CONVERGENCE OF UPWIND FINITE DIFFERENCE SCHEMES FOR A SCALAR CONSERVATION LAW WITH INDEFINITE DISCONTINUITIES IN THE FLUX FUNCTION *

SIDDHARTHA MISHRA†

Abstract. We consider the following scalar conservation law with flux function discontinuous in the space variable i.e.

$$u_t + (H(x)f(u) + ((1 - H(x))g(u))_x = 0 \quad \text{in} \quad \mathrm{R} \times \mathrm{R}_+$$
$$u(0,x) = u_0(x) \quad \text{in} \quad \mathrm{R} \tag{0.1}$$

where H is the Heaviside function and f and g are smooth with the assumptions that either f is convex and g is concave or f is concave and g is convex.We show the existence of a weak solution of (0.1) by showing that upwind finite difference schemes of the Godunov and the Enquist Osher type converge to a weak solution.Uniqueness follows from a Kruzkhov type entropy condition.We also provide explicit solutions to the Riemann problem for (0.1).At the level of numerics,we give easy to implement numerical schemes of the Godunov and the Enquist Osher type and report numerical experiments comparing our schemes with other existing schemes.The central feature of this paper is the modification of the singular mapping technique (the main analytical tool for these type of equations) which allows us to show that the numerical schemes converge.Equations of the type (0.1) with above hypothesis on the flux may occur when considering the following scalar conservation law with discontinuous flux,

$$\begin{array}{lcl} u_t + (k(x)f(u))_x & = & 0 \\ u(0,x) & = & u_0(x) \end{array} \tag{0.2}$$

with f convex and k of indefinite sign.

Key words. Conservation Laws,Discontinuous fluxes,Finite Differences,Singular mapping

AMS subject classifications. 35F25, 35L65, 65M06, 65M12, 76S05, 76M12, 76M20

1. Introduction. We are interested in the following scalar conservation law,

$$u_t + (f(k(x),u))_x = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}_+$$
$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R} \tag{1.1}$$

Here, the flux f is smooth in k and u but may depend discontinuously on the space variable through a co-efficient k.Note that the equations (0.2) and (0.1) are special cases of (1.1) and are referred to as the multiplicative case and the 2-flux case respectively.Observe that (1.1) can also be written as the following 2×2 system of conservation laws given by,

$$\begin{array}{lcl} u_t + (f(k,u))_x & = & 0 \\ k_t & = & 0 \end{array} \tag{1.2}$$

The special feature of this system is that if fails to be strictly hyperbolic which introduces difficulties into its analysis.

Equations of the type (1.1) are of great practical interest in several areas of Physics and Engineering like in the water flooding model in the Petroleum industry,in

*A part of this work appeared as a chapter in Masters Thesis of the Author titled "Scalar Conservation laws with Discontinuous flux",IISc,2003.

†Department of Mathematics,Indian Institute of Science,Bangalore,India and TIFR center,IISc campus,Bangalore-560012,India (`sid@math.tifrbng.res.in`).

modeling continuous sedimentation in the ideal Clarifier Thickener units used in waste water treatment plants,in traffic flows on highways with changing road conditions and in ion etching in the Semiconductor industry.They also serve as the building blocks for some dimensional splitting methods for solving multidimensional Hamilton Jacobi equations.For more details on some of these applications,see [27]

As in the continuous case i.e. when k is Lipschitz,weak solutions are sought for (1.1).For Definitions,see Section 2.As in the continuous case,we need to impose additional admissibility criteria or Entropy conditions in order to obtain stability and uniqueness for the weak solutions.We note that at present,there is no complete theory for the entropy solutions of (1.1).

Equation (0.1) with both f and g convex was investigated by Adimurthi and Gowda in [1],by considering the corresponding Hamilton Jacobi equation.They obtained an explicit Hopf Lax type formula for the solution and obtained explicit solutions to the Riemann problem for (0.1).Another interesting feature of [1] was the prescription of a proper geometric or characteristic entropy condition at the interface ($x = 0$) which entailed the omission of undercompressive waves ($f' > 0, g' < 0$) at the interface.Coupled with a Kruzkhov type entropy condition in the interior,the authors were able to show stability and uniqueness of weak solutions for (0.1).In [3],the authors used the solution of the Riemann problem and interface entropy condition developed in [1] to develop a Godunov type algorithm for approximating solutions of (0.1).They worked under the hypothesis that f and g have 1 minima (maxima) and no maxima (minima) in the phase space.It must be noted that the interface entropy condition of [3] is different in some cases from other prevailing entropy theories.In view of this,we are motivated to investigate (1.1) in detail by using [1],[3] as a starting point,the following questions have to be considered.

1.Can we extend the algorithm of [3] and the entropy theory of [1] to the more general equations of the form (1.1) ?
2.Can we extend the algorithm of [3] and the entropy theory of [1] to a larger class of fluxes including nonconvex and sign changing fluxes.?
3.Can we develop numerical schemes for (1.1) based on fluxes other than Godunov like the Enquist Osher and the Lax Friedricks Scheme and other higher order versions which are consistent with interface entropy conditions like that of [1]?
4.How does the entropy theory of [1],[3] compare with other entropy theories and what is the correct notion of the entropy solution of (1.1)?

These questions are being investigated in a series of papers by Adimurthi,Gowda and the author.This is the first paper in the series and looks at some aspects of questions 2 and 3.In [27],the author obtained explicit solutions of the Riemann problem for (0.1) when f is convex (concave) and g concave (convex).These solutions were used to construct a Godunov type algorithm for (0.1) with above hypothesis and convergence was shown.This paper builds on that work.

In this paper,we report the explicit solutions of the Riemann problem and the convergence of a Godunov type algorithm.As Enquist Osher type scheme is also suggested and shown to converge.As interesting feature of the shape of fluxes considered in this paper is that any weak solution of (0.1) satisfying a Kruzkhov entropy condition is stable and hence unique.Thus all existing entropy theories agree in this case and make it an ideal test case for various numerical schemes for equations of this type.It should be noted that a particular case of (0.2) is the case when $f = k_1 h$ and $g = k_2 h$ with h convex and k_1, k_2 of different signs.Thus,this case will serve as building block for developing numerical methods for (1.1) when k is of indefinite sign.This will be done in a forthcoming paper [4].

Equations of the type (1.1) have been dealt with extensively in literature both from the theoretical as well as the numerical point of view.One of the earliest references in Mathematical literature was in Lyons,[26] where some problems (particularly with the correct definition of entropy solutions) was highlighted.In [13],[14],Gimse and Risebro obtained solutions for the Riemann problem of (1.2) when f was convex and k positive and used it to develop a front tracking algorithm to show existence of a weak solution.Uniqueness was obtained by minimizing $|u_+ - u_-|$,the jump at the interface.Deihl investigated equations of the type (0.1) with applications in the Clarifier thickener unit in [9],[10],[11].He obtained solutions of the Riemann problem under extremely general hypothesis on the fluxes and showed uniqueness for the Riemann problem by using a variation condition which he termed as the Γ condition.Klingenberg and Risebro studied (0.2) by using the front tracking method in [20] for convex fluxes and showed stability as well as studied asymptotic behavior.Klausen and Risebro developed a front tracking algorithm for (1.1) in [19] and showed its convergence.In [21],the authors developed a front tracking algorithm for a specific example of nonconvex f.

Temple et al have studied (1.2) extensively is [28],[24],[25].The main convergence tool of singular mapping technique for these kind of problems was developed by Temple in [28].So far,all the schemes for (1.1) were based on the solutions of 2×2 Riemann problems of the form (1.2) but this was changed by Towers in [29],[30].In [29],Towers developed a staggered grid scheme for (0.2) which just used the solution of a scalar Riemann problem for continuous flux.In that paper,he developed staggered versions of both the Godunov as well as the Enquist Osher schemes for (0.2) when f is convex and k is assumed to be strictly positive.In [30],the author considered (0.2) with nonconvex fluxes and proved convergence for a staggered version of the Enquist Osher scheme.These works are a motivation for this and other forthcoming papers where we develop and analyze numerical schemes for (1.1) based on exact or approximate Riemann solvers.In this paper,we tackle the question of indefinite sign of the coefficient which was left open in [29].

More recently,there have been a series of papers on (1.1).In [16],Karlsen,Risebro and Towers study (1.1) with an added degenerate parabolic term by using a Enquist Osher type scheme.In [17],the authors consider the vanishing viscosity limit of (0.2) and show that it exists by using compensated compactness.They also include a degenerate parabolic term.A general entropy theory for (1.1) with degenerate parabolic terms is developed in [18] where wellposedness is shown for fluxes satisfying a certain "crossing condition" and a modified Kruzkhov type entropy condition which agrees with that of [3] for (0.1) expect in the undercompressive intersections case.Among other works are those of Burger et al on the Clarifier Thickener model in [5],[6] ,of Klingenberg,Karlsen and Risebro in [22] where a relaxation scheme for (0.2) is proposed and shown to converge.Other recent works include that of Coclite and Risebro on the time dependent discontinuous coefficient k for (1.1) in [8].

The case of sign changes in the discontinuous coefficient k has received less attention in literature.To illustrate the difficulties,we quote the authors in [22] as "Also,sign changes in k are usually ruled out with the singular mapping due to the additional analytical difficulties".A way out was suggested in [22] and [17] by using the compensated compactness approach but with an added assumption that the flux f should be genuinely nonlinear.In the above references,the authors show convergence for the relaxation and the vanishing viscosity approximations.In this paper,we give the first proof of convergence for Godunov and Enquist Osher type schemes for the indefinite fluxes case by modifying the singular mapping for this case.This forms the core of our paper.We have organized this paper as follows-In section 2,we will describe the continuous problem in detail i.e,a precise description of the hypothesis on the fluxes f and g and the initial data.We also define the entropy conditions and show that the entropy solutions of (0.1) if they exist,are unique.In section 3,we give explicit solutions of the Riemann problem for (0.1) in this case satisfying the entropy condition.In section 4,we describe our finite difference schemes of both the Godunov and the Enquist Osher type and investigate some of their properties.Section 5 deals with the convergence of the schemes and is the core of this paper.Section 6 describes various numerical experiments comparing our schemes with those of Towers in [29].In Section 7,we derive certain conclusions from this paper.

2. The Continuous Problem. As noted earlier,in this section we will describe the continuous problem (0.1) in some detail.We begin with a precise description of the various hypothesis in this paper.First we give the hypothesis on the fluxes f and g.Let $s < S \in \mathbb{R}$,such that $[s, S]$ is the domain of definition of the fluxes (the phase space).The fluxes satisfy the following hypothesis,

(H_1) $f, g : [s, S] \to \mathbb{R}$ are lipschitz continuous.
(H_2) $f(s) = g(s)$,$f(S) = g(S)$
(H_3) Either f is convex and g is concave on $[s, S]$ or f is concave and g is convex on $[s, S]$

As in [3],we can easily the extend the convexity hypothesis to the following,

$(\overline{H_3})$ Either f has 1 minima and no maxima on and g has 1 maxima and no minima on $[s, S]$ or f has 1 maxima and no minima on and g has 1 minima and no minima on $[s, S]$.

We remark that the hypothesis H_2 is mostly to ensure that the solutions are bounded in L^∞ but this is a sufficient condition and is by no means necessary for boundedness and can be relaxed.The key hypothesis on the fluxes are H_3 or $\overline{H_3}$ which make the flux geometry of the concave-convex or the mixed type.We also need the following constant

$$M = \max\{Lip(f), Lip(g)\} \tag{2.1}$$

For any fixed [s,S],we can have the 4 possible cases of the fluxes which we enumerate as follows,

Case A_1:f is convex and g is concave with $g(s) \le g(S)$.
Case A_2:f is convex and g is concave with $g(s) > g(S)$.
Case B_1:f is concave and g is convex with $g(s) \le g(S)$.
Case B_2:f is concave and g is convex with $g(s) > g(S)$.

Note that the hypothesis on the fluxes implies that the fluxes f and g only intersect at the endpoints of the domain of definition i.e s and S.Figure 1 depicts the various cases as given above.

Now we state our assumptions on the initial data.As in [3],we will need an estimator of the variation of the initial data which we denote as $\overline{N}(f, g, u_0)$,For precise definitions of this quantity,refer to section 4.The hypothesis on the initial data are,

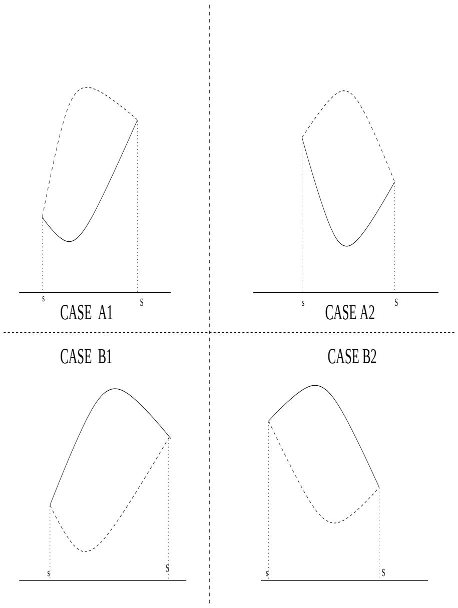

FIG. 2.1. *Various Cases of Fluxes f and g satisfying the hypothesis of this paper with the thick line representing f and the dashed line g*

$(IN_1) : s \leq u_0(x) \leq S, \quad \forall x \in I\!R$
$(IN_2) : \overline{N}(f, g, u_0) \leq C < +\infty.$

We now come to the definition of the entropy solution of (0.1),we start by defining a weak solution of (0.1) as a function $u \in L^\infty_{\rm loc}(I\!R \times I\!R_+)$ such that $\forall \varphi \in C_0^\infty(\overline{I\!R \times I\!R_+})$, the following holds.

$$\int_{I\!R}\int_{I\!R+} u\varphi_t + \int_{I\!R}\int_{I\!R+} (H(x)f(u) + (1 - H(x))g(u))\varphi_x dxdt + \int_{I\!R} u_0(x)\varphi(0, x)dx = 0\,. \tag{2.2}$$

It is easy to check that u is a weak solution of (0.1) iff it satisfies in the weak sense,

$$\begin{array}{lcl} u_t + (g(u))_x & = & 0, x < 0, t > 0 \\ u_t + (f(u))_x & = & 0, x > 0, t > 0 \\ u(0, x) & = & u_0(x), \forall x \in I\!R \end{array} \tag{2.3}$$

and the following interface Rankine Hugonoit condition,

$$f(u^+(t)) = g(u^-(t)), \qquad a.e.\ t \tag{2.4}$$

$$u^+(t) = \lim_{x \to 0+} u(x, t) \qquad\qquad u^-(t) = \lim_{x \to 0-} u(x, t) \tag{2.5}$$

In the above set of identities,(2.3) is the Rankine Hugonoit condition at the interface $x = 0$.We also need the specify a Kruzkhov type interior entropy condition,for

that we define,
Entropy pairs: $\rightarrow \{\varphi_i, \psi_i\}_{i=1,2}$ is said to be a entropy pair for (0.1) if φ_i is convex in $[s, S]$ and $\psi_1'(\theta) = \phi_1'(\theta) f'(\theta), \psi_2'(\theta) = \phi_2'(\theta) g'(\theta)$.

Let $u_0 \in L^\infty(\mathbb{R})$ is the initial data with $s \leq u_0(x) \leq S, \forall x \in \mathbb{R}$, then u is said to satisfy the interior entropy condition. If for any entropy pairs $(\varphi_i, \psi_i)_{i=1,2}, u$ satisfies in the sense of distributions.

$$\begin{array}{lll} \frac{\partial \varphi_1(u)}{\partial t} & + & \frac{\partial \psi_1(u)}{\partial x} \leq 0, \quad \forall x > 0, t > 0. \\ \frac{\partial \varphi_2(u)}{\partial t} & + & \frac{\partial \psi_2(u)}{\partial x} \leq 0, \quad \forall x < 0, t > 0 \end{array} \tag{2.6}$$

Now we define the **entropy solution** of (0.1) as $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is said to be an entropy solution for (0.1) if the following holds,
1.u is a weak solution of (0.1)
2.For all $t > 0$,$u^+(x, t)$ and $u^-(x, t)$ exists for a.e x,
3.u satisfies the interior entropy condition (2.5).
Equipped with the above notation,we state the main existence and uniqueness theorem of this paper,

Theorem 2.1 Let u_0 satisfy $(IN_1), (IN_2)$ and the fluxes f and g satisfy H_1, H_2 and H_3 respectively,then there exists a function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ such that u is a weak solution of (0.1) and the following holds,
1.For almost all t and x $\in \mathbb{R}$,$u(x-, t)$ and $u(x+, t)$ exist,
2.u satisfies the interior entropy condition (2.6) and is hence unique.

We make a few remarks regarding Theorem 2.1,
1.Note that unlike in [1],[3],we do not require any interface entropy condition at $x = 0$ for the solutions.This is an interesting outcome of the flux geometry that we are considering which leads to the fact that any solution that satisfies the interior entropy condition and the Rankine Hugonoit condition at the interface is unique.Note that the lack of any extra interface entropy condition enables us to relax the assumptions of regularity that were needed in [3] i.e u should have a discrete set of Lipschitz curves as its set of discontinuities.
2.As we will observe later also,the lack of an interface entropy condition in this case forces all the entropies theories like those in [9],[18] etc to agree in this case.

We will show the existence of a weak solution by showing that upwind finite difference schemes of the Godunov and the Enquist Osher type converge to it.Currently,we show the uniqueness of the entropy solutions for (0.1).For that as in [1],[3] ,we need to define the following interface entropy functional as,

$$\begin{aligned} I(u, v, t) = & (sgn(u^-(t) - u^+(t))(g(u^-) - g(v^-(t)) \\ & - (sgn(u^+(t) - (v^+(t))(f(u^+(t) - f(v^+(t)) \end{aligned}$$

We have the following Lemma regarding the sign of I,

Lemma 2.1:Let u and v be 2 entropy solutions of (0.1),then for almost all $t > 0$,we have that $I(u, v, t) \equiv 0$.

Proof:Let u and v be 2 entropy solutions of (0.1).By definition,$u^-(t), u^+(t), v^-(t), v^+(t)$ exist for almost all t and the Rankine Hugonoit condition (2.3) holds.From (2.3),I is equivalent to the following,

$$I = ((sgn(u^-(t) - (u^+(t)) - (sgn(u^+(t) - (v^+(t))(f(u^+(t) - f(v^+(t))$$

We fix t in the subsequent calculations and shall omit it for notational convenience.We shall proof the lemma assuming case A_1,the other cases follow similarly.First,from the shape of the fluxes in case A_1,we have that there exists unique $\overline{s}, \overline{S}$ such that
1.$\overline{s} \in (\theta_f, S)$ such that $f(s) = f(\overline{s})$. 2.$\overline{S} \in (s, \theta_g)$ such that $g(\overline{S}) = f(S)$.
From (2.3),it is easy to check that for any entropy solution u with limits at the interface as u^-, u^+ ,then $s \leq u^- \leq \overline{S}$ or $u^- = S$ and $u^+ = s$ or $\overline{s} \leq u^+ \leq S$.We have the following cases,

Case1:$u^- < v^-$
In this case,from the above bounds on the location of u^+, v^+,and the monotonicity of the fluxes,we have that either $u^+ < v^+$ or $u^+ = v^+ = S$.In the first case,it follows that $I \equiv 0$ and in the second case,from the endpoint intersection of the fluxes,we have $f(u^+) = f(v^+)$ and again $I \equiv 0$.

Case2:$u^- = v^-$
In this case,from (2.3) it follows that $f(u^+) = g(u^-)$ and $I \equiv 0$.

Case3:$u^- > v^-$
By symmetry of $I(u, v) = I(v, u)$,this case follows as case 1.So,in all the 3 possible cases,we have shown the lemma 2.1.

Now we are in a position to prove the following uniqueness theorem,

Theorem 2.2: Let $u, v \in L^\infty(I\!R \times I\!R_+)$ with $s \leq u, v \leq S$ be 2 solutions of (0.1) with initial data u_0, v_0 respectively,assume that
(i) For almost every t,$u(x+, t), v(x+, t), u(x-, t)$ and $v(x-, t)$ exists,and
(ii) u,v satisfy the interior entropy condition (2.6).
Then for any $\overline{M} \geq M,\ a < 0, b > 0,\ b - a \geq 2\overline{M}t$ the function

$$t \mapsto \int_{a+\overline{M}t}^{b-\overline{M}t} |u(x,t) - v(x,t)| dx$$

is non increasing and if $u_0 = v_0$ a.e ,then it follows that $u = v$ a.e.

Proof:The proof of the above theorem follows exactly as in [1] and is also outlined in [3].This proof is based on a standard doubling of variables argument introduced by Kruzkhov in [23].The interior entropy condition reduces the proof to that of a nonnegative sign for the interface entropy functional which is precisely the content of Lemma 2.1.We skip the details of the proof and refer the reader to [3].

So,in this section we have have established the uniqueness part of the proof of Theorem 2.1 without any extra assumptions on the behavior of the solution at the interface.In the rest of the paper,we will show the existence part of Theorem 2.1.

3. Solutions to the Riemann Problem. In this section,we give explicit solutions to the Riemann problem for (0.1) which satisfy both the interior entropy condition and the Rankine Hugonoit condition (2.3) at the interface.For the sake of simplicity,we will deal only with fluxes satisfying H_3.For fluxes satisfying $\overline{H_3}$,the solutions to the Riemann problem can be similarly worked out.We give a complete description of the Riemann problem in all the cases.

We are dealing with the (0.1) with the following Riemann Data,

$$u(x,0) = u_0(x) = \begin{cases} u_r & \text{if} \quad x > 0 \\ u_l & \text{if} \quad x < 0 \end{cases} \tag{3.1}$$

We start with the following case,
CaseA_1:We have to distinguish the following subcases,
Case 1:$g(u_l) < g(S)$
From the shape of the fluxes,it is easy to see that $\exists$ a unique $\theta \in (\theta_f, S]$ such that $f(\theta) = g(u_l)$,Define the following,
$s_0 = \frac{f(\theta)-f(u_r)}{\theta - u_r}$, $s_1 = \frac{g(u_l)-g(S)}{u_l - S}$,$s_2 = \frac{f(S)-f(u_r)}{S-u_r}$
This case has 2 possible subcases namely,
Case 1.1:In this case,$\theta \leq u_r$.
In this case,the solution is given by

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq 0 \\ \theta & \text{if} \quad 0 \leq x \leq tf'(\theta) \\ (f')^{-1}(\frac{x}{t}) & \text{if} \quad tf'(\theta) \leq x \leq tf'(u_r) \\ u_r & \text{if} \quad x \geq tf'(u_r). \end{cases} \tag{3.2}$$

Case1.2:In this case,$\theta > u_r$ and the solution is given by

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq 0 \\ \theta & \text{if} \quad 0 \leq x \leq s_0 t \\ u_r & \text{if} \quad x \geq s_0 t. \end{cases} \tag{3.3}$$

Case2:$g(u_l) \geq g(S)$ and the solution is given by

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq s_1 t \\ S & \text{if} \quad s_1 t \leq x \leq s_2 t \\ u_r & \text{if} \quad x \geq s_2 t. \end{cases} \tag{3.4}$$

Case A_2
Again,we have to consider the following subcases,
Case 1:$f(u_r) > f(S)$,
From the shape of the fluxes,it is easy to check that $\exists$ a unique $\eta \in [\theta_g, S]$ such that $g(\eta) = f(u_r)$.Also define the following,
$s_0 = \frac{g(\eta)-g(u_l)}{\eta - u_l}$, $s_1 = \frac{g(u_l)-g(S)}{u_l - S}$,$s_2 = \frac{f(S)-f(u_r)}{S-u_r}$
Case 1 has 2 possible cases namely,
Case 1.1:$u_l > \eta$ and the solution is given by

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq g'(u_l)t \\ (g')^{-1}(\frac{x}{t}) & \text{if} \quad g'(u_l)t \leq x \leq g'(\eta)t \\ \eta & \text{if} \quad g'(\eta)t \leq x \leq 0 \\ u_r & \text{if} \quad x \geq 0 \end{cases} \tag{3.5}$$

Case 1.2:$f(u_r) \leq f(S)$ and the solution is given by,

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq s_0 t \\ \eta & \text{if} \quad s_0 t \leq x \leq 0 \\ u_r & \text{if} \quad x \geq 0. \end{cases} \tag{3.6}$$

Case 2:$f(u_r) \leq f(S)$ and the solution is given by,

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq s_1 t \\ S & \text{if} \quad s_1 t \leq x \leq s_2 t \\ u_r & \text{if} \quad x \geq s_2 t. \end{cases} \tag{3.7}$$

Case B_1: Again,we have to consider the following cases,
Case 1:$g(u_l) \geq g(s)$.In this case,it is easy to see that $\exists$ a unique $\theta \in [s, \theta_f]$ such that $f(\theta) = g(u_l)$.We define the following,
$s_0 = \frac{f(\theta)-f(u_r)}{\theta - u_r}$,$s_1 = \frac{g(s)-g(u_l)}{s-u_l}$ and $s_2 = \frac{f(s)-f(u_r)}{s-u_r}$
We have the following subcases,
Case 1.1:$u_r < \theta$ and the solution is given by,

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq 0 \\ \theta & \text{if} \quad 0 \leq x \leq tf'(\theta) \\ (f')^{-1}(\frac{x}{t}) & \text{if} \quad tf'(\theta) \leq x \leq tf'(u_r) \\ u_r & \text{if} \quad x \geq tf'(u_r). \end{cases} \tag{3.8}$$

Case 1.2:$u_r > \theta$ and the solution is given by,

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq 0 \\ \theta & \text{if} \quad 0 \leq x \leq s_0 t \\ u_r & \text{if} \quad x \geq s_0 t. \end{cases} \tag{3.9}$$

Case 2:$g(u_l) < g(s)$ and the solution is given by,

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq s_1 t \\ s & \text{if} \quad s_1 t \leq x \leq s_2 t \\ u_r & \text{if} \quad x \geq s_2 t. \end{cases} \tag{3.10}$$

Case B_2:We have to consider the following cases,
Case 1:$f(u_r) \leq f(s)$,In this case t is easy to see that $\exists$ a unique $\eta \in [s, \theta_g]$ such that $f(u_r) = g(\eta)$.We define the following,
$s_0 = \frac{g(\eta)-g(u_l)}{\eta - u_l}$,$s_1 = \frac{g(s)-g(u_l)}{s-u_l}$ and $s_2 = \frac{f(s)-f(u_r)}{s-u_r}$
We have the following subcases,
Case 1.1:$u_l < \eta$ and the solution is given by

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq g'(u_l)t \\ (g')^{-1}(\frac{x}{t}) & \text{if} \quad g'(u_l)t \leq x \leq g'(\eta)t \\ \eta & \text{if} \quad g'(\eta)t \leq x \leq 0 \\ u_r & \text{if} \quad x \geq 0 \end{cases} \tag{3.11}$$

Case 1.2:$u_l \geq \eta$ and the solution is given by,

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq s_0 t \\ \eta & \text{if} \quad s_0 t \leq x \leq 0 \\ u_r & \text{if} \quad x \geq 0. \end{cases} \tag{3.12}$$

Case 2:$f(u_r) > f(s)$ and the solution is given by

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq s_1 t \\ s & \text{if} \quad s_1 t \leq x \leq s_2 t \\ u_r & \text{if} \quad x \geq s_2 t. \end{cases} \tag{3.13}$$

So we have given explicit solutions of the Riemann Problem for (0.1) in all cases.It is easy to check that the solutions satisfy the Rankine Hugonoit condition at the interface and the interior entropy condition which reduces to the standard Lax-Oleinik entropy condition.The crucial fact about the solutions is that the solutions in some cases are the endpoints (s,S) which are undercompressive in the sense that at (s,S),we have $f' > 0, g' < 0$.For example in case A_1,S is the undercompressive solution and as a consequence of theorem 2.2,it is unique.This is a strange result of the flux geometry considered here that we cannot avoid undercompressive waves at the interface which is quite striking given that the Interface entropy condition of [1],[3] essentially implies omitting undercompressive waves at the interface.

4. Description of the Finite Difference Schemes.. In this section,we seek to develop Finite difference schemes for (0.1) which are of the Godunov as well as the Enquist Osher type.As in [3],the key is to define the interface numerical fluxes,we start with the Godunov Flux.

4.1 Godunov Flux

Let h be a Lipschitz continuous function defined on $[s, S]$ and we use the standard Godunov flux H defined by,

$$\begin{aligned} H(a,b) &= \min_{\theta \in [a,b]} h(\theta), \quad \text{if } a \leq b \\ &= \max_{\theta \in [b,a]} h(\theta), \quad \text{if } a \geq b. \end{aligned}$$

We recall that H as defined above is Lipschitz in both its variables,nondecreasing in the first variable and nonincreasing in the second variable.We also note that for any h,the flux is obtained from the solution of the following Riemann problem given by,

$$\begin{aligned} w_t + (h(w))_x &= 0 \\ w(0,x) &= a \quad \text{if} \quad x < 0 \\ &= b \quad \text{if} \quad x > 0 \end{aligned} \tag{4.1}$$

Then $H(a,b) = w(0,a,b)$ where $w(\frac{x}{t},a,b)$ is the solution of the above equation (4.1).We use a similar method to define the interface Godunov flux.We proceed as in [3] and use the solution of the Riemann problem that were constructed in section 3 to define the Interface Godunov flux.Let F and G be the standard Godunov flux corresponding to the fluxes f and g respectively.We define the interface flux $\overline{F}$ in each case as follows,

Case A_1

$$\overline{F}(a,b) = \min\{G(a,S), F(S,b)\} \tag{4.2}$$

Case A_2

$$\overline{F}(a,b) = \max\{G(a,S), F(S,b)\} \tag{4.3}$$

Case B_1

$$\overline{F}(a,b) = \max\{G(a,s), F(s,b)\} \tag{4.4}$$

Case B_2

$$\overline{F}(a,b) = \min\{G(a,s), F(s,b)\} \tag{4.5}$$

Infact,we can more explicit formulas in each case than the formulas given above.We give a more explicit formula for the case A_1.It is easy to check that $\exists$ a unique $\overline{S} \in [s, \theta_g]$ such that $g(\overline{S} = g(S)$,then (4.2) can be written as

$$\overline{F}(a,b) = g(\min(a, \underline{S}))$$

Another way to writing down the above formulas is by defining the following function,which we define only in the case A_1 as,
Let $\overline{f} : [s, S] \mapsto I\!R$,such that

$$\begin{aligned} \overline{f}(\theta) &= g(\theta) \quad \forall \quad s \le \theta \le \overline{S} \\ &= g(S), \quad \text{otherwise} \end{aligned} \tag{4.6}$$

We call the above defined function $\overline{f}$ as the interface function and claim that the interface Godunov flux $\overline{F}$ is the standard Godunov flux corresponding to $\overline{f}$.This fact is easy to check.The interface function corresponding to other cases can be easily constructed.

4.2 Enquist Osher Flux

As seen before,the Godunov flux involves the exact solution of the Riemann problem which can be exhaustive.To reduce the computational complexity involved,various upwind schemes have been developed for the scalar conservation law which are based on approximate solutions of the Riemann problem.One of the more popular of these solvers is the Enquist Osher flux developed in [?].Let h be a lipschitz function,then Enquist Osher flux $\tilde{H}$ is given by

$$\tilde{H}(a,b) = \frac{1}{2}(h(a) + h(b) - \int_a^b |h'(\xi)| d\xi) \tag{4.7}$$

For a convex flux,there is a close relationship between the Godunov and the Enquist Osher(EO) fluxes which implies that in all cases expect in the overcompressive case i.e $h'(a) > 0 > h'(b)$,the 2 fluxes coincide.We can write a more explicit representation for the EO flux in the convex case by

$$\tilde{H}(a,b) = h(\max(a,\theta)) + h(\min(\theta, b)) - h(\theta) \tag{4.8}$$

Where θ is the unique minimum of the function h.We will use this formula (4.8) for our numerical experiments.We note that the EO flux is lipschitz,nondecreasing in the first variable and nonincreasing in the second variable.Also,check that the EO flux is upwind and hence has good resolution at the shocks.The development of an Enquist Osher algorithm for convex fluxes is undertaken in a forthcoming paper of Adimurthi and Gowda,[2].As in the above reference,we will employ the interface function $\overline{f}$ to construct the interface Enquist Osher flux.Let $\tilde{F}$ and $\tilde{G}$ be the EO fluxes corresponding to f and g respectively.We construct the interface EO flux only for the case A_1,other cases can be similarly treated.We define the interface EO flux $(\overline{\tilde{F}})$ in this case to be the standard EO flux corresponding to the interface function $\overline{f}$.Same construction can be followed in other cases.We recall that the Interface Godunov flux $(\overline{F})$ can be identified as the standard Godunov flux corresponding to the interface function $(\tilde{f})$ and we have following the same identification for defining the interface EO flux but the concave-convex flux geometry in this case forces us to have the following proposition,which is easy to check,

Proposition 4.1:Let $\overline{F}$ and $\overline{\tilde{F}}$ be as defined above,then $\overline{F}(a,b) = \overline{\tilde{F}}(a,b), \forall a,b \in [s,S]$.

Henceforth,we will also refer to the interface EO flux as $\overline{F}$.We collect some easy to verify facts regarding the interface flux in the following proposition,

Proposition 4.2:Let $\overline{F}$ be as given above and $a,b \in [s,S]$, then the following holds,
a) $\overline{F}$ is Lipschitz in each variable with

$$|\overline{F}(a,b) - \overline{F}(a,c)| \leq M|b-c|$$
$$|\overline{F}(a,b) - \overline{F}(a,b)| \leq M|a-c|$$

b)$\overline{F}$ is nondecreasing in a and non increasing in b
c) $\overline{F}(s,s) = f(s) = g(s), \overline{F}(S,S) = f(s) = g(S)$.
d) $\overline{F}$ is not consistent, i.e,$\exists \quad a \in [s,S]$ such that $\overline{F}(a,a) \neq f(a) \neq g(a)$.

Proof: (c) and (d) are easy to check from the definition of $\overline{F}$.(a) and (b) follow from the fact that $\overline{F}$ is the standard Godunov (EO) flux corresponding to the Lipschitz continuous function $\overline{f}$.

We remark that in general,the interface Godunov and EO fluxes maynot agree as is seen in the case when the fluxes are convex but in this case,on account of the flux geometry,they are exactly the same.We now proceed to describe our upwind finite difference schemes for (0.1),Before that,we need to define the estimator on the Total Variation norm $\overline{N}(f,g,u_0)$ that we had mentioned in Section 2 as follows,

$$\begin{aligned}
N_h(f,g,u_0) = &\sum_{i<-1} |G(u_i^0,u_{i+1}^0) - G(u_{i-1}^0,u_i^0)| + \sum_{i>1} |F(u_i^0,u_{i+1}^0) - F(u_{i-1}^0,u_i^0)| \\
&+ |\overline{F}(u_{-1}^0,u_1^0) - G(u_{-2}^0,u_{-1}^0)| + |F(u_1^0,u_2^0) - \overline{F}(u_{-1}^0,u_1^0)| \\
\tilde{N}_h(f,g,u_0) = &\sum_{i<-1} |\tilde{G}(u_i^0,u_{i+1}^0) - \tilde{G}(u_{i-1}^0,u_i^0)| + \sum_{i>1} |\tilde{F}(u_i^0,u_{i+1}^0) - \tilde{F}(u_{i-1}^0,u_i^0)| \\
&+ |\overline{F}(u_{-1}^0,u_1^0) - \tilde{G}(u_{-2}^0,u_{-1}^0)| + |\tilde{F}(u_1^0,u_2^0) - \overline{F}(u_{-1}^0,u_1^0)|
\end{aligned}$$

$$\begin{aligned}
N(f,g,u_0) &= \sup_{h>0} N_h(f,g,u_0) \\
\tilde{N}(f,g,u_0) &= \sup_{h>0} \tilde{N}_h(f,g,u_0) \\
\overline{N}(f,g,u_0) &= \sup\{N(f,g,u_0), \tilde{N}(f,g,u_0)\}
\end{aligned}$$

It is easy to see that if $u_0 \in BV(I\!R)$ then $N(f,g,u_0) \leq C||u_0||_{BV}$, where C is a constant depending only on the Lipschitz constants of f and g.We describe our grid as follows,

Discretization in Space.: Let $h > 0$ be the mesh size and we define the grid points in space as

$$x_{-1/2} = x_{1/2} = 0, x_{j+1/2} = jh, \text{ for } j \geq 0$$
$$x_{j-1/2} = jh, \text{ for } j \leq 0.$$

We also use the midpoints of the intervals given by

$$\begin{aligned}
x_j &= \left(\tfrac{2j-1}{2}\right) h, \text{ for } j \geq 1. \\
x_j &= \left(\tfrac{2j+1}{2}\right) h, \text{ for } u \leq -1.
\end{aligned}$$

Thus, we are using a uniform grid in space.
Time discretization. We use a uniform time step given by Δt and let the nth time level as $t^n = n\Delta t$.
Initial data. We use the averaging of the initial data as

$$\begin{aligned} u^0_{j+1} &= \frac{1}{h}\int\limits_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)dx, \quad \text{if } j \geq 0 \\ u^0_{j-1} &= \frac{1}{h}\int\limits_{x_{j-1/2}}^{x_{j-1/2}} u_0(x)dx \quad \text{if } j \leq 0. \end{aligned}$$

Note that $u_0 \in L^\infty(I\!R)$ makes the above values well defined.
CFL condition. For stability of the scheme, we use the following CFL condition. Let $\lambda = \frac{\Delta t}{h}$, and $M = \max\{\text{ Lip } f, \text{ Lip } g\}$, then

$$2\lambda M \leq 1 \quad (CFL)$$

Now, we are in a position to describe our Godunov type scheme as follows. For every time level $(n+1)$ we calculate the discrete u^{n+1}_j as

$$\begin{aligned} u^{n+1}_j &= u^n_j - \lambda(F(u^n_j, u^n_{j+1}) - F(u^n_{j-1}, u^n_j)), \quad \forall \quad j \geq 2 \\ u^{n+1}_1 &= u^n_1 - \lambda(F(u^n_1, u^n_2) - \overline{F}(u^n_{-1}, u^n_1)) \\ u^{n+1}_{-1} &= u^n_{-1} - \lambda(\overline{F}(u^n_{-1}, u^n_1) - G(u^n_{-2}, u^n_{-1})) \\ u^{n+1}_j &= u^n_j - \lambda(G(u^n_i, u^n_{j+1}) - G(u^n_{j-1}, u^n_j)), \quad \forall \quad j \leq -2. \end{aligned} \tag{4.9}$$

Similarly we define the Enquist Osher type scheme for (0.1) as

$$\begin{aligned} u^{n+1}_j &= u^n_j - \lambda(\tilde{G}(u^n_j, u^n_{j+1}) - \tilde{G}(u^n_{j-1}, u^n_j)), \quad \forall \quad j \leq -2 \\ u^{n+1}_{-1} &= u^n_{-1} - \lambda(\overline{\tilde{F}}(u^n_{-1}, u^n_1) - \tilde{G}(u^n_{-2}, u^n_{-1})) \\ u^{n+1}_1 &= u^n_1 - \lambda(\tilde{F}(u^n_1, u^n_2) - \overline{\tilde{F}}(u^n_{-1}, u^n_1)) \\ u^{n+1}_j &= u^n_j - \lambda(\tilde{F}(u^n_j, u^n_{j+1}) - \tilde{F}(u^n_{j-1}, u^n_j)) \quad \forall \quad j \geq 2 \end{aligned} \tag{4.10}$$

We now define the approximations in terms of the following piecewise constant functions,

$$\begin{aligned} u^h(x,t) &= u^n_j, \quad \text{if} \quad (x,t) \in I^n_j \\ \tilde{u}^h(x,t) &= \tilde{u}^n_j, \quad \text{if} \quad (x,t) \in I^n_j \\ I^n_j &= [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times [t^n, t^{n+1}) \end{aligned} \tag{4.11}$$

In the next section,we shall show that these approximations u^h and $\tilde{u}^h$ are compact in the appropriate topology and converge to a weak solution of (0.1).In addition,the limit satisfies the interior entropy condition and hence unique.

5. Convergence Analysis. In this section,we carry out the convergence analysis for our schemes (4.9) and (4.10).We begin by showing that the solutions are bounded.This is done by using the monotonicity of the scheme and the consistency of the fluxes at the endpoints.This gives compactness in the weak star topology which is not enough to pass to the limit in the nonlinear flux terms.In order to do so,we have to obtain derivative type estimates which for equations of the type (1.1) is generally measured via the singular mapping introduced by Temple.This tool has been adopted in several papers on these equations.See [20],[29],[30].We follow closely the tools of [3].We note that for the Godunov scheme (4.9),the convergence was shown in [27].The proof presented here differs significantly from that of [3] and [27].To start with,we give some preliminary lemmas for convergence.We define the following functions to write our schemes in conservation form,

$$\begin{aligned}
H_1(X,Y.Z) &= Y-\lambda(F(Y,Z)-F(X,Y)),\\
H_{-1}(X,Y.Z) &= Y-\lambda(G(Y,Z)-G(X,Y)),\\
H_2(X,Y.Z) &= Y-\lambda(F(Y,Z)-\overline{F}(X,Y)),\\
H_{-2}(X,Y.Z) &= Y-\lambda(\overline{F}(Y,Z)-G(X,Y)),\\
\tilde{H}_1(X,Y.Z) &= Y-\lambda(\tilde{F}(Y,Z)-\tilde{F}(X,Y)),\\
\tilde{H}_{-1}(X,Y.Z) &= Y-\lambda(\tilde{G}(Y,Z)-\tilde{G}(X,Y)),\\
\tilde{H}_2(X,Y.Z) &= Y-\lambda(\tilde{F}(Y,Z)-\overline{F}(X,Y)),\\
\tilde{H}_{-2}(X,Y.Z) &= Y-\lambda(\overline{F}(Y,Z)-\tilde{F}(X,Y)),
\end{aligned} \tag{5.1}$$

By using the definitions in (5.1),we have the following lemma,

Lemma 5.1:Let $2\lambda M \leq 1$ and $a \in [s,S]$,we have the following
1.$H_{\pm 1}(a,a,a) = \tilde{H}_{\pm 1}(a,a,a) = a$,$H_{\pm 2}(s,s,s) = \tilde{H}_{\pm 2}(s,s,s) = s$,$H_{\pm 2}(S,S,S) = \tilde{H}_{\pm 2}(S,S,S) = S$.
2.$H_i, \tilde{H}_i$ are nondecreasing in each of their variables and hence the schemes (4.9),(4.10) are monotone.

Proof:1 follows directly from the definition of the fluxes and from proposition 4.2,For 2,check that $H_{\pm 1}, \tilde{H}_{\pm 1}$ are monotone due to standard arguments.We check monotonicity only for H_2,the other cases follow similarly.As H_2 is Lipschitz,we differentiate it with respect to the variables to get that,

$$\frac{\partial H_2}{\partial X} = \lambda \frac{\partial(\overline{F})(X,Y)}{\partial a} \geq 0 \tag{5.2}$$

(As $\overline{F}$ is nondecreasing with respect to the first variable)

$$\frac{\partial H_2}{\partial Z} = -\lambda \frac{\partial \overline{F}(Y,Z)}{\partial b} \geq 0$$

(As $\overline{F}$ is nonincreasing with respect to the second variable)
Differentiating with respect to Y, we get that

$$\frac{\partial H_2}{\partial Y} = 1 - \lambda \frac{\partial F(Y,Z)}{\partial a} + \frac{\partial \overline{F}(X,Y)}{\partial b} \geq 1 - 2\lambda M \geq 0$$

(By the CFL condition and the monotonicity of the Numerical Fluxes),Hence we have shown that H_2 is monotone in each variable.

Now,we prove that the approximate solutions are uniformly bounded in L^{∞} by the following invariant region principle,

Lemma 5.2:Let $u_0 \in L^\infty(I\!R, [s, S])$ be the initial data and let $u_j^n, \tilde{u}_j^n$ be the corresponding solutions calculated by the Godunov scheme (4.9) and the Enquist Osher scheme (4.10) respectively,then

$$s \leq u_j^n \leq S \text{ for all } j, n, \tag{5.3}$$

$$s \leq \tilde{u}_j^n \leq S \text{ for all } j, n, \tag{5.4}$$

Proof:We will only prove (5.3) as (5.4) follows in an identical manner.,Since $s \leq u_0 \leq S$ and hence for all j, $s \leq u_j^0 \leq S$. By induction, assume that (5.3) holds for n. Then from (1) and (2) of lemma (5.1), we have

$$\begin{array}{rcl} s &=& H_{-1}(s,s,s) \leq H_{-1}(u_{j-1}^n, u_j^n, u_{j+1}^n) = u_j^{n+1} \leq H_{-1}(S,S,S) = S \text{ if } j \leq -2 \\ s &=& H_1(s,s,s) \leq H_1(u_{j-1}^n, u_j^n, u_{j+1}^n) = u_j^{n+1} \leq H_1(S,S,S) = S \quad \text{if } j \geq 2 \\ s &=& H_{-2}(s,s,s) \leq H_{-2}(u_{-2}^n, u_{-1}^n, u_1^n) = u_{-1}^{n+1} \leq H_{-2}(S,S,S) = S \\ s &=& H_2(s,s,s) \leq H_2(u_{-1}^n, u_1^n, u_2^n) = u_1^{n+1} \leq H_2(S,S,S) = S \end{array}$$

This proves (5.2).

As in the standard analysis of numerical schemes for scalar conservation laws,we use the monotonicity of the approximate solutions to obtain a discrete version of L^1 contractivity of the solutions which is the content of the following lemma,

Lemma 5.3:$u_0, v_0 \in L^\infty(I\!R,\ [s, S])$ be initial datas and let $\{u_j^n\}$ and $\{v_j^n\}$ be the corresponding solutions calculated by the Godunov type scheme (4.9) and and let $\tilde{u_j^n}$ and $\tilde{v_j^n}$ be the corresponding solutions calculated by the Enquist Osher scheme (4.10),then the following holds

$$\sum_{j\neq 0} |u_j^{n+1} - u_j^n| \leq \sum_{j\neq 0} |u_j^n - u_j^{n-1}| \tag{5.5}$$

$$\sum_{j\neq 0} |\tilde{u}_j^{n+1} - \tilde{u}_j^n| \leq \sum_{j\neq 0} |\tilde{u}_j^n - \tilde{u}_j^{n-1}| \tag{5.6}$$

Proof:To prove (5.5),(5.6),we can use the Crandall Tartar lemma as u_j^n and $\tilde{u}_j^n$ are given by a monotone and conservative scheme and we can apply it as in [15].

Now,as in [3],[27] we define the singular mappings for the scheme.To start with,we use the following standard notation $a \in I\!R$, then $a_+ = \max\{a, 0\}, a_- = \min\{a, 0\}, a = a_+ + a_-, |a| = a_+ - a_-$.
The singular mappings are given by,

$$\psi_1(\theta) = \int_\alpha^\theta |g'(\xi)| d\xi \tag{5.7}$$

$$\psi_2(\theta) = \int_\alpha^\theta |f'(\xi)| d\xi \tag{5.8}$$

Where $\alpha \in [s, S]$ is some number.Recall that we use the same form of the singular mapping as used in [3] expect that the fact that we center it at an arbitrary point in the domain of definition unlike the above reference where it is centered at a maxima (minima).This leads to a greater simplicity in the proof as compared to the above reference.Now,we are in a position to define the transformed schemes for the discrete values of the solution.We define it as

$$\begin{aligned} z_j^n &= \psi_1(u_j^n), \quad \text{if} \quad j \leq -1 \\ &= \psi_1(u_{-1}^n), \quad \text{if} \quad j \geq -1 \\ w_j^n &= \psi_2(u_j^1), \quad \text{if} \quad j \leq 1 \\ &= \psi_2(u_j^n), \quad \text{if} \quad j \geq 1 \end{aligned} \tag{5.9}$$

Similarly,we define the singular mapping for the Enquist Osher approximations as

$$\begin{aligned} \tilde{z}_j^n &= \psi_1(\tilde{u}_j^n), \quad \text{if} \quad j \leq -1 \\ &= \psi_1(\tilde{u}_{-1}^n), \quad \text{if} \quad j \geq -1 \\ \tilde{w}_j^n &= \psi_2(\tilde{u}_j^1), \quad \text{if} \quad j \leq 1 \\ &= \psi_2(\tilde{u}_j^n), \quad \text{if} \quad j \geq 1 \end{aligned} \tag{5.10}$$

We remark that unlike in [3],we define 2 sets of transformed variables inorder to obtain the normalized variation bounds.This is a key difference in our analysis which enables us to show the convergence for finite difference approximations for the concave-convex case.This feature of using a set of singular mappings instead of the one considered so far in literature appears to be novel and will be employed by us in the more general nonconvex case.We will show that the total variation of the approximate solutions under the above transformations is bounded in terms of the variation of the numerical fluxes.This is done in the following lemma which gives a bound on the normalized variation in each cell.

Lemma 5.4:$\forall j \in$,$\forall n \in I\!N$,we have the following inequalities,
$\forall j \leq -3$,we have that

$$\begin{aligned} (z_j^n - z_{j+1}^n)_+ \leq\ & |G(u_j^n, u_{j+1}^n) - G(u_{j-1}^n, u_j^n)| \\ & + |G(u_{j+1}^n, u_{j+2}^n) - G(u_j^n, u_{j+1}^n)| \end{aligned} \tag{5.11}$$

$\forall j \geq 2$,we have that,

$$\begin{aligned} -(w_j^n - w_{j+1}^n)_- \leq\ & |F(u_j^n, u_{j+1}^n) - F(u_{j-1}^n, u_j^n)| \\ & + |F(u_{j+1}^n, u_{j+2}^n) - F(u_j^n, u_{j+1}^n)| \end{aligned} \tag{5.12}$$

In addition,we have the following estimates for the other terms,

$$\begin{aligned} (z_{-2}^n - z_{-1}^n)_+ \leq\ & |G(u_{-2}^n, u_{-1}^n) - \overline{F}(u_{-1}^n, u_1^n)| \\ & + |G(u_{-3}^n, u_{-2}^n) - G(u_{-2}^n, u_{-1}^n)| \end{aligned} \tag{5.13}$$

$$\begin{aligned} -(w_1^n - w_2^n)_- \leq\ & |F(u_1^n, u_2^n) - \overline{F}(u_{-1}^n, u_1^n)| \\ & + |F(u_2^n, u_3^n) - F(u_1^n, u_2^n)| \end{aligned} \tag{5.14}$$

The above estimates also hold for the Enquist Osher approximations by replacing u_j^n, z_j^n, w_j^n by $\tilde{u}_j^n, \tilde{z}_j^n, \tilde{w}_j^n$ respectively.We call these analogous estimates as (5.11(a)),(5.12(a),(5.13(a)) and (5.14(a)) respectively.

Proof:We provide proofs for (5.11),(5.13) and (5.12(a)).The other inequalities can be similarly proved.

Proof of (5.11)
We fix t in the subsequent calculations and drop the superscript n in the notation.We have to consider 3 separate cases,First check that $(z_j - z_{j+1})_+ > 0$ iff $u_{j+1} < u_j$.

Case 1:$s \leq u_{j+1} < u_j < \theta_g$:-In this case,it is easy to check that we have the following,

$$(z_j - z_{j+1})_+ = g(u_j) - g(u_{j+1}) \tag{5.15}$$
$$G(u_j, u_{j+1}) = g(u_j) \tag{5.16}$$

(5.15) and (5.16) follow from the definition of G.For any $u_{j+2} \in [s, S]$,we have the following.If $u_{j+2} < u_{j+1}$,then we can check from the definition of G that $G(u_{j+1}, u_{j+2}) = g(u_{j+1})$ and if $u_{j+2} \geq u_{j+1}$,then by the fact that G is nonincreasing in the second variable and its consistency,we get that $G(u_{j+1}, u_{j+2}) \leq G(u_{j+1}, u_{j+1}) = g(u_{j+1})$.In either case,we have that

$$G(u_{j+1}, u_{j+2}) \leq g(u_{j+1}) \tag{5.17}$$

Now by combining (5.15),(5.16) and (5.17),we get that

$$(z_j^n - z_{j+1}^n)_+ \leq |G(u_{j+1}^n, u_{j+2}^n) - G(u_j^n, u_{j+1}^n)| \tag{5.18}$$

This gives the desired inequality in this case.

Case 2:$s \leq u_{j+1} < \theta_g \leq u_j$:In this case,we get directly from the definition that

$$(z_j - z_{j+1})_+ = g(\theta_g) - g(u_j) + g(\theta_g) - g(u_{j+1}) \tag{5.19}$$
$$G(u_j, u_{j+1}) = g(\theta_g) \tag{5.20}$$

For any $u_{j-1} \in [s, S]$,we have the following.If $u_{j-1} > u_j$,then we can check from the definition of G that $G(u_{j-1}, u_j) = g(u_j)$ and if $u_{j-1} \leq u_j$,then by the fact that G is nondecreasing in the first variable and its consistency,we get that $G(u_{j-1}, u_j) \leq G(u_j, u_j) = g(u_j)$.In either case,we have that

$$G(u_{j-1}, u_j) \leq g(u_j) \tag{5.21}$$

Now by combining (5.17),(5.20),(5.19) and (5.21),we get (5.11).

Case 3:$\theta_g \leq u_{j+1} < u_j \leq S$;In this case,from direct calculations we have that

$$(z_j - z_{j+1})_+ = g(u_{j+1}) - g(u_j) \tag{5.22}$$
$$G(u_j, u_{j+1}) = g(u_{j+1}) \tag{5.23}$$

Just by using (5.20),(5.22) and (5.23),we get the desired inequality that

$$(z_j - z_{j+1})_+ \leq |G(u_j^n, u_{j+1}^n) - G(u_{j-1}^n, u_j^n)| \tag{5.24}$$

This completes the proof of (5.11) in all the cases.
Next,we prove (5.13),As in the previous case,we have that $(z_{-2} - z_{-1})_+ > 0$ iff $u_{-2} > u_{-1}$ and we have to consider the following cases,

Case 1:$\theta_g \leq u_{-1} < u_{-2} \leq S$-The estimate (5.10) follows exactly as in case 3 of the previous case and we get the inequality,(5.24) as

$$(z_{-2} - z_{-1})_+ \leq |G(u^n_{-2}, u^n_{-1}) - G(u^n_{-3}, u^n_{-2})| \tag{5.25}$$

Case 2:$u_{-1} < u_{-2} \leq \theta_g$.In this case,from direct calculations we get that,

$$(z_{-2} - z_{-1})_+ = g(u_{-2}) - g(u_{-1}) \tag{5.26}$$

$$G(u_{-2}, u_{-1}) = g(u_{-2}) \tag{5.27}$$

Now for any $u_1 \in [s, S]$,we have from the definition of the Interface Godunov flux and its monotonicity properties that if $u_{-1} \leq \overline{S}$,$\overline{F}(u_{-1}, u_1) \leq \overline{F}(u_{-1}, s) = g(u_{-1})$ and similarly in the case $u_{-1} > \overline{S}$,we have $\overline{F}(u_{-1}, u_1) \leq g(S)$.In either case,we get that

$$\overline{F}(u_{-1}, u_1) \leq g(u_{-1}) \tag{5.28}$$

By combining (5.26),(5.27) and (5.28),we get that

$$(z_{-2} - z_{-1})_+ \leq |G(u_{-2}, (u_{-1})| - \overline{F}(u_{-1}, u_1)| \tag{5.29}$$

Case 3:$u_{-1} \leq \theta_g < u_{-2}$-In this case,we verify that

$$(z_{-2} - z_{-1})_+ = g(\theta_g) - g(u_{-2}) + g(\theta_g) - g(u_{-1}) \tag{5.30}$$

$$G(u_{-2}, u_{-1}) = g(\theta_g) \tag{5.31}$$

As in the previous cases,by combining (5.30),(5.31),(5.28) and (5.20),we prove (5.13).

Proof of 5.12(a):First check that $(\tilde{w}_j - \tilde{w}_{j+1})_- < 0$ iff $\tilde{u}_j < \tilde{u}_{j+1}$.We have to distinguish between 3 separate cases given by,

Case 1:$s \leq \tilde{u}_j < \tilde{u}_{j+1} \leq \theta_f$ In this case,we verify from direct calculations and the definition of the numerical flux $\tilde{F}$ that the following holds

$$-(\tilde{w}_j - \tilde{w}_{j+1})_- = f(\tilde{u}_j) - f(\tilde{u}_{j+1}) \tag{5.32}$$

$$\tilde{F}(\tilde{u}_j, \tilde{u}_{j+1}) = f(\tilde{u}_{j+1}) \tag{5.33}$$

For any $\tilde{u}_{j-1} \in [s, S]$,we have the following 2 cases i.e if $\tilde{u}_{j-1} < \tilde{u}_j$,then from the definition of the Enquist Osher flux it follows that $\tilde{F}(\tilde{u}_{j-1}, \tilde{u}_j) = f(\tilde{u}_j)$ and if $\tilde{u}_{j-1} > \tilde{u}_j$,then by the fact that the EO flux is nonincreasing in the first variable,it follows that $\tilde{F}(\tilde{u}_{j-1}, \tilde{u}_j) \geq \tilde{F}(\tilde{u}_j, \tilde{u}_j) = f(\tilde{u}_j)$.In either case we have,

$$\tilde{F}(\tilde{u}_{j-1}, \tilde{u}_j) \geq f(\tilde{u}_j) \tag{5.34}$$

So by combining (5.32),(5.33) and (5.34),we get that

$$-(\tilde{w}_j - \tilde{w}_{j+1})_- \leq |\tilde{F}(\tilde{u}_{j-1}, \tilde{u}_j) - \tilde{F}(\tilde{u}_j, \tilde{u}_{j+1}| \tag{5.35}$$

Thus proving the desired inequality (5.12(a)) in this case.

Case 2:$\theta_f \leq \tilde{u}_j \leq \tilde{u}_{j+1} \leq S$. In this case,we verify from direct calculations and the definition of the numerical flux $\tilde{F}$ that the following holds

$$-(\tilde{w}_j - \tilde{w}_{j+1})_- = f(\tilde{u}_{j+1}) - f(\tilde{u}_j) \tag{5.36}$$

$$\tilde{F}(\tilde{u}_j, \tilde{u}_{j+1}) = f(\tilde{u_j}) \tag{5.37}$$

For any $\tilde{u}_{j+2} \in [s, S]$,we have the following 2 cases i.e if $\tilde{u}_{j+1} < \tilde{u}_{j+2}$,then from the definition of the Enquist Osher flux it follows that $\tilde{F}(\tilde{u}_{j+1}, \tilde{u}_{j+2}) = f(\tilde{u}_{j+1})$ and if $\tilde{u}_{j+2} \leq \tilde{u}_{j+1})$,then by the fact that the EO flux is nondecreasing in the first variable,it follows that $\tilde{F}(\tilde{u}_{j+1}, \tilde{u}_{j+2}) \geq \tilde{F}(\tilde{u}_{j+1}, \tilde{u}_{j+1}) = f(\tilde{u}_{j+1})$.In either case we have,

$$\tilde{F}(\tilde{u}_{j+1}, \tilde{u}_{j+2}) \geq f(\tilde{u}_{j+1}) \tag{5.38}$$

So by combining (5.36),(5.37) and (5.38),we get that

$$-(\tilde{w}_j - \tilde{w}_{j+1})_- \leq |\tilde{F}(\tilde{u}_{j+1}, \tilde{u}_{j+2}) - \tilde{F}(\tilde{u}_j, \tilde{u}_{j+1})| \tag{5.39}$$

This again proves (5.12(a))

Case 3:$\tilde{u}_j \leq \theta_f \leq \tilde{u}_{j+1}$.In this case,we have that

$$\begin{aligned} -(\tilde{w}_j - \tilde{w}_{j+1})_- &= f(\tilde{u}_j) - f(\theta_f) + f(\tilde{u}_{j+1}) - f\theta_f) && (5.40)\\ \tilde{F}(\tilde{u}_j, \tilde{u}_{j+1}) &= f(\theta_f) && (5.41) \end{aligned}$$

Now by combining (5.40),(5.41),(5.34) and (5.38),we get (5.12(a)) i.e

$$\begin{aligned} -(\tilde{w}_j - \tilde{w}_{j+1})_- \leq\ & |\tilde{F}(\tilde{u}_{j+1}, \tilde{u}_{j+2}) - \tilde{F}(\tilde{u}_j, \tilde{u}_{j+1})| \\ & + |\tilde{F}(\tilde{u}_{j-1}, \tilde{u}_j) - \tilde{F}(\tilde{u}_j, \tilde{u}_{j+1})| \end{aligned}$$

Thus,we have shown the required estimates in all cases.Other estimates can be proved similarly and we can complete the proof of Lemma (5.4).

We use the above cell normalized variation inequalities in order to get the bounds on the total variation of the transformed schemes.More precisely,we have the following lemma,

Lemma 5.5:Let $z_j^n, w_j^n, \tilde{z}_j^n, \tilde{w}_j^n$ be as defined above,then the following holds,

$$\begin{aligned} \max\{TV(z_j^n), TV(\tilde{z}_j^n)\} &\leq \frac{4}{\lambda}\max\{\sum_{j\neq 0}|u_j^1 - u_j^0|, \sum_{j\neq 0}|\tilde{u}_j^1 - \tilde{u}_j^0|\} \\ \max\{TV(\tilde{w}_j^n), TV(w_j^n)\} &\leq \frac{4}{\lambda}\max\{\sum_{j\neq 0}|u_j^1 - u_j^0|, \sum_{j\neq 0}|\tilde{u}_j^1 - \tilde{u}_j^0|\} \end{aligned} \tag{5.42}$$

Proof:We prove the above estimate for the sequence z_j^n.The other sequences are shown satisfy the above inequality in the same way.First observe that $\forall j \geq -1$,we have

$$(z_j^n - z_{j+1}^n)_+ \equiv 0$$

So we have that,

$$TV(z_j^n) = \sum |(z_j^n - z_{j+1}^n| = 2\sum (z_j^n - z_{j+1}^n)_+$$

By adding (5.11),(5.12) and the above inequality over all j,we get that

$$\begin{aligned} \sum (z_j^n - z_{j+1}^n)_+ = 2(&\sum_{j\leq -2}|G(u_j^n, u_{j+1}^n) - G(u_{j-1}^n, u_j^n)| + |G(u_{-2}^n, u_{-1}^n) - \overline{F}(u_{-1}^n, u_1^j)| \\ &+ |\overline{F}(u_{-1}^n, u_1^n) - F(u_1^n, u_2^n)| + \sum_{j\geq 2}|F(u_j^n, u_{j+1}^n) - F(u_{j-1}^n, u_j^n)| \\ &= \frac{2}{\lambda}\sum_{j\neq 0}|u_j^{n+1} - u_j^n| \end{aligned} \tag{5.43}$$

From (5.5),we can get the required estimate on the right hand side of (5.42) and the inequality follows.In a similar way,we can get the bounds for the other transformed sequences.

Now we define piecewise constant functions based on the transformed sequences and write some estimates on these functions which follow in a straight forward way from Lemma 5.5, Let $z^h, w^h, \tilde{z}^h, \tilde{w}^h$ be defined as $z^h(x,t) = z_j^n, w^h(x,t) = w_j^n, \tilde{z}^h(x,t) = \tilde{z}_j^n, \tilde{w}^h(x,t) = \tilde{w}_j^n, \quad \forall \quad (x,t) \in I_j^n$ be piecewise constant functions for the transformed sequences,then the following estimates hold,

Lemma 5.6:With the functions defined as above and $\forall t \in I\!R_+$,we have

$$\max\{TV(z^h), TV(w^h), TV(\tilde{z}^h), TV(\tilde{w}^h)\} \leq \frac{4}{\lambda}\overline{N}(f,g,u_0) \tag{5.44}$$

Proof:Follows directly from Lemma 5.5.

We need time continuity estimates in L^1 for the approximations u^h which is given in the following lemma,

Lemma 5.7:Let $u_0, v_0 \in L^\infty(I\!R, [s,S])$ such that $N(f,g,u_0) < \infty, N(f,g,v_0) < \infty$ be initial datas and let u_h and v_h be the corresponding solutions obtained by the Godunov scheme (4.9) and $\tilde{u}_h$ and $\tilde{v}_h$ be the corresponding solutions given by the Enquist Osher type scheme (4.10),then

$$s \leq u_h(x,t), \tilde{u}_h(x,t) \leq S \qquad \forall \ (x,t) \in I\!R \times I\!R_+ \tag{5.45}$$

$$\int_{I\!R} |u_h(x,t) - u_h(x,\tau)| dx \leq N_h(f,g,u_0)(2\Delta t + |t-\tau|) \tag{5.46}$$

$$\int_{I\!R} |\tilde{u}_h(x,t) - \tilde{u}^h(x,\tau)| dx \leq N_h(f,g,u_0)(2\Delta t + |t-\tau|) \tag{5.47}$$

Proof:(5.45) follows directly from (5.3) and (5.4).We prove (5.46),as (5.47) follows in a similar way.
Let $t_n \leq t < t_{n+1}$ and $t_m \leq \tau < t_{m+1}$, so

$$|n-m|\Delta t = |t_n - t_m| \leq |t_n - t| + |t-\tau| + |\tau - t_m| \leq 2\Delta t + |t-\tau|.$$

Hence from lemma (5.3), we obtain

$$\begin{aligned}
\int_{I\!R} |u_h(x,t) - u_h(x,\tau)| dx &= h \sum_{j\neq 0} |u_j^n - u_j^m| \\
&\leq h \sum_{j\neq 0} \sum_{i=0}^{n-m+1} |u_j^{n-i} - u_j^{n-i-1}| \\
&\leq h|n-m| \sum_{j\neq 0} |u_j^1 - u_j^0| \\
&\leq \tfrac{\Delta t|n-m|}{\lambda} \sum_{j\neq 0} |u_j^1 - u_j^0| \\
&\leq (2\Delta t + |t-\tau|) N_h(f,g,u_0)
\end{aligned}$$

We complete the proof of (5.46).

Now equipped with the above lemmas,we proceed to state and prove our main convergence theorem,

Theorem 5.1:Assume that λ, M satisfy the CFL condition $2\lambda M \leq 1$ and u_0 satisfy the hypothesis $(IN_1),(IN_2)$.Let $u_h, \tilde{u}_h$ be approximate solutions defined above,then there exists a subsequence (still denoted by h) such that u_h converge almost everywhere to a weak solution u of (0.1).In fact $u_h \to u$ in $L^\infty_{\text{loc}}(I\!R_+, L^1_{\text{loc}}(I\!R))$ as h goes to 0.Similarly,along a further subsequence still denoted by h,u_h converge almost everywhere to a weak solution $\tilde{u}$ of (0.1) and similarly,$\tilde{u_h} \to \tilde{u}$ in $L^\infty_{\text{loc}}(I\!R_+, L^1_{\text{loc}}(I\!R))$ as h goes to 0.Furthermore,both u and $\tilde{u}$ satisfy the Kruzkhov entropy condition and hence identical to the entropy solution of (0.1).

Proof:This is the main convergence theorem for our Godunov and Enquist Osher type schemes (4.9),(4.10).The proof follows by the classical arguments of the Lax Wendroff theorem (See [15]) and the modifications introduced in [3].We omit some of the details and refer to the above quoted paper for them.Also we will only sketch the proof for the u_h case.The Enquist Osher case follows exactly as that of the Godunov case.

For any fixed $t > 0$,we have the BV bounds from the Lemma 5.5 and by using the standard Rellich compactness theorem that upto subsequences (still denoted by h),we get that $z_h(.,t), w_h(.,t)$ converge in L^1 and for almost all x to $z(.,t)$ and $w(.,t)$ respectively.

For fixed t and for almost all $x < 0$,then from the convergence we have that $\psi(u_h) \to z$.Note that ψ is monotone for $x < 0$ and we get that $u_h(x,t) \to \psi^{-1}(z(x,t)) = u(x,t)$.Thus for all $x < 0$,we get that $u_h(.,t)$ converges to $u(.,t)$ for almost all $x < 0$.Similarly,for $x > 0$,we define $u(x,t) = \psi_2^{-1}(u_h(x,t)$ and get the a.e convergence.Now we use the standard density argument along with the time continuity estimate (5.46),to get that

$$u_h \to u \quad \in L^\infty_{\text{loc}}((0,T), L^1_{\text{loc}}) \tag{5.48}$$

Once,we have the above convergence we can use the standard arguments of the Lax-Wendroff type in order to show that u_h converges to a weak solution of (0.1).The proof follows exactly as in [3] and we refer the reader to this reference for details.Similarly,the consistency of the scheme with the interior entropy condition is shown by using the numerical entropy fluxes of Crandall-Majda (see [7]).Check [3] for the details.

We reiterate that the key step of the proof of convergence is the Lemma 5.4 which used the set of singular mappings in order to get the normalized variation bounds.

6. Numerical Results. In this section,we will provide 2 numerical examples in order to compare our version of the Godunov scheme (4.9) with the staggered version of the Godunov scheme developed by Towers in [29].We add that we have also tested with the Enquist Osher type schemes on these examples but omit the results as they are very close to those obtained by the Godunov type schemes.

In both examples,we will consider the following flux functions for (0.1),

$$\begin{aligned} f(u) &= 2u^2 - u \\ g(u) &= 3u - 2u^2 \end{aligned}$$

Check that f is convex g is concave on $[0,1]$ and that $f(0) = g(0), \quad f(1) = g(1)$ thus satisfying the hypothesis for the fluxes considered in this paper.We have computed using our Godunov type scheme (4.9) which we will be referring as ERS and the staggered version of the Godunov scheme used by Towers which we term as TS.

In the first experiment,we consider the following initial data for (0.1).

$$\begin{aligned} u_0(x) &= 0.25 \quad \text{if} \quad \mathrm{x} < 0 \\ &= 0.5 \quad \text{if} \quad \mathrm{x} > 0 \end{aligned} \tag{6.1}$$

From section 3,we obtain the exact solution consists of a steady shock at the interface ($x = 0$) connecting 0.25 and 0.8624 and a right moving shock connecting 0.8624 and 0.5.The numerical results on various meshes are illustrated in figures (6.1) and (6.2) where the solutions are compared with the exact solution at times $t = 0.6$ and $t = 1.2$ respectively.The L^1 errors for different mesh sizes are shown in Table 6.1.From the above,we conclude that both ERS and TS show similar behavior with an equal amount of numerical dissipation at the interior shock.We show the results obtained with mesh sizes of $h = 0.1$ and $h = 0.025$ in order to illustrate the fact that the dissipation reduces as the mesh size shrinks to zero in accordance with the convergence theory developed in this paper.There is ver y little to differentiate between both schemes in this example altough ERS gives a slightly better approximation.This is due to the better resolution at the interface shock which we manage by solving the Riemann problem exactly.

In the second example,we consider the following initial data given by

$$\begin{aligned} u_0(x) &= 0.25 \quad \text{if} \quad \mathrm{x} < 0 \\ &= 1 \quad \text{if} \quad \mathrm{x} > 0 \end{aligned} \tag{6.2}$$

Again from section 3,we note that the exact solution consists of a steady shock at the interface connecting 0.5 with 0.8624 and a rarefaction wave connecting 0.8624 with 1.Figures (6.3) and (6.4) illustrate the numerical results obtained by us at $t = 0.6$ and $t = 0.9$ respectively and the L^1 errors are shown in Table (6.1).Again,our scheme resolves the solution quite well with the dissipation reducing as the mesh size reduces to zero.The numerical dissipation of TS is similar to that of ERS but as is clear from figure(6.3),it introduces a spurious traveling wave like solution at the beginning of the rarefaction wave.This wave is certainly not a part of the exact solution and its presence requires an explanation.We remark that the amplitude of the traveling wave goes to zero as the mesh size shrinks indicating that TS also converges in this case but this is a qualitative difference between our scheme and the staggered mesh versions.Also,from Table (6.1) note that our scheme gives a consistently lower errors than TS on account of the better resolution of the interface shock and the presence of the spurious traveling wave solutions.

We note that similar phenomena is also exhibited by the Enquist Osher versions of both schemes.The biggest advantage of the staggered schemes over 2×2 Riemann solvers is its simplicity.Al tough,we also exactly solve the Riemann problem for (0.1),yet our scheme is as simple to implement as the staggered schemes on account of the explicit formulas for the numerical fluxes obtained in section 4.In fact,any code for implementing a Godunov or Enquist Osher scheme can be easily modified for using our scheme.The comparable simplicity and slightly better resolution in certain situations make our schemes useful in situations where the staggered schemes are used.A further investigation is needed in order to compare our scheme with the staggered mesh schemes and this is the subject of a forthcoming paper.

TABLE 6.1
L^1 errors for ERS and TS in both Case 1 and Case 2.

h	Case 1 ERS	Case 1 TS	Case 2 ERS	Case 2 TS
0.1	0.0715	0.0903	0.0370	0.0872
0.05	0.0340	0.0437	0.0233	0.0444
0.025	0.0147	0.0210	0.0149	0.0238
0.0125	0.0077	0.0105	0.0098	0.0136

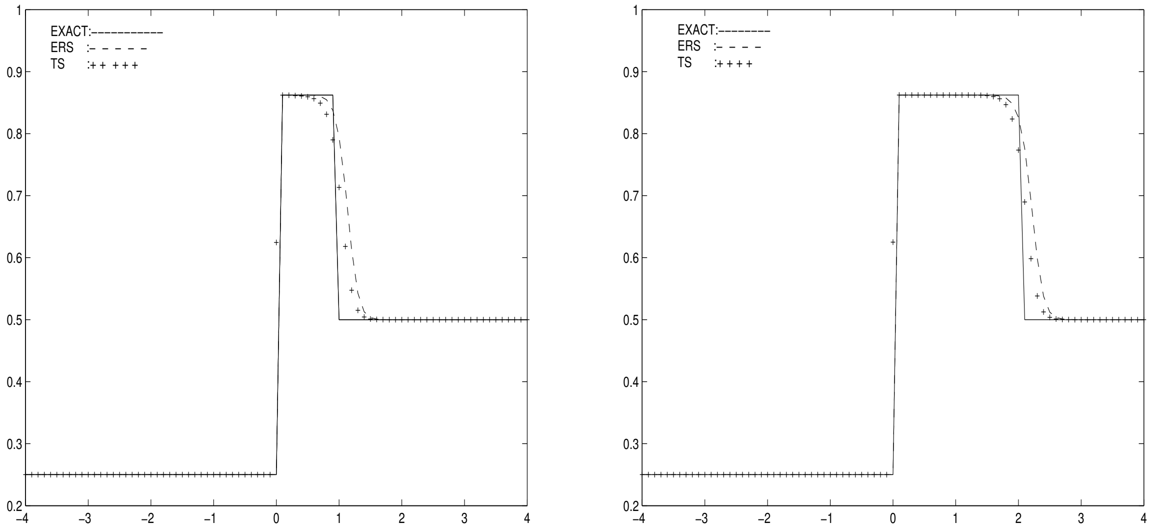

FIG. 6.1. *Computed Solutions for Case 1 using ERS and TS with h=0.1 at times t=0.6 and t=1.2 respectively*

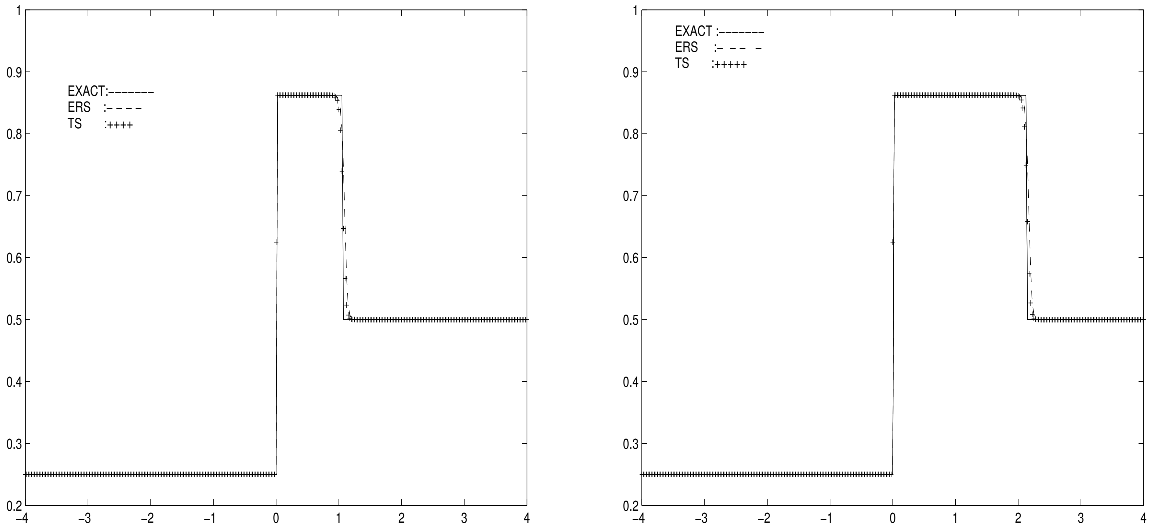

FIG. 6.2. *Computed Solutions for Case 1 using ERS and TS with h=0.025 at times t= 0.6 and t=1.2 respectively*

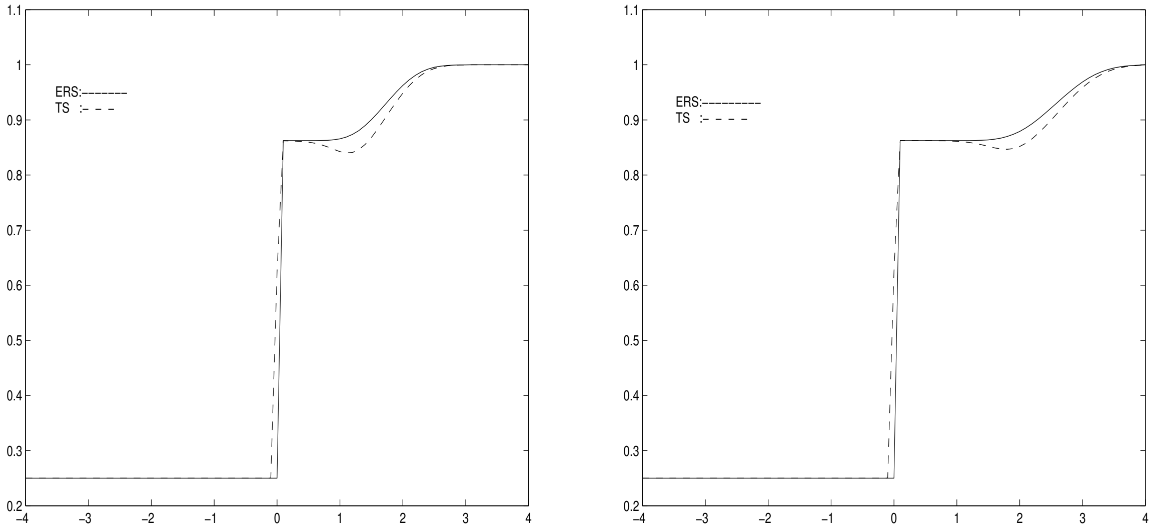

FIG. 6.3. *Computed Solutions for Case 2 using ERS and TS with h=0.1 at times t=0.6 and t=0.9 respectively*

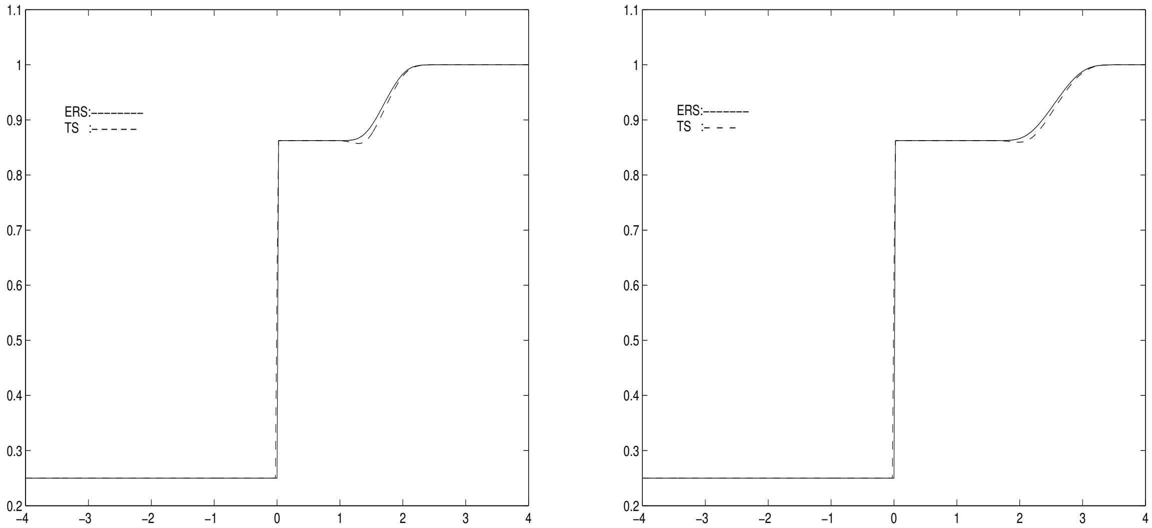

FIG. 6.4. *Computed Solutions for Case 2 using ERS and TS with h=0.025 at times t= 0.6 and t=1.2 respectively*

7. Conclusion. Scalar conservation laws with discontinuous flux occur very frequently in physical and engineering applications and hence we need to develop a wellposedness theory and stable numerical methods to approximate their solutions.An interesting case of such equations is when the one of the fluxes is convex and the other concave which can arise when the discontinuous coefficient is of indefinite sign.This case has received less attention in literature on account of the difficulties in adapting the principal convergence tool of singular mapping.

In this paper,we have studied a scalar conservation law with discontinuous flux having the concave-convex flux geometry.The uniqueness of entropy solutions is shown by using a Kruzkhov type argument.The strange feature of the flux geometry considered in this case is that we donot need to impose any additional entropy conditions at the interface.We give explicit solutions to the Riemann problem and use it to construct upwind finite difference schemes of the Godunov and Enquist Osher type,These schemes are shown to converge to the entropy solution by using a singular mapping technique.The novel feature of this paper is the use of a combination of singular mappings in order to show the convergence of our schemes to the entropy solutions.Numerical results are presented,which show that our schemes resolve the solutions quite well.Comparison with the staggered mesh schemes indicate better performance of our schemes in some situations.We also mention that the algorithm generalizing our scheme from the 2-flux case to that of the more general discontinuous case (1.1) is contained in a forthcoming paper.

Acknowledgments. The author wishes to thank Professor Adimurthi and Professor G.D.Veerappa Gowda for their useful suggestions and discussions.

REFERENCES

[1] Adimurthi and G.D.Veerappa Gowda,*Conservation Laws with Discontinuous flux*, Journal of Mathematics,Kyoto univ ,to appear.

[2] Adimurthi and G.D.Veerappa Gowda,*Extensions of Godunov scheme for conservation laws with flux function discontinuous in the space variable*,Preprint,(2003)

[3] Adimurthi,J.Jaffre and G.D.Veerappa Gowda,*Godunov type methods for Scalar Conservation Laws with Flux function discontinuous in the space variable*, SIAM J. Numer. Anal.,to appear.

[4] Adimurthi,G.D.Veerappa Gowda and Siddhartha Mishra,*Well-posedness for Scalar Conservation laws with Discontinuous Flux*, in preparation.

[5] R.Burger,K.H.Karlsen,C.Klingenberg and N.H.Risebro,A front tracking approach to a model of continuous sedimentation in ideal clarifier-thickener units, Nonlinear Analysis, Real world applications, 4(3):457-481,2003.

[6] R.Burger,K.H.Karlsen,N.H.Risebro and J.D.Towers,*Well-posedness in BV_t and convergence of a difference scheme for continuous sedimentation in ideal clarifier thickener units*, Numer. Math, to appear.

[7] M.G.Crandall and A.Majda,*Monotone difference approximations for scalar conservation laws*, Math. Comp,34 (1980),pp. 1-21.

[8] G.M.Coclite and N.H.Risebro,*Conservation Laws with time dependent discontinuous coefficients*,Preprint.

[9] S.Diehl,*On scalar conservation laws with point source and discontinuous flux function modeling continuous sedimentation*, SIAM J. Math. Anal., 26(6) (1995),pp. 1425-1451.

[10] S.Diehl,*A conservation law with point source and discontinuous flux function modelling continuous sedimentation*, SIAM J. Appl. Math., 56(2) (1996),pp. 388-419.

[11] S.Diehl,*Scalar conservation laws with discontinuous flux function.1-The viscous profile condition*, Communications in Mathematical Physics, 176 (1996),pp. 23-44.

[12] B.Enquist and S.Osher,*One sided difference approximations for nonlinear conservation laws*, Math. Comp., 34, (1980), pp. 321-351.

[13] T.Gimse and N.H.Risebro,*Riemann problems with discontinuous flux function*, In Proc. 3rd Internat. Conf. Hyperbolic problems Studentlitteratur, Uppsala,1991,pp. 488-502.

[14] T.Gimse and N.H.Risebro,*Solution of Cauchy problem for a conservation law with discontinuous flux function*, SIAM J. Math. Anal, 23(3), (1992),pp. 635-648.

[15] E.Godlewski and P.A.Raviart,*Hyperbolic systems of Conservation laws*, Mathematiques et Applications,Ellipses, Paris,1991.

[16] K.H.Karlsen,N.H.Risebro and J.D.Towers,*Upwind difference approximations for degenerate parabolic convection-diffusion equations with a discontinuous coefficient*, IMA J. Numer. Anal.,22(4), (2002),pp. 623-664.

[17] K.H.Karlsen,N.H.Risebro and J.D.Towers,*On a nonlinear degenerate parabolic transport-diffusion equation with a d iscontinuous coefficient*, Electron. J. Differential Equations,No 93, (2002),23 pages (electronic)

[18] K.H.Karlsen,N.H.Risebro and J.D.Towers,*L^1 stability for entropy solution of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients*, Skr. K. Nor. Vidensk. Selsk.,no 3, (2003) 49 pages.
[19] R.A.Klausen and N.H.Risebro,*Stability of conservation laws with discontinuous coefficients*, J. Differential Equations,157(1), (1999),pp. 41-60.
[20] C.Klingenberg and N.H.Risebro,*Convex conservation laws with discontinuous coefficients,Existence, Uniqueness and Asymptotic Behaviour*, Comm. Partial Differential Equations, 20(11-12), (1995),pp. 1959-1990.
[21] C.Klingenberg and N.H.Risebro,*Stability of a resonant system of conservation laws modeling polymer flow with gravitation*, J. Differential Equations, 1970(2), (2001),pp. 344-380.
[22] K.H.Karlsen,C.Klingenberg and N.H.Risebro,*A relaxation scheme for conservation laws with discontinuous coefficients*, Math. Comp.,to appear
[23] S.N.Kruzkhov,*First order quasilinear in several independent variables*, Math. USSR Sbornik, 10(2), (1970) ,pp. 217-243.
[24] L.W.Lin,B.J.Temple and J.H.Wang,*A comparison of the convergence rates for the Godunov's method and Glimm's method in resonant nonlinear system of conservation laws*,SIAM J. Numer. Anal.,32(3), (1995),pp. 824-840.
[25] L.W.Lin,B.J.Temple and J.H.Wang,*Suppression of oscillations in Godunov's method for a resonant nonstrictly hyperbolic system.*,SIAM J. Numer. Anal.,32(3), (1995),pp. 841-865.
[26] W.K.Lyons,*Conservation laws with sharp inhomogeneities*,Quart. Appl. Math.,40(4), (1983), pp. 385-393.
[27] Siddhartha Mishra,*Scalar Conservation Laws with Discontinuous flux*, M.S thesis,Indian Institute of Science,Bangalore,India, 2003.
[28] B.Temple,*Global solution of the Cauchy problem for a class of 2×2 nonstrictly hyperbolic conservation laws*, Adv. in Appl. Math.,3(3), (1982),pp. 335-375.
[29] J.D.Towers,*Convergence of a difference scheme for conservation laws with a discontinuous flux*,SIAM J. Numer. Anal.,38(2), (2000),pp. 681-698.
[30] J.D.Towers,*A difference scheme for conservation laws with a discontinuous flux-the nonconvex case*,SIAM J. Numer. Anal.,39(4), (2001),pp. 1197-1218.