Some Results on General System of Isentropic Gas Dynamics

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Abstract

In this paper, we construct a sequence of regular hyperbolic systems (1.1) to approximate the general system of isentropic gas dynamics (1.2). First, for each fixed approximation parameter δ, we establish the existence of entropy solutions for the Cauchy problem (1.1) with bounded initial date (1.4). Second, letting $\varepsilon = o(\delta)$, we obtain a simple proof of the H^{-1}_{loc} compactness of weak entropy pairs of system (1.2) in the form $\eta(\rho, u) = \rho H(\rho, u)$ constructed in [CL1, CL2].

1 Introduction

In this paper, we construct a sequence of regular hyperbolic systems

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho, \delta))_x = 0, \end{cases} \tag{1.1}$$

to approximate the system of isentropic gas dynamics in Eulerian coordinates

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = 0, \end{cases} \tag{1.2}$$

where ρ is the density of gas, u the velocity, $P = P(\rho)$ the pressure, and $\delta > 0$ in (1.1) denotes a regular perturbation constant and the perturbation pressure

$$P_1(\rho, \delta) = \int_{2\delta}^{\rho} \frac{t - 2\delta}{t} P'(t) dt. \tag{1.3}$$

For the polytropic gas, P takes the special form $P(\rho) = c\rho^{\gamma}$, where $\gamma > 1$ and c is an arbitrary positive constant.

The study of the existence of global weak solutions for the Cauchy problem (1.2) has a long history. For the polytropic gas in the Lagrangian coordinates, the first large data existence theorem with locally finite total variation, for $\gamma = 1$ was obtained by Nishida [Ni], and for $\gamma \in (1, 1+\delta)$, δ small, obtained by Nishida and Smoller [NS]. The method used in [Ni, NS] (see also [Liu] for the non-isentropic gas) is the Glimm's scheme method (cf. [Gl]).

Using the compensated compactness ideas developed by Tartar and Murat [Ta],[Mu], DiPerna [Di2] established a global existence theorem for the Cauchy problem (1.2) with large bounded initial data for $\gamma = 1 + \frac{2}{N}, N \geq 5$ odd, with the aid of the viscosity method. Ding, Chen, and Luo [DCL] and Chen [Chen] proved the convergence of the Lax-Friedrichs scheme and the existence of global solutions with L^{∞} large initial data with adiabatic exponent $\gamma \in (1, \frac{5}{3}]$. Lions, Perthame and Tadmor [LPT] proved the global existence of a weak solution for $\gamma \geq 3$ by applying the kinetic formulation coupled with the compensated compactness method. Finally, the method in [LPT] was successfully extended by Lions, Perthame and Souganidis [LPS] to fill up the gap $\gamma \in (\frac{5}{3}, 3)$ as well as a new proof for whole $\gamma > 1$. Recently Huang and Wang [HW] studied the case of $\gamma = 1$. So the existence of a generalized solution for the Cauchy problem (1.3) with bounded initial data was completely resolved for the case of a polytropic gas.

The global smooth solution of the Cauchy problem (1.2) with a class of smooth initial data with the vacuum for more general pressure function $P(\rho)$ was obtained in [Lu2], where a sequence of nonstrictly hyperbolic systems is used to approximate system (1.2).

The global existence of L^{∞} entropy solutions for system (1.2) with general presure $P(\rho)$ and arbitrarily large L^{∞} initial data was established in [CL1, CL2], in which, roughly speaking, $P(\rho)$ takes the form

$$P(\rho) = \rho^{\gamma}(1 + p(\rho))$$

with some restrictions on the function $p(\rho)$. In [CL1, CL2], some new ideas and techniques are used to construct the weak entropy kernel of system (1.2). In the neighborhood of vacuum, the weak entropy constructed in [CL1, CL2] is in the form $\eta(\rho,u) = \rho H(\rho,u)$, however, the proof of its compactness in $H^{-1}_{loc}(R \times R^+)$ is not provided.

In this paper, for fixed $\delta > 0$, we first establish the existence of entropy solutions for the Cauchy problem (1.1) with bounded measurable initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \quad \rho_0(x) \geq 2\delta. \tag{1.4}$$

Theorem 1 *Let the initial data $(\rho_0(x), u_0(x))$ be bounded measurable and $\rho_0(x) \geq 2\delta, P(\rho) \in C^2(0,\infty), P'(\rho) > 0,$*

$$2P'(\rho) + \rho P''(\rho) > 0 \text{ for } \quad \rho > 0 \tag{1.5}$$

and

$$\int_c^\infty \frac{\sqrt{P'(\rho)}}{\rho} d\rho = \infty, \quad \int_0^c \frac{\sqrt{P'(\rho)}}{\rho} d\rho < \infty, \quad \forall c > 0. \tag{1.6}$$

Then the Cauchy problem (1.1), (1.4) has a global bounded entropy solution.

Second, with the help of the perturbation parameter $\delta, \varepsilon = o(\delta)$, we obtain a simple proof of the H^{-1} compactness in the following theorem.

Theorem 2 *Let all conditions about in Theorem 1 be satisfied and the limit*

$$\lim_{\rho \to 0} \frac{(P'(\rho))^{\frac{3}{2}}}{\rho P''(\rho)} = e, \tag{1.7}$$

where $e \geq 0$ is a constant. If the weak entropy-entropy flux pair $(\eta(\rho,u), q(\rho,u))$ of system (1.2) is in the form $\eta(\rho,u) = \rho H(\rho,u)$ and $H_u(\rho,u), H_{uu}(\rho,u), H_{uuu}(\rho,u)$ are continuous on $0 \leq \rho \leq M_1, |u| \leq M_1$, where M_1 is a positive constant given in (2.11), then

$$\eta_t(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)) + q_x(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)) \tag{1.8}$$

is compact in $H^{-1}_{loc}(R \times R^+)$ as $\varepsilon = o(\frac{P'(2\delta)}{2\delta})$ and δ tends to zero, with respect to the viscosity solutions $(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)$ of the Cauchy problem (2.6) and (1.4).

2 Proof of Theorem 1

In this section, we prove Theorem 1.

By simple calculations, two eigenvalues of system (1.1) are

$$\lambda_1 = \frac{m}{\rho} - \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)}, \quad \lambda_2 = \frac{m}{\rho} + \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)} \tag{2.1}$$

with corresponding right eigenvectors

$$r_1 = (1, u - \sqrt{P'(\rho)})^T, \quad r_2 = (1, u + \sqrt{P'(\rho)})^T \tag{2.2}$$

and Riemann invariants

$$z(u,v) = \frac{m}{\rho} - \int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s} ds, \quad w(u,v) = \frac{m}{\rho} + \int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s} ds, \tag{2.3}$$

where $m = \rho u$. Moreover

$$\nabla \lambda_1 \cdot r_1 = -\frac{4\delta}{\rho^2}\sqrt{P'(\rho)} - \frac{\rho - 2\delta}{2\rho^2\sqrt{P'(\rho)}}(2P'(\rho) + \rho P''(\rho)) \tag{2.4}$$

and

$$\nabla \lambda_2 \cdot r_2 = \frac{4\delta}{\rho^2}\sqrt{P'(\rho)} + \frac{\rho - 2\delta}{2\rho^2\sqrt{P'(\rho)}}(2P'(\rho) + \rho P''(\rho)). \tag{2.5}$$

Consider the Cauchy problem for the related parabolic system

$$\begin{cases} \rho_t + ((\rho - 2\delta)u)_x = \varepsilon \rho_{xx} \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho, \delta))_x = \varepsilon (\rho u)_{xx}, , \end{cases} \tag{2.6}$$

with the initial data (1.4). We multiply (2.6) by (w_ρ, w_m) and (z_ρ, z_m), respectively, to obtain

$$\begin{aligned} & w_t + \lambda_2 w_x \\ = \quad & \varepsilon w_{xx} + \tfrac{2\varepsilon}{\rho}\rho_x w_x - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} & z_t + \lambda_1 z_x \\ = \quad & \varepsilon z_{xx} + \tfrac{2\varepsilon}{\rho}\rho_x z_x + \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2. \end{aligned} \tag{2.8}$$

Then the assumptions on $P(\rho)$ yield

$$w_t + \lambda_2 w_x \leq \varepsilon w_{xx} + \frac{2\varepsilon}{\rho}\rho_x w_x \tag{2.9}$$

and

$$z_t + \lambda_1 z_x \geq \varepsilon z_{xx} + \frac{2\varepsilon}{\rho}\rho_x z_x. \tag{2.10}$$

If we consider (2.9) and (2.10) as inequalities about the variables w and z, then we can get the estimates $w(\rho^{\varepsilon,\delta}, m^{\varepsilon,\delta}) \leq M, z(\rho^{\varepsilon,\delta}, m^{\varepsilon,\delta}) \geq -M$ by applying the maximum principle to (2.9) and (2.10). From these, we know that $u^{\varepsilon,\delta} = m^{\varepsilon,\delta}/\rho^{\varepsilon,\delta}$ are uniformly bounded with respect to ε, δ. Then, using the first equation in (2.6), we get $\rho^{\varepsilon,\delta} \geq 2\delta$. Therefore, the region

$$\Sigma = \{(\rho, m) : w(\rho, m) \leq M,\ z(\rho, m) \geq -M, \rho \geq 2\delta\}$$

is an invariant region. Thus we obtain the estimates

$$2\delta \leq \rho^{\varepsilon,\delta} \leq M_1, \quad |u^{\varepsilon,\delta}| \leq M_1 \tag{2.11}$$

for a suitable constant M_1, being independent of ε, δ, since $\int_c^\infty \frac{\sqrt{P'(\rho)}}{\rho} d\rho = \infty$ and $\int_0^c \frac{\sqrt{P'(\rho)}}{\rho} d\rho < \infty$ for any constant $c > 0$.

Thus for fixed $\delta > 0$, it follows from (2.1) that system (1.1) is strictly hyperbolic in the domain $\{(x,t) : \rho > 2\delta\}$, while it is nonstrictly hyperbolic in the domain $\{(x,t) : \rho = 2\delta\}$, since $\lambda_1 = \lambda_2$ when $\rho = 2\delta$. However, from (2.4) and (2.5), both characteristic fields of system (1.1) are genuinely nonlinear in the range $\rho \geq 2\delta$.

For smooth solutions, system (1.1) is equivalent to the following system

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = 0 \\ u_t + (\frac{1}{2}u^2 + \int_{2\delta}^{\rho} \frac{(t-2\delta)P'(t)}{t^2} dt)_x = 0, \end{cases} \tag{2.12}$$

and particularly, both systems have the same entropy-entropy flux pairs. Thus any entropy-entropy flux pair $(\eta(\rho, m), q(\rho, m))$ of system (1.1) satisfies the additional system

$$q_\rho = u\eta_\rho + \frac{(\rho - 2\delta)P'(\rho)}{\rho^2}\eta_u, \quad q_u = (\rho - 2\delta)\eta_\rho + u\eta_u. \tag{2.13}$$

Eliminating the q from (2.13), we have

$$\eta_{\rho\rho} = \frac{P'(\rho)}{\rho^2}\eta_{uu}. \tag{2.14}$$

Therefore, system (1.1) and system (1.2) have the same entropies.

It is easy to check that system (1.1) has a convex entropy

$$\eta^\star = \frac{\rho u^2}{2} + \int_{2\delta}^{\rho} \frac{(\rho - t)P'(t)}{t} dt, \tag{2.15}$$

with corresponding entropy flux

$$q^\star = \frac{\rho u^3}{2} - \frac{\delta u^3}{3} + u(\rho - 2\delta) \int_{2\delta}^{\rho} \frac{P'(t)}{t} dt. \tag{2.16}$$

We multiply (2.6) by $(\eta^\star_\rho, \eta^\star_m)$ to obtain the boundedness of

$$\varepsilon(\rho_x, m_x) \cdot \nabla^2 \eta^\star(\rho, m) \cdot (\rho_x, m_x)^T \tag{2.17}$$

in $L^1_{loc}(R \times R^+)$. Then it follows that

$$\varepsilon \frac{P'(\rho)}{\rho} \rho_x^2 + \varepsilon \frac{1}{\rho} [\frac{m}{\rho} \rho_x - m_x]^2 = \varepsilon \frac{P'(\rho)}{\rho} \rho_x^2 + \varepsilon \rho u_x^2 \tag{2.18}$$

is bounded in $L^1_{loc}(R \times R^+)$.

Since $\rho \geq 2\delta$, we get the boundedness of

$$\varepsilon \rho_x^2, \quad \varepsilon u_x^2 \quad \text{in} \quad L^1_{loc}(R \times R^+) \tag{2.19}$$

for any fixed $\delta > 0$.

Thus for smooth entropy-entropy flux pairs $(\eta_i(\delta, \rho, u), q_i(\delta, \rho, u)), i = 1, 2,$ of system (1.1), the following measure equations or the communicate relations are satisfied

$$\begin{aligned} &< \nu^\delta_{(x,t)}, \eta_1(\delta) q_2(\delta) - \eta_2(\delta) q_1(\delta) > \\ &\quad =< \nu^\delta_{(x,t)}, \eta_1(\delta) >< \nu^\delta_{(x,t)}, q_2(\delta) > - < \nu^\delta_{(x,t)}, \eta_2(\delta) >< \nu^\delta_{(x,t)}, q_1(\delta) >, \end{aligned} \tag{2.20}$$

where $\nu^\delta_{(x,t)}$ is the family of positive probability measures with respect to the viscosity solutions $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ of the Cauchy problem (2.6) and (1.4).

To finish the proof of Theorem 1, it is enough to prove that Young measures given in (2.20) are Dirac measures.

For applying for the framework given by DiPerna in [Di1] to prove that Young measures are Dirac ones, we construct four families of entropy-entropy flux pairs of Lax's type in the following special form:

$$\eta^1_k = e^{kw}(a_1(\rho) + \frac{b_1(\rho, k)}{k}), \; q^1_k = \eta^1_k(\lambda_2 + \frac{c_1(\rho, k)}{k} + \frac{d_1(\rho, k)}{k^2}); \tag{2.21}$$

$$\eta^2_{-k} = e^{-kw}(a_2(\rho) + \frac{b_2(\rho,k)}{k}),\ q^2_{-k} = \eta^2_{-k}(\lambda_2 + \frac{c_2(\rho,k)}{k} + \frac{d_2(\rho,k)}{k^2}); \tag{2.22}$$

$$\eta^2_{k} = e^{kz}(a_3(\rho) + \frac{b_3(\rho,k)}{k}),\ q^2_{k} = \eta^2_{k}(\lambda_1 + \frac{c_3(\rho,k)}{k} + \frac{d_3(\rho,k)}{k^2}); \tag{2.23}$$

$$\eta^1_{-k} = e^{-kz}(a_4(\rho) + \frac{b_4(\rho,k)}{k}),\ q^1_{-k} = \eta^1_{-k}(\lambda_1 + \frac{c_4(\rho,k)}{k} + \frac{d_4(\rho,k)}{k^2}), \tag{2.24}$$

where w, z are the Riemann invariants of system (1.1) given by (2.3). Notice that all the unknown functions $a_i, b_i (i = 1, 2, 3, 4)$ are only of a single variable ρ. This special simple construction yields an ordinary differential equation of second order with a singular coefficient $1/k$ before the term of the second order derivative. Then the following necessary estimates for functions $a_i(\rho), b_i(\rho, k)$ are obtained by the use of the singular perturbation theory of ordinary differential equations:

$$0 < a_i(\rho) \le M, \quad |b_i(\rho,k)| \le M, \tag{2.25}$$

$$0 < c_i(\rho) \le M, \quad (\text{ or } -M \le c_i(\rho) < 0), \quad |d_i(\rho,k)| \le M \tag{2.26}$$

uniformly for $2\delta \le \rho \le M_1$, where $i = 1, 2, 3, 4$ and M is a positive constant independent of k.

In fact, substituting entropies $\eta^1_k = e^{kw}(a_1(\rho) + b_1(\rho, k)/k)$ into (2.14), we obtain that

$$k[2f(\rho)a_1' + f'(\rho)a_1] + a_1'' + 2f(\rho)b_1' + f'(\rho)b_1 + \frac{b_1''}{k} = 0, \tag{2.27}$$

where $f(\rho) = \frac{\sqrt{P'(\rho)}}{\rho}$. Let

$$2f(\rho)a_1' + f'(\rho)a_1 = 0 \tag{2.28}$$

and

$$a_1'' + 2f(\rho)b_1' + f'(\rho)b_1 + \frac{b_1''}{k} = 0. \tag{2.29}$$

Then

$$a_1 = f(\rho)^{-\frac{1}{2}} > 0 \text{ for } \rho \ge 2\delta. \tag{2.30}$$

The existence of $b_1(\rho, k)$ and its uniform bound $|b_1(\rho, k)| \le M$ on $2\delta \le \rho \le M_1$ with respect to k can be obtained by the following lemma (cf. [Ka]) (also see Lemma 10.2.1 in [Lu1]):

Lemma 3 *Let $Y(x) \in C^2[0,h]$ be the solution of the equation*

$$F(x, Y, Y') = 0,$$

and functions $f(x,y,z,\lambda), F(x,y,z)$ be continuous on the regions $0 \leq x \leq h, |y - Y(x)| \leq l(x), |z - Y'(x)| \leq m(x)$ for some positive functions $l(x), m(x)$ and $\lambda_0 > \lambda > 0$. In addition,

$$|f(x,y,z,\lambda) - F(x,y,z)| \leq \varepsilon,$$

$$|F(x,y_2,z) - F(x,y_1,z)| \leq M|y_2 - y_1|,$$
$$\frac{F(x,y,z_2) - F(x,y,z_1)}{z_2 - z_1} \geq L$$

for some positive constants ε, M and L.

If $y(x) = y(x,\lambda)$ is a solution of the following ordinary differential equation of second order:

$$\lambda y'' + f(x,y,y',\lambda) = 0,$$

with $y(0) = Y(0)$ and $y'(0)$ being arbitrary, then for sufficiently small $\lambda > 0, \varepsilon > 0$ and $P = |y'(0) - Y'(0)|$, $y(x)$ exists for all $0 \leq x \leq h$ and satisfies

$$|y(x,\lambda) - Y(x)| < \Big[\frac{\varepsilon}{M} + \lambda(\frac{P}{L} + \frac{N}{M})\Big] exp(\frac{Mx}{L}),$$

where $N = \max_{0 \leq x \leq h} |Y(x)|$.

Furthermore, we can use Lemma 3 again to obtain the bound of b_1' with respect to k if we differentiate Equation (2.29) with respect to ρ.

By the second equation in (2.13), a entropy flux q_k^1 corresponding to η_k^1 is provided by

$$q_k^1 = \lambda_2 \eta_k^1 + e^{kw}(\frac{(\rho - 2\delta)a_1' - a_1}{k} + \frac{(\rho - 2\delta)b_1' - b_1}{k^2}), \tag{2.31}$$

where

$$(\rho - 2\delta)a_1' - a_1 = -\frac{(\rho P'' + 2P')(\rho - 2\delta) + 8\delta P'}{4\rho P'' f(\rho)^{\frac{1}{2}}} < 0 \tag{2.32}$$

and $(\rho - 2\delta)b_1' - b_1$ both are bounded uniformly on $\rho \in [2\delta, M_1]$.

In a similar way, we can obtain another entropy-entropy flux pair of Lax type as follows:

$$\begin{aligned} \eta_{-k}^2 &= e^{-kw}(a_2(\rho) + \tfrac{b_2(\rho,k)}{k}), \\ q_{-k}^2 &= \lambda_2 \eta_{-k}^2 + e^{-kw}(\tfrac{a_2 - (\rho - 2\delta)a_2'}{k} + \tfrac{b_2 - (\rho - 2\delta)b_2'}{k^2}), \end{aligned} \tag{2.33}$$

where $a_2(\rho) = a_1(\rho)$ and $b_2(\rho, k)$ satisfies

$$a_1'' - 2f(\rho)b_2' - f'(\rho)b_2 + \frac{b_2''}{k} = 0; \tag{2.34}$$

$$\begin{aligned} &\eta_k^2 = e^{kz}(a_3(\rho) + \tfrac{b_3(\rho,k)}{k}), \\ &q_k^2 = \lambda_1 \eta_k^2 + e^{kz}(\tfrac{(\rho-2\delta)a_3' - a_3}{k} + \tfrac{(\rho-2\delta)b_3' - b_3}{k^2}), \end{aligned} \tag{2.35}$$

where $a_3(\rho) = a_1(\rho)$ and $b_3(\rho, k)$ satisfies

$$a_1'' - 2f(\rho)b_3' - f'(\rho)b_3 + \frac{b_3''}{k} = 0; \tag{2.36}$$

$$\begin{aligned} &\eta_{-k}^1 = e^{-kz}(a_4(\rho) + \tfrac{b_4(\rho,k)}{k}), \\ &q_{-k}^1 = \lambda_1 \eta_{-k}^1 + e^{-kz}(\tfrac{a_4 - (\rho-2\delta)a_4'}{k} + \tfrac{b_4 - (\rho-2\delta)b_4'}{k^2}), \end{aligned} \tag{2.37}$$

where $a_4(\rho) = a_1(\rho)$ and $b_4(\rho, k)$ satisfies

$$a_1'' + 2f(\rho)b_4' + f'(\rho)b_4 + \frac{b_4''}{k} = 0. \tag{2.38}$$

Using the argument in Lemma 3 in Equation (2.38), we can get the existence of b_4 and the uniform bounded estimates of b_4, b_4' with respect to k. If making an independent transformation $\rho_1 = \rho - M_1$ to Equations (2.34) and (2.36), where M_1 is the upper bound of ρ, we also obtain the existence of b_2, b_3 and the uniform bounded estimates of b_2, b_3, b_2' and b_3' by Lemma 3 again.

Then the estimates in (2.25)-(2.26) are obtained, and hence Theorem 1 is proved when we use these entropy-entropy flux pairs in (2.21)-(2.24) together with the theory of compensated compactness coupled with DiPerna's framework [Di1].

3 Proof of Theorem 2

In this section, we prove Theorem 2.

First of all, we recall the proof of Theorem 2 for the case of polytropic gas, in which any weak entropy can be represented by the following explicit formula:

$$\eta_0(\rho, u) = \rho \int_0^1 [\tau(1-\tau)]^\lambda g(u + \rho^\theta - 2\rho^\theta \tau) d\tau, \tag{3.1}$$

where $\theta = \frac{\gamma-1}{2}, \lambda = \frac{3-\gamma}{2(\gamma-1)}$ and g is a smooth function. Multiplying $(\eta_{0\rho}, q_{0\rho})$ to the following parabolic system

$$\begin{cases} \rho_t + m_x = \varepsilon\rho_{xx} \\ m_t + (\frac{m^2}{\rho} + \rho^\gamma)_x = \varepsilon m_{xx}, \end{cases} \tag{3.2}$$

we have

$$\eta_{0t} + q_{0x} = \varepsilon\eta_{0xx} - \varepsilon(\rho_x^\varepsilon, m_x^\varepsilon)\cdot\nabla^2\eta_0(\rho^\varepsilon, m^\varepsilon)\cdot(\rho_x^\varepsilon, m_x^\varepsilon)^T, \tag{3.3}$$

where q_0 is the entropy flux corresponding to η_0. Then using the strictly convex entropy

$$\eta_c = \frac{m^2}{2\rho} + \frac{1}{\gamma-1}\rho^\gamma, \tag{3.4}$$

we first obtain the boundedness of

$$\varepsilon(\rho_x^\varepsilon, m_x^\varepsilon)\cdot\nabla^2\eta_0(\rho^\varepsilon, m^\varepsilon)\cdot(\rho_x^\varepsilon, m_x^\varepsilon)^T \tag{3.5}$$

in $L^1_{loc}(R\times R^+)$ since the hessian of η_0 is controlled by the hessian of η_c with the help of the explicit formula (3.1), and hence the compactness in $W^{-1,\alpha}_{loc}(R\times R^+)$, for some $\alpha\in(1,2)$, by the Sobolev embedding theorems. Second, in the case of $1<\gamma<2$, since η_c is strictly convex and $\eta_{0\rho}, \eta_{0m}$ are uniformly bounded, we can prove that $\varepsilon\eta_{0xx}(\rho^\varepsilon, m^\varepsilon)$ is compact in $H^{-1}_{loc}(R\times R^+)$. Noticing that the left-hand side in (3.3) is uniformly bounded in $W^{-1,\infty}(R\times R^+)$ with respect to ε, we get the proof of Theorem 2 by using Murat's theorem [Mu] for the case of $1<\gamma<2$.

However, for the case of $\gamma>2$, the entropy η_c is not strictly convex and hence the compactness of $\varepsilon\eta_{0xx}(\rho^\varepsilon, m^\varepsilon)$ in $H^{-1}_{loc}(R\times R^+)$ is not obvious. To overcome this technical difficulty, the authors in [LPS] used the periodic viscosity solutions in the space variable x to obtain an auxiliary estimate (see $(I.53)$ in [LPS]):

$$\int_0^T dt\int_0^L dx\varepsilon^2(\rho_x^\varepsilon)^2 \to 0 \quad \text{as } \varepsilon\to 0. \tag{3.6}$$

In this section, with the help of the approximation parameter δ, the compactness of $\varepsilon\eta_{0xx}(\rho^\varepsilon, m^\varepsilon)$ in $H^{-1}_{loc}(R\times R^+)$ can be easily obtained.

Now, we prove Theorem 2 for more general pressure $P(\rho)$.

We rewrite system (2.6) by the following equivalent system

$$\begin{cases} \rho_t + ((\rho-2\delta)u)_x = \varepsilon\rho_{xx} \\ u_t + (\frac{1}{2}u^2 + \int_{2\delta}^\rho \frac{(t-2\delta)P'(t)}{t^2}dt)_x = \varepsilon u_{xx} + \frac{2\varepsilon}{\rho}\rho_x u_x. \end{cases} \tag{3.7}$$

Let $(\eta(\rho,u),q(\rho,u)),(\eta(\rho,u),q_1(\rho,u,\delta))$ be the entropy-entropy flux pairs of systems (1.1),(1.2) respectively since they have the same entropy equation (2.14), but different entropy fluxes.

Multiplying system (3.7) by (η_ρ,η_u), we have

$$\begin{aligned} & \eta(\rho^\varepsilon,m^\varepsilon)_t+q(\rho^\varepsilon,m^\varepsilon)_x \\ = \; & \varepsilon\eta(\rho^\varepsilon,m^\varepsilon)_{xx}-(q_1(\rho^\varepsilon,m^\varepsilon,\delta)-q(\rho^\varepsilon,m^\varepsilon))_x+\tfrac{2\varepsilon}{\rho}\eta_u\rho_x u_x \\ - \; & \varepsilon(\eta_{\rho\rho}\rho_x^2+2\eta_{\rho u}\rho_x u_x+\eta_{uu}u_x^2). \end{aligned} \tag{3.8}$$

By entropy equation (2.14), we have

$$\begin{aligned} \eta_\rho &= \textstyle\int_0^\rho \frac{P'(\tau)}{\tau^2}\eta_{uu}(\tau,u)d\tau+g(u) \\ &= \textstyle\int_0^\rho \frac{P'(\tau)}{\tau}H_{uu}(\tau,u)d\tau+g(u) \end{aligned} \tag{3.9}$$

since $\eta(\rho,u)=\rho H(\rho,u)$, where $g(u)$ is an arbitrary smooth function. Furthermore by integrating (3.9), we get

$$\eta=\int_0^\rho\int_0^t\frac{P'(\tau)}{\tau}H_{uu}(\tau,u)d\tau dt+g(u)\rho \tag{3.10}$$

since $\eta(0,u)=0$. Then

$$\eta_u=\int_0^\rho\int_0^t\frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau dt+g'(u)\rho \tag{3.11}$$

and

$$\eta_{\rho u}=\int_0^\rho\frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau+g'(u). \tag{3.12}$$

Substituting (3.11),(3.12) into (3.8) and using entropy equation (2.14), we get the following equality

$$\eta(\rho^\varepsilon,m^\varepsilon)_t+q(\rho^\varepsilon,m^\varepsilon)_x=I_1+I_2+I_3, \tag{3.13}$$

where

$$I_1=\varepsilon\eta(\rho^\varepsilon,m^\varepsilon)_{xx}-(q_1(\rho^\varepsilon,m^\varepsilon,\delta)-q(\rho^\varepsilon,m^\varepsilon))_x, \tag{3.14}$$

$$I_2=-\varepsilon(\frac{P'(\rho)}{\rho}H_{uu}(\rho,u)\rho_x^2+\rho H_{uu}u_x^2), \tag{3.15}$$

$$I_3=-2\varepsilon(\int_0^\rho\frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau-\frac{1}{\rho}\int_0^\rho\int_0^t\frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau dt)\rho_x u_x. \tag{3.16}$$

For any $\phi \in C_0^1(R\times R^+)$ with $S = \operatorname{supp}\phi$,

$$
\begin{aligned}
&|\int_0^\infty \int_{-\infty}^\infty \varepsilon\eta(\rho^\varepsilon, m^\varepsilon)_{xx}\phi dxdt| \\
&\leq \varepsilon|(\eta_\rho\rho_x^\varepsilon + H_u\rho^\varepsilon u_x^\varepsilon)\phi_x|dxdt \\
&\leq M[(\textstyle\int\int_S \varepsilon\frac{P'(\rho^\varepsilon)}{\rho^\varepsilon}(\rho_x^\varepsilon)^2\frac{\varepsilon\rho^\varepsilon}{P'(\rho^\varepsilon)}dxdt)^{\frac{1}{2}} \\
&+(\textstyle\int\int_S \varepsilon^2(\rho^\varepsilon)^2(u_x^\varepsilon)^2dxdt)^{\frac{1}{2}}](\int\int_S(\phi_x)^2dxdt)^{\frac{1}{2}} \to 0
\end{aligned}
\tag{3.17}
$$

since $\varepsilon = o(\frac{P'(2\delta)}{2\delta})$ or $\frac{\varepsilon\rho^\varepsilon}{P'(\rho^\varepsilon)} \to 0$ as $\varepsilon, \delta \to 0$. Since $q_1(\rho^\varepsilon, m^\varepsilon, \delta) - q(\rho^\varepsilon, m^\varepsilon)$ tends to zero as δ tends to zero, we get the compactness of I_1 in $H_{loc}^{-1}(R\times R^+)$. Using (2.18) again, we know that I_2 is bounded in $L_{loc}^1(R\times R^+)$, and hence compact in $W_{loc}^{-1,\alpha}(R\times R^+)$, for some $\alpha \in (1,2)$, by the Sobolev embedding theorems. Using the Vol'pert theorem and the limit given in (1.7), we have the following estimates

$$
\begin{aligned}
&|\textstyle\int_0^\rho \frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau| \leq M|\int_0^\rho \frac{P'(\tau)}{\tau}d\tau| \leq M_1\sqrt{P'(\rho)}, \\
&|\textstyle\int_0^\rho\int_0^t \frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau dt| \leq M|\int_0^\rho\int_0^t \frac{P'(\tau)}{\tau}d\tau dt| \leq M_1\rho\sqrt{P'(\rho)}.
\end{aligned}
\tag{3.18}
$$

Using these estimates together with (2.18), we get the boundedness of I_3 in $L_{loc}^1(R\times R^+)$ and hence the compactness in $W_{loc}^{-1,\alpha}(R\times R^+)$, for some $\alpha \in (1,2)$, by the Sobolev embedding theorems.

Therefore the right-hand side of (3.13) is compact in $W_{loc}^{-1,\alpha}(R\times R^+)$ for some $\alpha \in (1,2)$, but the left-hand side is bounded in $W^{-1,\infty}(R\times R^+)$. This implies the compactness of $\eta(\rho^\varepsilon, m^\varepsilon)_t + q(\rho^\varepsilon, m^\varepsilon)_x$ in $H_{loc}^{-1}(R\times R^+)$ and hence the proof of Theorem 2 by the Murat theorem [Mu].

Acknowledgments: This paper is partially supported by a grant, so called "One Hundred Talent Project" from the Chinese Academy of Sciences, and a Humboldt fellowship.

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