On Large Time Step Godunov Scheme for Hyperbolic Conservation Laws

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Abstract

In this paper we study the large time step (LTS) Godunov scheme proposed by LeVeque for nonlinear hyperbolic conservation laws. As we known, when the Courant number is larger than 1, the linear interactions of the elementary waves in this scheme will be much more complicated than those for Courant number less than 1. In this paper, we will show that for scalar conservation laws, for any fixed Courant number, all the possible wave interactions in each time step $t_j < t < t_{j+1}$ only happen in finite number of cells, and the number is bounded by a constant depending only on the Courant number for a given initial value problem if the initial data is BV. This implies that the total length of the linear superposition zone in x direction will go to zero as the spatial step size goes to 0. And as an application of the result mentioned above, we show that for any given Courant number, if the total variation of the initial data satisfies some conditions, the solutions of the LTS Godunov scheme will converge to the entropy solutions for the convex scalar conservation laws.

1 Introduction and Notation

We are concerned with initial value problems for nonlinear hyperbolic conservation laws

$$u_t + f(u)_x = 0, \quad -\infty < x < \infty, t \geq 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad -\infty < x < \infty, \tag{1.2}$$

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in which the initial data

$$u_0 \in L^\infty(\mathbb{R}) \bigcap BV(\mathbb{R}) \text{ with } \|u_0\|_\infty \leq M \tag{1.3}$$

the flux function $f(u)$ is a smooth function of u.

It is well known that solutions to (1.1) and (1.2) generally develop discontinuities even when $u_0(x)$ is smooth. Therefore we seek weak solutions, i.e., locally integrable function $u(x,t)$ satisfies

$$\iint_{\mathbb{R}\times(0,\infty)} [u\varphi_t + f(u)\varphi_x]\, dxdt + \int_{-\infty}^{\infty} u_0(x)\varphi(x)\, dx = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}\times(0,\infty))$.

Moreover, a weak solution u is called a entropy solution, if it satisfies

$$U(u)_t + F(u)_x \leq 0 \tag{1.4}$$

in the sense of distributions, that is

$$-\iint_{\mathbb{R}\times(0,\infty)} [U(u)\varphi_t + F(u)\varphi_x]\, dxdt \leq 0, \quad \text{for all } 0 \leq \varphi \in C_0^\infty(\mathbb{R}\times(0,\infty)), \tag{1.5}$$

where U is any strictly convex function,

$$0 < c_a \leq U''(u) \leq c_b, \quad \text{for } |u| \leq M.$$

$U(\cdot)$ is called an entropy of initial value problem (1.1) and (1.2), the function $F(u) = \int_0^u U'(\xi)f'(\xi)\, d\xi$ is the associated entropy flux. Any smooth solution u of (1.1) satisfies $U(u)_t + F(u)_x = 0$. The entropy condition (1.4) ensures uniqueness of weak solutions to the initial value problem (1.1) and (1.2).

In order to compute the numerical approximation of (1.1) and (1.2), we partition the x-axis into intervals of length h by the set of points $x_i = ih, i \in \mathbb{Z}$, and the positive time axis into intervals by the points $t_j = j\Delta t, j \in \mathbb{N}_0$. The grid points (x_i, t_j) define a rectangular mesh on $\mathbb{R}\times[0,\infty)$. We will always assume that the time step $\Delta t = \lambda h$ for some fixed mesh ratio $\lambda > 0$. We denote the Courant number C as

$$C = \frac{a\Delta t}{h}, \quad \text{where } a = \sup_{|u|\leq M} |f'(u)|. \tag{1.6}$$

Many approximate methods for (1.1) and (1.2) are based on solutions to Riemann problems. At each time step $t = t_j$, we just use a piece-wise constant function

$$u_h^j(x) = u_i^j, \quad x \in [x_i, x_{i+1}),$$

to approximate the true solution $u(x, t_j)$. Denote the Riemann solution at each grid point (x_i, t_j) with left and right states u_{i-1}^j and u_i^j as $u_{h,i}^j(x,t)$. In convex scalar equations, $u_{h,i}^j(x,t)$ is either a shock wave or a rarefaction wave. In general scalar equations, it is a composite wave consisting of admissible discontinuities (shocks, contact discontinuities) and rarefaction waves. As long as the Courant number $C < \frac{1}{2}$, the neighboring Riemann solutions will be separated by the intermediate constant states. Therefore, setting

$$u_h^j(x,t) = u_h^j(x) + \sum_{i \in \mathbb{Z}} [u_{h,i}^j(x,t) - u_{h,i}^j(x,t_j)] \tag{1.7}$$

gives an exact weak solution to (1.1) with initial data $u_h^j(x)$ in the strip $t_j < t < t_{j+1}$. When $C > \frac{1}{2}$ the waves issuing from different grid points may interact with each other. The use of the exact weak solution beyond the time of interaction would be computationally difficult and expensive particularly for systems, except for the Godunov scheme with Courant number less than 1, see [12]. In the large time step (LTS) Godunov scheme proposed by LeVeque [11], the solution (1.7) is taken despite $C > \frac{1}{2}$. This means that we let the waves simply pass through one another with no changes in speed or strength and with no creation of new waves, so they behave as the solutions in a linear system $u_t + Au_x = 0$. In other words, we use linear superposition formula (1.7) to approximate the nonlinear interaction. For convenience, we will still call each wave in (1.7) as shock, admissible discontinuity or rarefaction wave respectively, although it no longer is that beyond the linear superpositions.

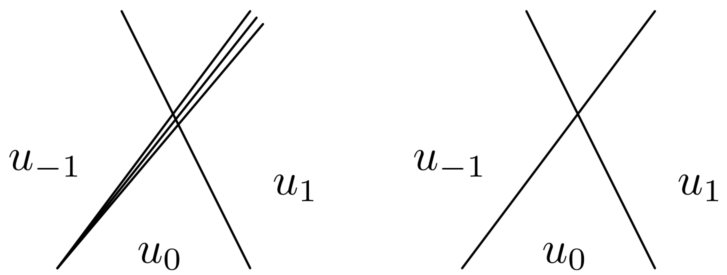
Although (1.7) will fail to be a weak solution of (1.1) on the strip $t_j \leq t < t_{j+1}$ beyond the interaction time, in [12], LeVeque showed that even for arbitrarily large Courant number, LTS Godunov scheme gives a consistent approximation for systems of conservation laws and convergent approximation to the initial value problem of (1.1). Brenier [2] and Wang [15] showed that LTS Glimm scheme gives a convergent approximation for Courant number less than or equal to 1, and consistent approximation for any given Courant number for system of conservation laws. In fact, the approximate solutions constructed by LTS Godunov and Glimm schemes are total variation diminishing (TVD) for the scalar conservation laws. Other LTS schemes had been introduced by Brenier [1], Toro and Billet [14]. In [12] LeVeque conjectured that his LTS Godunov schemes approximates an entropy solution. As we know, there does not exist any numerical evidence of entropy violating shocks for this schemes, see [11][8]. Surprisingly, it was found numerically that the LTS scheme with moderate value of the Courant number (larger than 1 but smaller than 3, say) has much higher resolution, see [11]. In [16] and [17] Wang and Warnecke proved the entropy consistency of the LTS Godunov and Glimm schemes for Courant numbers less than or equal to 1. If the flux function has constant curvature, the results extend to Courant numbers slightly larger than 1. If the flux function is monotone this holds for Courant number is 2 and for monotone initial data this is true for arbitrary Courant number. The entropy consistency of large time step schemes for the isetropic equations of gas dynamics was considered by Jiang and wang [9]. For $L^1(\mathbb{R})$ error estimate of the LTS schemes see [13] and [7]. To our knowledge, the entropy consistency results are essentially proved for Courant number $C \leq 1$ until now.

In this paper, we will study LTS Godunov scheme with arbitrary Courant number. In this case, the waves can travel C cells in the time interval (t_j, t_{j+1}). So the linear interactions of the elementary waves can not be confined in one cell. In section 2 we will prove that in the time interval (t_j, t_{j+1}), waves issuing from all the grid points $(x_i, t_j)(i \in \mathbb{Z})$ can be divided into several maximal connected sets (see definition 2.1), and each of them consists of finite number of waves. More specifically, if we denote a typical maximal connected set of waves as

$$A = \{W_i, W_{i+1}, \cdots, W_{i+K}\},$$

then, the integer K is independent of mesh size. Actually, We can show that the total number of cells including all the possible linear superpositions in the strip $t_j < t < t_{j+1}$ is bounded by a constant depending only on the Courant number C for a given initial value problem. From this result, we know that each maximal connected set of waves A is confined in a closed trapezoidal zone D, whose the top boundary is the line segment between $(x_{i-[C]-1}, t_{j+1})$ and $(x_{i+K+[C]+1}, t_{j+1})$, the bottom boundary is the line segment between (x_i, t_j) and (x_{i+K}, t_j), the left boundary is a broken line from (x_i, t_j) to $(x_{i-[C]-1}, t_{j+1})$, the right boundary is a broken line from (x_{i+K}, t_j) to $(x_{i+K+[C]+1}, t_{j+1})$ (see section 3 for the details about the left and right boundaries). This result plays an important role for proving entropy consistency of the scheme. As in this case, interactions of the waves involved can not be confined in a single grid cell, so one has to set up a generalized cell entropy inequality in D, which is named as a linear superposition zone. When the Courant number is fixed, the area of D will go to zero as the spatial step goes to 0 for general scalar conservation laws. The result is interesting in its own sake for the LTS Godnuve scheme, because this implies that the consistency error of the linear superposition could be "negligible", since the size of linear superposition zone is order of h for arbitrary given Courant number.

It is well known that if the initial value u_0 is in $L^\infty(\mathbb{R}) \bigcap BV(\mathbb{R})$, the solutions of LTS Godunov scheme are bounded uniformly in $L^\infty(\mathbb{R}\times(0,\infty)) \bigcap BV(\mathbb{R}\times(0,\infty))$; see [12]. Thus the entropy condition (1.5) is a distributional inequality for the signed Radon measure $\eta(u) = U(u)_t + F(u)_x$. This measure is called the dissipation measure by DiPerna [3]. In order to investigate entropy consistency of LTS Godunov scheme, we need to estimate the change of the dissipation measure through the linear superpositions. In section 3, we will study the change of dissipation measure on each linear superposition zone. Because the change of the dissipation measure across the linear superposition involving rarefaction waves is too complicated to estimate, we use a piecewise constant function to approximate the rarefaction wave. In the linear superpositions of all kinds of waves , only a shock interacting a rarefaction wave will produce increase for the dissipation measure. A rarefaction wave meeting another rarefaction wave, a shock interacting another shock will produce decrease for the dissipation measure. We will estimate the net increase amount. Furthermore, we assume that each rarefaction wave interacts with all the shocks in a maximal connected set of waves. Therefore, conditions on the initial data in the entropy consistency theorems is not optimal. The key estimation is (3.13) in Lemma 3.5, which gives a sufficient condition to ensure the dissipation measure is negative on a superposition zone D. In section 4, we prove a theorem (Theorem 4.1) on the entropy consistency for LTS Godunov scheme, the final result is: for any given Courant number C, if the product of increase total variation and decrease total variation of the initial data is bounded by a constant depending on the Courant



(a) linear superposition of a rarefaction wave with a neighboring admissible discontinuity

(b) linear superposition of two neighboring admissible discontinuities

Figure 2.1: the figure for Lemma 2.1

number C for a given initial value problem of a convex conservation law, the solution of the scheme will converge to the entropy solution.

2 The estimations on linear superposition zone

In this section, we are concerned with the initial value problem for general scalar conservation laws (1.1), (1.2) with the flux function f satisfying

$$\sup_{|u|\le M} |f''(u)| \le c_2 \tag{2.1}$$

where c_2 is a positive constant. And the initial data satisfies (1.3).

For LTS Godunov scheme proposed by LeVeque (1.7), waves issuing from different grid points at time level $t = t_j$ may interact(linear superposition) with each other in the time interval (t_j, t_{j+1}). We

definition 2.1. If a pair of neighboring waves interact with each other, we call it an interacting neighbor pair.

Because any two neighboring rarefaction waves can not interact with each other, in each interacting neighbor pair, at least one of the two waves is an admissible discontinuity.

definition 2.2. A pair of waves W_1 and W_2 is called connected, if there exist $W_{i_1}, W_{i_2}, \ldots, W_{i_k}$ in the same strip such that W_1 interacts with W_{i_1}, W_{i_1} interacts with W_{i_2}, $\cdots$, W_{i_k} interacts with W_2.

definition 2.3. A set of waves A in the strip $t_j < t < t_{j+1}$ is called a maximal connected set if any pair of waves in A is connected, and each wave in A doesn't interact with any waves not in A.

Evidently, a maximal connected set of waves must consist of some waves issuing from consecutive grid points at $t = t_j$. Then, the question arises of how many waves in each maximal connected set. In this section we will show that each maximal connected set consists of a finite number of consecutive waves, and the number is bounded by a constant depending on the Courant number for a given initial value problem. (ref. Theorem 2.4).

Lemma 2.1. *C is the Courant number (defined in* (1.6)*) of LTS Godunov scheme* (1.7)*.*

(1) If an interacting neighbor pair consists of a rarefaction wave $R(u_{-1}, u_0)$ and an admissible discontinuity $S(u_0, u_1)$ (ref. Figure 2.1(a)), then the strength of $S(u_0, u_1)$ is bounded below by $\frac{2a}{c_2 C}$, i.e.,

$$|u_0 - u_1| \geq \frac{2a}{c_2 C}. \tag{2.2}$$

(2) If an interacting neighbor pair consists of two admissible discontinuities $S(u_{-1}, u_0)$ and $S(u_0, u_1)$ (ref. Figure 2.1(b)), then,

$$|u_{-1} - u_1| \geq \frac{2a}{c_2 C}. \tag{2.3}$$

This implies that at least one of them has a strength bounded below by $\frac{a}{c_2 C}$., i.e.,

$$\text{either } |u_{-1} - u_0| \geq \frac{a}{c_2 C}, \ \text{ or } |u_0 - u_1| \geq \frac{a}{c_2 C}. \tag{2.4}$$

Proof : For (*1*), since the right boundary of the rarefaction wave $R(u_{-1}, u_0)$ travels with speed $f'(u_0)$, and the discontinuity $S(u_0, u_1)$ travels with speed $(f(u_0) - f(u_1))/(u_0 - u_1)$

$$\left| f'(u_0) - \frac{f(u_0) - f(u_1)}{u_0 - u_1} \right| \Delta t \tag{2.5}$$

$$= \left| \frac{f(u_1) - f(u_0) - f'(u_0)(u_1 - u_0)}{u_0 - u_1} \right| \Delta t \tag{2.6}$$

$$\leq \frac{(u_0 - u_1)^2 \int_0^1 \xi |f''(\xi u_0 + (1 - \xi) u_1)| \, d\xi}{|u_0 - u_1|} \Delta t \tag{2.7}$$

$$\leq \frac{c_2}{2} |u_0 - u_1| \Delta t \tag{2.8}$$

so, if

$$\frac{c_2}{2} |u_0 - u_1| \Delta t < h \tag{2.9}$$

then the rarefaction wave (left one) can not catch up the discontinuity (right one) within the time period $0 < t < \Delta t$. Hence, in order to let them interact, we must have

$$\frac{c_2}{2} |u_0 - u_1| \Delta t \geq h,$$

i.e.,

$$|u_0 - u_1| \geq \frac{2a}{c_2 C}. \tag{2.10}$$

For (*2*), since,

$$\left| \frac{f(u_{-1}) - f(u_0)}{u_{-1} - u_0} - \frac{f(u_1) - f(u_0)}{u_1 - u_0} \right| \Delta t \tag{2.11}$$

$$\leq \int_0^1 |f'(\xi u_{-1} + (1-\xi)u_0) - f'(\xi u_1 + (1-\xi)u_0)| \xi \Delta t \tag{2.12}$$

$$\leq \int_0^1 \int_0^1 |f''(\eta\xi(u_{-1} - u_1) + \xi u_{-1} + (1-\xi)u_0)||u_{-1} - u_1| \xi \, d\eta d\xi \Delta t \tag{2.13}$$

$$\leq c_2 |u_{-1} - u_1| \int_0^1 \int_0^1 \xi \, d\eta d\xi \Delta t \tag{2.14}$$

$$\leq \frac{c_2}{2} |u_{-1} - u_1| \Delta t \tag{2.15}$$

by a similar way, we must have

$$|u_{-1} - u_1| \geq \frac{2a}{c_2 C} \tag{2.16}$$

to ensure the two discontinuities interact with each other. If u_{-1}, u_0 and u_1 are three monotone numbers, then the total strength of the two discontinuities is equal to $|u_{-1} - u_1|$. If u_0 is not between u_{-1} and u_1, then

$$|u_{-1} - u_1| = ||u_{-1} - u_0| - |u_1 - u_0||.$$

So at least one of the discontinuity's strength is greater than $\frac{a}{c_2 C}$.

Lemma 2.2. *C is the Courant number (defined in* (1.6)*) of LTS Godunov scheme* (1.7)*. In the time interval $(t_j, t_j + \Delta t)$,*

(1) an admissible discontinuity can interact at most with $[2C]$ *other waves;*

(2) a rarefaction wave can interact at most with $2[2C]$ *other waves.*

proof: For the Courant number C, the two waves issuing from grid points (x_i, t_j) and $(x_{i+[2C]+1}, t_j)$ respectively can not meet each other within the time period $(t_j, t_j + \Delta t)$, since each wave travels at most $C\Delta t$ in this time interval.

Lemma 2.3. *C is the Courant number (defined in* (1.6)*) of LTS Godunov scheme* (1.7)*. A is a maximal connected set of waves in the strip $t_j < t < t_{j+1}$. B is any subset of A that consists of $2([2C]+1)$ consecutive waves in A. Then, there exists at least one interacting neighbor pair in B.*

Proof: Since each wave travels at most as far as Ch in the time interval (t_j, t_{j+1}), the wave issuing from the middle grid point in B can not interact with any waves not in B. We denote this wave as W_i. Since A is a maximal connected set, W_i must interact with another wave in B. If W_i interacts with more than one waves in B, we pick out the one which is nearest to W_i, and denote it as W_{i+l}. Without loss of generality, we can assume $l > 0$. If $l = 1$, the proof is completed. If $l > 1$, because of the selection method of W_{i+l}, all the waves between W_i and W_{i+l} must not interact with W_i, and must interact with W_{i+l}. So W_{i+l} and its left neighbor is an interacting neighbor pair.

Denote the total variation of a function $u(x)$ over (a, b) by $TV_{(a,b)}(u)$. We have the following important theorem on the size of the superposition zone.

Theorem 2.4. *C is the Courant number (defined in* (1.6)*) of LTS Godunov scheme* (1.7)*. Then, a maximal connected set of waves A contains finite number of consecutive waves $\{W_i, W_{i+1}, \cdots, W_{i+K}\}, i \in \mathbb{Z}, K \in \mathbb{N}$, and the number K satisfies*

$$K \leq \frac{4c_2C(C+1)}{a} TV_{(-\infty,\infty)}(u_0).$$

Proof: Let's first show that K must be a finite number. If A contains infinite consecutive waves, we can find disjoint subsets B_k of A with

$$B_k = \{W_{i_k}, \cdots, W_{i_k+2[2C]+1}\}, \quad i_{k+1} > i_k + 2[2C] + 1$$

by Lemma 2.3, there are at least one interacting neighboring pair in each B_k. By Lemma 2.1, we have

$$\frac{a}{c_2C} \leq TV_{[x_{i_k}, x_{i_k+2[2C]+1}]}(u_h(\cdot, t_j)).$$

Since we have infinite B_k, this contradicts with

$$TV(u_h(\cdot, t_j)) \leq TV(u_0) < \infty.$$

Suppose that there are $K+1$ ($K \geq 1$) consecutive waves $W_i, \cdots, W_{i+K}$ in A. If $K < 2[2C]+1$, there is at least one interacting neighbor pair in A. Otherwise, we find the positive integer m such that $2([2C]+1)(m-1) < K \leq (2[2C]+1)m$, and divide A into $m+1$ subsets

$$\begin{aligned}
&B_1 = \{W_i, \cdots, W_{i+2[2C]+1}\},\\
&B_2 = \{W_{i+2[2C]+1}, \cdots, W_{i+4[2C]+2}\},\\
&\cdots\\
&B_m = \{W_{i+2(m-1)[2C]+m-1}, \cdots, W_{i+2m[2C]+m}\},\\
&B_{m+1} = \{W_{i+2m[2C]+m}, \cdots, W_{K+1}\}.
\end{aligned}$$

By Lemma 2.3, we have at least m interacting neighbor pairs in A. There is possibility that two interacting neighbor pairs shares one admissible discontinuity.

For example, the interacting neighbor pair in B_1 is $\{W_{i+2[2C]}, W_{i+2[2C]+1}\}$, in B_2 is $\{W_{i+2[2C]+1}, W_{i+2[2C]+2}\}$, $W_{i+2[2C]}$ and $W_{i+2[2C]+2}$ are rarefaction waves, they share one admissible discontinuity $W_{i+2[2C]+1}$. In this case, the strength of the admissible discontinuity $W_{i+2[2C]+1}$ is bounded below by $\frac{2a}{c_2C}$. So we can regard $W_{i+2[2C]+1}$ as two admissible discontinuities whose strength are bounded below by $\frac{a}{c_2C}$. Then we have

$$\frac{a}{c_2C}m \le TV_{(-\infty,\infty)}(u_h(\cdot,t_j)) \le TV_{(-\infty,\infty)}(u_0),$$

which means

$$m \le \frac{c_2C}{a}TV_{(-\infty,\infty)}(u_0).$$

So,

$$K \le 2([2C]+1)m \le \frac{2c_2C([2C]+1)}{a}TV_{(-\infty,\infty)}(u_0) \le \frac{4c_2C(C+1)}{a}TV_{(-\infty,\infty)}(u_0)$$

which completes the proof.

Remark 2.5. *This theorem tells us that when we use a LTS scheme to approximate a general scalar conservation laws, as long as the initial data $u_0 \in BV(\mathbb{R})$, the number of grid cells in a linear superposition zone can be bounded by the Courant number for a given problem. So when the mesh size h is small, each linear superposition zone must be small.*

If the flux function in (1.1) *is convex, the total strength of the shocks is bounded by decrease total variation of the initial data, so the number of grid cells in a linear superposition zone can be bounded by the decrease total variation of the initial data u_0, i.e.*

$$K \le \frac{4c_2C(C+1)}{a}DTV(u_0) \tag{2.17}$$

It is easy to know from the proof of Theorem 2.4 that in each strip

$$\{(x,t)| -\infty < x < \infty, t_j \le t < t_{j+1}\},$$

all the possible wave interactions happen in a finite number of cells, and the number depends on the Courant number for a given initial value problem. So when the Courant number is fixed, the total length of all linear superposition zones in x direction from t_j to t_{j+1} is equal to $O(h)$, and will go to zero as h goes to 0.

3 Estimation on the change of dissipation measure in a linear superposition zone

In this section, we are concerned with convex scalar conservation laws (1.1), (1.2) with

$$0 < c_1 \leq f''(u) \leq c_2, \quad \text{for } |u| \leq M,$$

where c_1 and c_2 are constant numbers. Denote $Q(t)$ as

$$Q(t) = \int_{a(t)}^{b(t)} U(u_h(x,t))_t + F(u_h(x,t))_x dx, \quad t \in (t_j, t_{j+1}) \tag{3.1}$$

where $a(t)$ and $b(t)$ are the left and right boundaries of a typical linear superposition zone corresponding to a maximal connected set of waves A. $a(t)$ is formed by all the points on the most left waves in A, $b(t)$ is formed by all the points on the most right waves in A. Evidently, they are broken lines satisfying $a(t) \geq x_i + \frac{x_{i-[C]-1} - x_i}{\Delta t}(t - t_j)$ and $b(t) \leq x_{i+K} + \frac{x_{i-[C]-1} - x_i}{\Delta t}(t - t_j)$. From Theorem 2.4, $[a(t), b(t)]$ is a finite interval for a given initial value problem (1.1), (1.2) and mesh. In this section, we will estimate the change of $Q(t)$ on the time interval $(t_j, t_j + \Delta t)$.

In order to investigate the interaction of a rarefaction wave with a shock, we use a fan function of piecewise constants to approximate a rarefaction wave.

definition 3.1. For a rarefaction wave

$$R(x/t; u_l, u_r) = \begin{cases} u_l, & x/t \leq f'(u_l), \\ g(x/t), & f'(u_l) \leq x/t \leq f'(u_r), \\ u_r, & x/t \geq f'(u_r), \end{cases}$$

in which g is the inverse of function f'. Denote

$$\begin{aligned}
u_i &= u_l + i\frac{u_r - u_l}{n}, \quad i = 0, \dots, n, \\
u_{-\frac{1}{2}} &= u_0, \\
u_{i+1/2} &= \frac{1}{f'(u_{i+1}) - f'(u_i)} \int_{f'(u_i)}^{f'(u_{i+1})} u(\xi)\, d\xi, \quad i = 0, \dots, n-1, \\
&= \frac{f'(u_i)u_{i-1/2} - f'(u_{i+1})u_{i+3/2} - f(u_i) + f(u_{i+1})}{f'(u_i) - f'(u_{i+1})}, \\
u_{n+\frac{1}{2}} &= u_n, \\
AR(u_l, u_r; n) &= \begin{cases} u_0 = u_l, & x/t \leq f'(u_l), \\ u_{i+1/2}, & f'(u_i) \leq x/t \leq f'(u_{i+1}), \quad i = 0, \dots, n, \\ u_n = u_r, & x/t \geq f'(u_r). \end{cases}
\end{aligned}$$

Then we call $AR(u_l, u_r; n)$ is the approximate rarefaction wave for $R(x/t; u_l, u_r)$.

In this way, we approximate a rarefaction wave by n discontinuities with speed $f'(u_i)$

$$(u_0, u_{1/2}, f'(u_0)), (u_{1/2}, u_{i+3/2}, f'(u_1)), \cdots, (u_{n-1/2}, u_{n+\frac{1}{2}}, f'(u_n)).$$

In fact, $AR(u_l, u_r; n)$ can be considered as one kind of the approximate Riemann solver proposed by Harten, Lax and Van Leer [6], which is consistent with (1.1) and satisfies entropy condition in the sense given by them.

If there is only a single rarefaction wave in $(x_i, x_{i+K}) \times (t_j, t_j + \Delta t)$, then we have $Q(t) \equiv 0, t \in (t_j, t_j + \Delta t)$. When we replace $R(u_l, u_r; x/t)$ with $AR(u_l, u_r; n)$, this will lead to an approximation for $Q(t)$, let's call it $Q_n(t)$. First of all, we can show that $Q_n(t)$ is negative, and will tend to zero as n goes to infinity. In fact we have

$$
\begin{aligned}
Q_n = &(U(u_0) - U(u_{\frac{1}{2}}))f'(u_0) - F(u_0) + F(u_{\frac{1}{2}}) + \\
&+ (U(u_{\frac{1}{2}}) - U(u_{\frac{3}{2}}))f'(u_1) - F(u_{\frac{1}{2}}) + F(u_{\frac{3}{2}}) + \\
&+ \cdots + \\
&+ (U(u_{i-\frac{1}{2}}) - U(u_{i+\frac{1}{2}}))f'(u_i) - F(u_{i-\frac{1}{2}}) + F(u_{i+\frac{1}{2}}) + \\
&+ \cdots + \\
&+ (U(u_{n-\frac{1}{2}}) - U(u_n))f'(u_n) - F(u_{n-\frac{1}{2}}) + F(u_n) \\
&= U(u_0)f'(u_0) + U(u_{\frac{1}{2}})(f'(u_1) - f'(u_0)) + \\
&+ \cdots + \\
&+ U(u_{i+\frac{1}{2}})(f'(u_{i+1}) - f'(u_i)) + \\
&+ \cdots + \\
&+ U(u_{n-\frac{1}{2}})(f'(u_n) - f'(u_{n-1})) - \\
&- U(u_n)f'(u_n) - F(u_0) + F(u_n)
\end{aligned} \tag{3.2}
$$

By Taylor expansion, we have

$$U(u(\xi)) = U(u_{i+\frac{1}{2}}) - U'(u_{i+\frac{1}{2}})(u_{i+\frac{1}{2}} - u(\xi)) + \frac{1}{2}\,U''(\tilde{u})(u(\xi) - u_{i+\frac{1}{2}})^2.$$

If we set $\eta = u(\xi)$, then $f'(u(\xi)) = \xi, f'(\eta) = \xi, d\xi = f''(\eta)d\eta$. From the definition of $u_{i+\frac{1}{2}}$, we have

$$\int_{f'(u_i)}^{f'(u_{i+1})} (u_{i+\frac{1}{2}} - u(\xi))\, d\xi = 0.$$

Therefore,

$$\begin{aligned}
&\frac{1}{f'(u_{i+1})-f'(u_i)}\int_{f'(u_i)}^{f'(u_{i+1})} U(u(\xi))\,d\xi \\
&= U(u_{i+\frac12}) + \frac{1}{2(f'(u_{i+1})-f'(u_i))}\int_{f'(u_i)}^{f'(u_{i+1})} U''(\tilde u)(u(\xi)-u_{i+\frac12})^2\,d\xi \\
&= U(u_{i+\frac12}) + \frac{U''(\hat u)}{2(f'(u_{i+1})-f'(u_i))}\int_{u_i}^{u_{i+1}} f''(\eta)(\eta-u_{i+\frac12})^2\,d\eta \\
&= U(u_{i+\frac12}) + \frac{U''(\tilde u)f''(\hat\eta)}{2(f'(u_{i+1})-f'(u_i))}\frac{u_{i+1}-u_i}{3}\,[(u_{i+1}-u_{i+\frac12})^2+ \\
&\qquad + (u_{i+1}-u_{i+\frac12})(u_i-u_{i+\frac12}) + (u_i-u_{i+\frac12})^2] \\
&= U(u_{i+\frac12}) + \frac{U''(\tilde u)f''(\hat\eta)}{6f''(\tilde\eta)}\,[(u_{i+1}-u_{i+\frac12})^2+ \\
&\qquad + (u_{i+1}-u_{i+\frac12})(u_i-u_{i+\frac12}) + (u_i-u_{i+\frac12})^2].
\end{aligned}$$

Since

$$0 > (u_{i+1}-u_{i+\frac12})(u_i-u_{i+\frac12}) \ge -\frac{(u_{i+1}-u_{i+\frac12})^2+(u_i-u_{i+\frac12})^2}{2}$$

and

$$0 < c_1 \le f''(u) \le c_2, \qquad 0 < c_a \le U''(u) \le c_b,$$

we have

$$\frac{1}{f'(u_i)-f'(u_{i+1})}\int_{f'(u_i)}^{f'(u_{i+1})} U(u(\xi))\,d\xi \ge U(u_{i+\frac12}) + \frac{c_a\,c_1}{12c_2}[(u_{i+1}-u_{i+\frac12})^2+(u_i-u_{i+\frac12})^2].$$

Moreover,

$$\begin{aligned}
u_{i+1}-u_{i+\frac12} &= \frac{1}{f'(u_{i+1})-f'(u_i)}\int_{f'(u_i)}^{f'(u_{i+1})}(u_{i+1}-u(\xi))\,d\xi \\
&= \frac{1}{f'(u_{i+1})-f'(u_i)}\int_{u_i}^{u_{i+1}}(u_{i+1}-\eta)f''(\eta)\,d\eta \\
&= \frac{f''(\hat\eta)}{f'(u_{i+1})-f'(u_i)}\cdot\frac12(u_{i+1}-u_i)^2 \\
&\ge \frac{c_1}{2c_2}(u_{i+1}-u_i),
\end{aligned}$$

so we finally have

$$\frac{1}{f'(u_{i+1})-f'(u_i)}\int_{f'(u_i)}^{f'(u_{i+1})} U(u(\xi))\,d\xi \ge U(u_{i+\frac12}) + \frac{c_a\,c_1^3}{24c_2^3}(u_{i+1}-u_i)^2. \quad (3.3)$$

Hence,

$$U(u_{i+\frac12})(f'(u_{i+1})-f'(u_i)) \le \int_{f'(u_i)}^{f'(u_{i+1})} U(u(\xi))\,d\xi - \frac{c_a\,c_1^4}{24c_2^3}(u_{i+1}-u_i)^3. \quad (3.4)$$

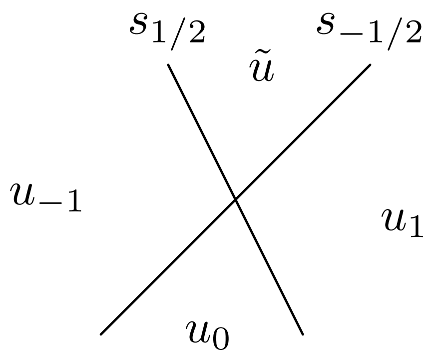


Figure 3.1: a pair of interacted discontinuities

Substituting (3.4) into (3.2) gives

$$Q_n \le \int_{f'(u_r)}^{f'(u_l)} U(u(\xi))\,d\xi - U(u_r)f'(u_r) + U(u_l)f'(u_l) - F(u_l) + F(u_r) - \\ - \frac{c_a\, c_1^4}{24c_2^3}\sum_{i=0}^{n-1}(u_{i+1}-u_i)^3.$$

Since $u(\xi)$ is a rarefaction wave, we have

$$\int_{f'(u_r)}^{f'(u_l)} U(u(\xi))\,d\xi - U(u_r)f'(u_r) + U(u_l)f'(u_l) - F(u_l) + F(u_r) = 0.$$

Notice that $u_{i+1} - u_i = \dfrac{u_r - u_l}{n}$, we obtain

$$Q_n \le -\frac{c_a\, c_1^4}{24c_2^3}\frac{(u_r-u_l)^3}{n^2} < 0 \tag{3.5}$$

Now let's estimate the change of $Q(t)$ when two discontinuities interact linearly with each other.

Lemma 3.1. *Suppose that a pair of neighboring discontinuities* $(u_{-1}, u_0, s_{-\frac{1}{2}})$ *and* $(u_0, u_1, s_{\frac{1}{2}})$ *interacts linearly with each other(ref. figure 3.1). Then, through the interaction,* $Q(t)$ *must be decreased when (i)* $u_{-1} > u_0 > u_1$*, or (ii)* $u_{-1} < u_0 < u_1$*; must be increased when (iii)* $u_{-1} < u_0$ *and* $u_0 > u_1$*, or (iv)* $u_{-1} > u_0$ *and* $u_0 < u_1$*.*

proof: Before the interaction,

$$Q_1 = (U(u_{-1}) - U(u_0))s_{-\frac{1}{2}} - F(u_{-1}) + F(u_0) + \\ + (U(u_0) - U(u_1))s_{\frac{1}{2}} - F(u_0) + F(u_1).$$

After the interaction,

$$Q_2 = (U(u_{-1}) - U(\tilde{u}))s_{\frac{1}{2}} - F(u_{-1}) + F(\tilde{u}) + \\ + (U(\tilde{u}) - U(u_1))s_{-\frac{1}{2}} - F(\tilde{u}) + F(u_1),$$

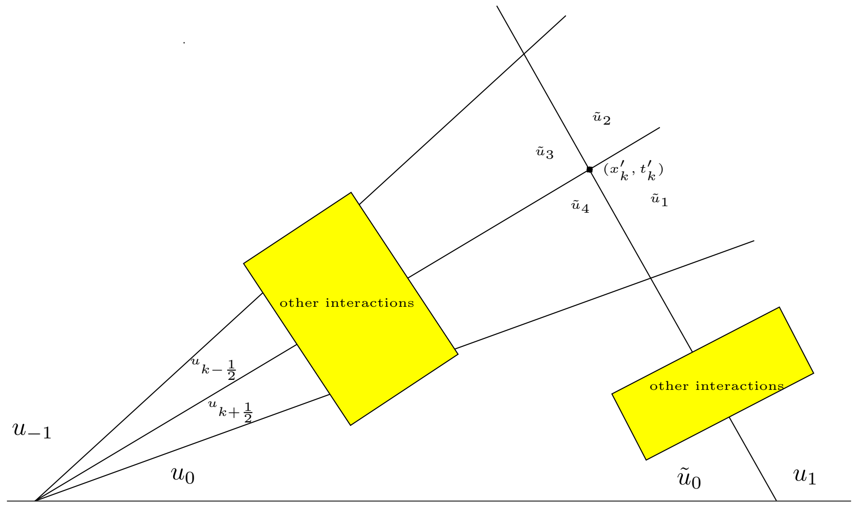


Figure 3.2: Figure for Lemma 3.2

where $\tilde{u} = u_{-1} + u_1 - u_0$.

We study their difference

$$
\begin{aligned}
-(Q_1 - Q_2) &= -(U(u_{-1}) - U(\tilde{u}) - U(u_0) + U(u_1))(s_{-\frac{1}{2}} - s_{\frac{1}{2}}) \\
&= -\int_{u_0}^{u_{-1}} \int_{u+(u_1-u_0)}^{u} U''(\eta)\, d\eta du \cdot (s_{-\frac{1}{2}} - s_{\frac{1}{2}}). \qquad (3.6)
\end{aligned}
$$

In order to make sure that the two discontinuities interact with each other, we must have $s_{\frac{1}{2}} < s_{-\frac{1}{2}}$. Thus,

$$
\begin{aligned}
-sign(Q_1 - Q_2) &= sign\left(\int_{u_0}^{u_{-1}} \int_{u+(u_1-u_0)}^{u} U''(\eta)\, d\eta du \right) \qquad (3.7) \\
&= -sign\left((u_{-1} - u_0)(u_1 - u_0)\right). \qquad (3.8)
\end{aligned}
$$

Here $U'' > 0$ is used. And the conclusion of the lemma follows from (3.8).

Now, let's consider a typical case for the linear interactions in a LTS scheme. Suppose that $AR(u_{-1}, u_0; n)$ and $S(\tilde{u}_0, u_1)$ are in a linear superposition zone D, and they interact with each other in D. Generally speaking, there are several shocks and rarefaction waves between $AR(u_{-1}, u_0; n)$ and $S(\tilde{u}_0, u_1)$. We denote the total strength of all the shocks between $AR(u_{-1}, u_0; n)$ and $S(\tilde{u}_0, u_1)$ as $\bar{S}$, ref. figure 3.2.

Lemma 3.2. *For the typical case mentioned above, Q_n will increase after the interaction of $AR(u_{-1}, u_0; n)$ and $S(\tilde{u}_0, u_1)$ happens, and the increased amount ΔQ_n satisfies*

$$
\Delta Q_n \le \frac{n+1}{n} \frac{c_2 c_b}{2} (u_0 - u_{-1})(\tilde{u}_0 - u_1) \left[2\bar{S} + (\tilde{u}_0 - u_1) \right].
$$

proof: $Q_n(t)$ remains unchanged until the interaction happens. Now let's look at the interaction point (x'_k, t'_k) of a discontinuity in $AR(u_{-1}, u_0; n)$

$$x - x_i = f'(u'_k)(t - t_j),$$

with the shock

$$x - x_{i+1} = \frac{f(\tilde{u}_0) - f(u_1)}{\tilde{u}_0 - u_1}(t - t_j).$$

From (3.6), we have

$$\begin{aligned}
&Q(t'_k + 0) - Q(t'_k - 0) \\
&= \int_{\tilde{u}_4}^{\tilde{u}_3} \int_{u+(\tilde{u}_1 - \tilde{u}_4)}^{u} U''(\eta)\, d\eta \left(\frac{f(\tilde{u}_0) - f(u_1)}{\tilde{u}_0 - u_1} - f'(u'_k) \right) du \qquad (3.9) \\
&\le \int_{\tilde{u}_3}^{\tilde{u}_4} \int_{u+(\tilde{u}_1 - \tilde{u}_4)}^{u} U''(\eta)\, d\eta \left(f'(u_0) - \frac{f(\tilde{u}_0) - f(u_1)}{\tilde{u}_0 - u_1} \right) du. \qquad (3.10)
\end{aligned}$$

Since

$$\begin{aligned}
&\left(f'(u_0) - \frac{f(\tilde{u}_0) - f(u_1)}{\tilde{u}_0 - u_1} \right) \\
&= \int_0^1 (f'(u_0) - f'(\tilde{u}_0 \xi + u_1(1 - \xi))) d\xi \\
&= \int_0^1 f''(\eta)[(u_0 - \tilde{u}_0)\xi + (u_0 - u_1)(1 - \xi)] d\xi \\
&\le \frac{c_2}{2} [(u_0 - \tilde{u}_0) + (u_0 - u_1)] \\
&\le \frac{c_2}{2} \left[2 \sum_j |S_j| + (\tilde{u}_0 - u_1) \right]
\end{aligned}$$

where $S'_j s$ are the shocks between $AR(u_{-1}, u_0; n)$ and $S(\tilde{u}_0, u_1)$, $|S_j|$ is the strength of S_j. So

$$Q(t'_k + 0) - Q(t'_k - 0) \le \frac{c_2 c_b}{2} (\tilde{u}_4 - \tilde{u}_1)(\tilde{u}_4 - \tilde{u}_3)[2 \sum_j |S_j| + (\tilde{u}_0 - u_1)].$$

Notice that

$$\tilde{u}_4 - \tilde{u}_3 = u_{k+\frac{1}{2}} - u_{k-\frac{1}{2}}, \quad \tilde{u}_4 - \tilde{u}_1 = \tilde{u}_0 - u_1.$$

and the definition of $u_{k+\frac{1}{2}}$, we have

$$Q(t'_k + 0) - Q(t'_k - 0) \le \frac{c_2 c_b}{2} \frac{u_0 - u_{-1}}{n} (\tilde{u}_0 - u_1)[2 \sum_j |S_j| + (\tilde{u}_0 - u_1)]$$

Summation with respect to k, we get the total increased amount satisfies

$$\begin{aligned}
\Delta Q_n &= \sum_k (Q(t'_k + 0) - Q(t'_k - 0)) \\
&\le \frac{c_2 c_b}{2} \frac{n+1}{n} (u_0 - u_{-1})(\tilde{u}_0 - u_1) \left[2 \sum_j |S_j| + (\tilde{u}_0 - u_1) \right]
\end{aligned}$$

which complete the proof.

Now we need a technical Lemma which can be deduced by Hölder inequality to estimate $Q_n(t)$ in a linear superposition zone.

Lemma 3.3. *Assume that $a_1, a_2, \dots, a_n$ are positive real numbers, then we have*

$$n^2 \sum_{i=1}^{n} a_i^3 \geq \Big(\sum_{i=1}^{n} a_i \Big)^3 \tag{3.11}$$

Lemma 3.4. *Assume that there are l shocks $S_1, S_2, \dots, S_l$ and m ($m \geq 1$) approximate rarefaction waves $AR_1, AR_2, \dots, AR_m$ in a superposition zone D, with*

$$S = \sum_{i=1}^{l} |S_i|, \quad R = \sum_{j=1}^{m} |R_j|$$

Then for any $t \in (t_j, t_j + \Delta t)$,

$$Q_n(t) \leq \frac{c_2 c_b}{2} S^2 \left[\frac{2(n+1)}{n} R - \frac{c_1 c_a}{c_2 c_b} l^{-2} S \right] \tag{3.12}$$

where $|R_j|$ is the strength of AR_j.

Proof: For $t \in (t_j, t_j + \Delta t)$, $Q_n(t)$ remains unchanged until the interaction happens. By Lemma 3.1, $Q_n(t)$ increases only when it passes a interaction point of a discontinuity in an approximate rarefaction wave and a shock. By Lemma 3.2, for a typical case (x_k', t_k') we have

$$\Delta Q_n \leq c_2 c_b \frac{n+1}{n} (u_0 - u_{-1})(\tilde{u}_0 - u_1) S.$$

Summation over all the interaction points from t_j to t, we have,

$$Q_n(t) - Q(t_j + 0) \leq c_2 c_b \frac{n+1}{n} S^2 R.$$

where

$$Q(t_j + 0) \geq -\frac{c_1 c_a}{2} \sum_{i=1}^{l} |S_i|^3.$$

So by the technical Lemma 3.3

$$\begin{aligned} Q_n(t) &\leq \frac{c_2 c_b}{2} \left[\frac{2(n+1)}{n} S^2 R - \frac{c_1 c_a}{c_2 c_b} \sum_{i=1}^{l} |S_i|^3 \right] \\ &\leq \frac{c_2 c_b}{2} \left[\frac{2(n+1)}{n} S^2 R - \frac{c_1 c_a}{c_2 c_b} l^{-2} S^3 \right] \end{aligned}$$

which complete the proof.

Lemma 3.5. *Assume that there are l shocks $S_1, S_2, \ldots, S_l$ and m rarefaction waves $R_1, R_2, \ldots, R_m$ in a maximal connected set A, and the corresponding linear superposition zone is D. Denote R as the total strength of $R_1, R_2, \ldots, R_m$, S as the total strength of $S_1, S_2, \ldots, S_l$, and*

$$< \eta, 1 > |_D = \int_{t_j}^{t_{j+1}} \int_{a(t)}^{b(t)} (U(u_h)_t) + F(u_h)_x) dxdt$$

Then

$$< \eta, 1 > |_D \leq \left(1 - \frac{1}{2C}\right) \Delta t \, c_2 c_b S^2 \left[R - \frac{c_1 c_a}{c_2 c_b} \, l^{-2} S\right]. \tag{3.13}$$

Proof: Denote u_n as the linear superposition of l shocks $S_1, S_2, \ldots, S_l$ and m approximate rarefaction waves $AR_1, AR_2, \ldots, AR_m$, and denote

$$\eta_n = U(u_n)_t + F(u_n)_x.$$

Then, because any pair of waves from (x_i, t_j) and (x_k, t_j) can't meet each other before

$$t = t_j + \frac{\Delta t}{2C},$$

by Lemma 3.3, (3.5) and a classical result for shock (see Lemma 2.1 in [16]), we have

$$\begin{aligned}
< \eta_n, 1 > |_D &\leq \int_{t_j + \frac{\Delta t}{2C}}^{t_{j+1}} \int_{a(t)}^{b(t)} \eta_n \, dxdt + \int_{t_j}^{t_j + \frac{\Delta t}{2C}} \int_{a(t)}^{b(t)} \eta_n \, dxdt \\
&\leq \int_{t_j + \frac{\Delta t}{2C}}^{t_{j+1}} Q_n(t) \, dt + \int_{t_j}^{t_j + \frac{\Delta t}{2C}} \left(-\frac{c_1 c_a}{2} \sum_{i=1}^{l} |S_i|^3\right) dt \\
&\leq \Delta t \left(1 - \frac{1}{2C}\right) \frac{c_2 c_b}{2} S^2 \left[\frac{2(n+1)}{n} R - \frac{c_1 c_a}{c_2 c_b} \, l^{-2} S\right] \\
&\quad - \frac{\Delta t}{2C} \frac{c_1 c_a}{2} \, l^{-2} S^3 \\
&\leq \Delta t \left(1 - \frac{1}{2C}\right) c_2 c_b \, S^2 \left[\frac{n+1}{n} R - \frac{c_1 c_a}{c_2 c_b} \frac{C}{2C-1} \, l^{-2} S\right].
\end{aligned}$$

Since $AR(u_l, u_r; n)$ converges to $R(u_l, u_r; x/t)$ point-wisely when $n \longrightarrow \infty$, we have

$$\begin{aligned}
< \eta, 1 > |_D &= \lim_{n \to \infty} < \eta_n, 1 > |_D \\
&\leq \left(1 - \frac{1}{2C}\right) \Delta t \, c_2 c_b S^2 \left[R - \frac{c_1 c_a}{c_2 c_b} \frac{C}{2C-1} \, l^{-2} S\right]
\end{aligned}$$

which complete the proof.

4 Entropy consistency of LTS Godunov scheme

In this section, we will prove a theorem on the entropy consistency of the LTS Godunov scheme. It is well known (ref. [4]) that for scalar conservation laws with a convex flux function a weak solution satisfies (1.5) for all convex entropy pairs $\{U(u), F(u)\}$ if it satisfies (1.5) for the special entropy pair

$$U(u) = \frac{u^2}{2}, \quad F(u) = \int^u uf'(u)\,du \tag{4.1}$$

Theorem 4.1. *Assume the initial data* $u_0(x)$ *satisfies* (1.3). *For LTS Godunov scheme, a sufficient condition to ensure entropy consistency for any given Courant number* C *is*

$$ITV(u_0)DTV(u_0) \leq \frac{a^2 c_1}{8c_2^3 C(4C^2-1)(C+1)} \tag{4.2}$$

Proof: By Theorem 2.4, in each linear superposition zone D, there are at most $K+1$ waves, and

$$K \leq \frac{4c_2 C(C+1)}{a} DTV(u_0).$$

If all of them are shocks, by Lemma 3.1, we have $< \eta, 1 > |_D < 0$. Otherwise, there are at most K shocks among the $K+1$ waves, by Lemma 2.1 and Lemma 2.3, there are at least m shocks ($m \geq \frac{K}{2[2C]+2}$) with strength bounded below by $\frac{a}{c_2 C}$ in D. By Lemma 3.5, we have $< \eta, 1 > |_D \leq 0$ if

$$\Delta = \sum_i |R_i| - \frac{c_1}{K^2 c_2} \frac{C}{2C-1} \sum_j |S_j| \leq 0.$$

Since

$$\sum_{R_i \in D} |R_i| \leq ITV(u_0), \quad \sum_{S_j \in D} |S_j| \geq m \frac{a}{c_2 C},$$

when $DTV(u_0) \neq 0$, we have,

$$\begin{aligned}
\Delta &\leq ITV(u_0) - \frac{c_1}{c_2^2} \frac{1}{K^2} \frac{a}{2C-1} m \\
&\leq ITV(u_0) - \frac{ac_1}{c_2^2} \frac{1}{K} \frac{1}{2C-1} \frac{1}{2([2C]+1)} \\
&\leq ITV(u_0) - \frac{a^2 c_1}{8c_2^3} \frac{1}{C(4C^2-1)(C+1)} \frac{1}{DTV(u_0)} \\
&\leq 0
\end{aligned}$$

in which we use (4.2) in the final inequality. If $DTV(u_0) = 0$, the initial data is monotone increasing, there are only rarefaction waves in the solution and there is no interaction, so the entropy consistency is guaranteed.

Finally, let's consider the relation between the inequality

$$< \eta, 1 > |_D \leq 0 \tag{4.3}$$

and the cell entropy inequality

$$\begin{aligned} &\int_{x_i}^{x_{i+1}} U(u(x, t_{j+1} + 0)dx - \\ &- \int_{x_i}^{x_{i+1}} U(u(x, t_j + 0)dx + \int_{t_i}^{t_{i+1}} (F(u(x_{i+1}, t) - F(u(x_i, t))\, dt \leq 0 \end{aligned}$$

where $\{U, F\}$ is the entropy pair (4.1).

On any cell $D_{ij} = (x_i, x_{i+1}) \times (t_j, t_{j+1})$, by the generalized Gauss-Green formula for BV functions [5],

$$\begin{aligned} &< \eta, 1 > |_{D_{ij}} \\ &= \int_{x_i}^{x_{i+1}} U(u(x, t_{j+1} - 0)dx - \int_{x_i}^{x_{i+1}} U(u(x, t_j + 0)dx \\ &\quad + \int_{t_i}^{t_{i+1}} (F(u(x_{i+1}, t) - F(u(x_i, t))\, dt, \end{aligned}$$

so

$$\begin{aligned} &\int_{x_i}^{x_{i+1}} U(u(x, t_{j+1} + 0)dx - \\ &- \int_{x_i}^{x_{i+1}} U(u(x, t_j + 0)dx + \int_{t_i}^{t_{i+1}} (F(u(x_{i+1}, t) - F(u(x_i, t))\, dt \\ &=< \eta, 1 > |_{D_{ij}} + \left[\int_{x_i}^{x_{i+1}} U(u(x, t_{j+1} + 0)dx - \int_{x_i}^{x_{i+1}} U(u(x, t_{j+1} - 0)dx \right]. \end{aligned}$$

By the construction of Godunov scheme, from Jensen's inequality, the second term in the right hand side is non-positive. So if $< \eta, 1 > |_{D_{ij}} \leq 0$ we can get the ordinary cell entropy inequality. But in the LTS Godunov scheme with Courant number $C > 1$, if D is a linear superposition zone, or there is no interaction happening in D, (4.3) is valid. From theorem 2.4, the number of waves in a linear superposition zone is finite and independent of the mesh size h, so we can add the neighboring cells, in which there are no interactions of waves, to one or several linear superposition zones, and form a rectangular zone D_i,

$$D_i = \{(x, t) | x_i \leq x < x_{i+k_i}, t_j \leq t < t_{j+1}\}.$$

By

$$\int_{t_i}^{t_{i+1}} F(u(x_i - 0, t))dt = \int_{t_i}^{t_{i+1}} F(u(x_i + 0, t))dt$$

we have

$$
\begin{aligned}
&\int_{x_i}^{x_{i+k_i}} U(u(x,t_{j+1}+0)dx- \\
&-\int_{x_i}^{x_{i+k_i}} U(u(x,t_j+0)dx+\int_{t_i}^{t_{i+1}} \left(F(u(x_{i+k_i},t)-F(u(x_i,t)\right) dt \\
&=<\eta,1>|_{D_i}+\left[\int_{x_i}^{x_{i+1}} U(u(x,t_{j+1}+0)dx-\int_{x_i}^{x_{i+1}} U(u(x,t_{j+1}-0)dx\right] \\
&+\left[\int_{x_{i+1}}^{x_{i+2}} U(u(x,t_{j+1}+0)dx-\int_{x_{i+1}}^{x_{i+2}} U(u(x,t_{j+1}-0)dx\right] \\
&\cdots\cdots \\
&+\left[\int_{x_{i+k_i-1}}^{x_{i+k_i}} U(u(x,t_{j+1}+0)dx-\int_{x_{i+k_i-1}}^{x_{i+k_i}} U(u(x,t_{j+1}-0)dx\right] \\
&\leq<\eta,1>|_{D_i}\leq 0. \qquad (4.4)
\end{aligned}
$$

This is in fact a cell entropy inequality on D_i. By the technique used in the Lax-Wendroff Theorem[10], this inequality also implies the entropy condition (1.5) for LTS Godunov scheme.

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