CONTINUOUS GLIMM-TYPE FUNCTIONALS AND SPREADING OF RAREFACTION WAVES

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Abstract. Several Glimm-type functionals for (piecewise smooth) approximate solutions of nonlinear hyperbolic systems have been introduced in recent years. In this paper, following a work by Baiti and Bressan on genuinely nonlinear systems we provide a framework to prove that such functionals can be extended to general functions with bounded variation and we investigate their lower semi-continuity properties with respect to the strong L^1 topology. In particular, our result applies to the functionals introduced by Iguchi-LeFloch and Liu-Yang for systems with general flux-functions, as well as the functional introduced by Baiti-LeFloch-Piccoli for nonclassical entropy solutions. As an illustration of the use of continuous Glimm-type functionals, we also extend a result by Bressan and Colombo for genuinely nonlinear systems, and establish an estimate on the spreading of rarefaction waves in solutions of hyperbolic systems with general flux-function.

1. Introduction

Major progress has been made in recent years on the well-posedness theory for *nonlinear hyperbolic systems* in one space variable

$$\partial_t u + A(u)\,\partial_x u = 0, \quad u = u(x,t) \in \mathbb{R}^N,\ x \in \mathbb{R},\ t > 0. \tag{1.1}$$

Here, the matrix $A(u)$ is assumed to be strictly hyperbolic, with distinct eigenvalues $\lambda_j(u)$ and left- and right-eigenvectors $l_j(u)$ and $r_j(u)$, and discontinuous solutions with small total variation are being considered. The existence theory for (1.1) started with Glimm's pioneering work [23], who introduced linear and quadratic functionals allowing one to control the total variation of solutions of (1.1) in the case of *genuinely nonlinear, conservative systems*, which correspond to

$$A(u) = Df(u), \quad \nabla\lambda_j(u)\cdot r_j(u) \neq 0 \tag{1.2}$$

for some flux-function $f : \mathbb{R}^N \to \mathbb{R}^N$. The Glimm functionals $\mathbf{V}(u^h(t)) + C\,\mathbf{Q}(u^h(t))$ (linear) and $\mathbf{Q}(u^h(t))$ (quadratic) decrease in time when evaluated on approximate solutions $u^h = u^h(x,t)$ constructed by the Glimm scheme.

The well-posedness theory covers the following issues:

- the existence of entropy solutions (Glimm [23], and [15, 19, 37, 33, 7, 42, 6, 28, 40, 5]),
- the uniqueness of these solutions (Bressan and LeFloch [11], and [3, 6]),
- the L^1 continuous dependence with respect to initial data (Bressan et al. [8, 10, 13, 7], LeFloch et al. [35, 26, 25, 32]), Liu and Yang [38, 39]),

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- and the regularity of solutions (Glimm and Lax [24], DiPerna [20], Dafermos [16], and [12]).

The references above restrict attention to the "entropy solutions" which contain compressive shocks satisfying Lax entropy inequalities. On the other hand, an extension of these results to "nonclassical" entropy solutions containing undercompressive shock waves is currently under development; see [32] and the references cited therein.

A generalization of the Glimm functionals was introduced by Schatzman [43] for piecewise Lipschitz continuous functions and extended to general functions with bounded variation (BV) in [1, 9]. In particular, Baiti and Bressan [1] established the important property that the standard Glimm functionals are lower semi-continuous. The generalization uses "nonconservative products" corresponding to the multiplication of a function with bounded variation by a bounded measure. Such products have been systematically studied [18] (see also [34]), and have been found useful in many applications [12, 25].

The objective of this paper is two-fold. On one hand, we check that the arguments in [1] apply to the Glimm-type functionals proposed by Iguchi and LeFloch [28] and Liu and Yang [40] (for classical entropy solutions and general flux-functions f) and by Baiti, LeFloch, and Piccoli [4] (for nonclassical entropy solutions) and, therefore, these functionals can be extended to general functions with bounded variation. In fact, a general framework for linear and for quadratic functionals is presented below in Section 2. We then discuss two applications, in Section 3 (classical entropy solutions) and in Section 4 (nonclassical entropy solutions).

On the other hand, in Section 5, to illustrate how continuous Glimm functionals are used and since such estimates are interesting in their own sake, we derive an estimate on the *spreading of rarefaction waves* for solutions with bounded variation of general systems of conservation laws. Our estimate is a "continuous" version of Liu's estimate on the spreading of rarefaction waves. Such estimates were proven earlier for *approximate* solutions of the Glimm scheme by Glimm and Lax [24] and Lax [30, 31] (genuinely nonlinear systems) and Liu [37] (piecewise genuinely nonlinear systems). A version of this estimate for *general BV solutions* of genuinely nonlinear systems was established by Bressan and Colombo [9]. We also refer to Dafermos [16] and Jenssen and Sinestrari [29] for scalar conservation laws with one inflection point.

The spreading estimate controls the decay of solutions for large times. Further important results on the large-time decay of solutions, especially for systems of two conservation laws, can be found in Dafermos ([17] and the references therein) and Trivisa [45]. Note that the results in the present paper remain valid for functions with large data whenever Glimm-type functionals for functions with large data are available.

2. A general framework

In this section, we establish general results concerning linear and quadratic functionals defined over sets of functions with bounded variation. The class of functionals considered here will contain the functionals encountered with the hyperbolic system (1.1), as we will see in the forthcoming sections. Note however that the hyperbolic system (1.1) does not appear explicitly in this section.

2.1. Linear functionals.

Given an open and convex subset $\Omega \subset \mathbb{R}^N$ and a mapping $\sigma : \Omega \times \Omega \to \mathbb{R}$ we consider the *generalized total variation functional*

$$\mathbf{V}(u) := \sum_{x \in \mathcal{J}(u)} \sigma(u(x-), u(x+)), \tag{2.1}$$

defined for all piecewise constant functions $u : \mathbb{R} \to \Omega$, where $\mathcal{J}(u)$ denotes the set of discontinuity points of u and $u(x\pm)$ are the left- and right-hand traces at the point x. We suppose that σ has the form

$$\sigma(u_-, u_+) := \chi(u_-, u_+)\, |u_+ - u_-|, \quad u_-, u_+ \in \Omega,$$

where, for some $c_* > 0$, the function $\chi : \Omega \times \Omega \to [c_*, +\infty)$ satisfies the Lipschitz condition (on the diagonal) :

$$L(\chi) := \sup_{(u_-,u_+)\neq(\overline{u},\overline{u})} \frac{|\chi(u_-, u_+) - \chi(\overline{u}, \overline{u})|}{|u_- - \overline{u}| + |u_+ - \overline{u}|} < \infty. \tag{2.2}$$

When $\chi \equiv 1$, $\mathbf{V}(u)$ reduces to the standard total variation functional which is well-known to be extendible to general functions with bounded variation and to be lower semi-continuous. This latter property need not hold for general χ, as we will see.

When $u : \mathbb{R} \to \Omega$ is a function with bounded variation, its distributional derivative u_x is a vector-valued measure which can be decomposed into a continuous part and an atomic part

$$u_x = u_x^c + u_x^a.$$

We denote by $|u_x|$ and $|u_x^c|$ the measures of total variation associated with u_x and u_x^c, respectively. Based on this decomposition we now introduce the **wave measure** $\mu(u)$ associated with the function u,

$$\mu(u) = \mu(u)^c + \mu(u)^a,$$

as follows. The continuous part is defined by

$$\mu(u)^c := \chi(u, u)\, |u_x^c|, \tag{2.3}$$

which makes sense since $|u_x^c|$ has no atom. The atomic part $\mu(u)^a$ is concentrated on the countable set $\mathcal{J}(u)$ of jump discontinuities. Specifically, for every $x \in \mathcal{J}(u)$ we set

$$\mu(u)^a(\{x\}) = \sigma(u(x-), u(x+)), \tag{2.4}$$

where σ is the mapping in (2.1).

By construction, the wave measure associated with the function u is non-negative, has finite total mass, and admits the representation

$$\mu(u) = \chi(u, u)\, |u_x^c| + \sum_{x\in\mathcal{J}(u)} \sigma(u(x-), u(x+))\, \delta_x,$$

where δ_x denotes the Dirac mass at the point x.

When restricting attention to functions such that $\|u\|_{L^\infty(\mathbb{R})} \le K$ for some fixed K and in view of (2.2), the generalized strength satisfies

$$c_*\, |u_+ - u_-| \le \sigma(u_-, u_+) \le c^*\, |u_+ - u_-|$$

where $c^* = \sup\big\{\chi(u, v) \,/\, |u|, |v| \le K\big\}$, which implies that (in the sense of measures)

$$c_*\, |u_x| \le \mu(u) \le c^*\, |u_x|. \tag{2.5}$$

The framework described in this section should be applicable to hyperbolic systems of conservation laws. It is worth pointing out that the wave front tracking scheme generates piecewise constant functions to which we will be able to apply the above setting. On the other hand, in order to handle solutions constructed by the Glimm scheme, it is necessary to extend the setting to a class of piecewise smooth functions. This is discussed later in this section.

Theorem 2.1. (Linear functional/piecewise constant functions.) *Consider the functional* $\mathbf{V}$ *defined by* (2.1) *on all piecewise constant functions. Then,* $\mathbf{V}$ *can be extended to all functions* $u : \mathbb{R} \to \Omega$ *with bounded variation by setting*

$$\begin{aligned} \mathbf{V}(u) &:= \mu(u)\big(\mathbb{R}\big) \\ &= \int_{\mathbb{R}} \chi(u,u)\,|u_x^c| + \sum_{x \in \mathcal{J}(u)} \sigma(u(x-), u(x+)), \end{aligned} \tag{2.6}$$

and is equivalent to the usual total variation functional, in the sense that for every $K > 0$ *there exists a constant* $c^* > 0$ *such that, for every* u *satisfying* $\|u\|_{L^\infty(\mathbb{R})} \le K$,

$$c_*\, TV(u) \le \mathbf{V}(u) \le c^*\, TV(u).$$

When σ *satisfies the triangle inequality*

$$\sigma(u_1,u_2) + \sigma(u_2,u_3) \ge \sigma(u_1,u_3), \tag{2.7}$$

for all u_1, u_2, u_3 *in* Ω*, this extension of the functional* $\mathbf{V}$ *is lower semi-continuous in the strong* L^1 *topology, that is*

$$\mathbf{V}(u) \le \liminf_{h \to 0} \mathbf{V}(u^h) \tag{2.8}$$

for every sequence of BV *functions* $u^h \to u$ *in* $L^1(\mathbb{R})$.

Remark 2.2. If (2.7) is violated for three given states u_1, u_2, u_3 then the functional $\mathbf{V}$ cannot be lower semi-continuous, as follows easily from the example

$$u^h(x) = \begin{cases} u_1, & x < -2h, \\ u_2, & -h < x < h, \\ u_3, & x > 2h. \end{cases} \tag{2.9}$$

Here, $\mathbf{V}(u^h) = \sigma(u_1,u_2) + \sigma(u_2,u_3)$ while the limit $u := \lim_{h\to 0} u^h$ satisfies $\mathbf{V}(u) = \sigma(u_1,u_3)$. This example shows also that, when (2.7) holds and thus Theorem 2.1 applies, the inequality (2.8) may well be a strict inequality. □

Proof of Theorem 2.1. We generalize the arguments used by Baiti and Bressan [1] for genuinely nonlinear systems.

Step 1. We start by showing that, if $u : \mathbb{R} \to \Omega$ is an arbitrary function with bounded variation then for every constant state $\overline{u}$ and for every open interval (a,b), we have the estimate

$$\Big| \int_{(a,b)} \chi(\overline{u},\overline{u})\, \varphi\, |u_x| - \int_{(a,b)} \varphi\, d\mu(u) \Big| \le L(\chi) \sup_{(a,b)} |u - \overline{u}| \int_{(a,b)} |\varphi| |u_x|, \tag{2.10}$$

for every continuous function $\varphi : (a,b) \to \mathbb{R}$.

Namely, by (2.2)-(2.4) we have

$$\begin{aligned} \Big| \int_{(a,b)} \chi(\overline{u},\overline{u})\, \varphi\, |u_x^c| - \int_{(a,b)} \varphi\, d\mu(u) \Big| &\le \int_{(a,b)} |\chi(\overline{u},\overline{u}) - \chi(u,u)|\, |\varphi|\, |u_x^c| \\ &\le L(\chi) \sup_{(a,b)} |u - \overline{u}| \int_{(a,b)} |\varphi|\, |u_x^c|. \end{aligned}$$

Similarly, for the points of discontinuity we can write

$$\Big| \sum_{x\in\mathcal{J}(u)\cap(a,b)} \chi(\overline{u},\overline{u})\,\varphi(x)\,|u(x+)-u(x-)| - \sum_{x\in\mathcal{J}(u)\cap(a,b)} \chi(u(x-),u(x+))\,\varphi(x)\,|u_+(x)-u_-(x)|\Big|$$
$$\le 2\,L(\chi)\,\sup_{(a,b)}|u-\overline{u}| \sum_{x\in\mathcal{J}(u)\cap(a,b)} |\varphi(x)|\,|u(x+)-u(x-)|,$$

which completes the proof of the statement of Step 1.

Step 2. We then check that when the triangle inequality (2.7) is satisfied, then for every function $u:\mathbb{R}\to\Omega$ with bounded variation and every interval $[a,b]$

$$\mathbf{V}(u) \ge \sigma(u(a),u(b)).$$

Indeed, consider a sequence u^h of piecewise constant functions that converge to u pointwise with $\mathbf{V}(u^h)\to\mathbf{V}(u)$. By using the triangle inequality inductively on the number of jumps we then have

$$\mathbf{V}(u^h) = \sum_{x\in(a,b)} \sigma(u^h(x-),u^h(x+)) \ge \sigma(u^h(a),u^h(b)),$$

therefore in the limit $h\to 0$.

Step 3. We only have to prove the lower semicontinuity property under the assumption that the triangle inequality (2.7) is satisfied. Consider any sequence of functions u^h that has uniformly bounded variation and converges in L^1 to some limit u. For the l.s.c. property we have to compare the waves measures:

$$\alpha^h := \mu(u^h), \qquad \alpha := \mu(u),$$

associated with u^h and with u, respectively. Without loss of generality, we can assume (by extracting a subsequence still denoted by u^h if necessary) that the following properties hold:

(i) $\lim_{h\to 0}\mathbf{V}(u^h)$ exists (that is, the whole sequence $\mathbf{V}(u^h)$ converges),
(ii) $u^h(x)\to u(x)$ at every point $x\in\mathbb{R}$ (by Helly's theorem),
(iii) and there exists a non-negative, bounded measure β such that $|u^h_x|\rightharpoonup\beta$ in the weak-star sense of measures.

Note also that, by the standard lower convergence property of the total variation of measures, we have

$$|u_x| \le \beta, \tag{2.11}$$

in the sense that $|u_x|(I)\le\beta(I)$ for every open interval I. Therefore, by (2.5),

$$\alpha \le c^*\,|u_x| \le c^*\,\beta.$$

Fix some $\epsilon>0$. Since the measure β has finite mass, we can extract finitely many points $x_1,x_2,\dots,x_a$ where β has a large point mass, so that $\beta(\{x\})<\epsilon$ for every $x\neq x_1,x_2,\dots,x_a$. We can then choose $r>0$ sufficiently small and introduce further points $x_{a+1},x_{a+2},\dots,x_b$ distinct from the previous x_i's, so that, by setting $I_n:=(x_n-r,x_n+r)$ for $n=1,2,\dots,b$

$$\begin{aligned}
&\beta(\{x\})<\epsilon \quad \text{for every } x\neq x_1,x_2,\dots,x_a,\\
&\beta\Big(I_n\setminus\{x_n\}\Big)<\epsilon/a,\\
&\Big(\bigcup_{n=1}^{b} I_n\Big)\subset[-L,+L], \quad \beta\Big(\mathbb{R}\setminus[-L,L]\Big)<\epsilon \text{ for some } L.
\end{aligned} \tag{2.12}$$

In addition, we require that any point x_n belongs to at most two intervals.

To compare α^h with α we distinguish between the points of large jump and the regions of small oscillations.

Consider first an interval I_n for $n \in \{1, \dots, a\}$ which contain a point x_n of large jump. By (2.11) and (2.12) we see that the mass of the measure α in this interval is close to its value at the point x_n, as is clear from

$$\begin{aligned} \alpha(I_n \setminus \{x_n\}) &\le c^* \, |u_x|(I_n \setminus \{x_n\}) \\ &\le c^* \, \beta(I_n \setminus \{x_n\}) \le c^* \, \epsilon/a. \end{aligned} \tag{2.13}$$

On the other hand, to estimate α^h we rely on the triangle inequality (2.7), together with the convergence property at the endpoints of the interval:

$$|u^h(x_n \pm r) - u(x_n \pm r)| \le \epsilon/a$$

for h sufficiently small. Therefore, since σ is Lipschitz continuous (at least), we find

$$|\sigma(u(x_n-), u(x_n+)) - \sigma(u^h(x_n - r), u^h(x_n + r))| \le C \, \epsilon/a \tag{2.14}$$

for all h sufficiently small. Now, using Step 2 we can squizze waves on the boundary of the interval I_n and we find

$$\alpha^h(I_n) \ge \sigma\big(u^h(x_n - r), u^h(x_n + r)\big) \tag{2.15}$$

By combining (2.14) and (2.15) and using that $\alpha(\{x_n\}) = \sigma\big(u(x_n-), u(x_n+)\big)$ we obtain

$$\alpha^h(I_n) \ge \alpha(I_n) - C\epsilon/a. \tag{2.16}$$

We consider now any interval I_n (for $n = a+1, \dots, b$) of small oscillation of the measure β. In this case, the weak convergence of the sequence $|u^h_x| \rightharpoonup \beta$ implies that there exists h_0 such that

$$|u^h_x|(I_n) < \epsilon$$

for all sufficiently small h. It is easy to see that the oscillation of both u^h and u is then small on the interval I_n, since for sufficiently small h and every $x, y \in I_n$

$$\begin{aligned} |u^h(x) - u^h(y)| &\le C \, |u^h_x|(I_n) \le C \, \epsilon, \\ |u(x) - u(y)| &\le \alpha(I_n) \le C \, \beta(I_n) < \epsilon. \end{aligned} \tag{2.17}$$

Therefore, setting $\overline{u}_n := u(x_n)$ and using the result in Step 1 above we obtain

$$\begin{aligned} &\Big| \int_{I_n} \varphi \, d\alpha - \int_{I_n} \varphi \, d\alpha^h \Big| \\ &\le \Big| \int_{I_n} \varphi \, d\alpha - \int_{I_n} \chi(\overline{u}_n, \overline{u}_n) \, \varphi \, |u_x| \Big| + \Big| \int_{I_n} \chi(\overline{u}_n, \overline{u}_n) \, \varphi \, |u_x| - \int_{I_n} \chi(\overline{u}_n, \overline{u}_n) \, \varphi \, |u^h_x| \Big| \\ &\quad + \Big| \int_{I_n} \chi(\overline{u}_n, \overline{u}_n) \, \varphi |u^h_x| - \int_{I_n} \varphi \, d\alpha^h \Big| \\ &\le C \, \|u(x) - \overline{u}_n\|_{L^\infty(I_n)} \int_{I_n} |\varphi| \, (|u_x| + |u^h_x|) + \int_{I_n} \chi(\overline{u}_n, \overline{u}_n) \, \varphi \, \big| |u_x| - |u^h_x| \big|, \end{aligned}$$

which, since the test-function φ is arbitrary, yields

$$\alpha(I_n) \le \liminf_{h \to 0} \alpha^h(I_n) + 2 \, C \, \epsilon \, \beta(I_n). \tag{2.18}$$

Now, by combining (2.16) and (2.18) we have

$$\mathbf{V}(u^h) \ge \sum_n |\alpha^h|(I_n) \ge \sum_n |\alpha|(I_n) - C\,\epsilon,$$

which implies that

$$\mathbf{V}(u) \le \liminf_{h\to 0} \mathbf{V}(u^h) + C\,\epsilon.$$

Since ϵ is arbitrary the lower semicontinuity of $\mathbf{V}$ is established and the proof of Theorem 2.1 is completed. □

Instead of defining the functional $\mathbf{V}$ for general piecewise constant functions, we can alternatively introduce a set of **admissible functions** $\mathcal{A}$, consisting of a set of piecewise Lipschitz continuous functions $u : \mathbb{R} \to \Omega$ with uniformly bounded total variation ($\mathcal{A}$ will be specified in the applications) and suppose that $\mathbf{V}(u)$ *is defined for* $u \in \mathcal{A}$ *only,* by the formula

$$\mathbf{V}(u) = \mu(u)\big(\mathbb{R}\big) := \int_{\mathcal{C}(u)} \chi(u,u)\,|u_x| + \sum_{x\in\mathcal{J}(u)} \sigma(u(x-), u(x+)), \tag{2.19}$$

where the integral is over the intervals of continuity $\mathcal{C}(u)$ of the function u. We impose on the set $\mathcal{A}$ properties that are satisfied by functions generated by the Glimm scheme. Recall however that the system (1.1) is not provided at this stage. It is convenient to impose that

$$\begin{aligned}&\text{For every } u_-, u_+ \in \Omega \text{ and } x_0 \in \mathbb{R} \text{ there exists a sequence } u^h \in \mathcal{A}\\ &\text{such that } u^h(x) = u_- \text{ for } x < x_0 - h, \quad u^h(x) = u_+ \text{ for } x > x_0 + h.\end{aligned} \tag{2.20}$$

This condition guarantees that the set $\mathcal{A}$ is sufficiently large to extend the functional uniquely to general functions. In particular, we see immediately that every piecewise constant function is limit of functions in $\mathcal{A}$ and, more generally, the following density property hold:

For every function u with bounded variation there exists a sequence $u^h \in \mathcal{A}$ such that $u^h \to u$ in L^1 locally.

In the applications, $\mathcal{A}$ will be strictly smaller than the set of all piecewise Lipschitz continuous functions. We observe that, at this stage, the strength $\sigma(u_-, u_+)$ has been prescribed when $(u_-, u_+) \in \mathcal{A}^{jump}$ only. In the application, an extension of $\sigma(u_-, u_+)$ to arbitrary pairs (u_-, u_+) will emerge naturally from the analysis. Theorem 2.1 will then be applied, showing in particular that, when (the extension of) σ satisfies (the Lipschitz condition stated at the beginning of Section 2.1 and) the triangle inequality (2.7) for all u_1, u_2, u_3, the functional $\mathbf{V}$ is lower semicontinuous in the strong L^1 topology.

Remark 2.3. The assumptions in Theorem 2.1 are quite strong and, in the precise form stated, apply to scalar conservation laws (in both the classical and the nonclassical contexts) and to special systems of equations for which a non-increasing linear functional is available. □

2.2. **Quadratic functionals.** We now consider a class of quadratic functionals given by

$$\mathbf{Q}(u) := \sum_{\substack{x,y\in\mathcal{J}(u)\\ x<y}} \sigma(u_-(x), u_+(x))\,\sigma(u_-(y), u_+(y))\,\Theta\big(u_-(x), u_+(x); u_-(y), u_+(y)\big), \tag{2.21}$$

defined for piecewise constant functions $u : \mathbb{R} \to \Omega$. Here we assume that the coefficient Θ satisfies

$$\Theta : \Omega \times \Omega \to \mathbb{R}_+ \text{ is Lipschitz continuous,} \tag{2.22}$$

The following property is critical to establish the lower semi-continuity property. To any three states $u_l, u_m, u_r \in \Omega$ and reals $\alpha < \gamma$ we associate the piecewise constant function

$$u(x) := \begin{cases} u_l, & x < \alpha, \\ u_m, & \alpha < x < \gamma, \\ u_r, & x > \gamma. \end{cases}$$

For β any real, we also introduce the step function

$$\tilde{u}(x) := \begin{cases} u_l, & x < \beta, \\ u_r, & x > \beta. \end{cases}$$

Then, the property $(\mathcal{P})$ imposes that for all such u, $\tilde{u}$

$$\mathbf{Q}(\tilde{u}) \le \mathbf{Q}(u).$$

Since, clearly $\mathbf{Q}(u)$ and $\mathbf{Q}(\tilde{u})$ are independent of the location of the waves (determined by α, β, γ) we can simply set $\mathbf{Q}(u_l, u_m, u_r) := \mathbf{Q}(u)$ and $\mathbf{Q}(u_l, u_r) := \mathbf{Q}(\tilde{u})$. We re-formulate the property in the compact form:

$(\mathcal{P})$: For every three states u_l, u_m, u_r we have

$$\mathbf{Q}(u_l, u_r) \le \mathbf{Q}(u_l, u_m, u_r).$$

Remark 2.4. By induction on the number of jump discontinuities we immediately deduce from the property $\mathcal{P}$ that, for every piecewise constant function $u = u(x)$ with jumps at points $x_l < x_{m_1} < \dots x_{m_N} < x_r$ we have that

$$\mathbf{Q}(u_l, u_r) \le \mathbf{Q}(u_l, u_{m_1}, \dots, u_{m_N}, u_r).$$

Intuitively, the property $\mathcal{P}$ says that the functional diminishes when waves within the interval (a, b) are squizzed to its end points. This property was used by Bressan and Baiti [1] in their analysis of genuinely nonlinear hyperbolic systems. This condition is necessary for the functional to be l.s.c. as follows from the example presented in Remark 2.2. □

We can also consider the functional (2.21) within the framework piecewise Lipschitz continuous functions. Let $\mathcal{A}$ be a set of admissible functions satisfying the condition (2.20) introduced earlier for the linear functionals. On the set $\mathcal{A}$ we then set

$$\begin{aligned} \mathbf{Q}(u) = \iint_{\{x<y\}} & \theta\big(u(x), u(x); u(y), u(y)\big)\, \chi(u(x), u(x))\, \chi(u(y), u(y))\, |u_x(x)|\, |u_y(y)| \\ & + \sum_{\substack{x,y \in \mathcal{J}(u) \\ x<y}} \sigma(u_-(x), u_+(x))\, \sigma(u_-(y), u_+(y))\, \Theta(u_-(x), u_+(x), u_-(y), u_+(y)). \end{aligned} \tag{2.23}$$

At this stage σ is known only for those pairs associated with functions in $\mathcal{A}$. Based on the condition (2.20) every pair (u_-, u_+) admits at least one approximating sequence u^h satisfying (2.20) and we can set

$$\sigma(u_-, u_+) := \inf \sum_x \sigma(u^h(x-), u^h(x+)),$$

where the infimum is over all approximating sequences. Similarly we extend the definition of the coefficient Θ by adopting the rather natural expression

$$\sigma(u_-, u_+)\, \sigma(\tilde{u}_-, \tilde{u}_+)\, \Theta(u_-, u_+, \tilde{u}_-, \tilde{u}_+) = \inf \sigma(u^h_-, u^h_+)\, \sigma(\tilde{u}^h_-, \tilde{u}^h_+)\, \Theta(u^h_-, u^h_+, \tilde{u}^h_-, \tilde{u}^h_+),$$

where the infimum is over all sequences $u^h, \tilde{u}^h \in \mathcal{A}$ satisfying

$$u^h(x) = u_- \text{ for } x < -h, \quad u^h(x) = u_+ \text{ for } x > h,$$

and

$$\tilde{u}^h(x) = \tilde{u}_- \text{ for } x < 1-h, \quad \tilde{u}^h(x) = \tilde{u}_+ \text{ for } x > 1+h.$$

Theorem 2.5. *Consider the functional* $\mathbf{Q}$ *defined either by* (2.21) *on all piecewise constant functions or by* (2.23) *for functions in some set* $\mathcal{A}$ *of piecewise Lipschitz continuous functions satisfying* (2.20). *The functional* $\mathbf{Q}$ *can be extended to the class of all functions* $u : \mathbb{R} \to \Omega$ *with bounded variation by setting*

$$\mathbf{Q}(u) = \iint_{\{x<y\}} a(x,y)\, |u^c_x(x)|\, |u^c_y(y)| + \sum_{\substack{x,y \in \mathcal{J}(u) \\ x<y}} b(x,y), \tag{2.24}$$

where

$$a(x,y) := \chi(u(x),u(x))\, \chi(u(y),u(y))\, \Theta(u(x),u(x),u(y),u(y))$$
$$b(x,y) := \sigma(u_-(x),u_+(x))\, \sigma(u_-(y),u_+(y))\, \Theta(u_-(x),u_+(x),u_-(y),u_+(y)).$$

Moreover, when the property $(\mathcal{P})$ *is satisfied, the functional* $\mathbf{Q}$ *is lower semi-continuous for the* L^1 *topology.*

The proof follows similar line of argument as in the ones in the proof of Theorem 2.1, while taking into consideration the key property $(\mathcal{P})$ and the quadratic character of the functional $\mathbf{Q}$. We omit the details.

3. Application to classical entropy solutions

In this section, we consider the Glimm-type functionals introduced in [28] to get uniform total variation bounds for (piecewise smooth) approximate solutions to systems of conservation laws. Based on the results in the previous section, our purpose here is to show that these functionals naturally extend to general functions with bounded variation and are lower semi-continuous.

Consider the system of nonlinear hyperbolic equations

$$\partial_t u + \partial_x f(u) = 0, \quad u = u(x,t) \in \mathbb{R}^N, \tag{3.1}$$

where we assume that all values u belong to a small neighborhood of the origin in $\mathbb{R}^N$ and, for each u, the matrix $A(u) := Df(u)$ has real and distinct eigenvalues $\lambda_1(u) < \ldots < \lambda_N(u)$ and, therefore, basis of left- and right-eigenvectors $l_j(u)$, $r_j(u)$, $1 \le j \le N$. We assume that the system is piecewise genuinely nonlinear (PGNL) in the sense that

$$\text{If } (\nabla\lambda_j \cdot r_j)(u) = 0 \text{ then } (\nabla(\nabla\lambda_j \cdot r_j)\cdot r_j)(u) \neq 0. \tag{3.2}$$

Following Oleinik (scalar equations), Wendroff (systems of two equations), and Liu (general systems) (see [37] for references), we are interested in entropy solutions to (3.1), satisfying the following entropy criterion: *the shock speed is non-increasing along each wave curve.*

The Riemann problem for (3.1) was solved in [37, 28] and wave curves in each family were constructed. It has been found convenient [28] to introduce a mapping $u \mapsto p_i(u)$ satisfying

$$\nabla p_i(u) \cdot r_i(u) > 0 \tag{3.3}$$

and to parameterize the i-wave curve $m \mapsto \psi_i(m; u_0)$ issuing from some (left-hand) state u_0 so that

$$p_i\big(\psi_i(m; u_0)\big) = m. \tag{3.4}$$

By definition, for any m, u_0 can be connected to the (right-hand) state $\psi_i(m; u_0)$ by using an i-wave fan made of a combination of i-shock waves and i-rarefaction fans.

In the following we denote by $\mathcal{S}_i(u_0)$ the set of all states u that can be connected to u_0 by a single shock in the i-family, and by $\mathcal{R}_i(u_0)$ the set of states that can be connected to u_0 by a single rarefaction. The shock speed is denoted by $\overline{\lambda}_i(u_0, u)$.

For u_-, u_+, we denote by $\sigma_i(u_-, u_+)$ the strength of the i-wave fan in the Riemann solution associated with the states u_- and u_+. The total wave strength in the Riemann solution associated with u_-, u_+ is defined as

$$\sigma(u_-, u_+) := \sum_i |\sigma_i(u_-, u_+)|.$$

To the total wave strength, we associate the **total variation functional** $\mathbf{V}(u)$ defined by

$$\mathbf{V}(u) := \sum_{x \in \mathcal{J}(u)} \sigma(u_-(x), u_+(x)). \tag{3.5}$$

In addition we can define the **i-wave variation functionals** $\mathbf{V}_i(u)$ by

$$\mathbf{V}_i(u) := \sum_{x \in \mathcal{J}(u)} \sigma_i(u(x-), u(x+)), \qquad i = 1, \dots, N. \tag{3.6}$$

Obviously we have $\mathbf{V}(u) = \sum_i \mathbf{V}_i(u)$.

Moreover, we define a quadratic functional $\mathbf{Q}(u)$, called the **generalized interaction potential,** which is based on a weight function Θ, called a **generalized angle**, defined as follows. First, we define the generalized angle associated with two admissible shocks or rarefaction fans, say an i-wave (u_-, u_+) located on the left-hand side of a j-wave (u'_-, u'_+), as follows :

$$\Theta_{ij}(u_-, u_+; u'_-, u'_+) := \begin{cases} 1, & i = j \text{ and } \sigma_i(u_-, u_+)\,\sigma_i(u'_-, u'_+) < 0, \\ \theta_{ij}(u_-, u_+; u'_-, u'_+), & \text{in all other cases,} \end{cases} \tag{3.7}$$

where

$$\theta_{ij}(u_-, u_+; u'_-, u'_+)$$
$$= \begin{cases} \Big(\overline{\lambda}_j(u'_-, u'_+) - \overline{\lambda}_i(u_-, u_+)\Big)^-, & u_+ \in \mathcal{S}_i(u_-),\ u'_+ \in \mathcal{S}_j(u'_-), \\ \frac{1}{p_+ - p_-} \int_{p_-}^{p_+} \Big(\overline{\lambda}_j(u'_-, u'_+) - \lambda_i(\psi_i(m; u_-, u_+))\Big)^- \, dm, & u_+ \in \mathcal{R}_i(u_-),\ u'_+ \in \mathcal{S}_j(u'_-), \\ \frac{1}{p'_+ - p'_-} \int_{p'_-}^{p'_+} \Big(\lambda_j(\psi_j(n; u'_-, u'_+)) - \overline{\lambda}_i(u_-, u_+)\Big)^- \, dn, & u_+ \in \mathcal{S}_i(u_-),\ u'_+ \in \mathcal{R}_j(u'_-), \\ \frac{1}{(p_+ - p_-)(p'_+ - p'_-)} \int_{p_-}^{p_+} \int_{p'_-}^{p'_+} \Big(\lambda_j(\psi_j(n; u'_-, u'_+)) - \lambda_i(\psi_i(m; u_-, u_+))\Big)^- \, dm dn, & \\ & u_+ \in \mathcal{R}_i(u_-),\ u'_+ \in \mathcal{R}_j(u'_-), \end{cases} \tag{3.8}$$

and $p_\pm = p_i(u_\pm)$ and $p'_\pm = p_j(u_\pm)$. Based on (3.7) the quadratic functional is then defined for functions u made of elementary waves only by the formula

$$\mathbf{Q}(u) := \sum_{\substack{i\text{-wave at } x \in \mathcal{J}(u) \\ j\text{-wave at } y \in \mathcal{J}(u)}} \sigma_i(u(x-), u(x+))\,\sigma_j(u(y-), u(y+))\,\Theta_{ij}\big(u(x-), u(x+); u(y-), u(y+)\big). \tag{3.9}$$

Note, if the function $u(x)$ is an arbitrary piecewise constant function then the above definition can be extended by solving a Riemann problem at each jump discontinuity $x \in \mathcal{J}(u)$. To each i-wave fan in the Riemann solution associated with the data $u_\pm := u_\pm(x)$ we associate its i-wave speed function $m \mapsto \Lambda_i(m; u_-, u_+)$ (the variable m is the global parameter introduced when solving the Riemann problem), which is a Lipschitz continuous and coincides either with the characteristic speed $\lambda_i(\psi(m; u_-, u_+))$ or is constant equal to a shock speed. Then we define

$$\mathbf{Q}(u) = \sum_{i,j} \sum_{\substack{x,y \in \mathcal{J}(u) \\ x<y}} Q_{ij}(x, y) \tag{3.10}$$

with

$$Q_{ij}(x, y) := \begin{cases} |\sigma_i(u(x-), u(x+))|\,|\sigma_i(u(y-), u(y+))|, & i = j \text{ and } \sigma_i(u_-, u_+)\,\sigma_i(u'_-, u'_+) < 0, \\ \iint \Big(\Lambda_j(n; u(y-), u(y+)) - \Lambda_i(m; u(x-), u(x+))\Big)^- \, dmdn & \text{in all other cases.} \end{cases} \tag{3.11}$$

We now present the generalization of the formulas (3.8) and (3.9) to general functions with bounded variation. Let $u : \mathbb{R} \mapsto \mathbb{R}^N$ be a function with bounded variation. The vector-valued measure $\frac{du}{dx}$ can be decomposed into continuous part u_x^c and an atomic part, the latter being supported on the countable set $\mathcal{J}(u)$ of jump discontinuities. For $i = 1, \dots, N$ we define the nonconservative product [18]

$$\big[\nabla p_i(u) \cdot u_x\big] =: \nu_i$$

as a signed measure as follows. If $B \subset \mathcal{J}(u)$ is a Borel set

$$\nu_i(B) := \int_B \nabla p_i(u) \cdot u_x^c.$$

The atomic part is the measure whose mass at x is the strength of the i-th wave in the solution of the Riemann problem with data $u_-(x)$ and $u_+(x)$:

$$\nu_i\big(\{x\}\big) = \sigma_i(u_-(x), u_+(x)).$$

These definitions lead us to the representation formula

$$\nu_i(B) = \int_B \nabla p_i(u) \Big(\frac{du}{dx}\Big)^c + \sum_{x \in \mathcal{J}(u)} \sigma_i(u(x-), u(x+)). \tag{3.12}$$

Call $\nu_i^\pm$ the positive and negative parts of the signed measure respectively, which are non-negative measures such that

$$\begin{aligned} \nu_i &= \nu_i^+ - \nu_i^- \\ |\nu_i| &= \nu_i^+ + \nu_i^-. \end{aligned} \tag{3.13}$$

It is important to notice that this decomposition is not associated with a “shock/rarefaction” decomposition of a solution (consider, for instance, a Riemann solution) as is the case for genuinely nonlinear systems. This decomposition is needed to distinguish between waves of the same and of different direction, which are handled differently in light of (3.11). The **total strength of waves** and **i-wave strength** in u are given by

$$\begin{aligned} V_i(u) &:= |\nu_i|\big(\mathbb{R}\big) = \int_{\mathbb{R}} \Big|\nabla p_i(u) \cdot u_x^c\Big| + \sum_{x \in \mathcal{J}(u)} \big|\sigma_i(u(x-), u(x+))\big|, \\ \mathbf{V}(u) &:= \sum_{i=1}^N V_i(u). \end{aligned} \tag{3.14}$$

On the other hand, the *generalized interaction potential* for u is given by

$$\mathbf{Q}(u) = \sum_{i,j} Q^c_{ij}\Big(\{x<y\}\Big) + \sum_{i,j} \sum_{\substack{x,y\in\mathcal{J}(u)\\ x<y}} Q_{ij}(x,y), \tag{3.15}$$

where the jump part was defined earlier and the continuous part is given by

$$Q^c_{ij} := \begin{cases} |\nu^c_{i,x}|\,|\nu^c_{j,y}|\,\Big(\lambda_j(y)-\lambda_i(x)\Big)^-, & \text{if } i\neq j,\\ \left(\nu^c_{i,x}\right)^+\left(\nu^c_{i,y}\right)^- + \left(\nu^c_{i,x}\right)^-\left(\nu^c_{i,y}\right)^+ & \\ \qquad +\Big(\left(\nu^c_{i,x}\right)^+\left(\nu^c_{i,y}\right)^+ + \left(\nu^c_{i,x}\right)^-\left(\nu^c_{i,y}\right)^-\Big)\Big(\lambda_j(y)-\lambda_i(x)\Big)^-, & \text{if } i=j, \end{cases}$$

where $\nu_{i,x} := \nabla p_i(u)\cdot u^c_x(x)$, etc.

We summarize our results for nonlinear hyperbolic systems (3.1), which follow immediately from the framework established in Section 2. We have introduced earlier in this section the function σ and Θ which allowed us to define functional $\mathbf{V}$ and $\mathbf{Q}$ of the form (2.1) and (2.21), respectively. Most importantly we observe that the key assumption ($\mathcal{P}$) is satisfied by the functionals $\mathbf{Q}$ and $\mathbf{V}+C\,\mathbf{Q}$ (for sufficiently large $C>0$) in [28], as this property follows immediately from the key interaction estimates therein. On the other hand, we take here the admissible set to be the set of functions obtained by combining shock and rarefaction waves together. It has been checked in [37, 28] that an arbitrary jump can always be decomposed using such admissible functions only (for instance by solving a Riemann problem); thus the property (2.20) hold. We conclude that:

Theorem 3.1. *Under the assumption and with the notation introduced in this section, the functionals $\mathbf{V}$ and $\mathbf{Q}$ can be extended to general functions with bounded variation. In addition, $\mathbf{Q}$ and $\mathbf{V}+C\,\mathbf{Q}$ are lower semi-continuous for the L^1 topology. In particular, if $u^h = u^h(x,t)$ is a sequence of solutions satisfying the tame variation condition* [11] *(for instance solutions constructed by the Glimm scheme) and converging to some limit $u=u(x,t)$ we have*

$$\begin{aligned} &\mathbf{Q}(u(t)) \le \mathbf{Q}(u(0)),\\ &\mathbf{V}(u(t)) + C\,\mathbf{Q}(u(t)) \le \mathbf{V}(u(0)) + C\,\mathbf{Q}(u(0)), \quad t\ge 0. \end{aligned} \tag{3.16}$$

The proof is immediate by applying Theorems 2.1 and 2.5.

4. Application to nonclassical entropy solutions

The framework in Section 2 can also be applied to the "nonclassical" functional introduced in [4]. We refer to [32] for a review on nonclassical entropy solutions and we content here with describing some implication of Theorem 2.1 in this context.

Consider the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \quad u = u(x,t)\in\mathbb{R}, \tag{4.1}$$

when the flux $f:\mathbb{R}\to\mathbb{R}$ is assumed to be a concave-convex function satisfying, by definition,

$$\begin{gathered} u\,f''(u) > 0 \quad (u\neq 0),\\ \lim_{|u|\to\infty} f'(u) = +\infty. \end{gathered}$$

We associate with f the functions $\varphi^\natural, \varphi^{-\natural}:\mathbb{R}\to\mathbb{R}$ defined by

$$f'\big(\varphi^\natural(u)\big) = \frac{f(u)-f\big(\varphi^\natural(u)\big)}{u-\varphi^\natural(u)}, \quad u\neq 0,$$

and $\varphi^{-\natural} = (\varphi^{\natural})^{-1}$.

We consider the Cauchy problem in the class of nonclassical entropy solutions. A Lipschitz continuous, *kinetic function* $\varphi^{\flat} : \mathbb{R} \to \mathbb{R}$ is prescribed such that

$$\varphi^{-\natural}(u) < \varphi^{\flat}(u) \le \varphi^{\natural}(u), \quad u > 0,$$
$$\varphi^{\natural}(u) \le \varphi^{\flat}(u) < \varphi^{-\natural}(u), \quad u < 0,$$
$$\varphi^{\flat} \text{ is monotone decreasing,}$$

and $\varphi^{\flat}$ satisfies the strict contraction property:

$$0 < \frac{\varphi^{\flat} \circ \varphi^{\flat}(u)}{u} < 1, \quad u \neq 0.$$

The so-called **nonclassical Riemann solver** is determined from this kinetic function. (See [32], Chap. 2.) A solution contains one or two waves, one of them at most being a rarefaction wave. In contrast with the classical Oleinik-Kruzkov's theory, a nonclassical solution may contain undercompressive shocks –which, precisely, are determined by the kinetic function.

Under some assumptions on the kinetic function ([32], Chap. VIII), a **generalized total variation functional** $\mathbf{V}(u)$, can be defined as follows. If $u : \mathbb{R} \mapsto \mathbb{R}$ is a piecewise constant function, then

$$\mathbf{V}(u) := \sum_x \sigma(u(x-), u(x+)),$$

where, as usual, the summation is over all points of discontinuity of u. The **generalized strength** $\sigma(u_-, u_+)$ is defined as follows, in order to handle nonclassical solutions,

$$\sigma(u_-, u_+) := \begin{cases} |u_+ - u_-|, & u_- u_+ \ge 0, \\ |u_- - (1 - K(u_-))\, u_+|, & u_- u_+ \le 0,\ |u_+| \le \big|\varphi^{\sharp}(u_-)\big|, \\ \big|u_- + \varphi^{\flat}(u_-) - (2 - K(u_-))\, \varphi^{\sharp}(u_-)\big|, & u_+ = \varphi^{\flat}(u_-), \end{cases} \tag{4.2}$$

where K is a Lipschitz continuous function satisfying some restrictions presented in [32]. Here, $\varphi^{\sharp}$ is the companion function associated with the kinetic function by

$$\frac{f(u) - f\big(\varphi^{\sharp}(u)\big)}{u - \varphi^{\sharp}(u)} = \frac{f(u) - f\big(\varphi^{\flat}(u)\big)}{u - \varphi^{\flat}(u)}, \quad u \neq 0.$$

It is shown in [32] that the wave front tracking approximations $u^h = u^h(x,t)$, based on the nonclassical Riemann solver determined by the kinetic function $\varphi^{\flat}$, are well defined globally in time and satisfy

$$\mathbf{V}(u^h(t)) \le \mathbf{V}(u^h(s)), \quad 0 \le s \le t. \tag{4.3}$$

More precisely, Baiti, LeFloch, and Piccoli [4] prove that the triangle inequality

$$\sigma(u_l, u_m) + \sigma(u_m, u_r) \ge \sigma(u_l, u_r) \tag{4.4}$$

hold for any three constant states u_l, u_m, u_r.

In view of these results, a direct application of Theorem 2.1 yields:

Theorem 4.1. *Under the assumption and with the notation introduced in this section, the functional* $\mathbf{V}$ *can be extended to general functions with bounded variation and is lower semi-continuous for the* L^1 *topology. In particular, if* $u^h = u^h(x,t)$ *is a sequence of approximate solutions constructed in* [11] *and converging to some limit* $u = u(x,t)$ *we have*

$$\mathbf{V}(u(t)) \le \mathbf{V}(u(0)), \quad t \ge 0. \tag{4.5}$$

5. Spreading of rarefaction waves

5.1. **Scalar conservation laws with general flux.** Consider the equation

$$\partial_t u + \partial_x f(u) = 0, \tag{5.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth mapping which need not be convex or concave. We are interested in entropy solution of (5.1) $u : \mathbb{R} \to \mathbb{R}$ which satisfies Oleinik's entropy condition: for a shock joining u_- and u_+ the following relation

$$\frac{f(u_0) - f(u_-)}{u_0 - u_-} \geq \frac{f(u_+) - f(u_-)}{u_+ - u_-} \geq \frac{f(u_+) - f(u_0)}{u_+ - u_0},$$

holds for every u_0 between u_- and u_+. Solutions under consideration have bounded variation and therefore admit left- and right-hand traces at discontinuity points.

In the spirit of Glimm and Lax [24], we let $\chi_1 = \chi_1(t)$ and $\chi_2 = \chi_2(t)$ be two characteristic lines associated with (5.1) and we denote by $D(t)$ the width of the strip bounded by $\chi_1(t)$ and $\chi_2(t)$: $D(t) := \chi_2(t) - \chi_1(t)$. Then we have the basic identity:

$$\frac{d}{dt} D(t) = \frac{d}{dt}\chi_2(t) - \frac{d}{dt}\chi_1(t) = f'(u(\chi_2(t), t)) - f'(u(\chi_1(t), t)). \tag{5.2}$$

For the sake of comparison, let us first recall Glimm-Lax's argument in the case where the flux f has no inflection points. Integrating the above identity with respect to t, and taking into consideration that u is constant along characteristics we get

$$D(T) = D(0) + T\left(f'(u(\chi_2(t), t)) - f'(u(\chi_1(t), t))\right). \tag{5.3}$$

Let us divide an arbitrary interval I_0 on the x-axis into subintervals $[\xi_{j-1}, \xi_j]$, $j = 1, \ldots, n$ on which $f'(u(x, 0))$ is monotone increasing for j odd, and monotone decreasing for j even. (One should assume, first, that there exists finitely many changes of monotonicity and, then, let the number of changes of monotonicity tend to infinity.)

We denote by $\xi_j(t)$ the characteristic issuing from ξ_j: by convention, this characteristic continues as a shock if it runs into a shock, and by $I(t) = [\xi_0(t), \xi_n(t)]$. Note that for all t, $f'(u(x, t))$ is monotone increasing for j odd, and monotone decreasing for j even.

Moreover, the Oleinik entropy condition implies that $f'(u)$ decreases across shocks that is $f'(u_l) > f'(u_r)$. Therefore, the total increasing variation of $f'(u(x, t))$ across $I(t)$ is:

$$\sum_{j\, odd} f'(u_j(t)) - f'(u_{j-1}(t)),$$

where $u_{j-1}^+(t)$ denotes the value of u on the right edge of $\xi_{j-1}(t)$, $u_j^-(t)$ the value of u on the left edge of $\xi_j(t)$. Denote now by $\chi_{j-1}(t), \chi_j(t)$ characteristics starting inside (ξ_{j-1}, ξ_j) which intersect $\xi_{j-1}(t)$ respectively $\xi_j(t)$ at $t = T$. Now, $u_j(t)$ is the constant value u on $\chi_j(t)$. Therefore,

$$D_j(T) = D_j(0) + T\left(f'(u_j(t)) - f'(u_{j-1}(t))\right).$$

Summing over j odd we get:

$$\sum_j D_j(T) = \sum_j D_j(0) + T\sum_j \Big(f'(u_j(t)) - f'(u_{j-1}(t))\Big).$$

The intervals $[\chi_{j-1}(t), \chi_j(t)]$ are disjoint and lie in $I(t)$, therefore the sum of their length can not exceed the length meas$(I(t))$. Hence, the total increasing variation of $f'(u(T)$ over $I(T)$

satisfies

$$TV^{inc}_{I(T)}(f'(u(T)) \leq \frac{\text{meas}(I(T))}{T}. \tag{5.4}$$

Remark 5.1. *The generalized characteristics associated with nonconvex conservation laws are not straight lines due to the presence of contact discontinuities and centers of rarefaction waves above the initial line $t = 0$. See the discussion in Dafermos* [16] *and Jenssen and Sinestrari* [29] *on the regularity and decay of solution to scalar conservation laws with a flux having one inflection point. See also the works by Cheng* [14] *and Hoff* [27].

Consider now a scalar conservation law and assume for clarity that the flux f has a finite (but arbitrarily large) number of inflection points. Our estimates will be independent of the number of inflection points and, so, it will be straightforward to let it tend to infinity. If we compute, as before, the distance between two characteristics $\chi_{j-i}(t), \chi_j(t)$ we find

$$\begin{aligned} \frac{d}{dt}D(t) &= f'(u(\chi_j(t),t)) - f'(u(\chi_{j-1}(t),t)) \\ &= \sum_{\{\alpha=i, x_\alpha \in I(t)\}} (f'(u(x_\alpha+)) - f'(u(x_a-))) =: M(t). \end{aligned} \tag{5.5}$$

The term $M(t)$ measure the amount of rarefaction waves. But,

$$M(T) - M(\tau) = \mathcal{O}(1)\,|\text{TV}(u(\cdot,T)) - \text{TV}(u(\cdot,\tau))|.$$

Therefore:

$$\frac{d}{dt}D(t) = f'(u_j(t)) - f'(u_{j-1}(t)) = M(T) - C\,|\text{TV}(u(\cdot,T)) - \text{TV}(u(\cdot,t))|. \tag{5.6}$$

If now we assume that

$$M(T) \geq 2C\,|\text{TV}(u(\cdot,T)) - \text{TV}(u(\cdot,t))|$$

then by (5.6) we have that

$$\frac{d}{dt}D(t) \geq \frac{M(T)}{2},$$

which, in turn, yields

$$M(T) \leq 2\frac{D(T)}{T}. \tag{5.7}$$

The alternative scenario is that

$$M(T) \leq 2C\,|\text{TV}(u(\cdot,T)) - \text{TV}(u(\cdot,t))|. \tag{5.8}$$

Combining (5.7) and (5.8) we have that

$$M(T) \leq 2\frac{D(T)}{T} + 2C\,\Big|\sum_\tau (\text{TV}(u(\cdot,\tau+)) - \text{TV}(u(\cdot,\tau-))\Big|.$$

Repeating the above procedure in a finite number of disjoint open intervals $I_j = (a_j, b_j)$ in $I(\cdot)$ we obtain that

$$\sum_{j=1}^m M_j(T) \leq C\Big(\sum_j \frac{b_j - a_j}{T} + (\text{TV}(u(\cdot,0)) - \text{TV}(u(\cdot,T)))\Big).$$

If we denote by ρ_T the *measure of rarefaction waves* in $u(T,\cdot)$ defined by

$$\rho_T(I) := \int_I (f'(u)_x)^{c+} + \sum_{x\in I}[f'(u(x,T))]^+,$$

$(f'(u)_x)^{c+}$ being the positive part of the absolutely continuous measure $f'(u(T)_x)^c$, we then have established:

Theorem 5.2. *Consider a scalar conservation law in one space dimension and solutions with range included in a given compact set. Then, there exists a constant C such that for every entropy solution with bounded variation and for every interval $I \subset \mathbb{R}$ the following estimate holds:*

$$\rho_T(I) \leq C\Big(\frac{meas(I(T))}{T} + (TV(u(\cdot,0)) - TV(u(\cdot,T)))\Big). \tag{5.9}$$

Proof. The discussion before the statement of the theorem shows that if $u^h \to u$ is a sequence of approximate solutions (given the Glimm scheme, say) and use the lowersemicontinuity property of the variation functional we get

$$\begin{aligned}\rho_T(I) \quad &\leq \liminf_{h\to 0}\Big(\rho^+_{h,T}(I) + C\,TV(u^h(T))\Big) - C\,TV(u(T)) \\ &\leq C\ \liminf_{h\to 0}\Big(\frac{\text{meas}(I)}{T} + TV(u^h(0))\Big) - C\,TV(u(T)) \\ &\leq C\Big(\frac{\text{meas}(I)}{T} + TV(u(0)) - TV(u(T))\Big).\end{aligned}$$

□

Remark 5.3. *The estimate in Theorem 5.2 was first established by Cheng* [14] *and was applied recently by Lellis and Riviere* [36] *to derive new regularity results for one-dimensional conservation laws.*

5.2. **Systems of conservation laws with general flux.** Consider now a system of N conservation laws in one space dimension which is strictly hyperbolic and piecewise genuinely nonlinear in the sense of [28]. We will establish an estimate on the spreading of the rarefaction waves. Roughly speaking, the rarefaction waves in any i-family within an interval $[s,t]$ are of three types :

(1) Waves already present at time s and which have propagated up to time s: these waves had enough time to decay and they are of the order of
$$\mathcal{O}(1)\,\frac{\text{meas}(I)}{t-s},$$
on any interval $I \subset \mathbb{R}$.

(2) Waves generated by interactions which took place during the time interval $[s,t]$: these waves can be estimated by the decrease in the interaction potential:
$$\mathcal{O}(1)\,\Big(\mathbf{Q}(s) - \mathbf{Q}(t)\Big).$$

(3) Waves cancelled during the time interval $[s,t]$, which are bounded by the change in total variation:
$$\mathcal{O}(1)\,|\mathbf{V}(s) - \mathbf{V}(t)|.$$

We define the *measure of i-rarefaction* in $u(\cdot,T)$ by

$$\rho^i_T(I) := \int_I (\lambda_i(u(T)_x)^{c+} + \sum_{x\in I}[\lambda_i(u(x,T))]^+,$$

the positive parts of the absolutely continuous measure $(\lambda_i(u(T)_x)^c$ and the jumps $[\lambda_i(u(x,T))]$, respectively. The main result of this section is then stated as follows:

Theorem 5.4. *Consider any system of N conservation laws in one space dimension* (3.1) *that is strictly hyperbolic and piecewise genuinely nonlinear, and entropy solutions with range in a small neighborhood of a constant state. Then, there exists a constant C such that, for every solution u with tame variation* [11] *(obtained by Glimm's random choice scheme, for instance), the following estimate holds:*

$$\rho^i_T(I) \le C\Big(\frac{meas(I)}{t-s} + (\mathbf{Q}(s) - \mathbf{Q}(t)) + |\mathbf{V}(s) - \mathbf{V}(t)|\Big) \tag{5.10}$$

for every interval $I \subset \mathbb{R}$.

Proof. The proof follows closely the presentation given by Bressan and Colombo [9]. The original ideas are due to Glimm and Lax [24] and Lax [30] and [31] (for genuinely nonlinear systems) and to Liu [37] (for general systems).

Step 1. Let us denote by $\chi_1(\cdot)$ and $\chi_2(\cdot)$ two (generalized) i-characteristics $t \to \chi_1(t), t \to \chi_2(t)$ passing through the points $a = \chi_1(T)$ and $b = \chi_2(T)$ at time $t = T$, and consider the interval $I(t) = [a(t), b(t)]$ for any $t \in [0, T]$. A weak solution of (3.1) is associated with the notion of a *generalized characteristic*, which is a Lipschitz curve with special properties, which allow us to think of waves as objects having a *global* identity (see [24, 17, 37]).

Here, we consider a BV solution obtained as limit of solutions u^h that satisfy the Bressan-LeFloch's tame variation condition, for instance solutions constructed by the random choice method of Glimm or by the wave front tracking algorithm. In the framework of front tracking algorithm the notion of generalized characteristic resembles the one presented in [17, 9] and is in the spirit of Filippov's theory [22]. In the context of Glimm's scheme one needs adopt the notion of a generalized characteristic introduced by Liu [37], which relies on the nonlinear superposition of wave patterns following the wave tracing method. In that sense, generalized characteristics are curves that move along the i-waves in the approximate solution. One realizes immediately that these curves fail to be Lipschitz continuous (in the case of rarefaction waves in the approximate solution, the characteristics move either always on the right edge of the wave or always along the left edge of the rarefaction wave). In this setting we think of generalized characteristics as piecewise Lipschitz continuous curves that jump to the right or left at times determined by the approximating scheme, according to the rule (iv) presented in Theorem 9.1 of Liu [37].

Let

$$D(t) := \chi_2(t) - \chi_1(t)$$

denote the distance between these two characteristics at time t, we call k_α the family of the wave located at the point x_a and $[\lambda_i(u(x_\alpha))]$ the jump of the i-characteristic speed at this point, that is

$$[\lambda_i(u(x_\alpha))] = \lambda_i(u(x_\alpha+)) - \lambda_i(u(x_\alpha-)). \tag{5.11}$$

Now differentiating $D(t)$ with respect to t we have that

$$\begin{aligned} \dot D(t) &= \dot\chi_2(t) - \dot\chi_1(t) = \lambda_i(u(\chi_2(t), t)) - \lambda_i(u(\chi_1(t), t)) \\ &= \sum_{\{x_\alpha \in I(t)\}} \lambda_i(u(x_\alpha+)) - \lambda_i(u(x_\alpha-)), \end{aligned} \tag{5.12}$$

and therefore,

$$\dot D(t) = M(t) + \mathcal{O}(1)\, K(t) \ \text{ a.e.,} \tag{5.13}$$

with

$$M(t) = \sum_{\{k_\alpha=i, x_a\in I(t)\}} [\lambda_i(u(x_\alpha))] \text{ and } K(t) = \sum_{\{k_\alpha\neq i, x_a\in I(t)\}} |[\lambda_i(u(x_\alpha))]|$$

denote the total amount of the (signed) elementary i-waves of $u(t,\cdot)$ contained in the interval $I(t)$ and the total strength of the k_α-waves, with $k_\alpha \neq i$ contained in $I(t)$.

In order to control the total strength of waves of families $k_\alpha \neq i$ contained in $I(t)$, we follow a similar line of arguments to the one in [9]. We introduce a piecewise Lipschitz continuous Φ having a finite number of discontinuities at points of interaction. At those points, the change in $\Phi(\cdot)$ can be controlled by the change in the interaction potential $\mathbf{Q}(\cdot)$,

$$\Phi(\tau+) - \Phi(\tau-) = C\Big(\mathbf{Q}(u(\tau-)) - \mathbf{Q}(u(\tau+))\Big). \tag{5.14}$$

Roughly speaking, $\Phi(t)$ represents the cumulated strength of the waves which do not approach the interval $I(t)$. Thanks to the strict hyperbolicity of the system it can be shown [9] that Φ is non-decreasing outside the interaction times. Furthermore,

$$K(t) \leq \frac{\dot{\Phi}(t)\, D(t)}{A}. \tag{5.15}$$

Step 2. Next, we provide an estimate for $M(t)$. We remark that since (generalized) characteristics of the same family do not cross, any change in $M(\cdot)$ is due to interactions and cancellations of elementary waves within the interval $[\chi_1(t), \chi_2(t)]$. We introduce now the following notation, given any quantity $G = G(\cdot)$, we denote $\Delta G(\tau) = G(\tau+) - G(\tau-)$. Then, at a given time $t = \tau$,

$$\Delta M(\tau) = \mathcal{O}(1)\, \Delta\mathbf{Q}(\tau) + \mathcal{O}(1)\, |\Delta\mathbf{V}(\tau)|,$$

which implies that

$$M(T) - M(t) = \mathcal{O}(1) \sum_\tau (\Delta\mathbf{Q}(\tau) + |\Delta\mathbf{V}(\tau)|), \tag{5.16}$$

where the summation is over all the interaction times in $[0,T]$ within $[\chi_1(\cdot), \chi_2(\cdot)]$.

Taking into consideration estimates (5.15) and (5.16), relation (5.13) gives

$$\dot{D}(t) + C\dot{\Phi}(t)D(t) \geq M(T) - C\Big(\sum_\tau (|\Delta\mathbf{Q}(\tau)| + |\Delta\mathbf{V}(\tau)|)\Big) \text{ a.e. in } t. \tag{5.17}$$

Since, Φ is uniformly bounded, and decreases at interaction times (5.14) (see also [9]), we have that $\mathrm{TV}(\Phi(t))$ is also uniformly bounded, which yields

$$\int_0^T \dot{\Phi}(t)dt \leq L, \tag{5.18}$$

with L being some constant.

Now, if we assume that

$$M(T) \geq 2C\Big(\sum_\tau (|\Delta\mathbf{Q}(\tau)| + |\Delta\mathbf{V}(\tau)|)\Big),$$

then (5.16) gives

$$\dot{D}(T) + C\,\dot{\Phi}(t)D(t) \geq \frac{M(T)}{2}. \tag{5.19}$$

Taking into consideration that Φ is, almost everywhere, not decreasing, (5.18) yields that

$$\frac{d}{dt}\Big(\exp\Big\{\int_0^t C\,\dot{\Phi}(s)ds\Big\}D(t)\Big) \geq \exp\Big\{\int_0^t C\dot{\Phi}(s)ds\Big\}\,\frac{M(T)}{2} \geq \frac{M(T)}{2}. \tag{5.20}$$

Now taking into account (5.18) and noticing that $D(0) \geq 0$, (5.20) yields that

$$D(T) \geq e^{-CL}\,\frac{M(T)}{2}\,T,$$

which implies that

$$M(T) \leq 2e^{-CL}\,\frac{D(T)}{T} + 2C\,\Big(\sum_\tau |\Delta\mathbf{Q}(\tau)| + |\Delta\mathbf{V}(\tau)|\Big). \tag{5.21}$$

If, on the other hand,

$$M(T) < 2C\,\Big(\sum_\tau |\Delta\mathbf{Q}(\tau)| + |\Delta\mathbf{V}(\tau)|\Big),$$

the estimate (5.21) still holds true.

Repeating the above procedure for any finite number of disjoint intervals of the form $I_q = [A_q, B_q)$ we get

$$\sum_{q=1}^k M_q(T) \leq C\,\Big(\frac{D_q(T)}{T} + \mathbf{Q}(u(0)) - \mathbf{Q}(u(T)) + |\mathbf{V}(u(0)) - \mathbf{V}(u(T))|\Big) \tag{5.22}$$

for some constant $C > 0$.

Step 3. We consider a partition of a line segment (a, b) on the upper-half plane, in several open intervals $I_q = (a_q, b_q)$, $q = 1, \dots, k$ that are bound to satisfy the following rules:

1. Each rarefaction wave in $u(T, \cdot)$ contained in the line segment (a, b) falls inside in one of the intervals I_q.
2. No shock generated at the initial line falls inside the intervals I_q.

The measure ρ^i_T of positive i-waves in $u(T, \cdot)$ satisfies

$$\rho^i_T((a, b)) = \sum_q M_q(T) + \mathcal{O}(1)\,\Big(\mathbf{Q}(u(0)) - \mathbf{Q}(u(T)) + |\mathbf{V}(u(0) - \mathbf{V}(u(T))|\Big).$$

We should remark here that the only negative i-waves contained in the union of the intervals I_q are those created from the interactions within $(0, T]$ and can be estimated by the decrease in the interaction potential and the amount of waves canceled within this interval. Therefore,

$$\rho^i_T((a, b)) \leq C\,\Big(\frac{b-a}{T} + (\mathbf{Q}(u(0)) - \mathbf{Q}(u(T))) + |\mathbf{V}(u(0) - \mathbf{V}(u(T))|\Big).$$

Moreover, as stated in Theorem 3.1 for the i-wave strengths, one can check that that the measure of rarefaction satisfies the following estimate, for every finite union of open intervals $I = I_1 \cup \cdots \cup I_m$,

$$\rho^i(I) + C\,\mathbf{Q}(u) \leq \liminf_{h\to 0}\,\Big(\rho^{i,h}(I) + C\,\mathbf{Q}(u^h)\Big)$$

as well as

$$\rho^i(I) + C\,(\mathbf{Q}(u) + C_0\mathbf{V}(u)) \leq \liminf_{h\to 0}\,\Big(\rho^{i,h}(I) + C\,(\mathbf{Q}(u^h) + C_0\mathbf{V}(u^h)))\Big).$$

Step 4. Consider now a sequence of approximate solutions u^h satisfying the tame variation condition (for instance solutions constructed Glimm's scheme). Using the lower semi-continuity property of the interaction potential $\mathbf{Q}$, and of $\mathbf{V} + C\,\mathbf{Q}$ (recall that $\mathbf{V}$ itself is not lower semi-continuous), which both are decreasing in time [28], we obtain

$$
\begin{aligned}
&\rho_T^i((a,b)) \leq \liminf_{h\to 0} \Big(\rho_T^{i,h}((a,b)) + C\,(\mathbf{V}(u^h(T)) + C_0\mathbf{Q}(u^h(T)))\Big) - C\,(\mathbf{V}(u(T)) + C_0\mathbf{Q}(u(T)))\\
&\leq \liminf_{h\to 0} \Big(C'\,\frac{b-a}{T} + C'\,(\mathbf{V}(u^h(0)) + C_0\mathbf{Q}(u^h(0))) - C\,(\mathbf{V}(u^h(T)) + C_0\,\mathbf{Q}(u^h(T)))\\
&\qquad\qquad + C'\,(\mathbf{V}(u^h(T)) + C_0\mathbf{Q}(u^h(T)))\Big) - C\,(\mathbf{V}(u) + C_0\mathbf{Q}(u))\\
&\leq C'\,\frac{b-a}{T} + C'\,(\mathbf{V}(u(0)) + C_0\mathbf{Q}(u(0))) - (C - C')\,(\mathbf{V}(u(T)) + C_0\mathbf{Q}(u(T)))\\
&\qquad\qquad - C\,(\mathbf{V}(u(T)) + C_0\mathbf{Q}(u(T)))\\
&= C'\,\frac{b-a}{T} + C'\,(\mathbf{V}(u(0)) + C_0\mathbf{Q}(u(0))) - C'\,(\mathbf{V}(u(T)) + C_0\mathbf{Q}(u(T))),
\end{aligned}
$$

provided we arrange the constants so that $C > C'$. This completes the proof of Theorem 5.4. □

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