big(s^n(t)+o(1)\big). \tag{6.15}$$

# 7 Control of the large values of $(\rho, u)$: proof of (5.39)

## 7.1 Preparations

In the present section we prove (5.39). First we replace $\frac{1}{n}\sum_{j\in\mathbb{T}^n}\cdots$ by $\int_{\mathbb{T}}\cdots\, dx$. Note that given a smooth function $F : \mathbb{T} \to \mathbb{R}$

$$\left|\frac{1}{n}\sum_{j\in\mathbb{T}^n} F\Big(\frac{j}{n}\Big) - \int_{\mathbb{T}} F(x)\, dx\right| \le \frac{1}{n}\left(\int_{\mathbb{T}} \left|\partial_x F(x)\right|^2\, dx\right)^{1/2}. \tag{7.1}$$

Hence it follows that

$$\mathbf{E}\left(\int_0^t \frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big\{\big(\partial_x v\big)\,\big(n^{3\beta}\widehat{\psi}^n\big)\, J^n(\widehat{\rho}^n,\widehat{u}^n)\Big\}\Big(s,\frac{j}{n}\Big)\, ds\right) = \tag{7.2}$$

$$\mathbf{E}\left(\int_0^t \int_{\mathbb{T}}\Big\{\big(\partial_x v\big)\,\big(n^{3\beta}\widehat{\psi}^n\big)\, J^n(\widehat{\rho}^n,\widehat{u}^n)\Big\}(s,x)\, dx\, ds\right) + A^n_{13},$$

where $A^n_{13}$ is again a simple numerical error term:

$$\begin{aligned}
|A^n_{13}| \;&\le\; Cn^{3\beta-1}\Bigg\{1 + \sqrt{\mathbf{E}\left(\int_0^s \int_{\mathbb{T}} \left|\partial_x \widehat{\psi}^n(s,x)\right|^2\, dx\, ds\right)} + \sqrt{\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\partial_x \widehat{\rho}^n(s,x)\right|^2\, dx\, ds\right)} \\
&\qquad + \sqrt{\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\partial_x \widehat{u}^n(s,x)\right|^2\, dx\, ds\right)}\Bigg\} \\
&=\; \mathcal{O}(n^{5\beta} l^{-1}) = o(1).
\end{aligned} \tag{7.3}$$

In the last step we use the most straightforward gradient bound (4.12). (Using the gradient bound (6.10) we could obtain the much better upper bound

$$|A_{13}^n| = \mathcal{O}(n^{(-1-\delta+7\beta)/2}) = o(n^{5\beta}l^{-1}),$$

but we do not need this sharper estimate at this stage.)

So, we have to prove that the first term on the right hand side of (7.2) is negligible. Recall that $J^n = S^n_\rho$. We start with the application of the martingale identity:

$$\begin{aligned}
\mathbf{E}\Big(\int_{\mathbb{T}}\Big\{\{vS^n(\widehat{\rho}^n,\widehat{u}^n)\}(t,x) - \{vS^n(\widehat{\rho}^n,\widehat{u}^n)\}(0,x) - \int_0^t \{(\partial_t v)S^n(\widehat{\rho}^n,\widehat{u}^n)\}(s,x)\,ds\Big\}\,dx\Big) = \\
\mathbf{E}\Big(\int_0^t\int_{\mathbb{T}} v(s,x)\,\Big(n^{1+\beta}L^nS^n(\widehat{\rho}^n(x),\widehat{u}^n(x))\Big)\,(\mathcal{X}^n_s)\,dx\,ds\Big) \\
+\mathbf{E}\Big(\int_0^t\int_{\mathbb{T}} v(s,x)\,\Big(n^{1+\beta+\delta}K^nS^n(\widehat{\rho}^n(x),\widehat{u}^n(x))\Big)\,(\mathcal{X}^n_s)\,dx\,ds\Big)
\end{aligned} \tag{7.4}$$

### 7.2 The left hand side of (7.4)

From (5.31), (5.32), and (5.37), we conclude that

$$|S^n(\rho,u)| \le C\,(\rho + |u|\,)\,1\!\!1_{\{\rho\vee|u|>M\}}.$$

Hence, using the large deviation bound (6.3) it follows that, by choosing $M$ sufficiently large we obtain

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big|S^n(\widehat{\rho}^n,\widehat{u}^n)(s,\frac{j}{n})\Big|\Big) \le \varepsilon\, h^n(s) + o(1).$$

Hence, applying again (7.1) we get

$$|\,\text{l.h.s. of (7.4)}\,| \le \frac{1}{2}\,h^n(t) + C\int_0^t h^n(s)\,ds + o(1). \tag{7.5}$$

**Remark:** Note that this is the point where $M$ and thus the lower edge of the cutoff is fixed. Also note the importance of the factor $1/2$ in fromt of $h^n(t)$ on the right hand side.

### 7.3 The right hand side of (7.4): first computations

First we compute how the infinitesimal generators $n^{1+\beta}L^n$ and $n^{1+\beta+\delta}K^n$ act on the function $\underline{\omega} \mapsto S^n(\widehat{\rho}^n(x),\widehat{u}^n(x))$:

$$\begin{aligned}
&n^{1+\beta}L^nS^n(\widehat{\rho}^n(x),\widehat{u}^n(x)) = \\
&\qquad\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big(n^{3\beta}\partial_x\widehat{\psi}^n\big) + S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(n^{2\beta}\partial_x\widehat{\phi}^n\big)\Big\}(x) + A^n_{14}(x),
\end{aligned} \tag{7.6}$$

$$\begin{aligned}
&n^{1+\beta+\delta}K^nS^n(\widehat{\rho}^n(x),\widehat{u}^n(x)) = \\
&\qquad n^{-1+\beta+\delta}\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big(n^{2\beta}\partial_x^2\widehat{\kappa}^n\big) + S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(n^{\beta}\partial_x^2\widehat{\chi}^n\big)\Big\}(x) + A^n_{15}(x),
\end{aligned} \tag{7.7}$$

where $A^n_{14}(x)$ and $A^n_{15}(x)$ are the following *numerical error terms*:

$$
\begin{aligned}
A^n_{14}(x) = A^n_{14}(\underline{\omega}, x) &:= n^{1+\beta} \sum_{j\in\mathbb{T}^n} \sum_{\omega',\omega'\in\Omega} r(\omega_j, \omega_{j+1}; \omega', \omega') \times \\
&\qquad \Big\{ S^n\big(\widehat{\rho}^n(x) + \frac{n^{2\beta}}{l}\big(a\big(\frac{nx-j}{l}\big) - a\big(\frac{nx-j-1}{l}\big)\big)(\eta' - \eta_j), \\
&\qquad\qquad \widehat{u}^n(x) + \frac{n^{\beta}}{l}\big(a\big(\frac{nx-j}{l}\big) - a\big(\frac{nx-j-1}{l}\big)\big)(\zeta' - \zeta_j)\big) \\
&\qquad -S^n\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big) \\
&\qquad -S^n_\rho\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big)\frac{n^{2\beta}}{l^2}a'\big(\frac{nx-j}{l}\big)(\eta' - \eta_j) \\
&\qquad -S^n_u\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big)\frac{n^{\beta}}{l^2}a'\big(\frac{nx-j}{l}\big)(\zeta' - \zeta_j)\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
A^n_{15}(x) = A^n_{15}(\underline{\omega}, x) &:= n^{1+\beta+\delta} \sum_{j\in\mathbb{T}^n} \sum_{\omega',\omega'\in\Omega} s(\omega_j, \omega_{j+1}; \omega', \omega') \times \\
&\qquad \Big\{ S^n\big(\widehat{\rho}^n(x) + \frac{n^{2\beta}}{l}\big(a\big(\frac{nx-j}{l}\big) - a\big(\frac{nx-j-1}{l}\big)\big)(\eta' - \eta_j), \\
&\qquad\qquad \widehat{u}^n(x) + \frac{n^{\beta}}{l}\big(a\big(\frac{nx-j}{l}\big) - a\big(\frac{nx-j-1}{l}\big)\big)(\zeta' - \zeta_j)\big) \\
&\qquad -S^n\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big) \\
&\qquad -S^n_\rho\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big)\frac{n^{2\beta}}{l^2}a'\big(\frac{nx-j}{l}\big)(\eta' - \eta_j) \\
&\qquad -S^n_u\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big)\frac{n^{\beta}}{l^2}a'\big(\frac{nx-j}{l}\big)(\zeta' - \zeta_j)\Big\} \\
&\quad +S^n_\rho\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big) \times \\
&\quad \frac{n^{1+3\beta+\delta}}{l^2} \sum_{j\in\mathbb{T}^n} \Big\{ a'\big(\frac{nx-j}{l}\big)(\kappa_{j+1} - \kappa_j) + \frac{1}{l}a''\big(\frac{nx-j}{l}\big)\kappa_j \Big\} \\
&\quad +S^n_u\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big) \times \\
&\quad \frac{n^{1+2\beta+\delta}}{l^2} \sum_{j\in\mathbb{T}^n} \Big\{ a'\big(\frac{nx-j}{l}\big)(\chi_{j+1} - \chi_j) + \frac{1}{l}a''\big(\frac{nx-j}{l}\big)\chi_j \Big\}.
\end{aligned}
$$

These error terms are easily estimated: using the fact that the second partial derivatives of $S^n$ are uniformly bounded and $\zeta$ and $\eta$ are bounded, by simple Taylor expansion after tedious but otherwise straightforward computations we find:

$$
\sup_{\underline{\omega}\in\Omega^n} \sup_{x\in\mathbb{T}} |A^n_{14}(\underline{\omega}, x)| \le Cn^{1+3\beta}l^{-2} = o(1), \tag{7.8}
$$

$$
\sup_{\underline{\omega}\in\Omega^n} \sup_{x\in\mathbb{T}} |A^n_{15}(\underline{\omega}, x)| \le Cn^{1+5\beta+\delta}l^{-3} = o(1). \tag{7.9}
$$

No probabilistic arguments are involved in these bounds. The global (averaged and integrated) error introduced by these terms will be of the same order.

Next we do some further transformations on the main terms coming from the right hand sides of (7.6) and (7.7). Performing integrations by part, introducing the macroscopic fluxes and using (5.28) we obtain:

$$
\begin{aligned}
-\int_{\mathbb{T}} v(x)\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\partial_x\big(n^{3\beta}\widehat{\psi}^n\big) + S^n_u(\widehat{\rho}^n,\widehat{u}^n)\partial_x\big(n^{2\beta}\widehat{\phi}^n\big)\Big\}(x)\,dx = \\
\int_{\mathbb{T}} \partial_x v(x)\Big\{\big(n^{3\beta}\widehat{\psi}^n\big)S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\Big\}(x)\,dx \\
+\int_{\mathbb{T}} \partial_x v(x)\Big\{F^n(\widehat{\rho}^n,\widehat{u}^n) - \Psi^n(\widehat{\rho}^n,\widehat{u}^n)S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\Big\}(x)\,dx \\
+\int_{\mathbb{T}} \partial_x v(x)\Big\{n^{2\beta}S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(\widehat{\phi}^n - \Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(x)\,dx \\
+\int_{\mathbb{T}} v(x)\Big\{\ n^{3\beta}S^n_{\rho\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(\widehat{\psi}^n - \Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\big) \\
+n^{3\beta}S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(\widehat{\psi}^n - \Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\big) \\
+n^{2\beta}S^n_{u\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(\widehat{\phi}^n - \Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big) \\
+n^{2\beta}S^n_{uu}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(\widehat{\phi}^n - \Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(x)\,dx
\end{aligned}
$$

Note that, since $J^n = S^n_\rho$, the first term on the right hand side is exactly the expression in the main term on the right hand side of (7.2). Estimating the other terms on the right hand side of (7.10) is the object of the next subsection.

Now we turn to the main term on the right hand side of (7.7). Here, straightforward integration by parts yields

$$
\begin{aligned}
-\int_{\mathbb{T}} v(x)\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big(n^{2\beta}\partial_x^2\widehat{\kappa}^n\big) + S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(n^{\beta}\partial_x^2\widehat{\chi}^n\big)\Big\}(x)\,dx = \qquad (7.10)\\
\int_{\mathbb{T}} \partial_x v(x)\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big(n^{2\beta}\partial_x\widehat{\kappa}^n\big) + S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(n^{\beta}\partial_x\widehat{\chi}^n\big)\Big\}(x)\,dx \\
+\int_{\mathbb{T}} v(x)\Big\{\quad S^n_{\rho\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(n^{2\beta}\partial_x\widehat{\kappa}^n\big) \\
+ S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\Big(\big(\partial_x\widehat{u}^n\big)\big(n^{2\beta}\partial_x\widehat{\kappa}^n\big) + \big(\partial_x\widehat{\rho}^n\big)\big(n^{\beta}\partial_x\widehat{\chi}^n\big)\Big) \\
+ S^n_{uu}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(n^{\beta}\partial_x\widehat{\chi}^n\big)\Big\}(x)\,dx
\end{aligned}
$$

We will estimate the terms emerging from the right hand side in the next subsection.

### 7.4 The right hand side of (7.4): bounds

#### 7.4.1

We note that

$$\big|F^n(\widehat{\rho}^n,\widehat{u}^n)-\Psi^n(\widehat{\rho}^n,\widehat{u}^n)S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big|\le C\big(1+|\widehat{u}^n|^2\big)1\!\!1_{\{\widehat{\rho}^n\vee|\widehat{u}^n|>M\}},$$

see (5.36) and (5.37). Hence, applying the large deviation bounds (6.3) and (6.4) we obtain

$$\mathbf{E}\left(\int_{\mathbb{T}}\Big|\Big\{F^n(\widehat{\rho}^n,\widehat{u}^n)-\Psi^n(\widehat{\rho}^n,\widehat{u}^n)S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\Big\}(s,x)\Big|\,dx\right)\tag{7.11}$$
$$\le C\,h^n(s)+o(1).$$

#### 7.4.2

We use

$$|S^n_u(\widehat{\rho}^n,\widehat{u}^n)|\le C,$$

see (5.32) and the first block replacement bound (6.9) to obtain:

$$\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{2\beta}S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(\widehat{\phi}^n-\Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(s,x)\Big|\,dx\,ds\right)\tag{7.12}$$
$$\le C\,l\,n^{(-1-\delta+\beta)/2}=o(1).$$

#### 7.4.3

Next we use

$$\big|S^n_{\rho\rho}(\widehat{\rho}^n,\widehat{u}^n)\big|\le\frac{C}{\log(\overline{r}/\underline{r})}\frac{1}{\underline{r}+\widehat{\rho}^n},\qquad \big|S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\big|\le\frac{C}{\log(\overline{r}/\underline{r})}\frac{1}{\sqrt{\underline{r}}+\sqrt{\widehat{\rho}^n}}$$
$$|S^n_{uu}(\widehat{\rho}^n,\widehat{u}^n)|\le\frac{C}{\log(\overline{r}/\underline{r})},\tag{7.13}$$

see (5.33), (5.34), respectively, (5.35), and note that here *we do not exploit* the fact that the constant factors on the right hand side are actually small. These, together with the block replacement bounds (6.9), the gradient bounds (6.12), (6.14) and the bound (4.1) on the relative entropy $s^n(t)$ yield the following four estimates:

$$\begin{aligned}
&\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{3\beta}S^n_{\rho\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(\widehat{\psi}^n-\Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(s,x)\Big|\,dx\,ds\right)\le C\,l\,n^{\beta-\delta}=o(1),\\
&\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{3\beta}S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(\widehat{\psi}^n-\Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(s,x)\Big|\,dx\,ds\right)\le C\,l\,n^{\beta-\delta}=o(1),\\
&\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{2\beta}S^n_{u\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(\widehat{\phi}^n-\Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(s,x)\Big|\,dx\,ds\right)\le C\,l\,n^{-\delta}=o(1),\qquad(7.14)\\
&\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{2\beta}S^n_{uu}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(\widehat{\phi}^n-\Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(s,x)\Big|\,dx\,ds\right)\le C\,l\,n^{-\delta}=o(1).
\end{aligned}$$

#### 7.4.4

Using

$$\left|S_\rho^n(\widehat{\rho}^n, \widehat{u}^n)\right| \le C, \qquad |S_u^n(\widehat{\rho}^n, \widehat{u}^n)| \le C,$$

see (5.31) and (5.32), and the gradient bounds (6.12), (6.13) we obtain the following two bounds:

$$\begin{aligned} n^{-1+\beta+\delta}\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{S_\rho^n(\widehat{\rho}^n, \widehat{u}^n)\big(\partial_x \widehat{\rho}^n\big)\right\}(s,x)\right| \, dx\, ds\right) \\ \le C\, n^{(-1+\delta+3\beta)/2} = o(1), \\ n^{-1+\beta+\delta}\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{S_u^n(\widehat{\rho}^n, \widehat{u}^n)\big(\partial_x \widehat{u}^n\big)\right\}(s,x)\right| \, dx\, ds\right) \\ \le C\, n^{(-1+\delta+\beta)/2} = o(1). \end{aligned} \tag{7.15}$$

#### 7.4.5

The following bounds are of paramount importance and they are sharp. We use (7.13) again and note that here we exploit it in its *full power*: the constant factor on the right hand side is small. These and the gradient bounds (6.12) and (6.14) yield the following three bounds:

$$\begin{aligned} n^{-1+\beta+\delta}\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{S_{\rho\rho}^n(\widehat{\rho}^n, \widehat{u}^n)\big(\partial_x \widehat{\rho}^n\big)\big(n^{2\beta}\partial_x \widehat{\kappa}^n\big)\right\}(s,x)\right| \, dx\, ds\right) \\ \le c\, s^n(t) + o(1), \\ n^{-1+\beta+\delta}\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{S_{\rho u}^n(\widehat{\rho}^n, \widehat{u}^n)\big(\partial_x \widehat{u}^n\big)\big(n^{2\beta}\partial_x \widehat{\kappa}^n\big)\right\}(s,x)\right| \, dx\, ds\right) \\ \le c\, s^n(t) + o(1), \\ n^{-1+\beta+\delta}\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{S_{\rho u}^n(\widehat{\rho}^n, \widehat{u}^n)\big(\partial_x \widehat{\rho}^n\big)\big(n^{\beta}\partial_x \widehat{\chi}^n\big)\right\}(s,x)\right| \, dx\, ds\right) \\ \le c\, s^n(t) + o(1), \\ n^{-1+\beta+\delta}\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{S_{uu}^n(\widehat{\rho}^n, \widehat{u}^n)\big(\partial_x \widehat{u}^n\big)\big(n^{\beta}\partial_x \widehat{\chi}^n\big)\right\}(s,x)\right| \, dx\, ds\right) \\ \le c\, s^n(t) + o(1). \end{aligned} \tag{7.16}$$

The ratio $\overline{r}/\underline{r}$ is chosen so large that

$$c \sup_{(t,x)\in[0,T]\times\mathbb{T}} |v(t,x)| < \frac{1}{2}. \tag{7.17}$$

### 7.5 Sumup

The identities (7.6), (7.7), (7.10), (7.10) and the bounds (7.8), (7.9), (7.11), (7.12), (7.14), (7.15), (7.16) yield

$$\left|\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left\{(\partial_x v)\,(n^{3\beta}\widehat{\psi}^n)\,S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\right\}(s,x)\,dx\,ds\right) - \left(\text{r.h.s. of (7.4)}\right)\right| \\ \leq \frac{1}{2}\,s^n(t) + c\int_0^t h^n(s)\,ds + o(1). \tag{7.18}$$

Finally, from (7.2), (7.3), (7.4), (7.5) and (7.18) we obtain (5.39).

## 8 Control of the small values of $(\rho, u)$: proof of the bounds (5.40) to (5.43)

### 8.1 Proof of (5.40)

We exploit the straightforward inequality

$$|J^n(\widehat{\rho}^n,\widehat{u}^n)| = \left|S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\right| \leq C\,\mathbb{1}_{\{\widehat{\rho}^n\vee|\widehat{u}^n|>M\}},$$

see (5.31) and (5.37), and boundedness of the functions $\rho(t,x)$, $u(t,x)$, $\partial_x v(t,x)$. Thus, applying the large deviation bound (6.3) we readily obtain (5.40).

### 8.2 Proof of (5.41)

This is very similar to what has been done in various parts of subsection 7.4. We use the block replacement bound (6.9) and the bound

$$|I^n(\widehat{\rho}^n,\widehat{u}^n)| = \left|1 - S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\right| \leq C \tag{8.1}$$

which follows from (5.31). We readily obtain

$$\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{n^{3\beta}\left(\widehat{\psi}^n - \Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\right)\,|I^n(\widehat{\rho}^n,\widehat{u}^n)|\right\}(s,x)\right|\,ds\,dx\right) \\ \leq C\,l\,n^{(-1-\delta+3\beta)/2} = o(1),$$

which proves (5.41).

### 8.3 Proof of (5.42)

We write

$$I^n(\widehat{\rho}^n,\widehat{u}^n) = \mathbb{1}_{\{\widehat{\rho}^n\vee|\widehat{u}^n|\leq M\}} + \mathbb{1}_{\{\widehat{\rho}^n\vee|\widehat{u}^n|>M\}} I^n(\widehat{\rho}^n,\widehat{u}^n), \tag{8.2}$$

and note that, by Taylor expansion of the function $(\rho,u)\mapsto\Psi(\rho,u)$

$$|\Psi^n(\widehat{\rho}^n,\widehat{u}^n) - \widehat{\rho}^n\widehat{u}^n|\,\mathbb{1}_{\{\widehat{\rho}^n\vee|\widehat{u}^n|\leq M\}} \leq C\,n^{-2\beta}.$$

On the other hand

$$|\Psi^n(\widehat{\rho}^n, \widehat{u}^n)| \le C\,\widehat{\rho}^n\ |\widehat{u}^n|$$

and

$$\widehat{\rho}^n\ |I^n(\widehat{\rho}^n, \widehat{u}^n)| \le C(1 + |\widehat{u}^n|), \tag{8.3}$$

see (5.16) and (5.31). Thus

$$|\Psi^n(\widehat{\rho}^n, \widehat{u}^n) - \widehat{\rho}^n\widehat{u}^n|\,|I^n(\widehat{\rho}^n, \widehat{u}^n)| \le C\left(n^{-2\beta} + \big(\,|\widehat{u}^n| + |\widehat{u}^n|^2\,\big)\,1\!\!1_{\{\widehat{\rho}^n\vee|\widehat{u}^n|>M\}}\right).$$

From this, using the large deviation bounds (6.3) and (6.4) we obtain (5.42).

### 8.4 Proof of (5.43)

We use again (8.2) and (8.3) and get

$$\begin{aligned}\big|\big(\widehat{\rho}^n - \rho\big)\big(\widehat{u}^n - u\big)\,I^n(\widehat{\rho}^n, \widehat{u}^n)\big| \le \quad & \big|\big(\widehat{\rho}^n - \rho\big)\big(\widehat{u}^n - u\big)\big|\,1\!\!1_{\{\widehat{\rho}^n\vee|\widehat{u}^n|\le M\}} \\ & + C\left(1 + |\widehat{u}^n| + |\widehat{u}^n|^2\right)1\!\!1_{\{\widehat{\rho}^n\vee|\widehat{u}^n|>M\}}\end{aligned}$$

Now the fluctuation bounds (6.5), (6.6), and the large deviation bounds (6.3), (6.4) together yield (5.43).

## 9 Construction of the cutoff function: proofs

### 9.1 Proof of Lemma 2

*Proof.* We sketch the proof for $u \ge 0$ and leave the very similar $u \le 0$ case for the reader. Let $\rho_1 > 0$, $u_1 > 0$ be so chosen that for $(\rho, u) \in [0, \rho_1] \times [0, u_1]$ the following bounds hold with a fixed $c > 0$:

$$|\Phi_u(\rho, u) - \Psi_\rho(\rho, u) - (2\gamma - 1)u| \le cu(u^2 + \rho),$$

$$|\Phi_\rho(\rho, u) - 1| \le c(u^2 + \rho),$$

$$|\Psi_u(\rho, u) - \rho| \le c\rho(u^2 + \rho),$$

and

$$\Phi_u(\rho, u) - \Psi_\rho(\rho, u) \ne 0 \quad \text{for } (\rho, u) \ne (0, 0).$$

This can be done due to the $(\rho, u) \to (0, 0)$ asymptotics of the macroscopic fluxes $\Phi$ and $\Psi$.

It follows that as long as $(\sigma(u; r), u) \in [0, \rho_1] \times [0, u_1]$

$$\frac{d\rho}{du} \le \frac{2\rho(1 + c'(\rho + u^2))}{\sqrt{(2\gamma - 1)^2u^2 + 4\rho} + (2\gamma - 1)u}.$$

This implies

$$\sigma(u;r) \le r + C\left(\sqrt{ru} \wedge r^{\frac{4\gamma-3}{4\gamma-2}} u^{\frac{1}{2\gamma-1}}\right)$$

with a positive C, as long as $(\sigma(u;r), u) \in [0, \rho_1] \times [0, u_1]$.

From our assumptions it also follows that for $\rho \le \rho_1$ and $u > u_1$

$$\frac{d\rho}{du} \le b\,\rho,$$

where

$$b := \sup_{\substack{\rho<\rho_1 \\ u>u_1}} \frac{2\Psi_u(\rho,u)}{\rho\left(\Phi_u(\rho,u) - \Psi_\rho(\rho,u)\right)} < \infty.$$

Hence it follows that for $u \ge u_1$

$$\sigma(u;r) \le \sigma(u_1;r)\exp\{b(u-u_1)\}$$

as long as $\sigma(u;r) \le \rho_1$.

Putting these two arguments together the upper bound

$$\sigma(u;r) \le r + C_1\left(\sqrt{ru} \wedge r^{\frac{4\gamma-3}{4\gamma-2}} u^{\frac{1}{2\gamma-1}}\right), \quad u \ge 0, \quad r < r_0,$$

follows with

$$r_0 := \sup\left\{r : \left(r + C\left(\sqrt{ru_1} \wedge r^{\frac{4\gamma-3}{4\gamma-2}} u_1^{\frac{1}{2\gamma-1}}\right)\right)\exp\{b(u^*-u_1)\} \le \rho_1\right\},$$

and

$$C_1 = \frac{r_0^{\frac{1}{4\gamma-2}}\left(\exp\{b(u^*-u_1)\} - 1\right)}{u_1} + C\exp\{b(u^*-u_1)\}.$$

□

## 9.2 Proof of Lemma 3

Note first that given the bounds (3.9) and condition (H) of subsection 3.1, (5.24) follows directly from (5.23), (5.22) and (5.11). So, we shall concentrate on (5.20)-(5.23) only.

By differentiating in the pde (5.11) and applying straightforward transformations we obtain the following differential equations for $S_\rho$, $S_u$, $S_{\rho\rho}$, and $S_{\rho u}$, respectively:

$$\begin{aligned}\Psi_u(S_\rho)_{\rho\rho} + \left(\Phi_u - \Psi_\rho\right)(S_\rho)_{\rho u} - \Phi_\rho(S_\rho)_{uu} + \\ +\Phi_\rho\left(\frac{\Psi_u}{\Phi_\rho}\right)_\rho (S_\rho)_\rho + \Phi_\rho\left(\frac{\Phi_u - \Psi_\rho}{\Phi_\rho}\right)_\rho (S_\rho)_u = 0,\end{aligned} \tag{9.1}$$

$$\begin{aligned}\Psi_u(S_u)_{\rho\rho} + \left(\Phi_u - \Psi_\rho\right)(S_u)_{\rho u} - \Phi_\rho(S_u)_{uu} + \\ +\Psi_u\left(\frac{\Phi_u - \Psi_\rho}{\Psi_u}\right)_u (S_u)_\rho - \Psi_u\left(\frac{\Phi_\rho}{\Psi_u}\right)_u (S_u)_u = 0,\end{aligned} \tag{9.2}$$

$$\Psi_u (S_{\rho\rho})_{\rho\rho} + \big(\Phi_u - \Psi_\rho\big)(S_{\rho\rho})_{\rho u} - \Phi_\rho (S_{\rho\rho})_{uu} + \tag{9.3}$$
$$+2\Phi_\rho \left(\frac{\Psi_u}{\Phi_\rho}\right)_\rho (S_{\rho\rho})_\rho + 2\Phi_\rho \left(\frac{\Phi_u - \Psi_\rho}{\Phi_\rho}\right)_\rho (S_{\rho\rho})_u - \Phi_\rho \left(\frac{\Psi_u}{\Phi_\rho}\right)_{\rho\rho} S_{\rho\rho}$$
$$= -\Phi_\rho \left(\frac{\Phi_u - \Psi_\rho}{\Phi_\rho}\right)_{\rho\rho} S_{\rho u},$$

$$\Psi_u (S_{\rho u})_{\rho\rho} + \big(\Phi_u - \Psi_\rho\big)(S_{\rho u})_{\rho u} - \Phi_\rho (S_{\rho u})_{uu} + \tag{9.4}$$
$$+\left\{ \Psi_u \left(\frac{\Phi_u - \Psi_\rho}{\Psi_u}\right)_u + \Phi_\rho \left(\frac{\Psi_u}{\Phi_\rho}\right)_\rho \right\} (S_{\rho u})_\rho +$$
$$+\left\{ \Phi_\rho \left(\frac{\Phi_u - \Psi_\rho}{\Phi_\rho}\right)_\rho - \Psi_u \left(\frac{\Phi_\rho}{\Psi_u}\right)_u \right\} (S_{\rho u})_u +$$
$$+\left\{ \Psi_u \left(\frac{\Phi_\rho}{\Psi_u}\right)_u \left(\frac{\Phi_u - \Psi_\rho}{\Phi_\rho}\right)_\rho + \Phi_\rho \left(\frac{\Phi_u - \Psi_\rho}{\Phi_\rho}\right)_{\rho u} \right\} S_{\rho u}$$
$$= -\left\{ \Psi_u \left(\frac{\Phi_\rho}{\Psi_u}\right)_u \left(\frac{\Psi_u}{\Phi_\rho}\right)_\rho + \Phi_\rho \left(\frac{\Psi_u}{\Phi_\rho}\right)_{\rho u} \right\} S_{\rho\rho}.$$

Because of the conditions outlined in subsection 3.1 all the coefficients are smooth functions, if $\rho$ is small enough.

The respective initial conditions are

$$S_\rho(\rho,0) = \frac{\log(r/\underline{r})}{\log(\overline{r}/\underline{r})} \mathbb{1}_{\{\rho\in[\underline{r},\overline{r}]\}} + \mathbb{1}_{\{\rho\in[\overline{r},r_0]\}}, \quad (S_\rho)_u(\rho,0) = 0, \tag{9.5}$$

$$S_u(\rho,0) = 0, \quad (S_u)_u(\rho,0) = \frac{1}{\log(\overline{r}/\underline{r})} \frac{\Psi_u(\rho,0)}{\rho\,\Phi_\rho(\rho,0)} \mathbb{1}_{\{\rho\in[\underline{r},\overline{r}]\}}, \tag{9.6}$$

$$S_{\rho\rho}(\rho,0) = \frac{\rho^{-1}}{\log(\overline{r}/\underline{r})} \mathbb{1}_{\{\rho\in[\underline{r},\overline{r}]\}}, \quad (S_{\rho\rho})_u(\rho,0) = 0, \tag{9.7}$$

$$S_{\rho u}(\rho,0) = 0,$$
$$(S_{\rho u})_u(\rho,0) = \frac{1}{\log(\overline{r}/\underline{r})} \left(\frac{\Psi_u(\rho,0)}{\rho\,\Phi_\rho(\rho,0)}\right)_\rho + \frac{1}{\log(\overline{r}/\underline{r})} \frac{\Psi_u(\rho,0)}{\rho\,\Phi_\rho(\rho,0)} \left\{\delta(\rho-\underline{r}) - \delta(\rho-\overline{r})\right\}. \tag{9.8}$$

Observe, that because of the asymptotics (3.9) we have

$$\frac{\Psi_u(\rho,0)}{\rho\,\Phi_\rho(\rho,0)} = 1 + \mathcal{O}(\rho), \quad \left(\frac{\Psi_u(\rho,0)}{\rho\,\Phi_\rho(\rho,0)}\right)_\rho (\rho,0) = \mathcal{O}(1). \tag{9.9}$$

In order to understand the pdes (9.1)-(9.4) first we analyze in general the pde

$$\Psi_u f_{\rho\rho} + \big(\Phi_u - \Psi_\rho\big) f_{\rho u} - \Phi_\rho f_{uu} + A f_\rho + B f_u + G f = 0, \tag{9.10}$$

the functions $\{A\,,\,B\,,\,G\} = \{A\,,\,B\,,\,G\}(\rho,u)$ being given on the left hand side of the pdes (9.1)-(9.4). It is easy to check that $A$ and $G$ are even, $B$ is odd with respect to $u$ and also that $\kappa := A(0,0)$ is $1, 2\gamma-1, 2, 2\gamma$, respectively, in the four cases.

We solve in $\widetilde{\mathcal{D}}$ the Cauchy problem (9.10) with the initial condition

$$f(\rho,0)=s(\rho), \qquad f_u(\rho,0)=t(\rho), \qquad \rho\in[0,r_0]. \tag{9.11}$$

The functions $s(\rho)$ and $t(\rho)$ will be identified with the various expressions in (9.5)-(9.8). Then, in $\mathcal{D}_3(\overline{r})\setminus\widetilde{\mathcal{D}}$ we solve the Goursat problem (9.10) with boundary conditions

$$f(\rho,u)=\begin{cases} f(r_1,0) & \text{on} \quad \partial\mathcal{D}_3(\overline{r})\setminus\widetilde{\mathcal{D}}, \\ \begin{array}{l}\text{given by the solution of}\\ \text{the previous Cauchy problem}\end{array} & \text{on} \quad \partial\widetilde{\mathcal{D}}\cap\mathcal{D}_3(\overline{r}). \end{cases} \tag{9.12}$$

Mind that $f(r_1,0)=1$ in the case of (9.1) and $f(r_1,0)=0$ in the cases (9.2)-(9.4).

The pde (9.10) is hyperbolic in the domains considered. Its Jacobian matrix is

$$D=D(\rho,u):=\begin{pmatrix} \Psi_\rho & \Psi_u \\ \Phi_\rho & \Phi_u \end{pmatrix}. \tag{9.13}$$

The eigenvalues of $D(\rho,u)$ are

$$\left.\begin{array}{c}\lambda\\ \mu\end{array}\right\}=\pm\frac{1}{2}\left\{\sqrt{\left(\Phi_u-\Psi_\rho\right)^2+4\Phi_\rho\Psi_u}\mp\left(\Phi_u-\Psi_\rho\right)\right\}+\Phi_u \tag{9.14}$$

Mind that from the Onsager relation (2.14) it follows that for any $(\rho,u)\in\mathcal{D}$ $\left(\Phi_u-\Psi_\rho\right)^2+4\Phi_\rho\Psi_u\geq 0$.

The characteristic coordinates (or Riemann invariants) $w=w(\rho,u)$, $z=z(\rho,u)$ of the pde (9.10) are determined, up to a functional relation

$$\widetilde{w}(\rho,u)=g(w(\rho,u)), \qquad \widetilde{z}(\rho,u)=h(z(\rho,u)), \tag{9.15}$$

by the eigenvalue equations

$$(w_\rho,w_u)D=\lambda\,(w_\rho,w_u), \qquad (z_\rho,z_u)D=\mu\,(z_\rho,z_u). \tag{9.16}$$

Due to the gauge invariance (9.15) we can choose the characteristic coordinates $w$ and $z$ so that

$$w(0,u)=u1\!\!1_{\{u>0\}}, \qquad z(0,u)=-u1\!\!1_{\{u<0\}}.$$

This choice determines uniquely the characteristic coordinates. Observe, that as a corollary of Lemma 2 we have

$$c_1\left(\sqrt{\rho}+u\right) \;<\; w(\rho,u)<c_2\left(\sqrt{\rho}+u\right), \tag{9.17}$$

$$\rho \;<\; c\,z(\rho,u)^{\frac{4\gamma-3}{2\gamma-1}}\,w(\rho,u)^{\frac{1}{2\gamma-1}}. \tag{9.18}$$

We denote

$$w^\circ:=w(r_0,0)=z(r_0,0)=:z^\circ$$

$$\underline{w}:=w(\underline{r},0)=z(\underline{r},0)=:\underline{z}$$

$$\overline{w}:=w(\overline{r},0)=z(\overline{r},0)=:\overline{z}$$

In characteristic coordinates

$$
\begin{aligned}
&\widetilde{\mathcal{D}} = [0, w^\circ] \times [0, z^\circ],\\
&\widetilde{\mathcal{D}} \cap \{u = 0\} = [0, w^\circ] \times [0, z^\circ] \cap \{w = z\},\\
&\mathcal{D}_3(\overline{r}) \setminus \widetilde{\mathcal{D}} = [w^\circ, u_*] \times [0, \overline{z}] \cup [0, \overline{w}] \times [z^\circ, u_*],\\
&\partial\mathcal{D}_3(\overline{r}) \setminus \widetilde{\mathcal{D}} = \{(w, \overline{z}) : w \in [w^\circ, u_*]\} \cup \{(\overline{w}, z) : z \in [z^\circ, u_*]\}\\
&\partial\widetilde{\mathcal{D}} \cap \mathcal{D}_3(\overline{r}) = \{(w^\circ, z) : z \in [0, z^\circ]\} \cup \{(w, z^\circ) : w \in [0, w^\circ]\}.
\end{aligned}
$$

The pde (9.10) written in characteristic coordinates reads

$$
f_{wz} + \alpha f_w + \beta f_z + \nu f = 0, \tag{9.19}
$$

where

$$
\begin{aligned}
\alpha = \alpha(w, z) &:= \frac{\lambda_z - Au_z + B\rho_z}{\lambda - \mu}\\
\beta = \beta(w, z) &:= -\frac{\mu_w - Au_w + B\rho_w}{\lambda - \mu}\\
\nu = \nu(w, z) &:= \frac{G(\rho_w u_z - \rho_z u_w)}{\lambda - \mu}
\end{aligned} \tag{9.20}
$$

(Now all the functions on the right are understood as functions of $(w, z)$.) In characteristic coordinates the initial conditions of the Cauchy problem in $\widetilde{\mathcal{D}}$ are

$$
\begin{aligned}
&f(v, v) = s(\rho(v, v)) =: \widetilde{s}(v),\\
&\big(f_w - f_z\big)(v, v) = 2u_w(v, v)\, t(\rho(v, v)) =: \widetilde{t}(v).
\end{aligned} \tag{9.21}
$$

The Cauchy problem (9.19)+(9.21) in the domain $\widetilde{\mathcal{D}} \cap \{z \leq w\}$ is solved by

$$
\begin{aligned}
f(w_0, z_0) \quad = \quad & \frac{1}{2}\varphi(z_0, z_0)\widetilde{s}(z_0) + \frac{1}{2}\varphi(w_0, w_0)\widetilde{s}(w_0) \\
&+\frac{1}{2}\int_{z_0}^{w_0} \varphi(v, v)\widetilde{t}(v)dv + \frac{1}{2}\int_{z_0}^{w_0} \big(\varphi_w - \varphi_z\big)(v, v)\widetilde{s}(v)dv\\
&+\int_{z_0}^{w_0} \varphi(v, v)(\beta(v, v) - \alpha(v, v))\widetilde{s}(v)dv
\end{aligned} \tag{9.22}
$$

where the *Riemann function* $\varphi$ is a solution of the adjoint Goursat problem:

$$
\varphi_{wz} - (\alpha\varphi)_w - (\beta\varphi)_z + \nu\varphi = 0, \tag{9.23}
$$

with boundary conditions:

$$
\begin{cases}
\varphi(w_0, t) = \exp\int_{z_0}^{t} \alpha(w_0, v)dv, & t \in [z_0, w_0],\\
\varphi(s, z_0) = \exp\int_{w_0}^{s} \beta(v, z_0)dv, & s \in [z_0, w_0].
\end{cases} \tag{9.24}
$$

Actually, the Riemann function depends also on $(w_0, z_0)$: $\varphi(w,z) = \varphi(w_0, z_0; w, z)$. In order to avoid heavy typography we omit explicit notation of this dependence. Note that in our cases (because of the left-right reflection symmetry of the respective pde), we will have $\beta(v,v) = \alpha(v,v)$, thus on the right hand side of (9.22) the last term cancels.
Also, if we consider the non-homogeneous pde

$$h_{wz} + \alpha h_w + \beta h_z + \nu h = g$$

with the same initial conditions then it is solved by

$$h(w_0, z_0) = f(w_0, z_0) + \int\int_{\triangle_{w_0,z_0}} g(s,t)\varphi(s,t)dsdt, \tag{9.25}$$

where $\triangle_{w_0,z_0}$ is the triangle with vertices $(z_0, z_0), (w_0, w_0), (w_0, z_0)$.

For details see any advanced textbook on partial differential equations, e.g. [8], [7], [3], [20].

In order to estimate $f$ we will give a uniform estimate on the Riemann function $\varphi$.

**Proposition 5 (Bounds on the Riemann function).** *Let $\varphi$ be the Riemann function associated to the equation (9.10) (where the coefficients $A, B, G$ are given on the left hand side of (9.1), (9.2), (9.3) or (9.4)) and $w_0 > z_0 > 0$. Then the following bounds hold uniformly for $(w_0, z_0) \in \widetilde{\mathcal{D}}$ with $z_0 < t < s < w_0$ and $(s,t) \in \widetilde{\mathcal{D}}$:*

$$\begin{aligned} |\varphi(s,t)| &< c\left(\frac{s}{w_0}\right)^{\frac{\kappa-1}{2\gamma-1}} \\ |(\partial_w\varphi - \partial_z\varphi)(s,s)| &< c\frac{1}{w_0}\left(\frac{s}{w_0}\right)^{\frac{\kappa-1}{2\gamma-1}-1}. \end{aligned} \tag{9.26}$$

Using Proposition 5 with (9.22), the initial conditions (9.5)-(9.8) and with Lemma 2 we can estimate $f$ in $\widetilde{\mathcal{D}}$ which gives (5.20) and (5.21) in this domain.

The equations (9.3) and (9.4) are not closed for $S_{\rho\rho}$ and $S_{\rho u}$, respectively, with the previous method one can only prove the required estimates for the solution of the respective homogeneous pdes. However, with (9.25) it is easy to show that these estimates can be extended for $S_{\rho\rho}$ and $S_{\rho u}$, too.

As an example, we show how to get the bound (5.22) for the *homogeneous* solution $f$ of (9.3) with initial conditions (9.7).

For the initial conditions we have the following bounds (using (9.17),(9.18)):

$$\begin{aligned} |\widetilde{s}(v)| &< \frac{c}{\log(\overline{r}/\underline{r})v^2}\mathbb{1}_{\{v\in[\underline{z},\overline{z}]\}}, \\ \left|\widetilde{t}(v)\right| &= 0. \end{aligned}$$

From Proposition 5 we have that for any $z_0 < w_0$

$$|\varphi(v,v)| < c\left(\frac{v}{w_0}\right)^{\frac{1}{2\gamma-1}}, \quad |\partial_w\varphi(v,v) - \partial_z\varphi(v,v)| < \frac{c}{w_0}\left(\frac{v}{w_0}\right)^{\frac{1}{2\gamma-1}-1}.$$

Together with (9.22) we get

$$|f(w_0,z_0)| < \frac{c}{\log(\overline{r}/\underline{r})} w_0^{-\frac{1}{2\gamma-1}} \underline{z}^{2-\frac{1}{2\gamma-1}} 1\!\!1_{\{z_0\in[\underline{z},\overline{z}],\underline{z}\le w_0\}}, \tag{9.27}$$

for any $z_0 < w_0$. This means

$$|f(w_0,z_0)| < \frac{c}{\log(\overline{r}/\underline{r})} \underline{z}^2 1\!\!1_{\{z_0\in[\underline{z},\overline{z}],\underline{z}\le w_0\}},$$

which (using (9.17)) translates to the $(\rho,u)$ coordinates the following way:

$$|f(\rho,u)| < \frac{c}{\log(\overline{r}/\underline{r})} \frac{1}{\underline{r}} 1\!\!1_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u).$$

(We only get this for $u \ge 0$, but from the symmetry of the pde this is also true for $u < 0$.) Also from (9.27) we get

$$|f(w_0,z_0)| < \frac{c}{\log(\overline{r}/\underline{r})} w_0^{-\frac{1}{2\gamma-1}} z_0^{2-\frac{1}{2\gamma-1}} 1\!\!1_{\{z_0\in[\underline{z},\overline{z}],\underline{z}\le w_0\}},$$

which (using (9.18)) gives

$$|f(\rho,u)| < \frac{c}{\log(\overline{r}/\underline{r})} \frac{1}{\rho} 1\!\!1_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u).$$

Putting together the two bounds on $f$ we get the required estimate of (5.22).

Now back to the proof of Proposition 5. The following lemma will be a basic tool for our estimates:

**Lemma 7 (Goursat estimate).** *Suppose the functions $A(w,z), B(w,z), C(w,z)$ are defined on $[x_1,x_2]\times[y_1,y_2]$ where $0\le y_1 < y_2 \le x_1 < x_2 \le \infty$ and have the following properties:*

$$\begin{aligned} \sup_{z\in[y_1,y_2]} \int_{x_1}^{x_2} |B(s,z)|\,ds &< \frac{1}{6}, \\ \sup_{w\in[x_1,x_2]} \int_{y_1}^{y_2} |A(w,t)|\,dt &< \frac{1}{6}, \\ \int_{x_1}^{x_2}\int_{y_1}^{y_2} |C(s,t)|\,dsdt &< \frac{1}{6}. \end{aligned} \tag{9.28}$$

*Let $U$ be a solution of*

$$U_{wz} - (AU)_w - (BU)_z + CU = 0 \tag{9.29}$$

*on the rectangle $[x_1,x_2]\times[y_1,y_2]$ and let*

$$\sup_{y_1\le z\le y_2} |U(x_1,z)| + \sup_{x_1\le w\le x_2} |U(w,y_2)| = M.$$

*Then*

$$\sup_{(w,z)\in[x_1,x_2]\times[y_1,y_2]} |U(w,z)| \le 5M.$$

*Proof.* Denote $U(w, y_2) = f(w)$, $U(x_1, z) = g(z)$. Denote

$$\begin{aligned}\widetilde{f}(w) &= f(w) - \int_{x_1}^{w} B(s, y_2) f(s) ds,\\ \widetilde{g}(z) &= g(z) - \int_{y_2}^{z} A(x_1, t) g(t) dt.\end{aligned}$$

Then $U$ satisfies the following integral equation (for every $(w, z) \in [x_1, x_2] \times [y_1, y_2]$):

$$\begin{aligned}U(w,z) &= \widetilde{f}(w) + \widetilde{g}(z) - U(x_1, y_2) + \int_{y_2}^{w} A(w,t) U(w,t) dt + \int_{x_1}^{z} B(s,z) U(s,z) ds\\ &\quad - \int_{x_1}^{w}\int_{y_2}^{z} C(s,t) U(s,t) ds dt. \end{aligned} \tag{9.30}$$

Taking absolute values after some trivial estimates we get:

$$|U(w,z)| \le \frac{13}{6} M + \frac{1}{2} \sup_{(s,t)\in[x_1,w]\times[z,y_2]} |U(s,t)|,$$

from that the needed bound follows immediately. □

**Remark.** *Observe, that if we have some additional estimates on $A_z, B_w$ and $C$ then differentiating (9.30) with respect to $w$ or $z$ and using the Grönwall inequality, one could also get bounds on $|U_w|$ and $|U_z|$ in $[x_1, x_2] \times [y_1, y_2]$.*

*Proof of Proposition 5.*
(i) We first prove the proposition in the case when $\Psi(\rho, u) = \rho u$ and $\Phi(\rho, u) = \rho + \gamma u^2$. In that case the coefficient functions in (9.10) take the following form: $A = \kappa$, $B = C = 0$. Also, the Riemann invariants can be computed explicitly:

$$\begin{aligned} w(\rho,u) &:= \left(\frac{\sqrt{(2\gamma-1)u^2+4\rho}+(2\gamma-1)u}{4\gamma-2}\right)^{\frac{2\gamma-1}{4\gamma-3}} \left(\sqrt{(2\gamma-1)u^2+4\rho} - (2\gamma-2)u\right)^{\frac{2\gamma-2}{4\gamma-3}}\\ z(\rho,u) &:= \left(\frac{\sqrt{(2\gamma-1)u^2+4\rho}-(2\gamma-1)u}{4\gamma-2}\right)^{\frac{2\gamma-1}{4\gamma-3}} \left(\sqrt{(2\gamma-1)u^2+4\rho} + (2\gamma-2)u\right)^{\frac{2\gamma-2}{4\gamma-3}}. \end{aligned} \tag{9.31}$$

One can easily check that the equations (9.16) hold. We define

$$\beta_0(w) := \beta(w,0) \quad \text{and} \quad U(w,z) := \varphi(w,z) \exp\left(-\int_{w_0}^{w} \beta_0(s) ds\right) \tag{9.32}$$

(for $0 \le z < w$). From the definitions one can calculate that

$$\beta_0(w) = \frac{\kappa-1}{2\gamma-1}\frac{u_w(w,0)}{u(w,0)} = \frac{\kappa-1}{2\gamma-1} w^{-1}.$$

Thus, it is enough to prove, that

$$|U(s,t)| < C, \quad |\partial_w U(s,s)| < C\frac{1}{s}, \quad |\partial_z U(s,s)| < C \tag{9.33}$$

for $z_0 \le t \le s \le w_0$ uniformly, with a constant depending only on $\kappa$. To show (9.33) we will apply Lemma 7.

From (9.23), (9.24) and (9.32) we get that

$$U_{wz} - (\alpha U)_w - ((\beta - \beta_0)U)_z - \alpha\beta_0 U = 0, \tag{9.34}$$

and

$$\begin{aligned} U(w_0, z) &= \exp\left(\int_{z_0}^{z} \alpha(w_0, t)dt\right), \\ U(w, z_0) &= \exp\left(\int_{w_0}^{w} (\beta(s, z_0) - \beta(s, 0))ds\right). \end{aligned}$$

Using the explicit formulas for the Riemann-invariants one can get estimates for the integrals needed for Lemma 7. Suppose $[x_1, x_2] \times [y_1, y_2] \subseteq [v, w_0] \times [v, z_0]$ for some $z_0 < v < w_0$. Then it can be shown that for $z_0 \le z \le w \le w_0$

$$\begin{aligned} \left|\int_{y_1}^{y_2} \alpha(w, t)dt\right| &< c\frac{x_2 - x_1}{w}, \\ \left|\int_{x_1}^{x_2} \beta(s, z) - \beta(s, 0)ds\right| &< cz\left(\frac{1}{x_1} - \frac{1}{x_2}\right), \\ \left|\int_{x_1}^{x_2}\int_{y_1}^{y_2} (\alpha\beta_0)(s, t)ds\,dt\right| &< c\left(\frac{1}{x_1} - \frac{1}{x_2}\right)(y_2 - y_1), \end{aligned} \tag{9.35}$$

where $c$ only depends on $\kappa$. From the first and second inequality it follows that $|U(w_0, z)|$ and $|U(w, z_0)|$ can be bounded by a constant depending only on $\kappa$. Now fix $s, t$ with $z_0 < t < s < w_0$. Using (9.35) one can partition $[v, w_0] \times [z_0, v]$ into smaller rectangles in a way that the number of rectangles only depends on the value of $\kappa$ and on each small rectangle the conditions of Lemma 7 hold (with $A = \alpha$, $B = \beta - \beta_0$, $C = -\alpha\beta_0$). Applying successively Lemma 7 for the small rectangles (starting with the vertex $(w_0, z_0)$) one gets $|U(s, t)| < c(\kappa)$. Using the remark after Lemma 7 one can also get the results of (9.33) for the partial derivatives of $U$.
(ii) In the general case we do not know the explicit forms of the coefficients in (9.10) only their asymptotics:

$$\begin{aligned} A(\rho, u) &= \kappa(1 + \mathcal{O}(\rho + u^2)) \\ B(\rho, u) &= c_1 u(1 + \mathcal{O}(\rho + u^2)) \\ C(\rho, u) &= c_2(1 + \mathcal{O}(\rho + u^2)). \end{aligned}$$

We also do not have explicit formulas for the Riemann-invariants, but because $\Psi(\rho, u) = \rho u(1 + \mathcal{O}(\rho + u^2))$ and $\Phi(\rho, u) = (\rho + \gamma u^2)(1 + \mathcal{O}(\rho + u^2))$ if $\rho, |u| \ll 1$ the level-lines will approximate the respective level-lines of the system examined in (i). We will follow the steps of the proof for the specific case. We define $\beta_0$ and $U$ as in (9.32). From the asymptotics we have $\beta_0(w) = \frac{\kappa-1}{2\gamma-1}w^{-1} + \mathcal{O}(1)$ which means it is again enough to prove (9.33). We have the following equation for $U$:

$$U_{wz} - (\alpha U)_w - ((\beta - \beta_0)U)_z + (\nu - \alpha\beta_0)U = 0, \tag{9.36}$$

with

$$\begin{aligned}
U(w_0, z) &= \exp\left(\int_{z_0}^{z} \alpha(w_0, t)dt\right), \\
U(w, z_0) &= \exp\left(\int_{w_0}^{w} (\beta(s, z_0) - \beta(s, 0))ds\right).
\end{aligned}$$

If we can prove similar bounds for the integrals of coefficients as in (9.35) then using Lemma 7 the required estimates follow. From (9.20) we have that

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} \nu(s,t)ds\,dt = \int\int_{\Delta} \frac{G}{\lambda-\mu}d\rho\,du$$

where $\Delta$ is the domain corresponding to the $[x_1, x_2] \times [y_1, y_2]$ rectangle on the $(\rho, u)$ plane. If $r_0$ is small enough, then for $(\rho, u) \in \widetilde{\mathcal{D}}$

$$\left|\frac{C}{\lambda-\mu}\right| < c\frac{1}{\sqrt{u^2+\rho}}$$

and since the right-hand side is integrable this gives uniform bounds on the previous integral. To get bounds on the integrals

$$\int_{y_1}^{y_2} \alpha(w,t)dt,\ \int_{x_1}^{x_2} \beta(s,z) - \beta(s,0)ds,\ \int_{x_1}^{x_2}\int_{y_1}^{y_2} (\alpha\beta_0)(s,t)ds\,dt,$$

we observe that because of the asymptotics described earlier if $r_0 \to 0$ these integrals will be uniformly close to the respective integrals of the (i) case. Thus, again if $r_0$ is small enough (but fixed!) then the arguments of (i) may be repeated. □

We have proved that the bounds (5.20)-(5.23) hold in $\widetilde{\mathcal{D}}$ if $r_0$ is small enough, now we have to extend this to the domain $\mathcal{D}_3(\overline{r}) \setminus \widetilde{\mathcal{D}}$ for the solution of the respective Goursat-problems with boundary conditions (9.12). The solution of these Goursat-problems can be expressed by integral equations similar to (9.30) (see [8], [7]). Thus if we have some estimates for the solution on the boundary (which we have from (9.12) and the previous estimates of the Cauchy problem) then these may be extended (up to a constant multiplier) if we have uniform bounds on the integrals of the respective coefficients.

If $\overline{r}$ is small enough then for $(\rho, u) \in \mathcal{D}_3(\overline{r}) \setminus \widetilde{\mathcal{D}}$ we have

$$|\Phi_\rho(\rho, u) - \Psi_u(\rho, u)| > c > 0$$

which also implies that in this domain $\lambda - \mu > c' > 0$. From this it follows, that we can fix a small enough $\overline{r}_0 > 0$ such that in the domain $\mathcal{D}_3(\overline{r}_0) \setminus \widetilde{\mathcal{D}}$ all the respective coefficients are well-defined smooth functions. Moreover, since this domain is compact, they are all bounded with a fixed constant which means that we have uniform bounds on the respective integrals. Using similar arguments as in the estimate of the solution of the Cauchy-problem one can get the bounds (5.20)-(5.23) also in this domain which completes the proof of Lemma 3.

### 9.3 Proof of Lemma 4

*Proof.* Note that

$$\big(F - \Psi S_\rho\big)_\rho = \Phi_\rho S_u - \Psi S_{\rho\rho},$$
$$\big(F - \Psi S_\rho\big)_u = \Phi_u S_u - \Psi S_{\rho u}.$$

Also, there exists a constant $C < \infty$ such that for any $(\rho, u) \in \mathcal{D}$

$$|\Psi(\rho, u)| \le C\rho|u|, \quad |\Phi_\rho(\rho, u)| \le C, \quad |\Phi_u(\rho, u)| \le C|u|.$$

From these and the bounds (5.21), (5.22), (5.23) of Lemma 3 it follows that

$$\big|\big(F - \Psi S_\rho\big)_\rho\big|(\rho, u) \le \frac{C}{\log(\overline{r}/\underline{r})}\big(\sqrt{\overline{r}} + |u|\big)\, 1\!\!1_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho, u),$$
$$\big|\big(F - \Psi S_\rho\big)_u\big|(\rho, u) \le \frac{C}{\log(\overline{r}/\underline{r})}\big(\rho + \sqrt{\overline{r}}|u|\big)\, 1\!\!1_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho, u).$$

Integrating these and using (5.16), the bound (5.25) follows. □

## 10 Proof of the "Tools"

### 10.1 Proof of the large deviation bounds (Proposition 2)

Recall the definition (6.1) of $L$. The following lemma follows from simple coupling arguments.

**Lemma 8.** (Stochastic dominations)
*There exists a constant $C$ depending only on $\max_{(s,x)\in[0,T]\times\mathbb{T}} \rho(s, x)$ and $\max_{(s,x)\in[0,T]\times\mathbb{T}} |u(s, x)|$ such that for any fixed $(s, x) \in [0, T] \times \mathbb{T}$ the following stochastic dominations hold:*

$$\mathbf{P}_{\nu_s^n}\Big(\widehat{\rho}^n(x) > z\Big) \le \mathbf{P}\Big(\mathrm{POI}(L) > (z/C)L\Big), \tag{10.1}$$

$$\mathbf{P}_{\nu_s^n}\Big(\,|\widehat{u}^n(x)| > z\Big) \le \mathbf{P}\Big(\,|\mathrm{GAU}| > \big((z/C) - 1\big)\sqrt{L}\Big), \tag{10.2}$$

*where* $\mathrm{POI}(L)$ *is a Poissonian random variable with expectation $L$, and* GAU *is a standard Gaussian random variable.*

**Lemma 9.** (Large deviation bounds)
*(i) For any $\gamma < \infty$ there exists $M < \infty$, such that for any $n$, $j \in \mathbb{T}^n$ and $s \in [0, T]$*

$$\begin{aligned}
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L \widehat{\rho}^n(\frac{j}{n}) 1\!\!1_{\{\widehat{\rho}^n(\frac{j}{n}) > M\}}\big\}\Big) \le 1,\\
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L \widehat{\rho}^n(\frac{j}{n}) 1\!\!1_{\{|\widehat{u}^n(\frac{j}{n})| > M\}}\big\}\Big) \le 1,\\
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L \big|\widehat{u}^n(\frac{j}{n})\big| 1\!\!1_{\{\widehat{\rho}^n(\frac{j}{n}) > M\}}\big\}\Big) \le 1,\\
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L \big|\widehat{u}^n(\frac{j}{n})\big| 1\!\!1_{\{|\widehat{u}^n(\frac{j}{n})| > M\}}\big\}\Big) \le 1.
\end{aligned} \tag{10.3}$$

*(ii) For any $\gamma \in (0, 1/(8C^2))$ there exists $M < \infty$, such that for any $n$, $j \in \mathbb{T}^n$ and $s \in [0, T]$*

$$\begin{aligned} \log \mathbf{E}_{\nu_s^n}\Big( \exp\big\{\gamma L \big|\widehat{u}^n(\tfrac{j}{n})\big|^2 \mathbb{1}_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\big\}\Big) &\leq 1, \\ \log \mathbf{E}_{\nu_s^n}\Big( \exp\big\{\gamma L \big|\widehat{u}^n(\tfrac{j}{n})\big|^2 \mathbb{1}_{\{|\widehat{u}^n(\frac{j}{n})|>M\}}\big\}\Big) &\leq 1. \end{aligned} \tag{10.4}$$

*Proof.* (i) We prove the first bound of (10.3), the other ones are done very similarly.

Let $Z_L$ be a $POI(L)$-distributed random variable. Using the stochastic domination (10.1) we obtain

$$\begin{aligned} &\log \mathbf{E}_{\nu_s^n}\Big( \exp\big\{\gamma L \widehat{\rho}^n(\tfrac{j}{n}) \mathbb{1}_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\big\}\Big) \\ &\quad\leq \log\Big(1 + \mathbf{E}_{\nu_s^n}\Big( \exp\big\{\gamma L \widehat{\rho}^n(\tfrac{j}{n})\big\} \mathbb{1}_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\Big)\Big) \\ &\quad\leq \sqrt{\mathbf{E}_{\nu_s^n}\Big( \exp\big\{2\gamma L \widehat{\rho}^n(\tfrac{j}{n})\big\}\Big)} \sqrt{\mathbf{P}_{\nu_s^n}\Big(\widehat{\rho}^n(\tfrac{j}{n}) > M\Big)} \\ &\quad\leq \sqrt{\mathbf{E}\Big( \exp\big\{\gamma C Z_L\big\}\Big)} \sqrt{\mathbf{P}\Big((C/M) Z_L > L\Big)} \\ &\quad\leq \exp\big\{\frac{L}{2}\big((e^{(\alpha C)/M} - 1) + (e^{\gamma C} - 1) - \alpha\big)\big\}, \end{aligned}$$

where $\alpha$ is arbitrary positive number. In the last step Markov's inequality is being used. Now, choosing $\alpha > \exp(2\gamma C)$ and $M > (C\alpha)/(\ln 2)$ we obtain (10.3).

(ii) Again, we prove the first bound in (10.4). The other one is done in an identical way.

Let again $Z_L$ be a $POI(L)$-distributed and $X$ be a standard Gaussian random variable. Using the stochastic dominations (10.1) and (10.2) we obtain

$$\begin{aligned} &\log \mathbf{E}_{\nu_s^n}\Big( \exp\big\{\gamma L \big|\widehat{u}^n(\tfrac{j}{n})\big|^2 \mathbb{1}_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\big\}\Big) \\ &\quad\leq \log\Big(1 + \mathbf{E}_{\nu_s^n}\Big( \exp\big\{\gamma L \big|\widehat{u}^n(\tfrac{j}{n})\big|^2\big\} \mathbb{1}_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\Big)\Big) \\ &\quad\leq \sqrt{\mathbf{E}_{\nu_s^n}\Big( \exp\big\{2\gamma L \big|\widehat{u}^n(\tfrac{j}{n})\big|^2\big\}\Big)} \sqrt{\mathbf{P}_{\nu_s^n}\Big(\widehat{\rho}^n(\tfrac{j}{n}) > M\Big)} \\ &\quad\leq \sqrt{\mathbf{E}\Big( \exp\big\{4\gamma C^2 (X^2 + L)\big\}\Big)} \sqrt{\mathbf{P}\Big(Z_L > (M/C) L\Big)} \\ &\quad\leq \big(1 - 8\gamma C^2\big)^{-1/4} \exp\big\{\frac{L}{2}\big(4\gamma C^2 + (e^{(\alpha C)/M} - 1) - \alpha\big)\big\}, \end{aligned}$$

where $\alpha$ is arbitrary positive number. Given $\gamma < 1/(8C^2)$, we choose $\alpha$ sufficiently large and $M > (C\alpha)/(\ln 2)$ to obtain (10.4). □

Now we turn to the proof of Proposition 2:

*Proof.* The bounds (6.3), respectively, (6.4) follow directly from the entropy inequality (6.2) of Lemma 5 and the bounds (10.3), respectively, (10.4) of Lemma 9. Recall that $L \gg 1$, as $n \to \infty$. □

### 10.2 Proof of the fluctuation bounds (Proposition 3)

Within this proof we need the notation

$$\begin{aligned}\widetilde{u}^n(s,x) &:= \frac{n^\beta}{l}\sum_k a\big(\frac{nx-k}{l}\big)\left(\zeta_k - n^{-\beta}u\big(s,\frac{k}{n}\big)\right) = \widehat{u}^n(x) - \mathbf{E}_{\nu_s^n}\big(\widehat{u}^n(x)\big),\\ \widetilde{\rho}^n(s,x) &:= \frac{n^{2\beta}}{l}\sum_k a\big(\frac{nx-k}{l}\big)\left(\eta_k - n^{-2\beta}\rho\big(s,\frac{k}{n}\big)\right) = \widehat{\rho}^n(x) - \mathbf{E}_{\nu_s^n}\big(\widehat{\rho}^n(x)\big).\end{aligned}$$

Since

$$\begin{aligned}&\Big|\,\big|\,\widehat{u}^n\big(s,\frac{j}{n}\big) - u\big(s,\frac{j}{n}\big)\,\big| - \big|\,\widetilde{u}^n\big(s,\frac{j}{n}\big)\,\big|\,\Big|\\ &\quad\leq \Big|\,1-\frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\,\Big|\,\big|\,u\big(s,\frac{j}{n}\big)\,\big| + \frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\,\big|\,u\big(s,\frac{j}{n}\big) - u\big(s,\frac{k}{n}\big)\,\big|\\ &\quad\leq C\left(\frac{1}{l}+\frac{l}{n}\right) = o(1),\end{aligned}$$

and, similarly

$$\begin{aligned}&\Big|\,\big|\,\widehat{\rho}^n\big(s,\frac{j}{n}\big) - \rho\big(s,\frac{j}{n}\big)\,\big| - \big|\,\widetilde{\rho}^n\big(s,\frac{j}{n}\big)\,\big|\,\Big|\\ &\quad\leq \Big|\,1-\frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\,\Big|\,\big|\,\rho\big(s,\frac{j}{n}\big)\,\big| + \frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\,\big|\,\rho\big(s,\frac{j}{n}\big) - \rho\big(s,\frac{k}{n}\big)\,\big|\\ &\quad\leq C\left(\frac{1}{l}+\frac{l}{n}\right) = o(1),\end{aligned}$$

we have to prove

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\big|\,\widetilde{u}^n\big(s,\frac{j}{n}\big)\,\big|^2\Big) \leq C\,h^n(s) + o(1), \tag{10.5}$$

respectively,

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\big|\,\widetilde{\rho}^n\big(s,\frac{j}{n}\big)\,\big|^2\,\mathbb{1}_{\{|\,\widetilde{\rho}^n(s,\frac{j}{n})\,|\leq M\}}\Big) \leq C\,h^n(s) + o(1). \tag{10.6}$$

**Lemma 10.** *(i) There exists $\gamma > 0$ (sufficiently small) such that for all $n$, $j\in\mathbb{T}^n$ and $s\in[0,T]$*

$$\log\mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L\,\big|\,\widetilde{u}^n\big(s,\frac{j}{n}\big)\,\big|^2\big\}\Big) \leq 1 \tag{10.7}$$

*(ii) For any $M<\infty$ there exists $\gamma>0$ (sufficiently small) such that for all $n$, $j\in\mathbb{T}^n$ and $s\in[0,T]$*

$$\log\mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L\,\big|\,\widetilde{\rho}^n\big(s,\frac{j}{n}\big)\,\big|^2\,\mathbb{1}_{\{|\,\widetilde{\rho}^n(s,\frac{j}{n})\,|\leq M\}}\big\}\Big) \leq 1. \tag{10.8}$$

*Proof.* (i) Let $X$ be a standard Gaussian random variable, which is independent of all other random variables appearing in this paper, and denote by $\langle \dots \rangle$ expectation with respect to $X$.

$$\begin{aligned}\log \mathbf{E}_{\nu_s^n}\Big(\exp\{\gamma L\,|\,\widetilde{u}^n(s,\frac{j}{n})\,|^2\}\Big) \qquad\qquad (10.9)\\ = \log \mathbf{E}_{\nu_s^n}\Big(\exp\{\frac{\gamma}{l}\,|\,\sum_k a(\frac{j-k}{l})\,(\zeta_k - \mathbf{E}_{\nu_s^n}(\zeta_k))\,|^2\}\Big)\\ = \log\Big\langle \mathbf{E}_{\nu_s^n}\Big(\exp\{X\sqrt{\frac{2\gamma}{l}}\sum_k a(\frac{j-k}{l})\,(\zeta_k - \mathbf{E}_{\nu_s^n}(\zeta_k))\}\Big)\Big\rangle.\end{aligned}$$

Now, note that the random variables $\zeta_k - \mathbf{E}_{\nu_s^n}(\zeta_k)$, $k \in \mathbb{T}^n$, are uniformly bounded and under the distribution $\mathbf{P}_{\nu_s^n}$ they are independent and have zero mean. Hence there exists a finite constant $C$ such that for any collection of real numbers $\lambda_k$, $k \in \mathbb{T}^n$

$$\mathbf{E}_{\nu_s^n}\Big(\exp\{\sum_k \lambda_k(\zeta_k - \mathbf{E}_{\nu_s^n}(\zeta_k))\}\Big) \le \exp\{C\sum_k \lambda_k^2\}.$$

Further on, there exists a finite constant $C$ such that for any $l$

$$\frac{1}{l}\sum_k |\,a(\frac{k}{l})\,|^2 \le C. \qquad (10.10)$$

From these it follows that for some finite constant $C$,

$$\text{r.h.s. of (10.9)} \ \le \log\big\langle \exp\{C\,\gamma\,X^2\}\big\rangle.$$

Choosing $\gamma$ sufficiently small in this last inequality we obtain (10.7).

(ii) Note first that, given $M < \infty$ fixed, there exists a zero mean bounded random variable $Y$ such that for any $r \in \mathbb{R}$

$$r^2 1\!\!1_{\{\,|\,r\,|\,\le M\,\}} \le \log \mathbf{E}\Big(\exp\{\,r\,Y\,\}\Big).$$

Let $Y_1, Y_2, \dots$ be i.i.d. copies of $Y$ which are also independent of all other random variables appearing in this paper, and denote by $\langle \dots \rangle$ expectation with respect to these. Then we have

$$\begin{aligned}\log \mathbf{E}_{\nu_s^n}\Big(\exp\{\gamma L\,|\,\widetilde{\rho}^n(s,\frac{j}{n})\,|^2 1\!\!1_{\{\,|\,\widetilde{\rho}^n(s,\frac{j}{n})\,|\,\le M\}}\}\Big) \qquad\qquad (10.11)\\ \le \log\Big\langle \mathbf{E}_{\nu_s^n}\Big(\exp\{\frac{\sum_{p=1}^{\lceil \gamma L\rceil} Y_p}{L}\sum_k a(\frac{j-k}{l})\,(\eta_k - \mathbf{E}_{\nu_s^n}(\eta_k))\}\Big)\Big\rangle.\end{aligned}$$

Next note that for any $\overline{\lambda} < \infty$ there exists a constant $C < \infty$ such that for any $n \in \mathbb{N}$, any $s \in [0,T]$ and any collection of real numbers $\lambda_k \in [-\overline{\lambda}, \overline{\lambda}]$, $k \in \mathbb{T}^n$

$$\mathbf{E}_{\nu_s^n}\Big(\exp\{\sum_k \lambda_k\,(\eta_k - \mathbf{E}_{\nu_s^n}(\eta_k))\}\Big) \le \exp\{Cn^{-2\beta}\sum_k \lambda_k^2\}.$$

Hence, using again (10.10),

$$\text{r.h.s. of (10.11)} \ \le \log \Big\langle \exp\big\{C\gamma\,\big((Y_1+\ldots+Y_{\lceil\gamma L\rceil})/\sqrt{\lceil\gamma L\rceil}\big)^2\big\}\Big\rangle.$$

Now, since the i.i.d. random variables $Y_1, Y_2, \ldots$ are bounded and have zero mean, choosing $\gamma$ sufficiently small this last expression can be made arbitrarily small, uniformly in $L$. Hence (10.8). □

Now back to the proof of Proposition 3.

*Proof.* From (6.2) and (10.7), respectively, from (6.2) and (10.8) we deduce (10.5), respectively, (10.6). Finally, these two bounds and the arguments at the beginning of the present subsection imply (6.5), respectively, (6.6). □

## 10.3 Proof of the block replacement and gradient bounds (Proposition 4)

### 10.3.1 An elementary probability lemma

Let $(\Omega,\pi)$ be a finite probability space and $\omega_i$, $i\in\mathbb{Z}$ i.i.d. $\Omega$-valued random variables with distribution $\pi$. Further on let

$$\begin{aligned} &\boldsymbol{\zeta}:\Omega\to\mathbb{R}^d, && \boldsymbol{\zeta}_i := \boldsymbol{\zeta}(\omega_i),\\ &\xi:\Omega^m\to\mathbb{R}, && \xi_i := \xi(\omega_i\ldots,\omega_{i+m-1}). \end{aligned}$$

For $\boldsymbol{x}\in\mathrm{co}(\mathrm{Ran}(\boldsymbol{\zeta}))$ denote

$$\Xi(\boldsymbol{x}) := \frac{\mathbf{E}_\pi\big(\xi_1\exp\{\sum_{i=1}^m \boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_i\}\big)}{\mathbf{E}_\pi\big(\exp\{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_1\}\big)^m},$$

where $\mathrm{co}(\cdot)$ stands for 'convex hull' and $\boldsymbol{\lambda}\in\mathbb{R}^d$ is chosen so that

$$\frac{\mathbf{E}_\pi\big(\boldsymbol{\zeta}_1\exp\{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_1\}\big)}{\mathbf{E}_\pi\big(\exp\{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_1\}\big)} = \boldsymbol{x}.$$

For $l\in\mathbb{N}$ we denote *plain* block averages by

$$\overline{\boldsymbol{\zeta}}_l := \frac{1}{l}\sum_{j=1}^{l}\boldsymbol{\zeta}_j.$$

Finally, let $b:[0,1]\to\mathbb{R}$ be a fixed smooth function and denote

$$M(b) := \int_0^1 b(s)\,ds, \qquad V(b) := M(b^2)-M(b)^2.$$

We also define the block averages *weighted by* $b$

$$\langle b\,,\boldsymbol{\zeta}\rangle_l := \frac{1}{l}\sum_{j=0}^{l} b(j/l)\boldsymbol{\zeta}_j, \quad \langle b\,,\xi\rangle_l := \frac{1}{l}\sum_{j=0}^{l} b(j/l)\xi_j,$$

The following lemma relies on elementary probability arguments:

**Lemma 11.** (Microcanonical exponential moments of block averages)
*There exists a constant $C < \infty$, depending only on $m$, on the joint distribution of $(\xi_i, \boldsymbol{\zeta}_i)$ and on the function $b$, such that the following bounds hold uniformly in $l \in \mathbb{N}$ and $\boldsymbol{x} \in (\mathrm{Ran}(\boldsymbol{\zeta}) + \cdots + \mathrm{Ran}(\boldsymbol{\zeta}))/l$:*
*(i) If $M(b) = 0$, then*

$$\mathbf{E}\Big( \exp\big\{\gamma\sqrt{l}\langle b\,,\,\xi\rangle_l\big\} \,\Big|\, \overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big) \le \exp\{C(\gamma^2 + \gamma/\sqrt{l})\}. \tag{10.12}$$

*(ii) If $M(b) = 1$ then*

$$\mathbf{E}\Big( \exp\big\{\gamma\sqrt{l}\big(\langle b\,,\,\xi\rangle_l - \Xi(\langle b\,,\,\boldsymbol{\zeta}\rangle_l)\big)\big\} \,\Big|\, \overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big) \le \exp\{C(\gamma^2 + \gamma/\sqrt{l})\}. \tag{10.13}$$

*Proof.* We prove the lemma with $m = 1$, that is with $(\xi_i)_{i=1}^l$ independent rather than $m$-dependent. The $m$-dependent case follows by applying Jensen's inequality in a rather straightforward way.
(i) In order to simplify the argument we make the assumption that the function $s \mapsto b(s)$ is odd:

$$b(1-s) = -b(s). \tag{10.14}$$

The same argument works if the function $s \mapsto b(s)$ can be rearranged (by permutation of finitely many subintervals of $[0,1]$) into a piecewise continuous odd function. This case is sufficient for our purposes. The proof of the fully general case — which goes through induction on $l$ — is more tedious and it is left as a fun exercise for the reader.

Assuming (10.14) we have

$$\sqrt{l}\langle b\,,\,\xi\rangle_l = l^{-1/2}\sum_{j=0}^{[l/2]} b(j/l)(\xi_j - \xi_{l-j})$$

and hence

$$\begin{aligned}
&\mathbf{E}\Big( \exp\big\{\gamma\sqrt{l}\langle b\,,\,\xi\rangle_l\big\} \,\Big|\, \overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big) \\
&\qquad = \mathbf{E}\Big(\mathbf{E}\Big( \exp\big\{\gamma\sqrt{l}\langle b\,,\,\xi\rangle_l\big\} \,\Big|\, \boldsymbol{\zeta}_j + \boldsymbol{\zeta}_{l-j} :\ j = 0, \dots, l\Big) \,\Big|\, \overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big) \\
&\qquad = \mathbf{E}\Big( \prod_{j=0}^{[l/2]} \mathbf{E}\Big( \exp\big\{\gamma l^{-1/2} b(j/l)(\xi_j - \xi_{l-j})\big\} \,\Big|\, \boldsymbol{\zeta}_j + \boldsymbol{\zeta}_{l-j}\Big) \,\Big|\, \overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big) \\
&\qquad \le \exp\big\{C\gamma^2 \sum_{j=1}^{[l/2]} l^{-1} b(j/l)^2\big\} \\
&\qquad = \exp\big\{C\gamma^2(V(b) + \mathcal{O}(1/l))\big\}.
\end{aligned}$$

In the second step we use the fact that the pairs $(\xi_j, \xi_{l-j})$, $j = 0, \dots, [l/2]$ are independent, given $\boldsymbol{\zeta}_j + \boldsymbol{\zeta}_{l-j}$, $j = 0, \dots, [l/2]$. In the third step we note that the variables $\xi_j$ are bounded

and $\mathbf{E}\big(\xi_j - \xi_{l-j}\big|\boldsymbol{\zeta}_j + \boldsymbol{\zeta}_{l-j}\big) = 0$.
(ii) Beside $\Xi(\boldsymbol{x})$ we also introduce the functions

$$\Xi_l : \big(\mathrm{Ran}(\boldsymbol{\zeta}) + \cdots + \mathrm{Ran}(\boldsymbol{\zeta})\big)/l \to \mathbb{R}, \qquad \Xi_l(\boldsymbol{x}) := \mathbf{E}\big(\xi_1\big|\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\big).$$

We shall exploit the following facts
(1) The functions $\Xi(\boldsymbol{x})$ and $\Xi_l(\boldsymbol{x})$ are uniformly bounded. This follows from the boundedness of $\xi_j$.
(2) The function $\boldsymbol{x} \mapsto \Xi(\boldsymbol{x})$ is smooth with bounded first two derivatives. This follows from direct computations.
(3) There exists a finite constant $C$, such that

$$|\Xi_l(\boldsymbol{x}) - \Xi(\boldsymbol{x})| \leq Cl^{-1}.$$

This follows from the so-called equivalence of ensembles (see e.g. Appendix 2 of [10]).

We write

$$\begin{aligned} \langle b\,,\,\xi\rangle_l - \Xi(\langle b\,,\,\boldsymbol{\zeta}\rangle_l) \quad &= \quad \big(\langle b\,,\,\xi\rangle_l - \overline{\xi}_l\big) + \big(\overline{\xi}_l - \Xi_l(\overline{\boldsymbol{\zeta}}_l)\big) \\ &\quad + \big(\Xi_l(\overline{\boldsymbol{\zeta}}_l) - \Xi(\overline{\boldsymbol{\zeta}}_l)\big) + \big(\Xi(\overline{\boldsymbol{\zeta}}_l) - \Xi(\langle b\,,\,\boldsymbol{\zeta}\rangle_l)\big)\,. \end{aligned} \tag{10.15}$$

By applying Jensen's inequality we conclude that we have to bound the exponential moments of type (10.13), separately for the four terms.

Bounding the first and last terms reduces directly to (10.12), the third term is uniformly $\mathcal{O}(l^{-1})$, so we only have to bound the exponential moments of the second term in (10.15). This is done by induction on $l$. Let $C(l)$ be the best constant such that for any $\gamma \in \mathbb{R}$

$$\mathbf{E}\Big(\exp\big\{\gamma\sqrt{l}\big(\overline{\xi}_l - \Xi_l(\overline{\boldsymbol{\zeta}}_l)\big)\big\}\,\Big|\,\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big) \leq \exp\{C(l)\gamma^2\}.$$

We prove that $C(l)$ stays bounded as $l \to \infty$.

The following identity holds

$$\begin{aligned} \sqrt{l+1}\big(\overline{\xi}_{l+1} - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big) \quad &= \quad \frac{l}{\sqrt{l+1}}\big(\overline{\xi}_l - \Xi_l(\overline{\boldsymbol{\zeta}}_l)\big) \\ &\quad + \frac{1}{\sqrt{l+1}}\big(\xi_{l+1} - \Xi_1(\boldsymbol{\zeta}_{l+1})\big) \\ &\quad + \frac{l}{\sqrt{l+1}}\big(\Xi_l(\overline{\boldsymbol{\zeta}}_l) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big) \\ &\quad + \frac{1}{\sqrt{l+1}}\big(\Xi_1(\boldsymbol{\zeta}_{l+1}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big) \end{aligned}$$

Thus

$$
\begin{aligned}
&\mathbf{E}\Big( \exp\{\gamma\sqrt{l+1}(\overline{\xi}_{l+1} - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1}))\} \Big| \overline{\boldsymbol{\zeta}}_{l+1} = \boldsymbol{x} \Big) = \\
&\quad \mathbf{E}\Big(\mathbf{E}\Big( \exp\{\gamma\sqrt{l+1}(\overline{\xi}_{l+1} - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1}))\} \Big| \overline{\boldsymbol{\zeta}}_{l}\,,\, \boldsymbol{\zeta}_{l+1} \Big) \Big| \overline{\boldsymbol{\zeta}}_{l+1} = \boldsymbol{x} \Big) = \\
&\quad \mathbf{E}\Big( \qquad \mathbf{E}\Big( \exp\{\frac{\gamma l}{\sqrt{l+1}}(\overline{\xi}_{l} - \Xi_{l}(\overline{\boldsymbol{\zeta}}_{l}))\} \Big| \overline{\boldsymbol{\zeta}}_{l} \Big) \times \\
&\qquad\qquad \mathbf{E}\Big( \exp\{\frac{\gamma}{\sqrt{l+1}}(\overline{\xi}_{l+1} - \Xi_{1}(\overline{\boldsymbol{\zeta}}_{l+1}))\} \Big| \boldsymbol{\zeta}_{l+1} \Big) \times \\
&\qquad\qquad \exp\{\frac{\gamma l}{\sqrt{l+1}}(\Xi_{l}(\overline{\boldsymbol{\zeta}}_{l}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1}))\} \times \\
&\qquad\qquad \exp\{\frac{\gamma}{\sqrt{l+1}}(\Xi_{1}(\boldsymbol{\zeta}_{l+1}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1}))\} \Big| \overline{\boldsymbol{\zeta}}_{l} = \boldsymbol{x} \Big).
\end{aligned}
$$

The terms

$$
\big(\overline{\xi}_{l+1} - \Xi_{1}(\overline{\boldsymbol{\zeta}}_{l+1})\big), \qquad l\big(\Xi_{l}(\overline{\boldsymbol{\zeta}}_{l}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big), \qquad \big(\Xi_{1}(\boldsymbol{\zeta}_{l+1}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big)
$$

are uniformly bounded and

$$
\begin{aligned}
\mathbf{E}\big(\overline{\xi}_{l+1} - \Xi_{1}(\overline{\boldsymbol{\zeta}}_{l+1}) \,\big|\, \boldsymbol{\zeta}_{l+1}\big) &= 0, \\
\mathbf{E}\big(\Xi_{l}(\overline{\boldsymbol{\zeta}}_{l}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1}) \,\big|\, \overline{\boldsymbol{\zeta}}_{l+1}\big) &= 0, \\
\mathbf{E}\big(\Xi_{1}(\boldsymbol{\zeta}_{l+1}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1}) \,\big|\, \overline{\boldsymbol{\zeta}}_{l+1}\big) &= 0.
\end{aligned}
$$

Using the induction hypothesis it follows that there exists a finite constant $B$ such that

$$
C(l+1) \le \frac{l}{l+1} C(l) + \frac{1}{l+1} B.
$$

Hence, $\limsup_{l\to\infty} C(l) \le B$ and the lemma follows. □

**Lemma 12.** (Microcanonical Gaussian bounds)
*There exists a $\gamma_0 > 0$, depending only on $m$, on the joint distribution of $(\xi_i, \boldsymbol{\zeta}_i)$ and on the function $b$, such that the following bounds hold uniformly in $l \in \mathbb{N}$ and $\boldsymbol{x} \in (\mathrm{Ran}(\boldsymbol{\zeta}) + \cdots + \mathrm{Ran}(\boldsymbol{\zeta}))/l$:*
*(i) If $M(b) = 0$, then*

$$
\log \mathbf{E}\Big( \exp\{\gamma_0 l \langle b\,,\,\xi\rangle_l^2\} \Big| \overline{\boldsymbol{\zeta}}_{l} = \boldsymbol{x} \Big) \le 1. \tag{10.16}
$$

*(ii) If $M(b) = 1$ then*

$$
\log \mathbf{E}\Big( \exp\{\gamma_0 l \big(\langle b\,,\,\xi\rangle_l - \Xi(\langle b\,,\,\boldsymbol{\zeta}\rangle_l)\big)^2\} \Big| \overline{\boldsymbol{\zeta}}_{l} = \boldsymbol{x} \Big) \le 1. \tag{10.17}
$$

*Proof.* This is actually a corollary of Lemma 11: The bounds (10.16) and (10.17) follow from (10.12), respectively, (10.13) by exponential Gaussian averaging (as in the proof of Lemma 10). □

### 10.3.2 Proof of Proposition 4

Now we turn to the proof of Proposition 4.

*Proof.* (i) In order to prove (6.9) first note that by simple numerical approximation (no probability bounds involved)

$$\left| \int_{\mathbb{T}} \left| \{ \widehat{\xi}^n - \Xi(\widehat{\eta}^n, \widehat{\zeta}^n) \}(x) \right|^2 dx - \frac{1}{n} \sum_{j \in \mathbb{T}^n} \left| \{ \widehat{\xi}^n - \Xi(\widehat{\eta}^n, \widehat{\zeta}^n) \}(\tfrac{j}{n}) \right|^2 \right|$$

$$\leq \frac{C}{l} = o\left( \frac{l^2}{n^{1+3\beta+\delta}} \right).$$

We apply Lemma 6 with

$$\mathcal{V} = \left| \{ \widehat{\xi}^n - \Xi(\widehat{\eta}^n, \widehat{\zeta}^n) \}(0) \right|^2 = \left| \langle a \,,\, \xi \rangle_l - \Xi\big( \langle a \,,\, \eta \rangle_l, \langle a \,,\, \zeta \rangle_l \big) \right|^2$$

We use the bound (10.17) of Lemma 12 with the function $b = a$. Note that $\gamma = \gamma_0 l$ can be chosen in (6.8). This yields the bound (6.9).

(ii) In order to prove (6.10) we start again with numerical approximation:

$$\left| \int_{\mathbb{T}} \left| \partial_x \widehat{\xi}^n(x) \right|^2 dx - \frac{1}{n} \sum_{j \in \mathbb{T}^n} \left| \partial_x \widehat{\xi}^n(\tfrac{j}{n}) \right|^2 \right| \leq C \frac{n^2}{l^3} = o(n^{1-3\beta-\delta}).$$

We apply Lemma 6 with

$$\mathcal{V} = \left| \partial_x \widehat{\xi}^n(0) \right|^2 = \frac{n^2}{l^2} \left| \langle a' \,,\, \xi \rangle_l \right|^2.$$

We use now the bound (10.16) of Lemma 12 with the function $b = a'$. Now we can choose $\gamma = \gamma_0 l^3/n^2$ and this will yield the bound (6.10).

(iii) Next we prove (6.11). We apply Lemma 6 with

$$\begin{aligned} \mathcal{V} &= \frac{\left| \partial_x \widehat{\xi}^n(0) \right|^2}{\widehat{\eta}^n(0)} = \frac{n^2}{l^3} \frac{\left| \sum_k a'(k/l) \xi_k \right|^2}{\sum_k a(k/l) \eta_k} \\ &= \frac{n^2}{2l^3} \frac{\left| \sum_k a'(k/l)(\xi_k - \xi_{-k}) \right|^2}{\sum_k a(k/l)(\eta_k + \eta_{-k})}, \end{aligned}$$

where in the last equality we use the fact that the weighting function $x \mapsto a(x)$ is *even*. We compute the exponential moment $\mathbf{E}_{N,Z}^{2l+1}\big(\exp\{\gamma \mathcal{V}\}\big)$. Let $X$ be a standard Gaussian random variable, which is independent of all other random variables appearing in this paper and denote by $\langle \dots \rangle$ averaging with respect to it. We have

$$\begin{aligned}
&\mathbf{E}_{N,Z}^{2l+1}\Big(\exp\{\gamma\mathcal{V}\}\Big)\\
&=\mathbf{E}_{N,Z}^{2l+1}\Big(\exp\{\gamma\frac{n^2}{2l^3}\frac{\left|\sum_k a'(k/l)(\xi_k-\xi_{-k})\right|^2}{\sum_k a(k/l)(\eta_k+\eta_{-k})}\}\Big)\\
&=\Big\langle\mathbf{E}_{N,Z}^{2l+1}\Big(\exp\{X\sqrt{\gamma}\frac{n}{l^{3/2}}\frac{\sum_k a'(k/l)(\xi_k-\xi_{-k})}{\sqrt{\sum_k a(k/l)(\eta_k+\eta_{-k})}}\}\Big)\Big\rangle\\
&=\Big\langle\mathbf{E}_{N,Z}^{2l+1}\Big(\mathbf{E}_{N,Z}^{2l+1}\Big(\exp\{X\sqrt{\gamma}\frac{n}{l^{3/2}}\frac{\sum_k a'(k/l)(\xi_k-\xi_{-k})}{\sqrt{\sum_k a(k/l)(\eta_k+\eta_{-k})}}\}\,\Big|\,\{\eta_k+\eta_{-k}\}_{k=0}^{l}\Big)\Big)\Big\rangle\\
&\leq\Big\langle\mathbf{E}_{N,Z}^{2l+1}\Big(\exp\{CX^2\gamma\frac{n^2}{l^3}\frac{\sum_k a'(k/l)^2(\eta_k+\eta_{-k})}{\sum_k a(k/l)(\eta_k+\eta_{-k})}\}\Big)\Big\rangle\\
&\leq\Big\langle\exp\{CX^2\gamma\frac{n^2}{l^3}\}\Big\rangle,
\end{aligned}$$

where we used the facts that the random variables $\eta_k$ are non-negative, $\Omega$ is finite and $\eta(\omega)=0$ implies $\xi(\omega)=0$. In the last step we used the inequality

$$a'(x)^2 \leq Ca(x),$$

which follows from the conditions on $a(x)$, see subsection 4.3.

From this bound it follows that in Lemma 6 we can choose $\gamma=\gamma_0 l^3/n^2$, with a small but fixed $\gamma_0$, and hence the second bound in (6.11) follows.

□

# 11 Appendix: Some details about the PDE (1.1)

Hyperbolicity: One has to analyze Jacobian the matrix

$$D:=\begin{pmatrix}(\rho u)_\rho & (\rho u)_u\\ (\rho+\gamma u^2)_\rho & (\rho+\gamma u^2)_u\end{pmatrix}=\begin{pmatrix}u & \rho\\ 1 & 2\gamma u\end{pmatrix}.$$

The eigenvalues with the corresponding right and left eigenvectors are:

$$Dr=\lambda r,\quad Ds=\mu s,\quad l^\dagger D=\lambda l^\dagger,\quad m^\dagger D=\mu m^\dagger,$$

($v^\dagger$ stands for the transpose of the column 2-vector $v$). The eigenvalues and eigenvectors are

$$\left.\begin{matrix}\lambda\\ \mu\end{matrix}\right\}=\pm\frac{1}{2}\left\{\sqrt{(2\gamma-1)^2u^2+4\rho}\pm(2\gamma+1)u,\right\}$$

and

$$\left.\begin{matrix}r^\dagger\\ s^\dagger\end{matrix}\right\}=\left(\frac{1}{2}\left\{\mp\sqrt{(2\gamma-1)^2u^2+4\rho}-(2\gamma-1)u\right\},1\right),$$

$$\left.\begin{matrix}l^\dagger\\ m^\dagger\end{matrix}\right\}=\left(1,-\frac{1}{2}\left\{\pm\sqrt{(2\gamma-1)^2u^2+4\rho}-(2\gamma-1)u\right\}\right).$$

So, we can conclude that the pde (1.1) is (strictly) hyperbolic in the domain

$$\begin{aligned} \gamma \neq 1/2: &\qquad \{(\rho,u) \in \mathbb{R}_+ \times \mathbb{R} : (\rho,u) \neq (0,0)\}, \\ \gamma = 1/2: &\qquad \{(\rho,u) \in \mathbb{R}_+ \times \mathbb{R} : \rho \neq 0\}. \end{aligned}$$

<u>*Riemann invariants:*</u> The Riemann invariants $w = w(\rho, u)$, $z = z(\rho, u)$ of the pde are given by the relations

$$(w_\rho, w_u) \cdot s = 0 = (z_\rho, z_u) \cdot r.$$

That is, the level lines $w =$ const., respectively $z =$ const. are determined by the ordinary differential equations

$$\left.\begin{array}{r} w = \text{const.} \\ z = \text{const.} \end{array}\right\}: \quad \frac{d\rho}{du} = \mp\frac{1}{2}\left\{\sqrt{(2\gamma-1)^2u^2+4\rho} \pm (2\gamma-1)u\right\}$$

Actually only the level lines of the functions $w(\rho,u)$, respectively, $z(\rho,u)$ are determined. In our case the Riemann invariants can be found explicitly. For $\gamma \neq 3/4$ we get

$$\left.\begin{array}{r} w(\rho,u) \\ z(\rho,u) \end{array}\right\} = F\left\{\left(\sqrt{(2\gamma-1)^2u^2+4\rho} \pm (2\gamma-1)u\right)^{\frac{2\gamma-1}{2\gamma-2}}\left(\sqrt{(2\gamma-1)^2u^2+4\rho} \mp (2\gamma-2)u\right)\right\}$$

Where $F : \mathbb{R} \to \mathbb{R}$ is an appropriately chosen bijection (mind, that only the level sets of the Riemann invariants are determined).

Note that due to the changes of sign of $2\gamma - 1$ and $2\gamma - 2$, the above expression gives rise to *qualitatively different* behavior of the Riemann invariants. The picture changes qualitatively at the critical values $\gamma = 1/2$, $\gamma = 3/4$ and $\gamma = 1$. In Figure 3 we present the qualitative picture of the level lines of $w(\rho,u)$ and $z(\rho,u)$ for $3/4 < \gamma < 1$, and $\gamma > 1$, respectively. (For economy reasons we omit the graphical representation of the other cases, but encourage the reader to sketch it.) In all cases the Riemann invariants satisfy the convexity conditions

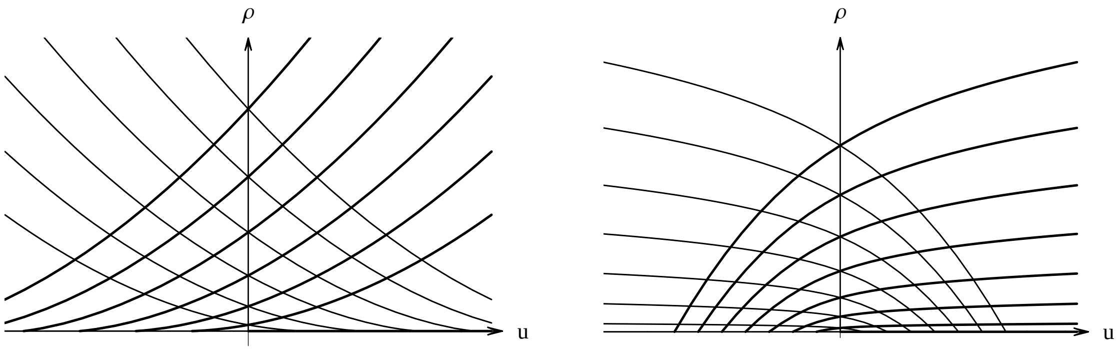
$$\begin{aligned} &w_{\rho\rho}w_u^2 - 2w_{\rho u}w_\rho w_u + w_{uu}w_\rho^2 \geq 0, \\ &z_{\rho\rho}z_u^2 - 2z_{\rho u}z_\rho z_u + z_{uu}z_\rho^2 \geq 0, \end{aligned} \tag{11.1}$$

in $\mathbb{R}_+ \times \mathbb{R}$ for all $\gamma$. (The sign of the function $F(\cdot)$ is so chosen, that these expressions be non-negative.) The inequalities are strict in the interior of $\mathbb{R}_+ \times \mathbb{R}$, except for the $\gamma = 1$ case, when these expressions identically vanish. These conditions are equivalent to saying that the level sets $\{(\rho,u) \in [0,\infty) \times (-\infty,\infty) : w(\rho,u) < c\}$ and $\{(\rho,u) \in [0,\infty) \times (-\infty,\infty) : z(\rho,u) < c\}$ be convex. See [11], [12] or [19] for the importance of these convexity conditions.

It is of crucial importance for our problem that the level curves $w(\rho,u) = w =$ const. expressed as $u \mapsto \rho(u,w)$ are convex for $\gamma < 1$, linear for $\gamma = 1$ and concave for $\gamma > 1$.

<u>*Genuine nonlinearity:*</u> Genuine nonlinearity holds if and only if

$$(\lambda_\rho, \lambda_u) \cdot r \neq 0 \neq (\mu_\rho, \mu_u) \cdot s.$$




(a) $3/4 < \gamma < 1$

(b) $1 < \gamma$

Figure 3: Level lines of Riemann-invariants

in the interior of the domain $\mathbb{R}_+ \times \mathbb{R}$. Elementary computations show that

$$\left.\begin{array}{r} (\lambda_\rho, \lambda_u) \cdot r = 0 \\ (\mu_\rho, \mu_u) \cdot s = 0 \end{array}\right\} \quad \Leftrightarrow \quad \rho = -\frac{4\gamma(2\gamma-1)^2}{(\gamma+1)^2} u^2 \text{ and } \left\{ \begin{array}{l} u \leq 0 \\ u \geq 0 \end{array} \right. . \tag{11.2}$$

Thus, for $\gamma \geq 0$, $\gamma \neq 0, 1/2$ the system is genuinely nonlinear on the closed domain $\mathbb{R}_+ \times \mathbb{R}$; for $\gamma = 0, 1/2$ it is genuinely nonlinear in the interior of $\mathbb{R}_+ \times \mathbb{R}$ (with genuine nonlinearity marginally lost on the boundary, $\rho = 0$). For $\gamma < 0$ genuine nonlinearity is lost in the interior of $\mathbb{R}_+ \times \mathbb{R}$.

*Lax entropies and entropy solutions:* Lax entropies of the pde (1.1) are solutions of the linear hyperbolic partial differential equation

$$\rho S_{\rho\rho} + (2\gamma - 1) u S_{\rho u} - S_{uu} = 0.$$

It turns out that the system is sufficiently rich in Lax entropies. In particular a Lax entropy globally convex in $\mathbb{R}_+ \times \mathbb{R}$ is

$$S(\rho, u) = \rho \log \rho + \frac{u^2}{2}. \tag{11.3}$$

Construction of other Lax entropies with particular features (e.g. possessing scale similarity, or polynomial in $\sqrt{\rho}$ and $u$, etc.) is a very instructive exercise.

*The Maximum Principle and positively invariant domains:* For $\gamma \geq 0$ our systems satisfy the conditions of the Lax's Maximum Principle proved in [11]. Namely: (i) they do posses a globally strictly convex Lax entropy bounded from below, see (11.3); (ii) the Riemann invariants $w(\rho, u)$ and $z(\rho, u)$ satisfy the convexity condition (11.1); (iii) they are genuinely nonlinear in the interior of $\mathcal{D}$, see (11.2).

Hence it follows that *convex domains bounded by level curves of $w(\rho, u)$ and $z(\rho, u)$ are positively invariant for entropy solutions.*

First we conclude, that $\mathcal{D}$ itself is positively invariant domain, as it should be.

Second: a very essential difference between the cases $\gamma < 1$, $\gamma = 1$ and $\gamma > 1$ follows, which is of crucial importance for the main result of the present paper. In the case $\gamma < 1$ all convex domains bounded by level curves of the Riemann invariants are *unbounded (non-compact)* and thus there is no a priori bound on the solutions. Even starting with smooth initial data with compact support nothing prevents the entropy solutions to blow up indefinitely after appearence of the shocks. On the other hand, if $\gamma \geq 1$ any compact subset of $\mathcal{D}$ is contained in a compact convex domain bounded by level sets of the Riemann invariants, which fact yields a priori bounds on the entropy solutions, given bounded initial data. A microscopic consequence of this fact is that the proof of our main theorem is valid only for $\gamma > 0$.

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