On isentropic solutions to first order quasilinear equations [1]

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Abstract

Conditions of existence of nontrivial isentropic solutions to first order conservation laws are found. Applications are given to the problem of functional dependence for partial derivatives of a smooth function of two variables.

Introduction. This paper arised in connection with the following question about functional dependence for partial derivatives of a function of two variables. Let $v = v(t,x)$ be a smooth nonlinear function in some domain $\Omega \subset \mathbb{R}^2$, and

$$v_t = \varphi(v_x) \quad \text{in } \Omega \tag{1}$$

where $\varphi(u)$ is a continuous function. The question is about properties of $\varphi(u)$. The derivatives v_t, v_x are only continuous and one could firstly conclude that the relation (1) can't give more than the natural property of continuity of $\varphi(u)$. But it turns out that $\varphi(u)$ must satisfy the rather strict property (*) in Theorem 7 below and in particular $\varphi(u)$ twice (!) differentiable on some set of positive measure. For the proof of this statement we use results of the theory of isentropic solutions to first order quasilinear equations, which were introduced in the thesis [7]. We present these results, which are interesting by itself, in § 1,2 below.

Remind firstly some notions of the theory of generalized entropy solutions to the Cauchy problem for a first order quasilinear equation

$$u_t + \mathrm{div}_x \varphi(u) = 0, \tag{2}$$

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$\varphi = (\varphi_1, \ldots, \varphi_n)$, $u = u(t,x)$, $(t,x) \in \Pi_T = (0,T) \times \mathbb{R}^n$, $0 < T \leq +\infty$, with initial condition

$$u(0,x) = u_0(x) \in L^\infty(\mathbb{R}^n). \tag{3}$$

We assume that the flux vector $\varphi(u)$ is only continuous: $\varphi(u) \in C(\mathbb{R}, \mathbb{R}^n)$.

Definition 1 (S.N.Kruzhkov, [2, 3]). A bounded measurable function $u = u(t,x) \in L^\infty(\Pi_T)$ is called a generalized entropy solution (briefly - g.e.s.) of the problem (2), (3) if

a) $\forall k \in \mathbb{R}$

$$|u-k|_t + \operatorname{div}_x[\operatorname{sign}(u-k)(\varphi(u) - \varphi(k))] \leq 0 \tag{4}$$

in the sense of distributions on Π_T (in $\mathcal{D}'(\Pi_T)$);

b) $\operatorname*{ess\,lim}_{t \to 0+} u(t,\cdot) = u_0$ in $L^1_{loc}(\mathbb{R}^n)$, i.e. there exists a set $\mathcal{E} \subset (0,T)$ of full Lebesgue measure such that $u(t,\cdot) \to u_0$ in $L^1_{loc}(\mathbb{R}^n)$ as $t \to 0+$, $t \in \mathcal{E}$.

Condition (4) means that for any test function $f = f(t,x) \in C_0^\infty(\Pi_T)$, $f \geq 0$

$$\int_{\Pi_T} \{|u-k| f_t + \operatorname{sign}(u-k)(\varphi(u) - \varphi(k), \nabla_x f)\} dtdx \geq 0.$$

Here and below $(\cdot,\cdot)$ denotes a scalar product in $\mathbb{R}^n$. We shall also use the notation $|\cdot|$ for the Euclidean norm of a finite-dimensional vector.

Setting in (4) $k = \pm\|u\|_\infty$ we easily derive that $u_t + \operatorname{div}_x \varphi(u) = 0$ in $\mathcal{D}'(\Pi_T)$ and $u(t,x)$ satisfies the equation (2) in the sense of distributions, i.e. $u(t,x)$ is really a generalized solution (g.s. for short).

As is known (see [2]-[8]) a g.e.s. of the problem (2), (3) always exists but, in the case under consideration when the flux vector is only continuous, can be nonunique if number of spatial variables $n > 1$. The corresponding examples were constructed in [4]-[7]. In the same papers some sufficient conditions of uniqueness were also found. Remark that for Lipschitz continuous flux vector uniqueness of g.e.s. is well-known (see [2, 3]).

§ 1. Isentropic solutions.

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open connected subset (a domain).

Definition 2 (see [7]). A function $u(t,x) \in L^\infty(\Omega)$ is called an isentropic solution (briefly - i.s.) of the equation (2) in the domain Ω if $\forall k \in \mathbb{R}$

$$|u-k|_t + \operatorname{div}_x[\operatorname{sign}(u-k)(\varphi(u) - \varphi(k))] = 0 \ \text{ in } \mathcal{D}'(\Omega), \tag{5}$$

i.e. the entropy condition (4) is satisfied in $\mathcal{D}'(\Omega)$ with equality sign.

Remark 1. As follows from Definition 2, $u = u(t,x)$ is an i.s. of (2) in Ω if and only if $-u$ is an i.s. of the equation $u_t - \mathrm{div}_x\varphi(-u) = 0$ in the same domain Ω.

Notate that in [7] more general isentropic measure valued solutions were studied. Condition (5) could be rewritten in more convenient form in the sense of the following proposition

Proposition 1. *The condition* (5) *is equivalent to any of following conditions:*

1) $\forall k \in \mathbb{R} \quad \max(u,k)_t + \mathrm{div}_x\varphi(\max(u,k)) = 0$ *in* $\mathcal{D}'(\Omega)$*;*

2) $\forall k \in \mathbb{R} \quad \min(u,k)_t + \mathrm{div}_x\varphi(\min(u,k)) = 0$ *in* $\mathcal{D}'(\Omega)$*.*

Proof. Observe firstly that

$$\begin{aligned} 2\max(u,k) &= |u-k| + (u+k),\ 2\varphi(\max(u,k)) = \\ &(\varphi(u)-\varphi(k))\mathrm{sign}\,(u-k) + (\varphi(u)+\varphi(k)); \end{aligned} \tag{6}$$

$$\begin{aligned} 2\min(u,k) &= -|u-k| + (u+k),\ 2\varphi(\min(u,k)) = \\ &-(\varphi(u)-\varphi(k))\mathrm{sign}\,(u-k) + (\varphi(u)+\varphi(k)). \end{aligned} \tag{7}$$

Then, the equality

$$u_t + \mathrm{div}_x\varphi(u) = 0 \ \text{ in } \mathcal{D}'(\Omega)$$

follows from (5) (for $|k| > \|u\|_\infty$), as well as from any of conditions 1), 2) (for $k < -\|u\|_\infty$, $k > \|u\|_\infty$ respectively). This together with equalities (6), (7) imply that conditions (5), 1) and 2) are equivalent. The proof is complete.

The above Proposition is the particular case of the following general statement.

Theorem 1. *If* $u_1 = u_1(t,x)$*,* $u_2 = u_2(t,x)$ *are i.s. of* (2) *in a domain* Ω *when the functions* $\max(u_1,u_2)$*,* $\min(u_1,u_2)$ *are i.s. of this equation in* Ω *as well.*

Proof. By condition 1) in Proposition 1 for any $k \in \mathbb{R}$

$$\max(u,k)_t + \mathrm{div}_x\varphi(\max(u,k)) = 0 \ \text{ in } \mathcal{D}'(\Omega). \tag{8}$$

Now we apply the method of "doubling variables" developed by S.N. Kruzhkov in papers [2, 3]. Consider the integral equality corresponding to (8) with $u = u_1(t,x)$, in which the test function $f(t,x;\tau,y) \in C_0^\infty(\Omega\times\Omega)$, and $k = \max(u_2(\tau, y), c)$, $c \in \mathbb{R}$. After integrating on $\Omega(\tau, y)$ this equality implies that

$$v_t + \mathrm{div}_x \varphi(v) = 0 \text{ in } \mathcal{D}'(\Omega \times \Omega), \tag{9}$$

$v = v(t, x; \tau, y) = \max(u_1(t, x), u_2(\tau, y), c)$. Changing places of function u_1, u_2 and variables (t, x), (τ, y) we derive the similar equality

$$v_\tau + \mathrm{div}_y \varphi(v) = 0 \quad \text{in } \mathcal{D}'(\Omega \times \Omega). \tag{10}$$

Combining (9) and (10) we obtain that

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\right) v + (\mathrm{div}_x + \mathrm{div}_y)\varphi(v) = 0 \text{ in } \mathcal{D}'(\Omega \times \Omega). \tag{11}$$

Choose a function $\beta(t, x) \in C_0^\infty(\mathbb{R}^{n+1})$, $\beta(t, x) \geq 0$, $\int \beta(t, x) dt dx = 1$ and set $\delta_\nu(t, x) = \nu^{n+1}\beta(\nu t, \nu x)$, $\nu \in \mathbb{N}$. Obviously, the sequence $\delta_\nu(t, x)$ converges to the Dirac δ-measure in $\mathcal{D}'(\mathbb{R}^{n+1})$ as $\nu \to \infty$.

Let $f(t, x) \in C_0^\infty(\Omega)$, $\nu \in \mathbb{N}$, and $g(t, x; \tau, y) = f(t, x)\delta_\nu(t - \tau, x - y)$. Clearly, for large enough $\nu \in \mathbb{N}$ the function $g(t, x; \tau, y) \in C_0^\infty(\Omega \times \Omega)$. Applying (11) to the test function g we derive the integral equality:

$$\int_{\Omega(t,x)} \int_{\Omega(\tau,y)} \{v f_t + (\varphi(v), \nabla_x f)\}\delta_\nu(t - \tau, x - y) d\tau dy dt dx = 0. \tag{12}$$

Let $w = w(t, x) = v(t, x; t, x) = \max(u_1(t, x), u_2(t, x), c)$,

$$I_\nu(t, x) = \int_{\Omega(\tau,y)} \{v f_t + (\varphi(v), \nabla_x f)\}\delta_\nu(t - \tau, x - y) d\tau dy;$$

$$I(t, x) = w(t, x) f_t(t, x) + (\varphi(w(t, x)), \nabla_x f(t, x)) = \int_{\Omega(\tau,y)} \{w f_t + (\varphi(w), \nabla_x f)\}\delta_\nu(t - \tau, x - y) d\tau dy.$$

Further, $|v(t, x; \tau, y) - w(t, x)| \leq |u_2(\tau, y) - u_2(t, x)|$, and for a.e. $(t, x, \tau, y) \in \Omega \times \Omega$ $|\varphi(v(t, x; \tau, y)) - \varphi(w(t, x))| \leq \omega(|u_2(\tau, y) - u_2(t, x)|)$ where $\omega(r) = \sup\{ \, |\varphi(u) - \varphi(v)| \mid u, v \in [-M, M], |u - v| < r \, \}$ is a continuity modulus of the vector $\varphi(u)$ on the segment $[-M, M]$ where $M = \max(\|u_1\|_\infty, \|u_2\|_\infty)$. Therefore, we derive an estimate

$$|I_\nu(t, x) - I(t, x)| \leq C_f \int_\Omega \Big[|u_2(\tau, y) - u_2(t, x)| + \omega(|u_2(\tau, y) - u_2(t, x)|)\Big]\delta_\nu(t - \tau, x - y) d\tau dy, \; C_f = \mathrm{const},$$

which implies that $I_\nu(t,x) \to I(t,x)$ for all $\nu \to \infty$ on the set of full measure consisting of Lebesgue points of the function $u_2(t,x)$. By boundedness of u_1, u_2, $\|I_\nu\|_\infty \le$ const. Applying the Lebesgue theorem on passing to the limit under an integral sign we derive the relation

$$\int_\Omega I_\nu(t,x)dtdx \underset{\nu\to\infty}{\to} \int_\Omega I(t,x)dtdx,$$

and since $\int_\Omega I_\nu(t,x)dtdx = 0$ by (12) then $\int_\Omega I(t,x)dtdx = 0$, that is

$$\int_\Omega [w(t,x)f_t(t,x) + (\varphi(w(t,x)), \nabla_x f(t,x))]dtdx = 0.$$

The latter equality together with arbitrariness of $f \in C_0^\infty(\Omega)$ means that $w_t + \operatorname{div}_x\varphi(w) = 0$ in $\mathcal{D}'(\Omega)$, i.e. for $p = p(t,x) = \max(u_1,u_2)$

$$\max(p,c)_t + \operatorname{div}_x\varphi(\max(p,c)) = 0 \ \text{ in } \mathcal{D}'(\Omega)\ \forall c \in \mathbb{R}.$$

By Proposition 1 $\quad p = \max(u_1,u_2)$ is an i.s. of (2) in Ω. Finally, by the equality $\min(u_1,u_2) = -\max(-u_1,-u_2)$ and Remark 1, $\min(u_1,u_2)$ is an i.s. of (2) as well. The proof is complete.

In the linear case when $\varphi(u) = (c,u)$, $c \in \mathbb{R}^n$ any generalized solution of the equation (2) is isentropic. This easily follows from the condition that a generalized solution remains constant along characteristics $x - ct =$ const. For nonlinear equation (2) existence of nontrivial i.s. imposes strong enough restrictions on character of nonlinearity (see § 2 below).

In general case the following sufficient condition of isentropicity holds

Theorem 2. Let $u(t,x) \in L^\infty(\Omega) \cap C(\Omega)$ be a continuous g.s. of the equation (2) in the domain Ω. Then $u(t,x)$ is an i.s. of (2) in Ω

Proof. Setting $u(t,x) = 0$ for $(t,x) \notin \Omega$ we may assume that $u(t,x)$ is determined on all $\mathbb{R}^{n+1}$. Consider sequences of averaged functions

$$u_\nu = (u * \delta_\nu)(t,x) = \int u(t-\tau, x-y)\delta_\nu(\tau,y)d\tau dy,$$

$$v_\nu = (v_{1\nu},\dots,v_{n\nu}) = \int \varphi(u(t-\tau,x-y))\delta_\nu(\tau,y)d\tau dy \in \mathbb{R}^n$$

where the sequence δ_ν, $\nu \in \mathbb{N}$ was defined in the proof of Theorem 1.

By known properties of averaged functions, $u_\nu, v_{i\nu} \in C^\infty(\Omega)$, $i = 1,\dots,n$ and, taking into account that $u(t,x)$ is continuous, $u_\nu(t,x) \to u(t,x)$,

$v_{i\nu} \to \varphi_i(u(t,x))$ as $\nu \to \infty$ uniformly on any compact in Ω. Let $P \subset \mathbb{R}$ be a set of common regular values of the functions $u_\nu(t,x)$ for all $\nu \in \mathbb{N}$. By Morse-Sard theorem (see, for instance, [1]) the set P has full measure and therefore is dense in $\mathbb{R}$. Then, assume that $k \in P$, $f = f(t,x) \in C_0^\infty(\Omega)$, $K = \operatorname{supp} f$. By our suppositions $u_t + \operatorname{div}_x\varphi(u) = 0$ in $\mathcal{D}'(\Omega)$, which implies that for large enough $\nu \in \mathbb{N}$

$$(u_\nu)_t + \operatorname{div}_x v_\nu = (u_t + \operatorname{div}_x\varphi(u)) * \delta_\nu = 0 \tag{13}$$

in some neighborhood $V \supset K$.

Introduce sets $S_\nu = \{ \ (t,x) \in V \ | \ u_\nu(t,x) = k \ \}$, $V_\nu^+ = \{ \ (t,x) \in V \ | \ u_\nu(t,x) \ge k \ \}$, $V_\nu^- = \{ \ (t,x) \in V \ | \ u_\nu(t,x) \le k \ \}$. Since k is a regular value of the function $u_\nu(t,x)$ then V_ν^+, V_ν^- are C^∞-smooth manifolds with common boundary $S_\nu = \partial V_\nu^+ = \partial V_\nu^-$. Denote by $\mathbf{n} = \mathbf{n}(t,x)$ an unitary normal vector on S_ν, external with respect to V_ν^+, $\mathbf{n} = (\mathbf{n}_t, \mathbf{n}_x)$ where $\mathbf{n}_x \in \mathbb{R}^n$ is a collection of spatial components of $\mathbf{n}$; $ds(t,x)$ is the Lebesgue-Hausdorff measure on the manifold S_ν. By Gauss-Ostrogradskii divergence theorem, we get that for large enough $\nu \in \mathbb{N}$

$$\begin{aligned}
&\int_\Omega [|u_\nu - k| f_t + \operatorname{sign}(u_\nu - k)(v_\nu - \varphi(k), \nabla_x f)] dtdx = \\
&\int_{V_\nu^+} [(u_\nu - k) f_t + (v_\nu - \varphi(k), \nabla_x f)] dtdx + \\
&\int_{V_\nu^-} [(k - u_\nu) f_t + (\varphi(k) - v_\nu, \nabla_x f)] dtdx = \\
&\int_{S_\nu} [(u_\nu - k)\mathbf{n}_t + (v_\nu - \varphi(k), \mathbf{n}_x)] f ds(t,x) - \\
&\int_{V_\nu^+} [(u_\nu)_t + \operatorname{div}_x v_\nu] f dtdx + \\
&\int_{S_\nu} [(u_\nu - k)\mathbf{n}_t + (v_\nu - \varphi(k), \mathbf{n}_x)] f ds(t,x) + \\
&\int_{V_\nu^-} [(u_\nu)_t + \operatorname{div}_x v_\nu] f dtdx = \\
&2\int_{S_\nu} [(u_\nu - k)\mathbf{n}_t + (v_\nu - \varphi(k), \mathbf{n}_x)] f ds(t,x)
\end{aligned} \tag{14}$$

taking account of equality (13). Then, $u_\nu = k$ on S_ν and $v_\nu - \varphi(u_\nu) \to 0$ as $\nu \to \infty$ uniformly on K. Therefore, right-hand side of the equality (14) tends to zero as $\nu \to \infty$. By the same reason

$$\lim_{\nu\to\infty} \int_\Omega [|u_\nu - k| f_t + \text{sign}\,(u_\nu - k)(v_\nu - \varphi(k), \nabla_x f)] dtdx =$$
$$\int_\Omega [|u - k| f_t + \text{sign}\,(u - k)(\varphi(u) - \varphi(k), \nabla_x f)] dtdx$$

and passing to the limit in (14) we derive that $\forall k \in P, f \in C_0^\infty(\Omega)$

$$\int_\Omega [|u - k| f_t + \text{sign}\,(u - k)(\varphi(u) - \varphi(k), \nabla_x f)] dtdx = 0. \tag{15}$$

At last, since the integral in (15) depends continuously on k, and P is dense in $\mathbb{R}$ then the identity (15) holds for all $k \in \mathbb{R}$, i.e. $\forall k \in \mathbb{R}$

$$|u - k|_t + \text{div}_x[\text{sign}\,(u - k)(\varphi(u) - \varphi(k))] = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Therefore $u(t,x)$ is an i.s. of the equation (2) in the domain Ω. The proof is complete.

In the case when the flux vector is Lipschitz, i.s. of the Cauchy problem for the equation (2) can be easily described. More exactly, suppose that $u_0(x) \in (a,b)$ a.e. in an open ball $V_r = \{\, x \in \mathbb{R}^n \mid |x| < r \,\}$, $0 < r \le \infty$ and

$$|\varphi(u) - \varphi(v)| \le N|u - v| \quad \forall u, v \in (a,b), \tag{16}$$

$\Omega = \Omega_r$ is a cone $\{\, (t,x) \in \Pi_T \mid |x| + Nt < r \,\}$ (remark that $\Omega = \Pi_T$ if $r = \infty$). Since the flux vector satisfies the Lipschitz condition (16), the vector of generalized derivatives $\varphi'(u)$ lies in $L^\infty([a,b])$ and $|\varphi'(u)| \le N$. Define for $(t,x) \in \Omega$ functions

$$u^+(t,x) = \inf\{\, u \in (a,b) \mid v > u_0(x - \varphi'(v)t) \ \text{ for a.e. } v \in (u,b) \,\},$$
$$u^-(t,x) = \sup\{\, u \in (a,b) \mid v < u_0(x - \varphi'(v)t) \ \text{ for a.e. } v \in (a,u) \,\}.$$

Observe that by the bound $|\varphi'(v)| \le N$, $x - \varphi'(v)t \in V_r$ for $(t,x) \in \Omega$. Let a function $u(t,x) \in L^\infty(\Omega)$ such that $u(0,x) = u_0(x)$ in the sense of relation

$$\operatorname*{ess\,lim}_{t\to 0} \int_{|x|<r-Nt} |u(t,x) - u_0(x)| dx = 0. \tag{17}$$

Then the following statement holds

Theorem 3. *A function $u(t,x)$ is an i.s. of the equation (2) in the domain Ω if and only if $u^+(t,x) = u^-(t,x) = u(t,x)$ a.e. in Ω.*

Proof. Set for $(t,x,v)\in\Omega\times(a,b)$ $U(t,x,v)=\theta(v-u(t,x))$, $U_0(x,v)=\theta(v-u_0(x))$ where $\theta(v)=\begin{cases}1 & , \quad v>0\\ 1/2 & , \quad v=0\\ 0 & , \quad v<0\end{cases}$. It is easy to verify that for $k\in\mathbb{R}$, $u=u(t,x)$

$$|u-k|=-\int_a^b \text{sign}\,(v-k)U(t,x,v)dv+|b-k|,$$

$$\text{sign}\,(u-k)(\varphi(u)-\varphi(k))=$$

$$-\int_a^b \text{sign}\,(v-k)\varphi'(v)U(t,x,v)dv+\text{sign}\,(b-k)(\varphi(b)-\varphi(k)).$$

Therefore, for $f=f(t,x)\in C_0^\infty(\Omega)$

$$\int_\Omega [|u-k|f_t+\text{sign}\,(u-k)(\varphi(u)-\varphi(k),\nabla_x f)]dtdx=$$

$$-\int_a^b \text{sign}\,(v-k)\left(\int_\Omega U(t,x,v)[f_t+(\varphi'(v),\nabla_x f)]dtdx\right)dv. \tag{18}$$

We take into account that values $c_0=|b-k|$ and $c=\text{sign}\,(b-k)(\varphi(b)-\varphi(k))\in\mathbb{R}^n$ don't depend on (t,x) and therefore $\int_\Omega [c_0 f_t+(c,\nabla_x f)]dtdx=0$. Besides, (17) implies that

$$\operatorname*{ess\,lim}_{t\to 0}\int_a^b\int_{|x|<r-Nt}|U(t,x,v)-U_0(x,v)|dxdv=0. \tag{19}$$

For

$$I_f(v)=\int_\Omega U(t,x,v)[f_t+(\varphi'(v),\nabla_x f)]dtdx$$

the relation (18) shows that $u(t,x)$ is an i.s. of the equation (2) in Ω if and only if $\forall k\in\mathbb{R}, f\in C_0^\infty(\Omega)$

$$\int_a^b \text{sign}\,(v-k)I_f(v)dv=0,$$

that is $I_f(v)=0$ a.e. on (a,b). Choose a countable dense set $S\subset C_0^\infty(\Omega)$. Then the set $E=\{\,v\in(a,b)\mid I_f(v)=0\;\forall f\in S\,\}$ has full measure. Since S is dense in $C_0^\infty(\Omega)$ (with respect to the standard locally convex topology) and for fixed $v\in E$ the map $f\to I_f(v)$ is continuous on $C_0^\infty(\Omega)$ then $I_f(v)=0$ $\forall f\in C_0^\infty(\Omega)$, i.e. for $v\in E$

$$U_t+(\varphi'(v),\nabla_x U)=0 \tag{20}$$

in $\mathcal{D}'(\Omega)$. We conclude that $U(t,x,v)$ is a g.s. of the linear equation (20) and therefore it remains constant along characteristics $x-\varphi'(v)t=\text{const}$. The latter means that $U(t,x,v)=g(x-\varphi'(v)t,v)$ for a.e. $(t,x)\in\Omega$ where $g(\cdot,v)$ is some bounded measurable function on V_r. Further, the limit relation (19) implies that $g(x,v)=U_0(x,v)$ a.e. on $V_r\times(a,b)$ and consequently

$$U(t,x,v)=U_0(x-\varphi'(v)t,v)=\theta(v-u_0(x-\varphi'(v)t)) \text{ a.e. on } \Omega\times(a,b). \quad (21)$$

Thus, the requirement that $u(t,x)$ is an i.s. of (2) is equivalent to equality (21). The latter in turns means that for a.e. $(t,x)\in\Omega$

$$v>u_0(x-\varphi'(v)t) \text{ for a.e. } v>u(t,x);\ v<u_0(x-\varphi'(v)t) \text{ for a.e. } v<u(t,x)$$

that is $u^+(t,x)=u^-(t,x)=u(t,x)$ a.e. in Ω. The proof is complete.

§ 2. The case of one spatial variable.

In the case $n=1$, $\Omega=\Pi_T$, $0<T\le\infty$ it is necessary for existence of an i.s. that the flux function is Lipschitz in the set of values of this e.s. More precisely, the following statement holds.

Theorem 4. *Let $u(t,x)$ be an i.s. of the Cauchy problem* (2), (3) *in the domain Ω, and $m=\operatorname{ess\,inf} u(t,x)$, $M=\operatorname{ess\,sup} u(t,x)$ in Ω. Then $\varphi(u)$ satisfies the Lipschitz condition on any segment $[a,b]\subset(m,M)$.*

Proof. Remark firstly that by maximum principle (see, for example, [8]), $m=\operatorname{ess\,inf} u_0(x)$, $M=\operatorname{ess\,sup} u_0(x)$. We carry out the proof by contradiction. Thus, assume that $\varphi(u)$ doesn't satisfy the Lipschitz condition on some segment $[a,b]\subset(m,M)$. Then there exist sequences $a_n,b_n\in[a,b]$, $a_n<b_n$, $n\in\mathbb{N}$ such that

$$|\varphi(b_n)-\varphi(a_n)|=\max_{u\in[a_n,b_n]}|\varphi(u)-\varphi(a_n)|>n|b_n-a_n|. \quad (22)$$

Clearly, $b_n-a_n\to 0$ as $n\to\infty$ and, after possible extracting of a subsequence, $a_n\to\alpha$, $b_n\to\alpha$ with some $\alpha\in[a,b]$. Denote $C_n=\varphi(b_n)-\varphi(a_n)$ and choose "entropies"

$$p_n(u)=(|u-a_n|-|u-b_n|+b_n-a_n)/(2C_n)=\begin{cases}0 & , \quad u<a_n\\ (u-a_n)/C_n & , \quad u\in[a_n,b_n]\\ (b_n-a_n)/C_n & , \quad u>b_n\end{cases}$$

with corresponding fluxes

$$q_n(u) = [(\varphi(u) - \varphi(a_n))\mathrm{sign}\,(u - a_n) - (\varphi(u) - \varphi(b_n))\mathrm{sign}\,(u - b_n) +$$
$$\varphi(b_n) - \varphi(a_n)]/(2C_n) = \begin{cases} 0 & , \quad u < a_n \\ (\varphi(u) - \varphi(a_n))/C_n & , \quad u \in [a_n, b_n] \\ 1 & , \quad u > b_n. \end{cases}$$

By (5), $\forall n \in \mathbb{N}$

$$p_n(u)_t + q_n(u)_x = 0 \ \text{ in } \mathcal{D}'(\Omega). \tag{23}$$

As follows from (22), $0 \le |p_n(u)| \le (b_n - a_n)/|C_n| < 1/n$, $|q_n(u)| \le 1$. After extracting an additional subsequence, we may assume that $q_n(\alpha) \to \beta \in [0, 1]$ as $n \to \infty$. Then $\forall u \in \mathbb{R}$ $q_n(u) \to \theta(u - \alpha)$ where $\theta(v) = \begin{cases} 1 & , \quad v > 0 \\ \beta & , \quad v = 0 \\ 0 & , \quad v < 0 \end{cases}$. By (23) with account of above relations we obtain in the limit as $n \to \infty$ that

$$\theta(u - \alpha)_x = 0 \ \text{ in } \mathcal{D}'(\Omega).$$

This equality implies that for a.e. $t \in (0, T)$ $\theta(u(t, \cdot) - \alpha)_x = 0$ in $\mathcal{D}'(\mathbb{R})$. Passing to the limit as $t \to 0$ and taking into account the initial condition we derive that $\theta(u_0(x) - \alpha)_x = 0$ in $\mathcal{D}'(\mathbb{R})$, but this contradicts to the condition $\alpha \in (m, M)$. The proof is complete.

Remark 2. In the previous Theorem the requirement $\Omega = \Pi_T$ is essential. Confirm it by the following example. Let $\varphi(u) = 2\sqrt{|u|}$. It is easy to verify that a function $u(t, x) = t|t|/x^2$ is a continuous and, therefore, isentropic solution of the equation $u_t + \varphi(u)_x = 0$ in the domain $\{x > |t|\}$ with set of values $(-1, 1)$ but the flux function $\varphi(u)$ doesn't satisfy the Lipschitz condition on any segment containing the point 0. In the case of an arbitrary domain Ω we prove the following theorem.

Theorem 5. *Let $u(t, x)$ be an i.s. of the equation (2) in a domain Ω, $m = \operatorname{ess\,inf} u(t, x)$, $M = \operatorname{ess\,sup} u(t, x)$ in Ω. Then there exists a closed nowhere dense set $F \subset (m, M)$ such that $\varphi(u)$ is locally Lipschitz on the complement $(m, M) \setminus F$.*

Proof. In the case of constant $u(t, x)$ we conclude that $m = M$ and the statement of Theorem is evident. So suppose that $m < M$ and consider a set U consisting of values $u \in (m, M)$ such that $\varphi(u)$ is Lipschitz in some vicinity of u. By construction U is open and $\varphi(u)$ is locally Lipschitz on U. Set $F = (m, M) \setminus U$. Then F is closed in (m, M) and to complete the proof it only remains to show that F is nowhere dense, i.e. $\operatorname{Int} F = \emptyset$. Assuming the contrary we can find a nondegenerate segment $[a, b] \subset F$. Introduce the truncated function $v(t, x) = \min(\max(u(t, x), a), b)$. Then $v(t, x) \in [a, b]$ and, by Theorem 1, $v(t, x)$ is an i.s. of the equation (2) in Ω. Besides, by inequalities $m < a < b < M$, sets $\{v(t, x) = a\}$, $\{v(t, x) = b\}$ have positive measure in Ω and, in particular, the i.s. $v(t, x)$ is not constant. Let $\Omega^t = \{\ x \in \mathbb{R} \mid (t, x) \in \Omega\ \}$ be a section of Ω, $I = \{\ t \in \mathbb{R} \mid \Omega^t \neq \emptyset\ \}$ be a projection of Ω onto the axis t, and

$$E = \{\ t \in I \mid \text{ for a.e. } x \in \Omega^t$$
$$(t, x) \text{ is a Lebesgue point of } v(t, x) \text{ and } v_x \neq 0 \text{ in } \mathcal{D}'(\Omega^t)\ \}.$$

If $E = \emptyset$ then for a.e. $t \in I$ $v(t, x)$ coincides a constant a.e. on any connected component of the section Ω^t, which implies that $v_x = \varphi(v)_x = 0$ in $\mathcal{D}'(\Omega)$. By the equation (2) we have also that $v_t = 0$ in $\mathcal{D}'(\Omega)$. Hence, with account of connectivity of Ω, $v(t, x) = c = \text{const}$ a.e. on Ω, which contradicts to our assumptions. Thus, $E \neq \emptyset$ and, therefore, we can find a value $t \in E$, an interval $(\alpha, \beta) \subset \Omega^t$, and a constant $c \in (a, b)$ such that sets $\{\ x \mid v(t, x) > c\ \}$, $\{\ x \mid v(t, x) < c\ \}$ have positive measure on (α, β). Since $[a, b] \subset F$ the function $\varphi(u)$ doesn't satisfy the Lipschitz condition on any vicinity of the point c. Therefore we can find sequences $a_n, b_n \in [a, b]$, $a_n < b_n$ in such way that $a_n, b_n \to c$ as $n \to \infty$ and

$$|\varphi(b_n) - \varphi(a_n)| = \max_{u \in [a_n, b_n]} |\varphi(u) - \varphi(a_n)| > n|b_n - a_n|,\ n \in \mathbb{N}.$$

Define the sequence of entropies $p_n(u)$ and the corresponding sequence of fluxes $q_n(u)$ in the same way as in the proof of Theorem 4 and pass to the limit in the entropy relation like (23) (with $u = v(t, x)$) as $n \to \infty$. We obtain that

$$\theta(v - c)_x = 0 \ \text{ in } \mathcal{D}'(\Omega)$$

where the function $\theta(v)$ is defined as in the proof of Theorem 4. This relation and the condition that for $t \in E$ for a.e. $x \in \Omega^t$ (t, x) is a Lebesgue point of $v(t, x)$ imply that $\theta(v - c)_x = 0$ in $\mathcal{D}'((\alpha, \beta))$. But the latter contradicts to requirement that sets where $v > c$ and where $v < c$ have positive measure on (α, β). The obtained contradiction proves that the set F is nowhere dense.

Lemma 1. *Let $f(y)$ be a function on a segment $[a,b]$, $y(x) \in C([\alpha,\beta])$, $y([\alpha,\beta]) = [a,b]$, and $E \subset [a,b]$. Assume that the superposition $g(x) = f(y(x))$ has bounded variation on $[\alpha,\beta] \cap y^{-1}(E)$. Then $f(y)$ is a function of bounded variation on E, moreover* $\operatorname{Var} f(y) \le \operatorname{Var} g(x)$.

Proof. Choose values $\alpha_1, \beta_1 \in [\alpha,\beta]$ such that $y(\alpha_1) = a$, $y(\beta_1) = b$ and assume for definiteness that $\alpha_1 < \beta_1$ (the opposite case $\alpha_1 > \beta_1$ reduces to the case above by replacing of $y(x)$ to $y(\alpha+\beta-x)$). Let points $y_i \in E$, $i = 1,\dots,m$; $y_i > y_{i-1}$, $i = 2,\dots,m$. We set $x_i = \max\{ \, x \in [\alpha_1,\beta_1] \mid y(x) = y_i \, \}$. By continuity of $y(x)$ we conclude that for $i > 1$ $y_i \in (y_{i-1}, b] \subset y((x_{i-1},\beta_1])$ and subsequently $x_i > x_{i-1}$, $i = 2,\dots,m$. By construction $x_i \in y^{-1}(E)$. Then

$$\sum_{i=2}^{m} |f(y_i) - f(y_{i-1})| = \sum_{i=2}^{m} |g(x_i) - g(x_{i-1})| \le \operatorname{Var} g(x).$$

Passing in this relation to supremum with respect to finite subsets $\{y_i\}$ of E, we get that $\operatorname{Var} f(y) \le \operatorname{Var} g(x)$, as required.

Proposition 2. *Let $n = 1$, $r > 0$; $u_0(x) \in C((-r,r))$ and the image $u_0((-r,r))$ contains an interval (a,b). Assume also that the flux function $\varphi(u)$ satisfies the Lipschitz condition (16), and $u = u(t,x)$ is a continuous i.s. of the Cauchy problem for the equation (2) in a trapezoid $\Omega = \Omega_r = \{ \, (t,x) \in \Pi_T \mid |x| + Nt < r \, \}$, with initial condition (17). Then, after possible "correction" of the derivative $\varphi'(u)$ on a set of null measure, $\varphi'(u)$ has a locally bounded variation on (a,b).*

Proof. Let $E \subset (a,b)$ be a set of full measure of Lebesgue points for the function $\varphi'(u)$, and $[c,d] \subset (a,b)$. By continuity of $u_0(x)$ there exists a segment $J = [\alpha,\beta] \subset (-r,r)$ such that $u_0(J) \supset [c,d]$. Clearly, for some small enough $h > 0$ a set $\{ \, (t,x) \, |\alpha - Nt \le x \le \beta + Nt, \ 0 < t \le h \, \} \subset \Omega$. Let's show that a function $f(y) = y + \varphi'(u_0(y))h$ increases on a set $J \cap u_0^{-1}(E)$. Assuming the contrary, we can choose values $y_1, y_2 \in (\alpha,\beta)$, $y_1 < y_2$ such that $u_1 = u_0(y_1), u_2 = u_0(y_2) \in E$, and $f(y_1) > f(y_2)$. This means that the characteristics $x_2(t) = y_2 + \varphi'(u_2)t$ and $x_1(t) = y_1 + \varphi'(u_1)t$ intersect for $t = t_0$ where

$$t_0 = [y_2 - y_1]/[\varphi'(u_1) - \varphi'(u_2)] \in (0,h). \tag{24}$$

We may also suppose that the function $u_0(y)$ is not constant in any vicinities of the points y_1, y_2. Indeed, in the opposite case we may replace y_1, y_2 to y_1', y_2' where values y_1', y_2' are defined by the condition that segments $[y_1, y_1']$, $[y_2', y_2]$

are maximal along segments $[y_1, z_1]$, $[z_2, y_2]$, on which the function $u_0(y)$ takes constant values u_1 and u_2 respectively. After such replacement the numerator of the fraction defining t_0 in the equality (24) may only become smaller, and the denominator doesn't change. Therefore, the condition $t_0 \in (0, h)$ of intersection for the characteristics remains valid.

"Perturbating" values y_1 and y_2, we can achieve that the functions $u_0(y)-u_1$ and $u_0(y)-u_2$ change sign in any vicinities of the points y_1 and y_2. Indeed, choose $\delta > 0$ such that $\alpha < y_1 - \delta < y_1 + \delta < y_2 - \delta < y_2 + \delta < \beta$ and

$$[y_2 - y_1 + 2\delta]/[\varphi'(u_1) - \varphi'(u_2) - 2\delta] \in (0, h). \tag{25}$$

Since the function $u_0(y)$ is continuous and isn't constant on the interval $I = (y_1 - \delta, y_1 + \delta)$ then the interior of its image $V = \operatorname{Int} u_0(I)$ is some nondegenerate interval. Further, the graph $\{u = u_0(y)\}$ has null measure on the plane and, therefore, for a.e. $u \in V$ the inverse image $u_0^{-1}(u)$ is a set of null measure on I. Taking into account also that $u_1 \in E$ is a Lebesgue point of the function $\varphi'(u)$ we can find a value $\bar{u}_1 \in V \cap E$ such that the set $I_0 = u_0^{-1}(\bar{u}_1)$ has null measure, and $|\varphi'(\bar{u}_1) - \varphi'(u_1)| < \delta$.

The sets $I_-, I_+ \subset I$, on which $u_0(y) < \bar{u}_1$ and $u_0(y) > \bar{u}_1$ respectively, are open, nonempty, and their closures cover I: $I = \operatorname{Cl} I_1 \cup \operatorname{Cl} I_2$ (in the opposite case $u_0(y) = \bar{u}_1$ on a nondegenerate interval). By connectivity of I $\exists \bar{y}_1 \in \operatorname{Cl} I_1 \cap \operatorname{Cl} I_2$. Then $u_0(\bar{y}_1) = \bar{u}_1$ and, because $\bar{y}_1$ is a limit point of both sets I_-, I_+, the function $u_0(y) - \bar{u}_1$ changes sign in any vicinity of $\bar{y}_1$. Analogously, we prove existence of a point $\bar{y}_2 \in (y_2 - \delta, y_2 + \delta)$ such that $\bar{u}_2 = u_0(\bar{y}_2) \in E$, $|\varphi'(\bar{u}_2) - \varphi'(u_2)| < \delta$, and the function $u_0(y) - \bar{u}_2$ changes sign in any vicinity of the point $\bar{y}_2$. Set

$$\bar{t}_0 = [\bar{y}_2 - \bar{y}_1]/[\varphi'(\bar{u}_1) - \varphi'(\bar{u}_2)].$$

Then, with account of (25), $\bar{t}_0 > 0$ and $\bar{t}_0 < [y_2 - y_1 + 2\delta]/[\varphi'(u_1) - \varphi'(u_2) - 2\delta] < h$. Hence, the characteristics $x = \bar{y}_2 + \varphi'(\bar{u}_2)t$ and $x = \bar{y}_1 + \varphi'(\bar{u}_1)t$ intersects at a point $(\bar{t}_0, \bar{x}_0)$ where $\bar{x}_0 = \bar{y}_1 + \varphi'(\bar{u}_1)\bar{t}_0$. Since $\bar{u}_1 \neq \bar{u}_2$, one of these values is different from $u(\bar{t}_0, \bar{x}_0)$. Let for definiteness $u(\bar{t}_0, \bar{x}_0) \neq \bar{u}_1$. Then two variants are possible: $u(\bar{t}_0, \bar{x}_0) > \bar{u}_1$ or $u(\bar{t}_0, \bar{x}_0) < \bar{u}_1$. We shall assume that $u(\bar{t}_0, \bar{x}_0) > \bar{u}_1$, the opposite variant is treated analogously. By continuity, $u(\bar{t}_0, x) > \bar{u}_1$ for all x from small enough vicinity $W = (\bar{x}_0 - \varepsilon, \bar{x}_0 + \varepsilon)$ of the point $\bar{x}_0$.

As follows from Theorem 3, for a.e. $(t, x, v) \in \Omega \times (a, b)$ $\theta(v - u(t, x)) = \theta(v - u_0(x - \varphi'(v)t))$ where the function $\theta(v)$ was defined in the proof of

Theorem 3. Therefore, this equality is valid on the set of common Lebesgue points of both right-hand and left-hand sides. For $x \in W$ values

$$y = y(x) = x - \varphi'(\bar{u}_1)\bar{t}_0 = x - \bar{x}_0 + \bar{x}_0 - \varphi'(\bar{u}_1)\bar{t}_0 = x - \bar{x}_0 + \bar{y}_1$$

run the interval $(\bar{y}_1 - \varepsilon, \bar{y}_1 + \varepsilon)$ and since the function $u_0(y) - \bar{u}_1$ changes sign on this interval $\exists x \in W \ u_0(y(x)) < \bar{u}_1$.

Taking into account that the function $u(t,x)$, $u_0(y)$ are continuous, that $\bar{u}_1$ is a Lebesgue point of the function $\varphi'(v)$, and that $u(\bar{t}_0, x) > \bar{u}_1$, $u_0(y(x)) < \bar{u}_1$, we conclude that $(t_0, x, \bar{u}_1)$ is a Lebesgue point of both functions $\theta(v - u(t,x))$ and $\theta(v - u_0(x - \varphi'(v)t))$. Therefore,

$$0 = \theta(\bar{u}_1 - u(\bar{t}_0, x)) = \theta(\bar{u}_1 - u_0(x - \varphi'(\bar{u}_1)\bar{t}_0)) = 1.$$

This contradiction proves that the function $f(y) = y + \varphi'(u_0(y))h$ increases on the set $J \cap u_0^{-1}(E)$. Consequently, $\varphi'(u_0(y)) = (f(y) - y)/h$ is a function of bounded variation as a difference of monotone functions. By Lemma 1, $\varphi'(u)$ is a function of bounded variation on $E \cap [c,d]$ for any segment $[c,d] \subset (a,b)$. In particular for all $u \in (a,b)$ there exist one-sided limits $\varphi'(u+) = \lim\limits_{v \to u+, v \in E} \varphi'(v)$ and the set $\{ \ u \in E \mid \varphi'(u) \neq \varphi'(u+) \ \}$ is at most countable. Taking into account that E is a set of full measure on (a,b) we derive that $\varphi'(u) = \varphi'(u+)$ a.e. on (a,b) and to complete the proof it only remains to observe that $\varphi'(u+)$ has a bounded variation on any segment $[c,d] \subset (a,b)$ (already, without the exception of a set of null measure).

Remark 3. One could easily deduce from the proof of Proposition 2 that the function $\varphi'(u)$ (after the possible correction on a set of null measure) has a bounded variation on segments like $u_0(J)$, $J = [\alpha, \beta] \subset (-r, r)$.

Now, we extend the result of Proposition 2 to the case of arbitrary domain $\Omega \subset \mathbb{R}^2$.

Theorem 6. *Let $u(t,x)$ be a continuous i.s. of the equation* (2) *in a domain Ω, $(a,b) = \operatorname{Int} u(\Omega)$, the flux function $\varphi(u)$ satisfies the Lipschitz condition* (16) *on (a,b). Then, after possible correction of the function $\varphi'(u)$ on a set of null measure, $\varphi'(u)$ has locally bounded variation on (a,b).*

Proof. Let's suppose that $u \neq \text{const}$, i.e. $a < b$, the case $a = b$ is trivial. Let $\bar{u} \in (a,b)$. Then sets

$$\Omega^- = \operatorname{Cl}\{ \ (t,x) \in \Omega \mid u(t,x) < \bar{u} \ \}, \ \Omega^+ = \{ \ (t,x) \in \Omega \mid u(t,x) \geq \bar{u} \ \}$$

are closed and nonempty. By connectivity of Ω there exists a point $(t_0, x_0) \in \Omega^- \cap \Omega^+$. Then $u(t_0, x_0) = \bar{u}$ and any vicinity U of the point (t_0, x_0) contains points (t, x) such that $u(t, x) < \bar{u}$. Choose

$$U = \{ \ (t, x) \mid |x - x_0| < r - N|t - t_0| \ \}$$

where $r > 0$ is so small that $U \subset \Omega$. Set $v_0(y) = u(t, x_0 + y)$, $y \in (-r, r)$. Then the functions $v = v^+(\tau, y) = u(t_0 + \tau, x_0 + y)$, $v = v^-(\tau, y) = u(t_0 - \tau, x_0 + y)$ are continuous i.s. the equations $v_\tau + \varphi(v)_y = 0$ and $v_\tau - \varphi(v)_y = 0$ respectively in the triangular domain $\{ \ (\tau, y) \mid \tau > 0, \ |y| < r - N\tau \ \}$ satisfying the same initial condition $v(0, y) = v_0(y)$ in the sense of relation (17). If $v_0(y) \geq \bar{u} \ \forall y \in (-r, r)$ then, as follows from Theorem 3, $v^\pm(\tau, y) \geq \bar{u}$, and $u(t, x) \geq \bar{u}$ in U, which contradicts our assumptions about the point (t_0, x_0). Thus, $\exists y \in (-r, r) \quad v_0(y) < \bar{u}$ and we can find values $h_1 > 0$, $\delta \in (0, r)$ such that $[\bar{u} - h_1, \bar{u}] \subset v_0([-r + \delta, r - \delta])$. By Proposition 2, with account of Remark 3, we conclude that the generalized derivative $\varphi'(u)$ has bounded variation on $E \cap [\bar{u} - h_1, \bar{u}]$ where E is a set of Lebesgue points of $\varphi'(u)$.

Choose the point $(t_0, x_0) \in \tilde{\Omega}^- \cap \tilde{\Omega}^+$ where

$$\tilde{\Omega}^- = \{ \ (t, x) \in \Omega \mid u(t, x) \leq \bar{u} \ \}, \ \tilde{\Omega}^+ = \mathrm{Cl}\{ \ (t, x) \in \Omega \mid u(t, x) > \bar{u} \ \},$$

are closed sets we derive by the same arguments as above that $\varphi'(u)$ has bounded variation on some set $E \cap [\bar{u}, \bar{u} + h_2]$, $h_2 > 0$. Hence, $\varphi'(u)$ has bounded variation on the union of the sets $E \cap [\bar{u} - h_1, \bar{u}]$ and $E \cap [\bar{u}, \bar{u} + h_2]$ that is on the set $E \cap [\bar{u} - h_1, \bar{u} + h_2]$. By arbitrarity of $\bar{u} \in (a, b)$ we conclude that $\varphi'(u)$ has locally bounded variation on $E \cap (a, b)$. To complete the proof we only need to correct $\varphi'(u)$ on a set of null measure replacing it, as in the proof of Proposition 2, to the function $\varphi'(u+) = \lim\limits_{v \to u+, v \in E} \varphi'(v)$.

Corollary. *Under the conditions of Theorem 6, the flux function $\varphi(u)$ has the first derivative on (a, b) with the exception of at most countable set of u, and has the second derivative a.e. on (a, b).*

Proof. By Theorem 6 we may suppose that the generalized derivative $\varphi'(u)$ has locally bounded variation on (a, b). Therefore (cf. [6]), $\varphi'(u)$ is continuous on (a, b) with the exception of at most countable set of discontinuity points of the first type, and it is differentiable a.e. on (a, b). This directly implies the statement of the Corollary.

Theorems 4-6 allow us to establish the following

Theorem 7. *Let $u(t,x)$ be a continuous g.s. of the equation* (2) *in a domain Ω, $(a,b) = \operatorname{Int} u(\Omega)$. Then*

(*) *there exists a closed nowhere dense set $F \subset (a,b)$ such that $\varphi(u)$ is locally Lipschitz on $U = (a,b) \setminus F$, and the generalized derivative $\varphi'(u)$ has locally bounded variation on U, after possible correction on a set of null measure.*

Besides, $F = \emptyset$ if Ω is a strip $(t_1, t_2) \times \mathbb{R}$.

Proof. By Theorem 2 $u(t,x)$ is an i.s. of the equation (2) in Ω. Then, by Theorem 5, there exists a closed nowhere dense set $F \subset (a,b)$ such that $\varphi(u)$ is locally Lipschitz on $U = (a,b) \setminus F$ and in particular the generalized derivative $\varphi'(u) \in L^\infty_{loc}(U)$. Besides, by Theorem 4, $F = \emptyset$ if Ω is a strip $(t_1, t_2) \times \mathbb{R}$. Let $\bar{u} \in U$. Choose a neighborhood $(c,d) \subset U$ of the point $\bar{u}$ such that $\varphi(u)$ satisfies the Lipschitz condition on (c,d). Then the function $\tilde{u} = \max(\min(u(t,x), d), c)$ is a continuous i.s. of the equation (2) in Ω (by Theorem 1) and satisfies assumptions of Theorem 6. By Theorem 6 $\varphi'(u)$ has, after possible correction on a set of null measure, locally bounded variation on (c,d). Because the point $\bar{u} \in U$ is arbitrary we conclude that $\varphi'(u)$ has locally bounded variation on U (after the corresponding correction). The proof is complete.

Now we are ready to give application of Theorem 7 to the problem of functional dependence of partial derivatives described in Introduction. Let $v = v(t,x)$ be a nonlinear smooth function on a domain $\Omega \subset \mathbb{R}^2$, which satisfies the relation (1): $v_t = \varphi(v_x)$ in Ω, where $\varphi(u)$ is a continuous function. We set $u = v_x \in C(\Omega)$. Differentiation of the equality (1) with respect to the variable x in $\mathcal{D}'(\Omega)$ gives that

$$u_t - \varphi(u)_x = 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{26}$$

i.e. $u = u(t,x)$ is a continuous g.s. of the equation (26). Let $(a,b) = \operatorname{Int} u(\Omega)$. Then $a < b$ (in the opposite case the derivatives v_t, v_x are constant and v is linear) and by Theorem 7 the function $\varphi(u)$ satisfies the condition (*). In particular, $\varphi(u)$ has the first derivative on $U = (a,b) \setminus F$ with the exception of at most countable set of u (this implies that the set of points where $\varphi(u)$ is not differentiable is a set of the I-st Bair category) , and has the second derivative a.e. on U.

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