Well-posedness of hyperbolic Initial Boundary Value Problems

Jean-François Coulombel
CNRS & Université Lille 1
Laboratoire de mathématiques Paul Painlevé
Cité scientifique
59655 VILLENEUVE D'ASCQ CEDEX, France
e-mail: jfcoulom@math.univ-lille1.fr

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Abstract

Assuming that a hyperbolic initial boundary value problem satsifies an a priori energy estimate with a loss of one tangential derivative, we show a well-posedness result in the sense of Hadamard. The coefficients are assumed to have only finite smoothness in view of applications to nonlinear problems. This shows that the weak Lopatinskii condition is roughly sufficient to ensure well-posedness in appropriate functional spaces.

AMS subject classification: 35L50, 35L40

1 Introduction

In this paper, we consider hyperbolic Initial Boundary Value Problems (IBVPs) in several space dimensions. Such problems typically read:

$$\begin{cases} \partial_t U + \sum_{j=1}^d A_j(t,x)\,\partial_{x_j} U + D(t,x)U = f(t,x)\,, & t \in]0,T[\,, \quad x \in \mathbb{R}^d_+\,, \\ B(t,y)\,U_{|x_d=0} = g(t,y)\,, & t \in]0,T[\,, \quad y \in \mathbb{R}^{d-1}\,, \\ U_{|t=0} = U_0(x)\,, & x \in \mathbb{R}^d_+\,. \end{cases} \tag{1}$$

The space variable x lies in the half-space $\mathbb{R}^d_+ := \{x = (x_1,\dots,x_d) \in \mathbb{R}^d / x_d > 0\}$, $y = (x_1,\dots,x_{d-1})$ denotes a generic point of $\mathbb{R}^{d-1}$, and $t = x_0$ is the time variable. The A_j's and D are square $n \times n$ matrices, while B is a $p \times n$ matrix of maximal rank (the integer p is given below). For simplicity, we shall only deal with noncharacteristic problems, but we point out that the analysis can be reproduced with only minor changes for uniformly characteristic problems (we shall go back to this in our final remarks).

To prove the well-posedness of (1), there are basically four steps (see e.g. [3] for a complete description). One first proves a priori energy estimates for *smooth* solutions. Then one defines a dual problem and shows the existence of *weak* solutions (this works because the original and the dual problems usually share the same stability properties). The third step is to show that *weak* solutions are *strong* solutions and thus satisfy the energy estimate. Eventually, one constructs solutions of the IBVP. The first step of this analysis is linked to the so-called (uniform) Lopatinskii condition (or uniform Kreiss-Lopatinskii condition), see [10]. Namely, the uniform Lopatinskii condition yields an energy estimate in L^2, with no loss of derivative from the source terms (f,g) to the solution U. The second step relies on Hahn-Banach and Riesz theorems (see [3]). One obtains *weak* solutions for which it is not possible to apply the a priori energy estimate. Thus, in the third step, one introduces a tangential mollifier, regularizes the *weak* solution, applies the a priori estimate to the regularized sequence, and passes to the limit. This procedure was introduced in [11]. The fourth step is to take into account the initial datum U_0, and it was first achieved in [18].

In all the above mentionned results, it is crucial that the first step yields an energy estimate **without loss of derivatives**. (We shall say that such problems are *stable* problems). Such an estimate holds either because the boundary conditions are maximally dissipative (or strictly dissipative, which is even better), either because the uniform Lopatinskii condition is satisfied. However, it is known that this stability condition is not met by some physically interesting problems. Examples of situations where the uniform Lopatinskii condition breaks down are provided by elastodynamics (with the well-known Rayleigh waves [22, 20]), shock waves or contact discontinuities in compressible fluid mechanics, see e.g. [12, 16]. For such *nonstable* problems, there is no L^2 estimate, but in some *weakly stable* situations, one can prove a priori energy estimates with a loss of one tangential derivative from the source terms to the solution. Without entering details, these problems are those for which the so-called Lopatinskii determinant vanishes at order 1 in the *hyperbolic* region of the cotangent of the boundary $T^*\mathbb{R}^d_{t,y} \simeq \mathbb{R}^d \times \mathbb{R}^d$. For noncharacteristic problems, such energy estimates with loss of one derivative have been derived by the author in [5], and for uniformly characteristic problems, similar energy estimates have been derived by P. Secchi and the author in [6]. (Note that for the Rayleigh waves problem, the Lopatinskii determinant vanishes in the *elliptic* region of the cotangent of the boundary, and the situation is slightly better, as shown in [20]).

In this paper, we show how to solve the IBVP for such *weakly stable* problems where losses of derivatives occur. More precisely, we show how to construct solutions of (1), with $U_0 = 0$, provided that we have an a priori estimate with a loss of one tangential derivative, both for the initial problem (1) and for a dual problem. The construction of a *weak* solution is quite classical, but still, it requires some attention. Then, we shall regularize our *weak* solution by using a tangential mollifier. Unlike in the case of *stable* problems, where **any** tangential mollifier is suitable, we shall show here that the choice of the mollifier is crucial in our context. Our result is that *weak* solutions are what we shall call *semi-strong* solutions. In the end, we shall prove a well-posedness result (in the sense of Hadamard) for the IBVP (1), when $U_0 = 0$. The case of general initial data is addressed in our final remarks.

The paper is organized as follows. In view of possible applications to nonlinear problems, we have chosen to work with *low regularity* coefficients. Of course, this choice will introduce technical difficulties, and we have found it appropriate to give in section 2 all the notations and results on paradifferential calculus that will be used throughout this paper. In section 3, we state precisely our *weak stability* assumption, and give our main result. In section 4, we prove that (1) admits *weak* solutions, and that these *weak* solutions are *semi-strong* solutions. Up to a few technical details, this ensures well-posedness for zero initial data. In section 5, we give some extensions of our results, and make a few comments.

2 Paradifferential calculus with a parameter

In this section, we collect some definitions and results on paradifferential calculus. We refer to the original works by Bony and Meyer [1, 15] and also to [14, 17] for the introduction of the parameter. The reader will find detailed proofs in these references. We first introduce some norms on the usual Sobolev spaces. For all $\gamma \geq 1$, and for all $s \in \mathbb{R}$, we equip the space $H^s(\mathbb{R}^d)$ with the following norm:

$$\|u\|_{s,\gamma}^2 := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \lambda^{2s,\gamma}(\xi)\, |\widehat{u}(\xi)|^2 \, d\xi \,, \quad \lambda^{s,\gamma}(\xi) := (\gamma^2 + |\xi|^2)^{s/2} \,.$$

We shall write $\|\cdot\|_0$ rather than $\|\cdot\|_{0,\gamma}$ for the (usual) L^2 norm.

The classification of paradifferential symbols (with a parameter) is the following:

Definition 2.1. *A paradifferential symbol of degree $m \in \mathbb{R}$ and regularity k ($k \in \mathbb{N}$) is a function $a(x,\xi,\gamma) : \mathbb{R}^d \times \mathbb{R}^d \times [1,+\infty[\to \mathbb{C}^{q\times q}$ such that a is $\mathcal{C}^\infty$ with respect to ξ and for all $\alpha \in \mathbb{N}^d$, there exists a constant C_α verifying*

$$\forall\,(\xi,\gamma)\,,\quad \|\partial_\xi^\alpha a(\cdot,\xi,\gamma)\|_{W^{k,\infty}(\mathbb{R}^d)} \le C_\alpha\,\lambda^{m-|\alpha|,\gamma}(\xi)\,.$$

The set of paradifferential symbols of degree m and regularity k is denoted by $\Gamma_k^m(\mathbb{R}^d)$. It is equipped with the obvious semi-norms. We denote by $\Sigma_k^m(\mathbb{R}^d)$ the subset of paradifferential symbols $a \in \Gamma_k^m(\mathbb{R}^d)$ such that for a suitable $\varepsilon \in]0,1[$ the partial Fourier transform of a satisfies

$$\forall\,(\xi,\gamma)\,,\quad \mathrm{Supp}\,\mathcal{F}_x\,a(\cdot,\xi,\gamma) \subset \{\zeta \in \mathbb{R}^d / \,|\zeta| \le \varepsilon\,(\gamma^2+|\xi|^2)^{1/2}\}\,.$$

Of course, the symbols in $\Sigma_k^m(\mathbb{R}^d)$ are $\mathcal{C}^\infty$ functions with respect to both variables x and ξ, and for all $a \in \Sigma_k^m(\mathbb{R}^d)$, we have the estimates

$$\forall\,(x,\xi,\gamma)\,,\quad |\partial_x^\beta\partial_\xi^\alpha a\,(x,\xi,\gamma)| \le C_{\alpha,\beta}\,\lambda^{m-|\alpha|+|\beta|,\gamma}(\xi)\,.$$

Thus any symbol $a \in \Sigma_k^m(\mathbb{R}^d)$ belongs to Hörmander's class $S_{1,1}^m$ [9] and defines an operator $\mathrm{Op}^\gamma(a)$ on the Schwartz' class $\mathcal{S}(\mathbb{R}^d)$ by the usual formula

$$\forall\,u \in \mathcal{S}(\mathbb{R}^d)\,,\quad \forall\,x \in \mathbb{R}^d\,,\quad \mathrm{Op}^\gamma(a)u(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\,a(x,\xi,\gamma)\,\widehat{u}(\xi)\,d\xi\,.$$

We shall use the following terminology:

Definition 2.2. *A family of operators $\{P^\gamma\}$ defined for $\gamma \ge 1$ will be said of order $\le m$ ($m \in \mathbb{R}$) if the operators P^γ are uniformly bounded from $H^{s+m}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for all s, independently of γ:*

$$\forall\,\gamma \ge 1\,,\quad \forall\,u \in H^{s+m}(\mathbb{R}^d)\,,\quad \|P^\gamma u\|_{s,\gamma} \le C_s\,\|u\|_{s+m,\gamma}\,.$$

The following Theorem is crucial:

Theorem 2.1. *If $a \in \Sigma_k^m(\mathbb{R}^d)$, $k \in \mathbb{N}$ and $m \in \mathbb{R}$, the family $\{\mathrm{Op}^\gamma(a)\}$ is of order $\le m$. More precisely, for all $s \in \mathbb{R}$, there exists a positive constant C such that*

$$\forall\,\gamma \ge 1\,,\quad \forall\,u \in H^{s+m}(\mathbb{R}^d)\,,\quad \|\mathrm{Op}^\gamma(a)u\|_{s,\gamma} \le C\,\|u\|_{s+m,\gamma}\,.$$

The constant C only depends on s, m, on the confinement parameter $\varepsilon \in]0,1[$, and on a finite number N of semi-norms of a (N only depends on s and m).

The regularization of symbols in the class $\Gamma_k^m(\mathbb{R}^d)$ is achieved by a convolution with admissible cut-off functions:

Definition 2.3. *Let $\psi : \mathbb{R}^d \times \mathbb{R}^d \times [1,+\infty[\to [0,+\infty[$ be a $\mathcal{C}^\infty$ function such that the following estimates hold for all $\alpha,\beta \in \mathbb{N}^d$:*

$$\forall\,(\zeta,\xi,\gamma),\quad |\partial_\zeta^\alpha\partial_\xi^\beta\psi\,(\zeta,\xi,\gamma)| \le C_{\alpha,\beta}\,\lambda^{-|\alpha|-|\beta|,\gamma}(\xi)\,.$$

We shall say that ψ is an admissible cut-off function if there exist real numbers $0 < \varepsilon_1 < \varepsilon_2 < 1$ satisfying

$$\begin{aligned}
\psi(\zeta,\xi,\gamma) = 1 \quad &\text{if} \quad |\zeta| \le \varepsilon_1(\gamma^2+|\xi|^2)^{1/2}\,,\\
\psi(\zeta,\xi,\gamma) = 0 \quad &\text{if} \quad |\zeta| \ge \varepsilon_2(\gamma^2+|\xi|^2)^{1/2}\,.
\end{aligned}$$

An example of cut-off function is the following: we choose a nonnegative $\mathcal{C}^\infty$ function χ_0 on $\mathbb{R}^d \times \mathbb{R}$ such that

$$|\xi_1|^2 + \gamma_1^2 \geq |\xi_2|^2 + \gamma_2^2 \Longrightarrow \chi_0(\xi_1, \gamma_1) \leq \chi_0(\xi_2, \gamma_2)\,,$$
$$\begin{cases} \chi_0(\xi,\gamma) = 1 & \text{if } \left(\gamma^2 + |\xi|^2\right)^{1/2} \leq 1/2, \\ \chi_0(\xi,\gamma) = 0 & \text{if } \left(\gamma^2 + |\xi|^2\right)^{1/2} \geq 1. \end{cases}$$

We define a function $\varphi_0(\xi,\gamma) := \chi_0(\xi/2,\gamma/2) - \chi_0(\xi,\gamma)$. Then the function ψ_0 defined by

$$\psi_0(\zeta,\xi,\gamma) := \sum_{p\geq 0} \chi_0(2^{2-p}\zeta, 0)\, \varphi_0(2^{-p}\xi, 2^{-p}\gamma) \tag{2}$$

is an admissible cut-off function (one can take $\varepsilon_1 = 1/16$ and $\varepsilon_2 = 1/2$).

If ψ is an admissible cut-off function, the inverse Fourier transform K^ψ of $\psi(\cdot,\xi,\gamma)$ satisfies

$$\forall\,(\xi,\gamma)\,, \quad \|\partial_\xi^\alpha K^\psi(\cdot,\xi,\gamma)\|_{L^1(\mathbb{R}^d)} \leq C_\alpha\, \lambda^{-|\alpha|,\gamma}(\xi)\,.$$

These L^1 bounds for the derivatives $\partial_\xi^\alpha K^\psi$ yield the following result:

Proposition 2.1. *Let ψ be an admissible cut-off function. The mapping*

$$a \longmapsto \sigma_a^\psi(x,\xi,\gamma) := \int_{\mathbb{R}^d} K^\psi(x-y,\xi,\gamma)\, a(y,\xi,\gamma)\, dy$$

is continuous from $\Gamma_k^m(\mathbb{R}^d)$ to $\Sigma_k^m(\mathbb{R}^d)$ for all m (the confinement parameter of σ_a^ψ is ε_2).

If $a \in \Gamma_1^m(\mathbb{R}^d)$, then $a - \sigma_a^\psi \in \Gamma_0^{m-1}(\mathbb{R}^d)$. In particular, if ψ_1 and ψ_2 are two admissible cut-off functions and $a \in \Gamma_1^m(\mathbb{R}^d)$, then $\sigma_a^{\psi_1} - \sigma_a^{\psi_2} \in \Sigma_0^{m-1}(\mathbb{R}^d)$.

Fixing an admissible cut-off function ψ, we define the paradifferential operator $T_a^{\psi,\gamma}$ by the formula

$$T_a^{\psi,\gamma} := \mathrm{Op}^\gamma(\sigma_a^\psi)\,.$$

If ψ_1 and ψ_2 are two admissible cut-off functions and $a \in \Gamma_1^m(\mathbb{R}^d)$, then Proposition 2.1 and Theorem 2.1 show that the family $\{T_a^{\psi_1,\gamma} - T_a^{\psi_2,\gamma}\}$ is of order $\leq (m-1)$.

The symbolic calculus is based on the following Theorem:

Theorem 2.2. *Let $a \in \Gamma_1^m(\mathbb{R}^d)$ and $b \in \Gamma_1^{m'}(\mathbb{R}^d)$. Then $ab \in \Gamma_1^{m+m'}(\mathbb{R}^d)$ and the family*

$$\{T_a^{\psi,\gamma} \circ T_b^{\psi,\gamma} - T_{ab}^{\psi,\gamma}\}_{\gamma\geq 1}$$

is of order $\leq m + m' - 1$ for all admissible cut-off function ψ.

Let $a \in \Gamma_1^m(\mathbb{R}^d)$. Then the family

$$\{(T_a^{\psi,\gamma})^* - T_{a^*}^{\psi,\gamma}\}_{\gamma\geq 1}$$

is of order $\leq m - 1$ for all admissible cut-off function ψ.

Let $a \in \Gamma_2^m(\mathbb{R}^d)$ and $b \in \Gamma_2^{m'}(\mathbb{R}^d)$. Then $ab \in \Gamma_2^{m+m'}(\mathbb{R}^d)$ and the family

$$\{T_a^{\psi,\gamma} \circ T_b^{\psi,\gamma} - T_{ab}^{\psi,\gamma} - T_{-i\sum_j \partial_{\xi_j} a \partial_{x_j} b}^{\psi,\gamma}\}_{\gamma\geq 1}$$

is of order $\leq m + m' - 2$ for all admissible cut-off function ψ.

Let $a \in \Gamma_2^m(\mathbb{R}^d)$. Then the family

$$\{(T_a^{\psi,\gamma})^* - T_{a^*}^{\psi,\gamma} - T_{-i\sum_j \partial_{\xi_j}\partial_{x_j} a^*}^{\psi,\gamma}\}_{\gamma\geq 1}$$

is of order $\leq m-2$ for all admissible cut-off function ψ.

An easy consequence of Theorem 2.2 is that, for any symbols $a \in \Gamma_2^m(\mathbb{R}^d)$ and $b \in \Gamma_2^{m'}(\mathbb{R}^d)$ that commute, the remainder

$$T_a^\gamma T_b^\gamma - T_b^\gamma T_a^\gamma - T_{-i\{a,b\}}^\gamma = [T_a^\gamma, T_b^\gamma] - T_{-i\{a,b\}}^\gamma$$

is of order $\leq m+m'-2$. Here above, the notation $\{a,b\}$ stands for the Poisson bracket of a and b:

$$\{a,b\} := \sum_j \partial_{\xi_j} a\, \partial_{x_j} b - \partial_{x_j} a\, \partial_{\xi_j} b\,.$$

We now study the case of paraproducts: they are defined by the particular choice of ψ_0 as cut-off function, where ψ_0 is defined by (2). We shall write T_a^γ instead of $T_a^{\psi_0,\gamma}$ for the associated paradifferential operators. We have the following important result:

Theorem 2.3. *Let $a \in W^{1,\infty}(\mathbb{R}^d)$, $u \in L^2(\mathbb{R}^d)$ and $\gamma \geq 1$. Then we have*

$$\|a\,u - T_a^\gamma u\|_0 \leq \frac{C}{\gamma}\,\|a\|_{W^{1,\infty}(\mathbb{R}^d)}\,\|u\|_0\,,\quad \|a\,\partial_{x_j} u - T_a^\gamma \partial_{x_j} u\|_0 \leq C\,\|a\|_{W^{1,\infty}(\mathbb{R}^d)}\,\|u\|_0\,,$$

$$\|a\,u - T_a^\gamma u\|_{1,\gamma} \leq C\,\|a\|_{W^{1,\infty}(\mathbb{R}^d)}\,\|u\|_0\,,$$

for a suitable constant C that is independent of (a,u,γ).

If in addition $a \in W^{2,\infty}(\mathbb{R}^d)$, we have

$$\|a\,u - T_a^\gamma u\|_{1,\gamma} \leq \frac{C}{\gamma}\,\|a\|_{W^{2,\infty}(\mathbb{R}^d)}\,\|u\|_0\,,$$

$$\|a\,\partial_{x_j} u - T_a^\gamma \partial_{x_j} u\|_{1,\gamma} \leq C\,\|a\|_{W^{2,\infty}(\mathbb{R}^d)}\,\|u\|_0\,,$$

for a suitable constant C that is independent of (a,u,γ).

We can extend the paradifferential calculus to symbols defined on a half-space in the following way: let Ω denote the half-space $\mathbb{R}^d \times]0,+\infty[= \mathbb{R}^{d+1}_+$. The space $L^2_{x_d}(H^s_{t,y})$ is equipped with the norm

$$|\!|\!| u |\!|\!|_{s,\gamma}^2 := \int_0^{+\infty} \|u(\cdot,x_d)\|_{s,\gamma}^2\,dx_d\,.$$

Again, we shall write $|\!|\!| \cdot |\!|\!|_0$ rather than $|\!|\!| \cdot |\!|\!|_{0,\gamma}$ when $s=0$ (that is, for the usual norm in $L^2(\Omega)$). We denote by $\Gamma_k^m(\Omega)$ the set of symbols $a(x_0,\dots,x_d,\xi,\gamma)$ defined on $\Omega\times\mathbb{R}^d\times[1,+\infty[$ such that the mapping $x_d \mapsto a(\cdot,x_d,\cdot)$ is bounded into $\Gamma_k^m(\mathbb{R}^d)$. We define the paradifferential operator T_a^γ by the formula

$$\forall\, u \in \mathcal{C}_0^\infty(\overline{\Omega})\,,\quad \forall\, x_d \geq 0\,,\quad (T_a^\gamma u)(\cdot,x_d) := T_{a(x_d)}^\gamma u(\cdot,x_d)\,.$$

Using Theorem 2.3 and integrating with respect to x_d, we obtain for all symbol $a \in W^{1,\infty}(\Omega)$ and all $u \in L^2(\Omega)$ the estimates:

$$|\!|\!| a\,u - T_a^\gamma u |\!|\!|_0 \leq \frac{C}{\gamma}\,\|a\|_{W^{1,\infty}(\Omega)}\,|\!|\!| u |\!|\!|_0\,,$$

$$|\!|\!| a\,\partial_{x_j} u - T_a^\gamma \partial_{x_j} u |\!|\!|_0 \leq C\,\|a\|_{W^{1,\infty}(\Omega)}\,|\!|\!| u |\!|\!|_0\,,\quad j=0,\dots,d-1\,.$$

When $a \in W^{2,\infty}(\Omega)$, one obtains an estimate with a gain of two tangential derivatives:

$$|\!|\!| a\,u - T_a^\gamma u |\!|\!|_{1,\gamma} \leq \frac{C}{\gamma}\,\|a\|_{W^{2,\infty}(\Omega)}\,|\!|\!| u |\!|\!|_0\,,$$

$$|\!|\!| a\,\partial_{x_j} u - T_a^\gamma \partial_{x_j} u |\!|\!|_{1,\gamma} \leq C\,\|a\|_{W^{2,\infty}(\Omega)}\,|\!|\!| u |\!|\!|_0\,,\quad j=0,\dots,d-1\,.$$

3 Statement of the result

Recall that Ω denotes the half-space $\mathbb{R}^d_{t,y} \times \mathbb{R}^+_{x_d}$. We first make the following assumption on the coefficients of (1):

Assumption 1. *The A_j's are defined on $\overline{\Omega}$ and belong to $W^{2,\infty}(\Omega)$.*

There exists $\delta > 0$ such that for all $(t,x) \in \Omega$ one has

$$|\det A_d(t,x)| \geq \delta \, .$$

The matrix B is defined on $\mathbb{R}^d$ and belongs to $W^{2,\infty}(\mathbb{R}^d)$. It has maximal rank p, where p equals the number of positive eigenvalues of A_d (that is, the number of incoming characteristics).

The system is symmetric hyperbolic, that is, there exists a (real) matrix valued mapping $S \in W^{2,\infty}(\Omega)$ verifying

$$\forall\, (t,x) \in \overline{\Omega} \, , \quad S(t,x) = S(t,x)^T \, , \quad S(t,x) \geq \delta\, I \, , \quad S(t,x)\, A_j(t,x) = A_j(t,x)^T\, S(t,x) \, .$$

We now make our first *weak stability* assumption on system (1):

Assumption 2. *For any $D_1 \in W^{1,\infty}(\Omega)$, and for any symbol $D_2 \in \Gamma^0_1(\Omega)$, there exists a constant C (that depends only on $\delta, \|A_j\|_{W^{2,\infty}(\Omega)}$, $\|D_1\|_{W^{1,\infty}(\Omega)}$, $\|B\|_{W^{2,\infty}(\mathbb{R}^d)}$ and on a finite number of seminorms of the symbol D_2) and there exists a constant $\gamma_0 \geq 1$ such that for all $U \in \mathcal{C}^\infty_0(\overline{\Omega})$ and for all $\gamma \geq \gamma_0$ one has*

$$\gamma\, \|\!|U|\!\|_0^2 + \|U_{|x_d=0}\|_0^2 \leq C \left(\frac{1}{\gamma^3}\, \|\!|f|\!\|_{1,\gamma}^2 + \frac{1}{\gamma^2}\, \|g\|_{1,\gamma}^2 \right) ,$$

$$\textit{where} \quad f := A_d^{-1} \left(\gamma U + \partial_{x_0} U + \sum_{j=1}^d A_j\, \partial_{x_j} U + D_1\, U \right) + T^\gamma_{D_2} U \, , \quad g := B\, U_{|x_d=0} \, .$$

Before stating our last assumption, we make a couple of remarks. In the derivation of energy estimates, one usually replaces the linear operator

$$U \longmapsto A_d^{-1} \left(\gamma U + \partial_{x_0} U + \sum_{j=1}^d A_j\, \partial_{x_j} U + D_1\, U \right) + T^\gamma_{D_2} U$$

by its paradifferential version

$$U \longmapsto T^\gamma_{(\gamma+i\xi_0)A_d^{-1}} U + \sum_{j=1}^{d-1} T^\gamma_{i\xi_j A_d^{-1} A_j} U + \partial_{x_d} U + T^\gamma_{A_d^{-1} D_1} U + T^\gamma_{D_2} U \, ,$$

and treats the errors as source terms[1]. These errors have the following form:

$$\gamma \left(A_d^{-1} U - T^\gamma_{A_d^{-1}} U \right) , \quad \text{or} \quad A_d^{-1} A_j\, \partial_{x_j} U - T^\gamma_{i\xi_j A_d^{-1} A_j} U \, , \quad \text{or} \quad A_d^{-1} D_1\, U - T^\gamma_{A_d^{-1} D_1} U \, .$$

To absorb these errors in an estimate with a loss of one tangential derivative, one needs the regularity stated in assumptions 1 and 2 for the coefficients (see Theorem 2.3 in the preceeding section).

[1]Recall that we use a tangential symbolic calculus for which x_d is seen as a parameter.

The crucial point in assumption 2 is that the energy estimate is **independent** of the lower order term in the interior equation. More precisely, if the energy estimate holds with $D_1 = D_2 = 0$, it is not clear whether it also holds for arbitrary D_1 and D_2. This is a major difference with the *stable* case where there is no loss of derivative (and therefore, one can treat lower order terms as source terms in energy estimates). In the framework of *weakly stable* problems, the lower order terms in the interior equations can not be neglected and one needs to pay special attention. One way to rephrase assumption 2 is the following: energy estimates with loss of one tangential derivative hold, independently of the lower order terms, and independently of their nature (meaning classical, or paradifferential, or a linear combination of the two).

We now turn to our last assumption, that is the analogue of assumption 2 for a dual problem. First recall the following definition:

Definition 3.1. *A dual problem for* (1) *is a linear problem that reads:*

$$\begin{cases} \partial_t V + \sum_{j=1}^d A_j^T \, \partial_{x_j} V + D_\sharp V = f_\sharp(t,x)\,, & t \in]0,T[\,, \quad x \in \mathbb{R}^d_+\,, \\ M_\sharp(t,y)\, V_{|x_d=0} = g_\sharp(t,y)\,, & t \in]0,T[\,, \quad y \in \mathbb{R}^{d-1}\,, \end{cases}$$

where $M_\sharp$ is a $(n-p)\times n$ matrix of maximal rank such that

$$\forall\,(t,y) \in \mathbb{R}^d\,, \quad B_\sharp(t,y)^T\, B(t,y) + M_\sharp(t,y)^T\, M(t,y) = A_d(t,y,0)\,, \tag{3}$$

for suitable $p\times n$ and $(n-p)\times n$ matrices $B_\sharp$ and M, and such that $B_\sharp, M_\sharp, M$ belong to $W^{2,\infty}(\mathbb{R}^d)$.

Our final assumption is that the energy estimate with loss of one tangential derivative is also satisfied by one dual problem[2], when the parameter γ is changed into $-\gamma$:

Assumption 3. *There exists a dual problem (that is, a matrix $M_\sharp$ satisfying* (3)*) such that for any $D_1 \in W^{1,\infty}(\Omega)$, and for any symbol $D_2 \in \Gamma^0_1(\Omega)$, there exists a constant C (that depends only on $\delta, \|A_j\|_{W^{2,\infty}(\Omega)}, \|D_1\|_{W^{1,\infty}(\Omega)}, \|M_\sharp\|_{W^{2,\infty}(\mathbb{R}^d)}$ and on a finite number of seminorms of the symbol D_2) and there exists a constant $\gamma_0 \geq 1$ such that for all $V \in \mathcal{C}_0^\infty(\overline{\Omega})$ and for all $\gamma \geq \gamma_0$ one has*

$$\gamma\, |\!|\!| V |\!|\!|_0^2 + \|V_{|x_d=0}\|_0^2 \leq C \left(\frac{1}{\gamma^3}\, |\!|\!| f_\sharp |\!|\!|_{1,\gamma}^2 + \frac{1}{\gamma^2}\, \|g_\sharp\|_{1,\gamma}^2 \right)\,,$$
$$\text{where} \quad f_\sharp := (A_d^T)^{-1} \left(\gamma V - \partial_{x_0} V - \sum_{j=1}^d A_j^T\, \partial_{x_j} V + D_1\, V \right) + T^\gamma_{D_2} V\,, \quad g_\sharp := M_\sharp\, V_{|x_d=0}\,.$$

In terms of the Lopatinskii condition, assumption 3 means that for one dual problem, the backward Lopatinskii condition degenerates at order 1 in the hyperbolic region of the cotangent of the boundary. In practice, one can usually compute explicit dual boundary conditions for which the Lopatinskii determinant equals that of the original problem (1). Thus the derivation of energy estimates for a dual problem is usually a direct consequence of energy estimates for the original problem.

In all what follows, we always make assumptions 1, 2 and 3. The result is the following:

Theorem 3.1. *Let $D \in W^{1,\infty}(\Omega)$, and let $T > 0$. Then, for all functions $f(t,x)$ and $g(t,y)$ verifying:*

$$f, \partial_t f, \partial_{x_1} f, \cdots, \partial_{x_{d-1}} f \in L^2(\Omega_T)\,, \quad \Omega_T :=]-\infty, T[\times \mathbb{R}^d_+\,,$$
$$g \in H^1(\omega_T)\,, \quad \omega_T :=]-\infty, T[\times \mathbb{R}^{d-1}\,,$$

[2] Note that, with our definition, the dual problem is not uniquely defined.

and such that f and g vanish for $t<0$, there exists a unique $U \in L^2(]-\infty,T[\times\mathbb{R}^d_+)$, whose trace on $\{x_d=0\}$ belongs to $L^2(]-\infty,T[\times\mathbb{R}^{d-1})$, that vanishes for $t<0$, and that is a solution to

$$\begin{cases} \partial_t U + \sum_{j=1}^d A_j(t,x)\,\partial_{x_j} U + D(t,x)U = f(t,x)\,, & t\in]-\infty,T[\,, \quad x \in \mathbb{R}^d_+\,, \\ B(t,y)\,U_{|_{x_d=0}} = g(t,y)\,, & t\in]-\infty,T[\,, \quad y\in\mathbb{R}^{d-1}\,. \end{cases}$$

In addition, $U \in \mathcal{C}([0,T];L^2(\mathbb{R}^d_+))$ and the following estimate holds for all $t\in[0,T]$ and all real number $\gamma \geq \gamma_0$:

$$\begin{aligned} &e^{-2\gamma t}\,\|U(t)\|^2_{L^2(\mathbb{R}^d_+)} + \gamma\,\|e^{-\gamma s}U\|^2_{L^2(\Omega_t)} + \|e^{-\gamma s}U_{|_{x_d=0}}\|^2_{L^2(\omega_t)} \\ &\quad \leq C\left(\frac{1}{\gamma}\|e^{-\gamma s}f\|^2_{L^2(\Omega_t)} + \frac{1}{\gamma^3}\|e^{-\gamma s}\nabla_{t,y}f\|^2_{L^2(\Omega_t)} + \|e^{-\gamma s}g\|^2_{L^2(\omega_t)} + \frac{1}{\gamma^2}\|e^{-\gamma s}\nabla g\|^2_{L^2(\omega_t)}\right)\,. \end{aligned}$$

The constant C and the parameter γ_0 only depend on δ, $\|A_j\|_{W^{2,\infty}(\Omega)}$, $\|D\|_{W^{1,\infty}(\Omega)}$, $\|B\|_{W^{2,\infty}(\mathbb{R}^d)}$ and $\|M_\sharp\|_{W^{2,\infty}(\mathbb{R}^d)}$.

4 Proof of the main result

In this section, we first show existence and uniqueness of solutions for the Boundary Value Problem, with source terms (f,g) in weighted spaces. For $\gamma\geq 1$, we define the spaces $L^2_\gamma(\Omega) := \exp(\gamma t)L^2(\Omega)$, $H^1_\gamma(\Omega) := \exp(\gamma t)H^1(\Omega)$. We also define the spaces

$$\begin{aligned} \mathcal{H}(\Omega) &:= \{v\in L^2(\Omega) \text{ s.t. } \partial_t v, \partial_{x_1}v,\dots,\partial_{x_{d-1}}v \in L^2(\Omega)\} = L^2(\mathbb{R}^+_{x_d};H^1(\mathbb{R}^d_{t,y}))\,, \\ \mathcal{H}_\gamma(\Omega) &:= \exp(\gamma t)\mathcal{H}(\Omega) = \{v\in\mathcal{D}'(\Omega) \text{ s.t. } \exp(-\gamma t)v\in\mathcal{H}(\Omega)\}\,, \\ \mathbb{H}(\Omega) &:= \{v\in\mathcal{H}(\Omega) \text{ s.t. } \partial_t v, \partial_{x_1}v,\dots,\partial_{x_d}v\in\mathcal{H}(\Omega)\}\,. \end{aligned} \tag{4}$$

The spaces $L^2_\gamma(\mathbb{R}^d)$ and $H^1_\gamma(\mathbb{R}^d)$ are defined in a similar way. The space $L^2_\gamma(\Omega)$ is equipped with the obvious norm:

$$\|v\|_{L^2_\gamma(\Omega)} := \|\!|\exp(-\gamma t)v|\!\|_0\,,$$

and the space $\mathcal{H}_\gamma(\Omega)$ is equipped with the norm

$$\|v\|_{\mathcal{H}_\gamma(\Omega)} := \|\!|\tilde{v}|\!\|_{1,\gamma} \quad \text{with } \tilde{v} := \exp(-\gamma t)v\,.$$

Similarly, the space $H^1_\gamma(\mathbb{R}^d)$ is equipped with the norm

$$\|w\|_{H^1_\gamma(\mathbb{R}^d)} := \|\tilde{w}\|_{1,\gamma} \quad \text{with } \tilde{w} := \exp(-\gamma t)w\,.$$

Some elementary, though useful, properties of the spaces $\mathcal{H}(\Omega)$ and $\mathbb{H}(\Omega)$ are collected in appendix A at the end of this paper. In particular, we show that elements of $\mathbb{H}(\Omega)$ admit a trace in $H^{3/2}(\mathbb{R}^d)$ (though they do not necesarily belong to $H^2(\Omega)$, since the definition (4) does not require $\partial^2_{x_d}f\in L^2(\Omega)$).

We consider a zero order coefficient $D\in W^{1,\infty}(\Omega)$, that we fix once and for all, and we wish to prove a well-posedness result for the following Boundary Value Problem:

$$\begin{cases} LU := \partial_t U + \sum_{j=1}^d A_j(t,x)\,\partial_{x_j}U + D(t,x)U = f(t,x)\,, & (t,x)\in\Omega\,, \\ B(t,y)\,U_{|_{x_d=0}} = g(t,y)\,, & (t,y)\in\mathbb{R}^d\,, \end{cases} \tag{5}$$

when the source terms f and g belong to $\mathcal{H}_\gamma(\Omega)$ and $H^1_\gamma(\mathbb{R}^d)$, and γ is large. In view of assumption 2, we expect to obtain a unique solution U in $L^2_\gamma(\Omega)$ whose trace on the boundary $\{x_d=0\}$ belongs to $L^2_\gamma(\mathbb{R}^d)$. In the end, we shall localize this result on a finite time interval.

For later use, we define the norm of the coefficients:

$$\mathbf{N} := \sum_{j=1}^{d} \|A_j\|_{W^{2,\infty}(\Omega)} + \|D\|_{W^{1,\infty}(\Omega)} + \|B\|_{W^{2,\infty}(\mathbb{R}^d)} + \|M_\sharp\|_{W^{2,\infty}(\mathbb{R}^d)}\,, \tag{6}$$

where $M_\sharp$ is given by assumption 3, and represents the dual boundary conditions.

4.1 Preliminary estimates

We first show that the original problem, as well as the dual problem, satisfy an energy estimate in $L^2(H^{-1})$ when the source terms are in L^2 (that is, we can *shift the indices of regularity*). More precisely, we have the following result:

Lemma 4.1. *Let $D_1 \in W^{1,\infty}(\Omega)$, and let $D_2 \in \Gamma_1^0(\Omega)$. There exists a constant C (that depends only on $\delta, \|A_j\|_{W^{2,\infty}(\Omega)}$, $\|D_1\|_{W^{1,\infty}(\Omega)}$, $\|B\|_{W^{2,\infty}(\mathbb{R}^d)}$ and on a finite number of seminorms of the symbol D_2) and there exists a constant $\gamma_1 \geq 1$ such that for all $U \in \mathcal{C}_0^\infty(\overline{\Omega})$ and for all $\gamma \geq \gamma_1$ one has*

$$\gamma\,|\!|\!|U|\!|\!|_0^2 + \|U_{|x_d=0}\|_0^2 \leq C\left(\frac{1}{\gamma^3}\,|\!|\!|f_1|\!|\!|_{1,\gamma}^2 + \frac{1}{\gamma^2}\,\|g_1\|_{1,\gamma}^2\right),$$

$$\textit{where}\quad f_1 := T^\gamma_{(\gamma+i\xi_0)A_d^{-1}}U + \sum_{j=1}^{d-1} T^\gamma_{i\xi_j A_d^{-1}A_j}U + \partial_{x_d}U + T^\gamma_{A_d^{-1}D_1+D_2}U\,,\quad g_1 := T^\gamma_B U_{|x_d=0}\,,$$

and one also has

$$\gamma\,|\!|\!|U|\!|\!|_{-1,\gamma}^2 + \|U_{|x_d=0}\|_{-1,\gamma}^2 \leq C\left(\frac{1}{\gamma^3}\,|\!|\!|f_2|\!|\!|_0^2 + \frac{1}{\gamma^2}\,\|g_2\|_0^2\right),$$

$$\textit{where}\ f_2 := A_d^{-1}\left(\gamma U + \partial_{x_0}U + \sum_{j=1}^{d} A_j\,\partial_{x_j}U + D_1\,U\right),\quad g_2 := B\,U_{|x_d=0}\,. \tag{7}$$

Proof. The first inequality is easily proved using the estimates given in Theorem 2.3:

$$\begin{aligned}
\|B\,U_{|x_d=0} - T^\gamma_B U_{|x_d=0}\|_{1,\gamma} &\leq C\,\|B\|_{W^{1,\infty}(\mathbb{R}^d)}\,\|U_{|x_d=0}\|_0\,,\\
|\!|\!|A_d^{-1}(\gamma U + \partial_{x_0}U) - T^\gamma_{(\gamma+i\xi_0)A_d^{-1}}U|\!|\!|_{1,\gamma} &\leq C\,\|A_d^{-1}\|_{W^{2,\infty}(\Omega)}\,|\!|\!|U|\!|\!|_0\,,\\
|\!|\!|A_d^{-1}A_j\,\partial_{x_j}U - T^\gamma_{i\xi_j A_d^{-1}A_j}U|\!|\!|_{1,\gamma} &\leq C\,\|A_d^{-1}A_j\|_{W^{2,\infty}(\Omega)}\,|\!|\!|U|\!|\!|_0\,,
\end{aligned}$$

thanks to assumption 1. Consequently, using assumption 2, the triangle inequality and choosing γ large enough, one can absorb the error terms in the left hand side of the inequality.

We now turn to the second estimate. Let $U \in \mathcal{C}_0^\infty(\overline{\Omega})$, and define

$$\begin{aligned}
\mathbf{f} &:= T^\gamma_{(\gamma+i\xi_0)A_d^{-1}}U + \sum_{j=1}^{d-1} T^\gamma_{i\xi_j A_d^{-1}A_j}U + \partial_{x_d}U + T^\gamma_{A_d^{-1}D_1}U\,,\\
\mathbf{g} &:= T^\gamma_B U_{|x_d=0}\,,\\
W &:= T^\gamma_{(\gamma^2+|\xi|^2)^{-1/2}}U = T^\gamma_{\lambda^{-1,\gamma}}U\,.
\end{aligned}$$

It is clear that we have

$$|\!|\!|W|\!|\!|_0 = |\!|\!|U|\!|\!|_{-1,\gamma}\,,\quad \text{and}\quad \|W_{|x_d=0}\|_0 = \|U_{|x_d=0}\|_{-1,\gamma}\,,$$

and we are thus led to derive an energy estimate of W in L^2. Thanks to assumption 1 and to Theorem 2.2, we compute:

$$
\begin{aligned}
T^\gamma_{(\gamma+i\xi_0)A_d^{-1}} W &+ \sum_{j=1}^{d-1} T^\gamma_{i\xi_j A_d^{-1} A_j} W + \partial_{x_d} W + T^\gamma_{A_d^{-1} D_1} W \\
&= T^\gamma_{\lambda^{-1,\gamma}} \mathbf{f} + T^\gamma_{-i\{(\gamma+i\xi_0)A_d^{-1},\lambda^{-1,\gamma}\}} U + \sum_{j=1}^{d-1} T^\gamma_{\{\xi_j A_d^{-1} A_j,\lambda^{-1,\gamma}\}} U + R^\gamma_{-2} U \\
&= T^\gamma_{\lambda^{-1,\gamma}} \mathbf{f} + T^\gamma_{-i\{(\gamma+i\xi_0)A_d^{-1},\lambda^{-1,\gamma}\}\lambda^{1,\gamma}} W + \sum_{j=1}^{d-1} T^\gamma_{\{\xi_j A_d^{-1} A_j,\lambda^{-1,\gamma}\}\lambda^{1,\gamma}} W + R^\gamma_{-1} W \,,
\end{aligned}
$$

where R^γ_{-2} is a family of order ≤ -2, and R^γ_{-1} is a family of order ≤ -1. In a similar way, we also compute

$$
T^\gamma_B W_{|x_d=0} = T^\gamma_{\lambda^{-1,\gamma}} \mathbf{g} + R^\gamma_{-1} W_{|x_d=0} \,,
$$

where, once again, R^γ_{-1} is a family of order ≤ -1. Now, we apply the first estimate of Lemma 4.1 with the symbol

$$
D_2 := i\{(\gamma+i\xi_0)A_d^{-1},\lambda^{-1,\gamma}\}\lambda^{1,\gamma} - \sum_{j=1}^{d-1} \{\xi_j A_d^{-1} A_j,\lambda^{-1,\gamma}\}\lambda^{1,\gamma} \in \Gamma^0_1(\Omega) \,.
$$

We get

$$
\gamma \, |||W|||_0^2 + \|W_{|x_d=0}\|_0^2 \leq C \left(\frac{1}{\gamma^3} \, |||T^\gamma_{\lambda^{-1,\gamma}} \mathbf{f} + R^\gamma_{-1} W|||_{1,\gamma}^2 + \frac{1}{\gamma^2} \, \|T^\gamma_{\lambda^{-1,\gamma}} \mathbf{g} + R^\gamma_{-1} W_{|x_d=0}\|_{1,\gamma}^2 \right) \,.
$$

Going back to the definition of W, and choosing γ large enough, we have already obtained the $L^2(H^{-1})$ estimate for the paradifferential problem, namely:

$$
\gamma \, |||U|||_{-1,\gamma}^2 + \|U_{|x_d=0}\|_{-1,\gamma}^2 \leq C \left(\frac{1}{\gamma^3} \, |||\mathbf{f}|||_0^2 + \frac{1}{\gamma^2} \, \|\mathbf{g}\|_0^2 \right) \,.
$$

The result now follows from Lemma 4.2 that we give just below. This Lemma enables us to control the distance (in L^2) between $\mathbf{f}$ and f_2, and the distance between $\mathbf{g}$ and g_2 (f_2 and g_2 are defined by (7)). $\square$

Lemma 4.2. *Let $\gamma \geq 1$, $a \in W^{1,\infty}(\mathbb{R}^d)$ and $v \in H^{-1}(\mathbb{R}^d)$. Then one has*

$$
\|(a - T^\gamma_a)\, v\|_0 \leq C \, \|a\|_{W^{1,\infty}(\mathbb{R}^d)} \, \|v\|_{-1,\gamma} \,,
$$

for a suitable constant C that does not depend on γ, a, v.

If, in addition, $a \in W^{2,\infty}(\mathbb{R}^d)$, one has

$$
\begin{aligned}
\|(a - T^\gamma_a)\, v\|_0 &\leq \frac{C}{\gamma} \, \|a\|_{W^{2,\infty}(\mathbb{R}^d)} \, \|v\|_{-1,\gamma} \,, \\
\|(a - T^\gamma_a)\, \partial_{x_j} v\|_0 &\leq C \, \|a\|_{W^{2,\infty}(\mathbb{R}^d)} \, \|v\|_{-1,\gamma} \,.
\end{aligned}
$$

Proof. We prove Lemma 4.2 for v in the Schwartz' class $\mathcal{S}(\mathbb{R}^d)$. The conclusion follows from a density/continuity argument. Decompose v as

$$
v = v_\natural + \sum_j \partial_{x_j} v_j \,, \quad \text{with} \quad \|v_\natural\|_0 \leq \gamma \, \|v\|_{-1,\gamma} \,, \quad \|v_j\|_0 \leq \|v\|_{-1,\gamma} \,.
$$

This decomposition holds with

$$\widehat{v_\natural}(\xi) := \frac{\gamma^2}{\gamma^2+|\xi|^2}\,\widehat{v}(\xi)\,, \quad \widehat{v_j}(\xi) := \frac{-i\xi_j}{\gamma^2+|\xi|^2}\,\widehat{v}(\xi)\,.$$

We easily get

$$\|a\,v - T^\gamma_a v\|_0 \le \|(a-T^\gamma_a)v_\natural\|_0 + \sum_j \|(a-T^\gamma_a)\,\partial_{x_j}v_j\|_0 \le C\,\|a\|_{W^{1,\infty}(\mathbb{R}^d)}\,\|v\|_{-1,\gamma}\,,$$

thanks to Theorem 2.3. The second part of Lemma 4.2 is proved in the same way, and we omit the details. □

When dealing with symbols defined on a half-space, one simply integrates the estimates of Lemma 4.2. The result is a gain of one or two tangential derivatives, depending on the regularity of the multiplicator. Then one can end the proof of Lemma 4.1. The details are left to the reader.

Of course, similar a priori estimates hold true for the dual problem (with γ changed into $-\gamma$), since the assumptions and the regularity of the coefficients are exactly the same. We therefore have:

Lemma 4.3. *Let $D_\sharp \in W^{1,\infty}(\Omega)$. There exists a constant C (that depends only on δ, $\|A_j\|_{W^{2,\infty}(\Omega)}$, $\|D_\sharp\|_{W^{1,\infty}(\Omega)}$ and $\|M_\sharp\|_{W^{2,\infty}(\mathbb{R}^d)}$) and there exists a constant $\gamma_1 \ge 1$ such that for all $V \in \mathcal{C}^\infty_0(\overline{\Omega})$ and for all $\gamma \ge \gamma_1$ one has*

$$\gamma\,\|\!|V|\!\|^2_{-1,\gamma} + \|V_{|x_d=0}\|^2_{-1,\gamma} \le C\left(\frac{1}{\gamma^3}\,\|\!|f^\sharp|\!\|^2_0 + \frac{1}{\gamma^2}\,\|g^\sharp\|^2_0\right)\,,$$

$$\textit{where}\quad f^\sharp := (A_d^T)^{-1}\left(\gamma V - \partial_{x_0}V - \sum_{j=1}^{d} A_j^T\,\partial_{x_j}V + D_\sharp\,V\right)\,, \quad g^\sharp := M_\sharp\,V_{|x_d=0}\,.$$

Recall that $M_\sharp$ represents the boundary conditions for the dual problem.

With the help of our $L^2(H^{-1})$ estimate, we are going to construct *weak* solutions of (5).

4.2 Existence of weak solutions

This paragraph is devoted to the proof of the following result:

Proposition 4.1. *There exists $\gamma_2(\mathbf{N},\delta) \ge 1$ such that for $\gamma \ge \gamma_2$, $f \in \mathcal{H}_\gamma(\Omega)$, and $g \in H^1_\gamma(\mathbb{R}^d)$, there exists $U \in L^2_\gamma(\Omega)$ satisfying $U_{|x_d=0} \in H^{-1/2}_\gamma(\mathbb{R}^d)$ and U is a solution to (5) (in the sense of distributions).*

Proof. We first commute (5) with the weight $\exp(-\gamma t)$, and we are led to search a function $\widetilde{U} \in L^2(\Omega)$ that is a solution to

$$\begin{cases} L^\gamma\widetilde{U} := \gamma\widetilde{U} + L\widetilde{U} = \widetilde{f}(t,x)\,, & (t,x)\in\Omega\,, \\ B(t,y)\,\widetilde{U}_{|x_d=0} = \widetilde{g}(t,y)\,, & (t,y)\in\mathbb{R}^d\,, \end{cases} \tag{8}$$

with $(\widetilde{f},\widetilde{g}) := \exp(-\gamma t)(f,g) \in \mathcal{H}(\Omega)\times H^1(\mathbb{R}^d)$. The formal adjoint $(L^\gamma)^*$ of the operator L^γ is defined by

$$(L^\gamma)^*V := \gamma V - \partial_t V - \sum_{j=1}^d A_j^T\,\partial_{x_j}V + \left(D^T - \sum_{j=1}^d \partial_{x_j}A_j^T\right)\,V\,.$$

Using relation (3) and formally integrating by parts, (8) reads

$$\forall V \in \mathcal{C}_0^\infty(\overline{\Omega})\,, \qquad \langle\!\langle \widetilde{U}, (L^\gamma)^* V \rangle\!\rangle_{L^2(\Omega)} = \langle\!\langle \widetilde{f}, V \rangle\!\rangle_{L^2(\Omega)} + \langle \widetilde{g}, B_\sharp V_{|_{x_d=0}} \rangle_{L^2(\mathbb{R}^d)} + \langle M\widetilde{U}_{|_{x_d=0}}, M_\sharp V_{|_{x_d=0}} \rangle_{L^2(\mathbb{R}^d)}\,,$$

where the scalar products are denoted as follows:

$$\langle\!\langle U_1, U_2 \rangle\!\rangle_{L^2(\Omega)} := \int_\Omega U_1(\mathbf{x}) \cdot \overline{U_2(\mathbf{x})}\, d\mathbf{x}\,, \quad \langle U_1, U_2 \rangle_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} U_1(t,y) \cdot \overline{U_2(t,y)}\, dtdy\,.$$

We define a set of appropriate test functions:

$$F := \Big\{ V \in \mathcal{C}_0^\infty(\overline{\Omega}) \text{ s.t. } M_\sharp V_{|_{x_d=0}} = 0 \Big\} \supset \mathcal{C}_0^\infty(\Omega)\,.$$

Thanks to assumption 3 (with $D_2 = 0$ and $D_1 = D^T - \sum \partial_{x_j} A_j^T$), we observe that the operator $(L^\gamma)^*$ is one-to-one in the vector space F. Consequently, we may define a linear form ℓ with the following formula:

$$\forall V \in F\,, \quad \ell[(L^\gamma)^* V] := \langle\!\langle \widetilde{f}, V \rangle\!\rangle_{L^2(\Omega)} + \langle \widetilde{g}, B_\sharp V_{|_{x_d=0}} \rangle_{L^2(\mathbb{R}^d)}\,. \tag{9}$$

The following estimate is now a consequence of Lemma 4.3:

$$\begin{aligned} |\ell[(L^\gamma)^* V]| &\le |\!|\!| \widetilde{f} |\!|\!|_{1,\gamma}\, |\!|\!| V |\!|\!|_{-1,\gamma} + C\, \|\widetilde{g}\|_{1,\gamma}\, \|V_{|_{x_d=0}}\|_{-1,\gamma} \\ &\le C\, \Big(|\!|\!| V |\!|\!|_{-1,\gamma} + \|V_{|_{x_d=0}}\|_{-1,\gamma} \Big) \le \frac{C}{\gamma^{3/2}}\, |\!|\!| (L^\gamma)^* V |\!|\!|_0\,, \end{aligned}$$

provided that γ is large enough, say $\gamma \ge \gamma_2(\mathbf{N}, \delta)$. Now we apply Hahn-Banach theorem and we can thus extend ℓ as a (continuous) linear form over the whole space $L^2(\Omega)$. Thanks to Riesz' theorem, we conclude that there exists a function $\widetilde{U} \in L^2(\Omega)$ verifying:

$$\forall V \in F\,, \quad \ell[(L^\gamma)^* V] = \langle\!\langle \widetilde{U}, (L^\gamma)^* V \rangle\!\rangle_{L^2(\Omega)}\,.$$

In particular, it is clear that $L^\gamma \widetilde{U} = \widetilde{f}$ in the sense of distributions. Using that A_d is invertible, the trace of $\widetilde{U}$ on the boundary $\{x_d = 0\}$ is well-defined and belongs to $H^{-1/2}(\mathbb{R}^d)$, see [3, chapter 7][3]. Moreover, the following Green's formula holds:

$$\forall V \in \mathcal{C}_0^\infty(\overline{\Omega})\,, \quad \langle\!\langle \widetilde{U}, (L^\gamma)^* V \rangle\!\rangle_{L^2(\Omega)} = \langle\!\langle \widetilde{f}, V \rangle\!\rangle_{L^2(\Omega)} + \langle A_d \widetilde{U}_{|_{x_d=0}}, V_{|_{x_d=0}} \rangle_{H^{-1/2}(\mathbb{R}^d), H^{1/2}(\mathbb{R}^d)}\,.$$

Combining with the definition of ℓ, see (9), we obtain:

$$\forall V \in F\,, \quad \langle \widetilde{g} - B\widetilde{U}_{|_{x_d=0}}, B_\sharp V_{|_{x_d=0}} \rangle_{H^{-1/2}(\mathbb{R}^d), H^{1/2}(\mathbb{R}^d)} = 0\,.$$

Using (3), we observe that the matrix

$$\begin{pmatrix} B_\sharp \\ M_\sharp \end{pmatrix} \in W^{2,\infty}(\mathbb{R}^d)$$

is invertible. We can therefore conclude that $B\widetilde{U}_{|_{x_d=0}} = \widetilde{g}$. This completes the proof of Proposition 4.1. □

[3] The proof in [3] is done with $\mathcal{C}^\infty$ bounded coefficients, but it extends to Lipschitzean coefficients by using paradifferential techniques to estimate commutators, see e.g. [4] and appendix B for such estimates.

We have constructed a solution $\widetilde{U} \in L^2(\Omega)$ of (8), whose trace belongs to $H^{-1/2}(\mathbb{R}^d)$ (so that the boundary conditions have a clear meaning). We point out that the $L^2(H^{-1})$ estimate given by Lemma 4.3 is crucial in order to obtain $\widetilde{U} \in L^2(\Omega)$. If we had only used the L^2 estimate given by assumption 3, we would have obtained a solution $\widetilde{U} \in L^2(H^{-1})$.

In view of assumption 2, we expect the function $\widetilde{U}$ to admit a trace in L^2 and to satisfy an appropriate energy estimate, namely:

$$\gamma \,\|\!|\widetilde{U}|\!\|_0^2 + \|\widetilde{U}_{|x_d=0}\|_0^2 \leq C \left(\frac{1}{\gamma^3} \,\|\!|\widetilde{f}|\!\|_{1,\gamma}^2 + \frac{1}{\gamma^2} \,\|\widetilde{g}\|_{1,\gamma}^2 \right) .$$

In the next paragraph, we show that this property holds. In particular, there exists a unique solution of (5) in $L^2_\gamma(\Omega)$, and its trace belongs to $L^2_\gamma(\mathbb{R}^d)$ when γ is large. Of course, such an existence-uniqueness result will hold independently of the zero order term D.

Before showing this result, we first state an analogue of Proposition 4.1 when the source terms are in L^2_γ. As a matter of fact, we want to solve the BVP for source terms with tangential derivatives in L^2_γ, but in the analysis, we shall see that we also need to solve BVPs with source terms that are only in L^2_γ.

Proposition 4.2. *There exists $\gamma_2(\mathbf{N},\delta) \geq 1$ such that for $\gamma \geq \gamma_2$, $f \in L^2_\gamma(\Omega)$, and $g \in L^2_\gamma(\mathbb{R}^d)$, there exists $U \in L^2(\mathbb{R}^+;H^{-1}_\gamma(\mathbb{R}^d))$ satisfying $U_{|x_d=0} \in H^{-3/2}_\gamma(\mathbb{R}^d)$ and U is a solution to* (5) *(in the sense of distributions).*

Proof. Most of the proof is similar to the proof of Proposition 4.1. Keeping the same notations for the linear form ℓ, and for the vector space F, and using assumption 2, we easily obtain the existence of $\widetilde{U} \in L^2(\mathbb{R}^+;H^{-1}(\mathbb{R}^d))$ such that

$$\forall\, V \in F\,, \quad \ell[(L^\gamma)^*V] = \langle\!\langle \widetilde{U},(L^\gamma)^*V \rangle\!\rangle_{L^2(H^{-1}),L^2(H^1)} \,.$$

In particular, one has $L^\gamma \widetilde{U} = \widetilde{f}$ in the sense of distributions. The problem is now to give a meaning to the boundary conditions. This is solved by a trace lemma, which we state in appendix C at the end of this paper. Using this result, we can conclude that the trace of $\widetilde{U}$ on $\{x_d = 0\}$ is well-defined and belongs to $H^{-3/2}(\mathbb{R}^d)$. Moreover, the following Green's formula holds:

$$\forall\, V \in \mathcal{C}_0^\infty(\overline{\Omega})\,, \quad \langle\!\langle \widetilde{U},(L^\gamma)^*V \rangle\!\rangle_{L^2(H^{-1}),L^2(H^1)} = \langle\!\langle \widetilde{f},V \rangle\!\rangle_{L^2(\Omega)} + \langle A_d \widetilde{U}_{|x_d=0}, V_{|x_d=0} \rangle_{H^{-3/2}(\mathbb{R}^d),H^{3/2}(\mathbb{R}^d)} \,.$$

Using a continuity/density argument, the equality holds for all functions $V \in H^2(\Omega)$ such that $M_\sharp V = 0$ on the boundary. As was done in the proof of Proposition 4.1, we obtain:

$$\langle \widetilde{g} - B\widetilde{U}_{|x_d=0}, B_\sharp V_{|x_d=0} \rangle_{H^{-3/2}(\mathbb{R}^d),H^{3/2}(\mathbb{R}^d)} = 0 \,,$$

provided that $V \in H^2(\Omega)$ and $M_\sharp V = 0$ on the boundary. Because $B_\sharp$ and $M_\sharp$ belong to $W^{2,\infty}(\mathbb{R}^d)$, for all function $\mu \in H^{3/2}(\mathbb{R}^d)$, there exists $V \in H^2(\Omega)$ such that $\mu = B_\sharp V_{|x_d=0}$, and $M_\sharp V_{|x_d=0} = 0$. We can therefore conclude that $\widetilde{g} = B\widetilde{U}_{|x_d=0}$. □

4.3 "Weak=semi-strong"

The result is the following:

Theorem 4.1. *Let $(\widetilde{f},\widetilde{g}) \in \mathcal{H}(\Omega) \times H^1(\mathbb{R}^d)$, and let $\widetilde{U} \in L^2(\Omega)$ be a solution to* (8) [4]*, for γ sufficiently large. Then there exist a sequence (U^ν) in $\mathbb{H}(\Omega)$, a bounded sequence (d^ν) in the set of symbols $\Gamma^0_1(\Omega)$, and a bounded sequence (b^ν) in the set of symbols $\Gamma^{-1}_1(\mathbb{R}^d)$, that satisfy the following properties:*

[4] Recall that the trace of $\widetilde{U}$ is automatically in $H^{-1/2}(\mathbb{R}^d)$ and the boundary conditions make sense.

$$U^\nu \longrightarrow \widetilde{U} \text{ in } L^2(\Omega),\ U^\nu_{|x_d=0} \longrightarrow \widetilde{U}_{|x_d=0} \text{ in } H^{-1/2}(\mathbb{R}^d),$$

$$L^\gamma U^\nu + A_d T^\gamma_{d^\nu} U^\nu \longrightarrow \widetilde{f} \text{ in } \mathcal{H}(\Omega),$$

$$B\, U^\nu_{|x_d=0} + T^\gamma_{b^\nu} U^\nu_{|x_d=0} \longrightarrow \widetilde{g} \text{ in } H^1(\mathbb{R}^d).$$

In particular, $\widetilde{U}_{|x_d=0}$ belongs to $L^2(\mathbb{R}^d)$ and the following energy estimate holds:

$$\gamma\, \|\widetilde{U}\|_0^2 + \|\widetilde{U}_{|x_d=0}\|_0^2 \leq C \left(\frac{1}{\gamma^3}\, \|\widetilde{f}\|_{1,\gamma}^2 + \frac{1}{\gamma^2}\, \|\widetilde{g}\|_{1,\gamma}^2 \right). \tag{10}$$

Recall that the space $\mathbb{H}(\Omega)$ is defined by (4)[5].

There is a similar result for solutions of (5) with source terms in L^2_γ. One simply needs to shift the indices (the regularized sequence belongs to $\mathcal{H}_\gamma(\Omega)$ and so on). The a priori estimate in $L^2(H^{-1}_\gamma)$ is the inequality (7) in Lemma 4.1. We omit the proof in this case, and focus on Theorem 4.1.

As detailed in the introduction, we are going to introduce a tangential mollifier in order to regularize $\widetilde{U}$. For all $\varepsilon \in]0,1]$, we define the following symbol ϑ_ε:

$$\forall\, (\xi,\gamma) \in \mathbb{R}^d \times [1,+\infty[\,, \quad \vartheta_\varepsilon(\xi,\gamma) := \frac{1}{\gamma^2 + \varepsilon|\xi|^2}\,,$$

as well as the corresponding Fourier multiplier:

$$\Theta^\gamma_\varepsilon := T^\gamma_{\vartheta_\varepsilon} = (\gamma^2 - \varepsilon \Delta_{t,y})^{-1}\,.$$

With slight abuse of notations, we let $\Theta^\gamma_\varepsilon$ act on functions defined over $\mathbb{R}^d$ and on functions defined over the half-space Ω (where we use symbolic calculus with respect to the tangential coordinates, and the Fourier transform has to be understood as a partial Fourier transform). This mollifier is exactly the one used in [7] (after introducing the parameter γ). As we shall see later on, it has some particularly nice commutation properties with the operator L^γ (these properties are expressed by relation (2.11) in [7]).

Elementary properties of the mollifier $\Theta^\gamma_\varepsilon$ are listed below:

Lemma 4.4. *Let $\gamma \geq 1$, $\varepsilon \in]0,1]$ and $s \in \mathbb{R}$. Then for all $v \in L^2(\mathbb{R}^+; H^s(\mathbb{R}^d))$, one has:*

$$\|\Theta^\gamma_\varepsilon v\|_{s,\gamma} \leq \frac{1}{\gamma^2}\, \|v\|_{s,\gamma}\,, \quad \|\Theta^\gamma_\varepsilon v\|_{s+1,\gamma} \leq \frac{1}{\gamma\sqrt{\varepsilon}}\, \|v\|_{s,\gamma}\,, \quad \|\Theta^\gamma_\varepsilon v\|_{s+2,\gamma} \leq \frac{1}{\varepsilon}\, \|v\|_{s,\gamma}\,.$$

If $v \in L^2(\mathbb{R}^+; H^{s+2}(\mathbb{R}^d))$, one has

$$\|\Theta^\gamma_\varepsilon v - v/\gamma^2\|_{s,\gamma} \leq \frac{\varepsilon}{\gamma^4}\, \|v\|_{s+2,\gamma}\,.$$

In particular, one has $\|\Theta^\gamma_\varepsilon v - v/\gamma^2\|_{s,\gamma} \to 0$ when $\varepsilon \to 0$, for all $v \in L^2(\mathbb{R}^+; H^s(\mathbb{R}^d))$.

The proof of Theorem 4.1 is based on several estimates of commutators. Before starting the proof, we recall a lemma by Friedrichs:

[5]Recall also that the trace on $\{x_d = 0\}$ of any element $v \in \mathbb{H}(\Omega)$ is well-defined and belongs to $H^{3/2}(\mathbb{R}^d)$. In particular, it belongs to $H^1(\mathbb{R}^d)$ and the third point of the Theorem makes sense, see Theorem A.1 in appendix A.

Lemma 4.5 (Friedrichs). *Let $a \in W^{1,\infty}(\Omega)$. There exists a constant C that depends only on $\|a\|_{W^{1,\infty}(\Omega)}$ such that, for all $\gamma \geq 1$, for all $\varepsilon \in]0,1]$, and for all $v \in L^2(\Omega)$, one has*

$$\|\!|[a, \Theta_\varepsilon^\gamma]\, v\|\!|_{1,\gamma} \leq C\, \|\!|v\|\!|_0 \,.$$

Furthermore, one has $\|\!|[a, \Theta_\varepsilon^\gamma]\, v\|\!|_{1,\gamma} \to 0$ when $\varepsilon \to 0$, for all $v \in L^2(\Omega)$.

Let $a \in W^{2,\infty}(\Omega)$ and $j \in \{0, \dots, d-1\}$. There exists a constant C that depends only on $\|a\|_{W^{2,\infty}(\Omega)}$ such that, for all $\gamma \geq 1$, for all $\varepsilon \in]0,1]$, and for all $v \in L^2(\Omega)$, one has

$$\|\!|[a\partial_{x_j} - T_a^\gamma \partial_{x_j}, \Theta_\varepsilon^\gamma]\, v\|\!|_{1,\gamma} \leq C\, \|\!|v\|\!|_0 \,.$$

Furthermore, one has $\|\!|[a\partial_{x_j} - T_a^\gamma \partial_{x_j}, \Theta_\varepsilon^\gamma]\, v\|\!|_{1,\gamma} \to 0$ when $\varepsilon \to 0$, for all $v \in L^2(\Omega)$.

We postpone the proof of Lemma 4.5 to appendix B (the first part is well-known), and we now give the proof of Theorem 4.1.

Proof. Define

$$U^\varepsilon := \Theta_\varepsilon^\gamma \widetilde{U} \in L^2(\mathbb{R}^+_{x_d}; H^2(\mathbb{R}^d_{t,y}))\,.$$

A direct computation yields

$$A_d^{-1} L^\gamma U^\varepsilon = \Theta_\varepsilon^\gamma (A_d^{-1} \widetilde{f}) + \gamma\, [A_d^{-1}, \Theta_\varepsilon^\gamma]\, \widetilde{U} + [A_d^{-1} D, \Theta_\varepsilon^\gamma]\, \widetilde{U} + \sum_{j=0}^{d-1} [A_d^{-1} A_j \partial_{x_j}, \Theta_\varepsilon^\gamma]\, \widetilde{U}\,,$$

where L^γ is defined by (8), and where we use the convention $A_0 = Id$. Thanks to Lemma 4.4 and to Lemma 4.5, we already have

$$\Theta_\varepsilon^\gamma (A_d^{-1} \widetilde{f}) + \gamma\, [A_d^{-1}, \Theta_\varepsilon^\gamma]\, \widetilde{U} + [A_d^{-1} D, \Theta_\varepsilon^\gamma]\, \widetilde{U} = \frac{1}{\gamma^2}\, A_d^{-1} \widetilde{f} + r_\varepsilon\,, \quad \|\!|r_\varepsilon\|\!|_{1,\gamma} \to 0\,.$$

This is because $A_d^{-1} \in W^{2,\infty}(\Omega)$, $D \in W^{1,\infty}(\Omega)$, and $A_d^{-1}\widetilde{f} \in L^2(\mathbb{R}^+; H^1(\mathbb{R}^d))$. Using the decomposition

$$[A_d^{-1} A_j \partial_{x_j}, \Theta_\varepsilon^\gamma]\, \widetilde{U} = [A_d^{-1} A_j \partial_{x_j} - T^\gamma_{iA_d^{-1} A_j \xi_j}, \Theta_\varepsilon^\gamma]\, \widetilde{U} + [T^\gamma_{iA_d^{-1} A_j \xi_j}, \Theta_\varepsilon^\gamma]\, \widetilde{U}\,,$$

and using Lemma 4.5 (recall that $A_d^{-1} A_j \in W^{2,\infty}(\Omega)$), we obtain

$$A_d^{-1} L^\gamma U^\varepsilon = \frac{1}{\gamma^2}\, A_d^{-1} \widetilde{f} + r_\varepsilon + \sum_{j=0}^{d-1} [T^\gamma_{iA_d^{-1} A_j \xi_j}, \Theta_\varepsilon^\gamma]\, \widetilde{U}\,, \tag{11}$$

where $\|\!|r_\varepsilon\|\!|_{1,\gamma}$ tends to 0. The remaining commutators are zero order terms in $\widetilde{U}$, uniformly with respect to ε. Therefore, one **cannot** neglect them and treat these commutators as source terms. What saves the day is that these commutators can be decomposed in the following way[6]:

$$[T^\gamma_{iA_d^{-1} A_j \xi_j}, \Theta_\varepsilon^\gamma]\, \widetilde{U} = T^\gamma_{d_{j,\varepsilon}} U^\varepsilon + r_\varepsilon\,,$$

where $d_{j,\varepsilon}$ is a symbol in $\Gamma_1^0(\Omega)$ that is uniformly bounded with respect to ε. Indeed, we use Theorem 2.2 to compute

$$[T^\gamma_{iA_d^{-1} A_j \xi_j}, \Theta_\varepsilon^\gamma]\, \widetilde{U} = T^\gamma_{\{A_d^{-1} A_j \xi_j, \vartheta_\varepsilon\}} \widetilde{U} + R_\varepsilon^\gamma \widetilde{U} = T^\gamma_{d_{j,\varepsilon} \vartheta_\varepsilon} \widetilde{U} + R_\varepsilon^\gamma \widetilde{U} = T^\gamma_{d_{j,\varepsilon}} U^\varepsilon + R_\varepsilon^\gamma \widetilde{U}\,, \tag{12}$$

[6] If ϑ_ε had compact support in ξ, such a decomposition would not hold. The choice of the mollifier ϑ_ε is therefore crucial.

where the symbol $d_{j,\varepsilon}$ is given by

$$d_{j,\varepsilon} = \sum_{k=0}^{d-1} \frac{2\varepsilon\, \xi_j \xi_k}{\gamma^2 + \varepsilon|\xi|^2}\, \partial_{x_k}(A_d^{-1}A_j) \in \Gamma_1^0(\Omega)\,, \tag{13}$$

and where R_ε^γ is of order ≤ -1, uniformly with respect to ε:

$$\|R_\varepsilon^\gamma V\|_{1,\gamma} \leq C\, \|V\|_0\,, \quad \forall V \in L^2(\Omega)\,, \quad \forall \varepsilon \in]0,1]\,.$$

The symbols $d_{j,\varepsilon}$ defined by (13) are uniformly bounded in $\Gamma_1^0(\Omega)$ with respect to ε.

We observe that $\vartheta_\varepsilon = \gamma^{-2} + \varepsilon\sigma_\varepsilon$, with σ_ε bounded in $\Gamma_1^2(\Omega)$. We also observe that $d_{j,\varepsilon} = \varepsilon\alpha_{j,\varepsilon}$, with $\alpha_{j,\varepsilon}$ bounded in $\Gamma_1^2(\Omega)$. Consequently, the remainder R_ε^γ in (12) satisfies

$$\lim_{\varepsilon\to 0} \|R_\varepsilon^\gamma V\|_{1,\gamma} = 0\,, \quad \forall V \in \mathcal{C}_0^\infty(\Omega)\,.$$

Thanks to the uniform bound on R_ε^γ, we may conclude that $\|R_\varepsilon^\gamma \widetilde{U}\|_{1,\gamma}$ tends to zero as ε tends to zero. Using (12) in equation (11), and defining

$$d^\varepsilon := -\sum_{j=0}^{d-1} d_{j,\varepsilon} \in \Gamma_1^0(\Omega)\,,$$

we obtain

$$A_d^{-1} L^\gamma U^\varepsilon + T_{d^\varepsilon}^\gamma U^\varepsilon = \frac{1}{\gamma^2}\, A_d^{-1} \widetilde{f} + r_\varepsilon\,, \quad \|r_\varepsilon\|_{1,\gamma} \to 0\,, \tag{14}$$

and the symbols d^ε are bounded in $\Gamma_1^0(\Omega)$. Note that (14) also reads

$$\partial_{x_d} U^\varepsilon = -A_d^{-1} \left(\gamma U^\varepsilon + \partial_{x_0} U^\varepsilon + \sum_{j=0}^{d-1} A_j\, \partial_{x_j} U^\varepsilon + D U^\varepsilon \right) - T_{d^\varepsilon}^\gamma U^\varepsilon + \frac{1}{\gamma^2}\, A_d^{-1} \widetilde{f} + r_\varepsilon \in \mathcal{H}(\Omega)\,,$$

and we thus have $U^\varepsilon \in \mathbb{H}(\Omega)$, with $\mathbb{H}(\Omega)$ defined by (4).

For the boundary conditions, one proceeds in an entirely similar way, and gets

$$B U^\varepsilon_{|x_d=0} + T_{b^\varepsilon}^\gamma U^\varepsilon_{|x_d=0} = \frac{1}{\gamma^2}\, \widetilde{g} + r_\varepsilon\,, \quad \|r_\varepsilon\|_{1,\gamma} \to 0\,, \tag{15}$$

with symbols b^ε bounded in $\Gamma_1^{-1}(\mathbb{R}^d)$.

Using (14) and (15), the first part of the Theorem is proved, provided that we define (with slight abuse of notations):

$$U^\nu := \gamma^2\, U^{\varepsilon_\nu} \in \mathbb{H}(\Omega)\,, \quad d^\nu := d^{\varepsilon_\nu}\,, \quad b^\nu := b^{\varepsilon_\nu}\,, \quad \varepsilon_\nu := \frac{1}{\nu+1}\,.$$

Now we show that $\widetilde{U}_{|x_d=0} \in L^2(\mathbb{R}^d)$ and that (10) holds. First note that the operators

$$V \longmapsto A_d^{-1} L^\gamma V + T_{d^\nu}^\gamma V \quad \text{and} \quad V \longmapsto B V_{|x_d=0}$$

are continuous from the space $\mathbb{H}(\Omega)$ into $\mathcal{H}(\Omega)$ and from $\mathbb{H}(\Omega)$ into $H^1(\mathbb{R}^d)$ (use Theorem A.1 in appendix A for the boundary operator). Since $\mathcal{C}_0^\infty(\overline{\Omega})$ is dense in $\mathbb{H}(\Omega)$ (see Proposition A.1 in appendix A), it is clear that the a priori energy estimate given by assumption 2 still holds when $U \in \mathbb{H}(\Omega)$ (and not only when $U \in \mathcal{C}_0^\infty(\overline{\Omega})$).

Thanks to assumption 2, and to the boundedness of d^ν in $\Gamma_1^0(\Omega)$, we know that there exists a constant $C = C(\mathbf{N}, \delta)$ and a positive number $\gamma_3(\mathbf{N}, \delta)$ such that

$$\gamma \, \| U^\nu \|_0^2 + \| U^\nu_{|x_d=0} \|_0^2 \le C \left(\frac{1}{\gamma^3} \, \| A_d^{-1} L^\gamma U^\nu + T^\gamma_{d^\nu} U^\nu \|_{1,\gamma}^2 + \frac{1}{\gamma^2} \, \| B U^\nu_{|x_d=0} \|_{1,\gamma}^2 \right) ,$$

for all $\nu \in \mathbb{N}$, and for all $\gamma \ge \gamma_3$. Decomposing

$$B U^\nu_{|x_d=0} = (B U^\nu_{|x_d=0} + T^\gamma_{b^\nu} U^\nu_{|x_d=0}) - T^\gamma_{b^\nu} U^\nu_{|x_d=0} \, ,$$

and using (14)-(15), we get (for γ large enough):

$$\gamma \, \| U^\nu \|_0^2 + \| U^\nu_{|x_d=0} \|_0^2 \le C \left(\frac{1}{\gamma^3} \, \| A_d^{-1} \widetilde{f} + r_\nu^{int} \|_{1,\gamma}^2 + \frac{1}{\gamma^2} \, \| \widetilde{g} + r_\nu^b \|_{1,\gamma}^2 \right) , \tag{16}$$

where

$$\| r_\nu^{int} \|_{1,\gamma} \longrightarrow 0 \, , \quad \| r_\nu^b \|_{1,\gamma} \longrightarrow 0 \, .$$

The sequence $(U^\nu_{|x_d=0})$ is thus bounded in $L^2(\mathbb{R}^d)$, and therefore, up to extracting a subsequence, it converges weakly in $L^2(\mathbb{R}^d)$ toward some function $u^\infty \in L^2(\mathbb{R}^d)$. Since the whole sequence $(U^\nu_{|x_d=0})$ converges toward $\widetilde{U}_{|x_d=0}$ in $H^{-1/2}(\mathbb{R}^d)$, this implies $\widetilde{U}_{|x_d=0} \in L^2(\mathbb{R}^d)$. Moreover, we know that the sequence (U^ν) converges strongly toward $\widetilde{U}$ in $L^2(\Omega)$, and (16) yields

$$\begin{aligned} \gamma \, \| \widetilde{U} \|_0^2 + \| \widetilde{U}_{|x_d=0} \|_0^2 &\le C \, \liminf \left(\frac{1}{\gamma^3} \, \| A_d^{-1} \widetilde{f} + r_\nu^{int} \|_{1,\gamma}^2 + \frac{1}{\gamma^2} \, \| \widetilde{g} + r_\nu^b \|_{1,\gamma}^2 \right) , \\ &\le C \left(\frac{1}{\gamma^3} \, \| \widetilde{f} \|_{1,\gamma}^2 + \frac{1}{\gamma^2} \, \| \widetilde{g} \|_{1,\gamma}^2 \right) . \end{aligned}$$

This completes the proof. □

We summarize Proposition 4.1 and Theorem 4.1 by the following well-posedness result for the Boundary Value Problem (5):

Theorem 4.2. *Let $D \in W^{1,\infty}(\Omega)$. There exists $\gamma_3(\mathbf{N}, \delta)$ such that for $\gamma \ge \gamma_3$, $f \in \mathcal{H}_\gamma(\Omega)$ and $g \in H^1_\gamma(\mathbb{R}^d)$, there exists a unique solution $U \in L^2_\gamma(\Omega)$ of the following system:*

$$\begin{cases} LU = \partial_t U + \sum_{j=1}^d A_j(t,x) \, \partial_{x_j} U + D(t,x) U = f(t,x) \, , & (t,x) \in \Omega \, , \\ B(t,y) \, U_{|x_d=0} = g(t,y) \, , & (t,y) \in \mathbb{R}^d \, , \end{cases}$$

This solution satisfies $U_{|x_d=0} \in L^2_\gamma(\mathbb{R}^d)$ and the following estimate holds:

$$\gamma \, \| U \|_{L^2_\gamma(\Omega)}^2 + \| U_{|x_d=0} \|_{L^2_\gamma(\mathbb{R}^d)}^2 \le C \left(\frac{1}{\gamma^3} \, \| f \|_{\mathcal{H}_\gamma(\Omega)}^2 + \frac{1}{\gamma^2} \, \| g \|_{H^1_\gamma(\mathbb{R}^d)}^2 \right) .$$

In addition, there exists a sequence (U^ν) in $H^1_\gamma(\Omega)$ that satisfies the following properties:

$$\begin{aligned} &U^\nu \longrightarrow U \quad \text{in} \quad L^2_\gamma(\Omega) \, , && U^\nu_{|x_d=0} \longrightarrow U_{|x_d=0} \quad \text{in} \quad L^2_\gamma(\mathbb{R}^d) \, , \\ &LU^\nu \longrightarrow f \quad \text{in} \quad L^2_\gamma(\Omega) \, , && B \, U^\nu_{|x_d=0} \longrightarrow g \quad \text{in} \quad L^2_\gamma(\mathbb{R}^d) \, . \end{aligned}$$

The last part of the Theorem is proved in [17], using Friedrichs' lemma (lemma 4.5). We make the following important comments: even though the source terms f and g have tangential derivatives in L^2_γ, there is no hope to prove, for instance, that LU^ν converges toward f in $\mathcal{H}_\gamma(\Omega)$, see Theorem 4.1. Eventually, the convergence of the traces in L^2_γ can be obtained because we already know (thanks to Theorem 4.1) that the trace of U belongs to $L^2_\gamma(\mathbb{R}^d)$.

When the source terms are only in L^2_γ, there is an analogous result:

Theorem 4.3. *Let $D \in W^{1,\infty}(\Omega)$. There exists $\gamma_3(\mathbf{N},\delta)$ such that for $\gamma \geq \gamma_3$, $f \in L^2_\gamma(\Omega)$ and $g \in L^2_\gamma(\mathbb{R}^d)$, there exists a unique solution $U \in L^2(\mathbb{R}^+_{x_d} : H^{-1}_\gamma(\mathbb{R}^d_{t,y}))$ of the following system:*

$$\begin{cases} \partial_t U + \sum_{j=1}^d A_j(t,x)\,\partial_{x_j} U + D(t,x)U = f(t,x)\,, & (t,x) \in \Omega\,, \\ B(t,y)\,U_{|_{x_d=0}} = g(t,y)\,, & (t,y) \in \mathbb{R}^d\,, \end{cases}$$

This solution satisfies $U_{|_{x_d=0}} \in H^{-1}_\gamma(\mathbb{R}^d)$ and the following estimate holds:

$$\gamma^3\,\|U\|^2_{L^2(\mathbb{R}^+;H^{-1}_\gamma(\mathbb{R}^d))} + \gamma^2\,\|U_{|_{x_d=0}}\|^2_{H^{-1}_\gamma(\mathbb{R}^d)} \leq C\left(\frac{1}{\gamma}\,\|f\|^2_{L^2_\gamma(\Omega)} + \|g\|^2_{L^2_\gamma(\mathbb{R}^d)}\right)\,.$$

One should also keep in mind that the dual problem admits similar well-posedness results.

4.4 Well-posedness with zero initial data. End of the proof

Now, we show that Theorem 4.2 and Theorem 4.3 yield a well-posedness result for the Initial Boundary Value Problem with zero initial data. We first prove that the classical *support lemma* extends to weakly stable problems:

Lemma 4.6. *There exists $\gamma_4(\mathbf{N},\delta)$ such that, if $\gamma \geq \gamma_4$, $(f,g) \in \mathcal{H}_\gamma(\Omega) \times H^1_\gamma(\mathbb{R}^d)$ vanish for $t < T_0$, then the solution $U \in L^2_\gamma(\Omega)$ of (5) vanishes for $t < T_0$. Moreover, if $\gamma \geq \gamma_4$, $f \in L^2_\gamma(\Omega)$, and $g \in L^2_\gamma(\mathbb{R}^d)$, then the solution $U \in L^2(\mathbb{R}^+;H^{-1}_\gamma(\mathbb{R}^d))$ of (5) also vanishes for $t < T_0$.*

Proof. We give the proof when the source terms are in $\mathcal{H}_\gamma(\Omega) \times H^1_\gamma(\mathbb{R}^d)$, but the proof is similar when the source terms are in L^2_γ. There is no loss of generality in assuming $T_0 = 0$. (Otherwise, use a translation $t \mapsto t - T_0$). We fix a function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that χ does not vanish and

$$\chi(t) = \begin{cases} 1\,, & \text{if } x \leq 0\,, \\ \exp(-t)\,, & \text{if } x \geq 1\,. \end{cases}$$

The function $(t,y,x_d) \in \Omega \mapsto \chi'(t)/\chi(t)$ belongs to $W^{1,\infty}(\Omega)$. Consequently, for all γ large enough, the only solution in $L^2_\gamma(\Omega)$ to the linear problem

$$\begin{cases} LV - \dfrac{\chi'(t)}{\chi(t)} V = 0\,, & (t,x) \in \Omega\,, \\ B(t,y)\,V_{|_{x_d=0}} = 0\,, & (t,y) \in \mathbb{R}^d\,, \end{cases} \tag{17}$$

is the trivial solution $V = 0$, thanks to Theorem 4.2 (we use the essential fact that Theorem 4.2 holds for any zero order term in $W^{1,\infty}(\Omega)$).

Consider some data $(f,g) \in \mathcal{H}_\gamma(\Omega) \times H^1_\gamma(\mathbb{R}^d)$ that vanish for $t < 0$. Then we have $(f,g) \in \mathcal{H}_{\gamma+j}(\Omega) \times H^1_{\gamma+j}(\mathbb{R}^d)$ for all integer j. Thanks to Theorem 4.2, we know that there exists a unique $U_j \in L^2_{\gamma+j}(\Omega)$ satisfying

$$\begin{cases} LU_j = f(t,x)\,, & (t,x) \in \Omega\,, \\ B(t,y)\,U_{j|_{x_d=0}} = g(t,y)\,, & (t,y) \in \mathbb{R}^d\,. \end{cases}$$

The function $\chi(U_{j+1} - U_j)$ belongs to $L^2_{\gamma+j}(\Omega)$ and one checks that it is a solution to (17). Therefore it equals zero, and $U_{j+1} = U_j = \cdots = U_0$. Furthermore, we know that

$$\sup_{j\in\mathbb{N}} \frac{1}{\gamma+j}\,\|f\|_{\mathcal{H}_{\gamma+j}(\Omega)} < +\infty \quad \text{and} \quad \sup_{j\in\mathbb{N}} \frac{1}{\gamma+j}\,\|g\|_{H^1_{\gamma+j}(\mathbb{R}^d)} < +\infty\,,$$

because f and g vanish for $t<0$. Thus Theorem 4.2 yields

$$\sup_j \|U_j\|_{L^2_{\gamma+j}(\Omega)} = \sup_j \|U_0\|_{L^2_{\gamma+j}(\Omega)} < +\infty \, .$$

This implies that U_0 vanishes for $t<0$. □

We introduce a few notations: for $T>0$, let $\Omega_T := \Omega \cap \{t<T\} =]-\infty, T[\times \mathbb{R}^d_+$, and let $\omega_T :=]-\infty, T[\times\mathbb{R}^{d-1}$. The spaces $L^2_\gamma(\Omega_T)$, $L^2_\gamma(\omega_T)$, and $\mathcal{H}_\gamma(\Omega_T)$ are defined similarly as $L^2_\gamma(\Omega)$ etc. The definition of the norms in $L^2_\gamma(\Omega_T)$ and $L^2_\gamma(\omega_T)$ is clear. As regards the norm in $\mathcal{H}_\gamma(\Omega_T)$, it is defined by

$$\|f\|^2_{\mathcal{H}_\gamma(\Omega_T)} := \gamma^2 \|f\|^2_{L^2_\gamma(\Omega_T)} + \sum_{j=0}^{d-1} \|\partial_{x_j} f\|^2_{L^2_\gamma(\Omega_T)} \, .$$

The norm of $H^1_\gamma(\omega_T)$ is defined in a similar way. We are now able to end the proof of Theorem 3.1.

Proof. We consider source terms $f \in \mathcal{H}(\Omega_T)$, and $g \in H^1(\omega_T)$, that vanish in the past. We note that f and g belong to $\mathcal{H}_\gamma(\Omega_T)$ and to $H^1_\gamma(\omega_T)$ for all $\gamma \geq 1$.

We extend f and g as functions $f_\flat \in \mathcal{H}(\Omega)$ and $g_\flat \in H^1(\mathbb{R}^d)$. Because $f_\flat$ and $g_\flat$ vanish for $t<0$, it is also straightforward that $f_\flat \in \mathcal{H}_\gamma(\Omega)$ and $g_\flat \in H^1_\gamma(\mathbb{R}^d)$ for all $\gamma \geq 1$. Consequently, for γ large enough, there exists a unique $U_\flat \in L^2_\gamma(\Omega)$ such that

$$\begin{cases} LU_\flat = f_\flat(t,x)\, , & (t,x) \in \Omega\, , \\ B(U_\flat)_{|x_d=0} = g_\flat(t,y)\, , & (t,y) \in \mathbb{R}^d\, , \end{cases}$$

and $U_\flat$ satisfies the corresponding energy estimate, see Theorem 4.2. Furthermore, $U_\flat$ vanishes in the past, thanks to Lemma 4.6. We also have

$$\begin{aligned} \gamma\,\|U_\flat\|^2_{L^2_\gamma(\Omega_T)} + \|(U_\flat)_{|x_d=0}\|^2_{L^2_\gamma(\omega_T)} &\leq \gamma\,\|U_\flat\|^2_{L^2_\gamma(\Omega)} + \|(U_\flat)_{|x_d=0}\|^2_{L^2_\gamma(\mathbb{R}^d)} \\ &\leq C\left(\frac{1}{\gamma^3}\|f_\flat\|^2_{\mathcal{H}_\gamma(\Omega)} + \frac{1}{\gamma^2}\|g_\flat\|^2_{H^1_\gamma(\mathbb{R}^d)}\right) \leq C'\left(\frac{1}{\gamma^3}\|f\|^2_{\mathcal{H}_\gamma(\Omega_T)} + \frac{1}{\gamma^2}\|g\|^2_{H^1_\gamma(\omega_T)}\right) . \end{aligned}$$

The restriction U of $U_\flat$ to Ω_T belongs to $L^2(\Omega_T)$ because $U_\flat$ vanishes in the past and $U_\flat \in L^2_\gamma(\Omega_T)$ when γ is large. We have thus constructed a solution in $L^2(\Omega_T)$ to the localized problem:

$$\begin{cases} LU = f(t,x)\, , & (t,x) \in \Omega_T\, , \\ B(t,y)\, U_{|x_d=0} = g(t,y)\, , & (t,y) \in \omega_T\, . \end{cases} \tag{18}$$

We now show uniqueness of such a solution. Let $U \in L^2(\Omega_T)$ have a trace in $L^2(\omega_T)$, vanish in the past, and satisfy

$$\begin{cases} LU = 0\, , & (t,x) \in \Omega_T\, , \\ BU_{|x_d=0} = 0\, , & (t,y) \in \omega_T\, . \end{cases}$$

Let $\varepsilon > 0$, and consider a function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\chi(t)=1$ if $t \leq T-2\varepsilon$, and $\chi(t)=0$ if $t \geq T-\varepsilon$. Define $U_\chi := \chi U$. Then $U_\chi \in L^2(\Omega)$ and U_χ vanishes in the past, so $U_\chi \in L^2_\gamma(\Omega)$ for all $\gamma \geq 1$. Moreover, we compute:

$$\begin{cases} LU_\chi = \chi'(t) U\, , & (t,x) \in \Omega\, , \\ BU_{\chi|x_d=0} = 0\, , & (t,y) \in \mathbb{R}^d\, . \end{cases}$$

Observe that $\chi' U \in L^2_\gamma(\Omega)$ for all $\gamma \geq 1$, and $\chi' U$ vanishes for $t < T - 2\varepsilon$. We can thus apply Lemma 4.6 (with source terms in $L^2_\gamma)^7$, and conclude that U_χ vanishes for $t < T - 2\varepsilon$. Consequently, U vanishes for $t < T$, that is, $U = 0$.

To end the proof of Theorem 3.1, we show the continuity with respect to the time variable. We consider source terms $f \in \mathcal{H}(\Omega_T)$ and $g \in H^1(\omega_T)$ that vanish in the past, and we continue these functions as $f_\flat \in \mathcal{H}(\Omega)$ and $g_\flat \in H^1(\mathbb{R}^d)$. We already know that the unique solution $U \in L^2(\Omega_T)$ of (18) that vanishes in the past, is the restriction to Ω_T of the solution $U_\flat \in L^2_\gamma(\Omega)$ to the global problem

$$\begin{cases} LU_\flat = f_\flat \,, & (t,x) \in \Omega \,, \\ BU_{\flat|_{x_d=0}} = g_\flat \,, & (t,y) \in \mathbb{R}^d \,, \end{cases}$$

Moreover, the trace of $U_\flat$ on $\{x_d = 0\}$ belongs to $L^2(\mathbb{R}^d)$, and vanishes in the past. To prove Theorem 3.1, it is sufficient to show the continuity of $U_\flat$ with respect to the time variable. Thanks to Theorem 4.2, we know that there exists a sequence (U^ν) in $H^1_\gamma(\Omega)$ verifying

$$\begin{aligned} &U^\nu \longrightarrow U_\flat \quad \text{in} \quad L^2_\gamma(\Omega) \,, && U^\nu_{|_{x_d=0}} \longrightarrow U_{\flat|_{x_d=0}} \quad \text{in} \quad L^2_\gamma(\mathbb{R}^d) \,, \\ &LU^\nu \longrightarrow LU_\flat = f_\flat \quad \text{in} \quad L^2_\gamma(\Omega) \,, && B\,U^\nu_{|_{x_d=0}} \longrightarrow g_\flat \quad \text{in} \quad L^2_\gamma(\mathbb{R}^d) \,. \end{aligned}$$

Using the Friedrichs symmetrizer S (see assumption 1), the classical energy estimate for symmetric hyperbolic systems in a half-space reads (see e.g. [11]):

$$e^{-2\gamma t} \, \|U^\nu(t)\|^2_{L^2(\mathbb{R}^d_+)} + \gamma \, \|U^\nu\|^2_{L^2_\gamma(\Omega_t)} \leq C \left(\frac{1}{\gamma} \|LU^\nu\|^2_{L^2_\gamma(\Omega_t)} + \|U^\nu{}_{|_{x_d=0}}\|^2_{L^2_\gamma(\omega_t)} \right) ,$$

and we also have

$$\begin{aligned} e^{-2\gamma t} \, &\|U^\nu(t) - U^{\nu'}(t)\|^2_{L^2(\mathbb{R}^d_+)} + \gamma \, \|U^\nu - U^{\nu'}\|^2_{L^2_\gamma(\Omega_t)} \\ &\leq C \left(\frac{1}{\gamma} \|LU^\nu - LU^{\nu'}\|^2_{L^2_\gamma(\Omega_t)} + \|U^\nu{}_{|_{x_d=0}} - U^{\nu'}{}_{|_{x_d=0}}\|^2_{L^2_\gamma(\omega_t)} \right) . \end{aligned}$$

Passing to the limit, we obtain the continuity of $U_\flat$ with respect to the time variable. The previous estimates for the trace of U on ω_t yields the estimate stated in Theorem 3.1. □

5 Some remarks

5.1 The IBVP with general initial data

Using Theorem 3.1, one would like to show well-posedness of the IBVP (1) with general initial data. Assume first that the coefficients A_j's and D, as well as the Friedrichs' symmetrizer S, are $\mathcal{C}^\infty$, bounded, and with bounded derivatives. Extend those coefficients to the whole space $\mathbb{R}^{d+1}$, so that the system remains symmetric hyperbolic. Then for all $f \in L^1(]0,T[; H^2(\mathbb{R}^d))$, and for all $U_0 \in H^2(\mathbb{R}^d)$, one can construct a solution $U^{(1)} \in \mathcal{C}^0([0,T]; H^2(\mathbb{R}^d)) \cap \mathcal{C}^1([0,T]; H^1(\mathbb{R}^d))$ to the Cauchy problem

$$\begin{cases} \partial_t U^{(1)} + \sum_{j=1}^d A_j(t,x)\, \partial_{x_j} U^{(1)} + D(t,x) U^{(1)} = f(t,x) \,, & t \in]0,T[\,, \quad x \in \mathbb{R}^d \,, \\ U^{(1)}_{|_{t=0}} = U_0(x) \,, & x \in \mathbb{R}^d \,. \end{cases}$$

For the IBVP (1), one seeks the solution under the form $U = U^{(1)} + U^{(2)}$, with $U^{(2)}$ solution to

$$\begin{cases} \partial_t U^{(2)} + \sum_{j=1}^d A_j(t,x)\, \partial_{x_j} U^{(2)} + D(t,x) U^{(2)} = 0 \,, & t \in]0,T[\,, \quad x \in \mathbb{R}^d_+ \,, \\ B(t,y) U^{(2)}_{|_{x_d=0}} = g - BU^{(1)}_{|_{x_d=0}} \,, & t \in]0,T[\,, \quad y \in \mathbb{R}^{d-1} \,, \\ U^{(2)}_{|_{t=0}} = 0 \,, & x \in \mathbb{R}^d \,. \end{cases}$$

[7] It is crucial here to have a well-posedness result for source terms in L^2_γ.

Consequently, if the source term g belongs to $H^1(]0,T[\times\mathbb{R}^{d-1})$, and if the initial data $U_0 \in H^2(\mathbb{R}^d_+)$ satisfy the compatibility condition

$$g_{|t=0} = B(0,y)(U_0)_{|x_d=0} \,,$$

then one can solve the IBVP (1) with a source term $f \in L^1(]0,T[;H^2(\mathbb{R}^d_+))$, thanks to Theorem 3.1.

However, this strategy hardly applies when the A_j's are in $W^{2,\infty}(\Omega)$ and D is only in $W^{1,\infty}(\Omega)$. The problem is to solve the Cauchy problem with initial data, for instance in $H^2(\mathbb{R}^d)$, and to obtain a solution on $]0,T[\times\mathbb{R}^d$, such that its trace on $]0,T[\times\mathbb{R}^{d-1}$ belongs to H^1. This does not seem possible with a zero order coefficient in $W^{1,\infty}$. We therefore prefer not to pursue this issue, which is a little beyond the scope of this paper. However, for $\mathcal{C}^\infty$ bounded coefficients, and with data that satisfy the above mentioned compatibility condition, the techniques of [18] should yield a well-posedness result for the IBVP (1). The result of [19] even suggests that, under this compatibility condition, the IBVP is well-posed with initial data in $H^1(\mathbb{R}^d_+)$.

When dealing with nonlinear problems, one usually solves the nonlinear equations by a sequence of linearized problems with zero initial data and source terms that vanish in the past, see e.g. [12, 14, 17]. This is another reason why we do not pursue the study of general initial data.

5.2 Uniformly characteristic IBVP

In applications, it often happens that the boundary is characteristic, that is, the determinant of the matrix A_d vanishes on the boundary. In many of these cases (at least in many physically relevant problems), the rank of A_d is constant only on the boundary, and the boundary conditions are maximally dissipative. In such situations, the corresponding IBVP has been studied in great details by many authors, even at the level of quasilinear equations, see e.g. [8, 21] and the references cited therein. When the boundary conditions satisfy the uniform Lopatinskii condition, and when the rank of A_d is constant in a neighborhood of the boundary, the linear IBVP was studied in [13].

When losses of tangential derivatives occur, and when the boundary is uniformly characteristic (this happens for instance in the study of contact discontinuities, see [6]), one can reproduce the analysis developed above. More precisely, assume that there exist two invertible matrices $Q_{1,2}(t,x) \in W^{2,\infty}(\Omega)$, such that

$$\forall\,(t,x) \in \Omega\,, \quad Q_1(t,x)A_d(t,x)Q_2(t,x) = \begin{pmatrix} 0_{n_0} & & \\ & I_{n_+} & \\ & & -I_{n_-} \end{pmatrix}\,,$$

where I_k is the identity matrix in $\mathbb{R}^k$, and $n_0, n_\pm$ are fixed integers. Then the problem

$$\begin{cases} \partial_t U + \sum_{j=1}^d A_j(t,x)\,\partial_{x_j} U + D(t,x)U = f(t,x)\,, & (t,x) \in \Omega_T\,, \\ B(t,y)\,U_{|x_d=0} = g(t,y)\,, & (t,y) \in \omega_T\,, \end{cases}$$

with source terms $f \in \mathcal{H}(\Omega_T)$ and $g \in H^1(\omega_T)$ that vanish in the past, satisfies a well-posedness result that is entirely analogous to Theorem 3.1, provided that the analogues of assumptions 2 and 3 for characteristic problems are satisfied. The only difference is that we can control only the noncharacteristic part of the trace of the solution U on the boundary.

With the help of our analysis, the verification of the well-posedness of the linearized equations for the vortex sheets problem (as studied in [6]) is thus essentially reduced to the calculation of the Lopatinskii determinant for a suitable dual problem.

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A Some properties of anisotropic Sobolev spaces

In $\mathbb{R}^{d+1}$, a generic point is denoted by $x = (x_0, \dots, x_d)$. We use the notation $x = (x', x_d)$ with $x' \in \mathbb{R}^d$ and $x_d \in \mathbb{R}$. We also use the notation $\Omega = \mathbb{R}^{d+1}_+ = \{x \in \mathbb{R}^{d+1} \text{ s.t. } x_d > 0\}$. We define the following spaces

$$\begin{aligned}
\mathcal{H}(\mathbb{R}^{d+1}) &:= \{f \in L^2(\mathbb{R}^{d+1}) \text{ s.t. } \partial_{x_0} f, \dots, \partial_{x_{d-1}} f \in L^2(\mathbb{R}^{d+1})\}\,,\\
\mathbb{H}(\mathbb{R}^{d+1}) &:= \{f \in \mathcal{H}(\mathbb{R}^{d+1}) \text{ s.t. } \partial_{x_0} f, \dots, \partial_{x_d} f \in \mathcal{H}(\mathbb{R}^{d+1})\}\,,\\
\mathcal{H}(\Omega) &:= \{f \in L^2(\Omega) \text{ s.t. } \partial_{x_0} f, \dots, \partial_{x_{d-1}} f \in L^2(\Omega)\}\,,\\
\mathbb{H}(\Omega) &:= \{f \in \mathcal{H}(\Omega) \text{ s.t. } \partial_{x_0} f, \dots, \partial_{x_d} f \in \mathcal{H}(\Omega)\}\,.
\end{aligned}$$

The spaces $\mathcal{H}(\mathbb{R}^{d+1})$ and $\mathbb{H}(\mathbb{R}^{d+1})$ are equipped with the following norms[8]:

$$\begin{aligned}
\|f\|^2_{\mathcal{H}(\mathbb{R}^{d+1})} &:= \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} (1 + |\xi'|^2) |\widehat{f}(\xi)|^2 \, d\xi\,,\\
\|f\|^2_{\mathbb{H}(\mathbb{R}^{d+1})} &:= \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} (1 + |\xi|^2)(1 + |\xi'|^2) |\widehat{f}(\xi)|^2 \, d\xi\,,
\end{aligned}$$

where we have decomposed $\xi = (\xi', \xi_d) \in \mathbb{R}^d \times \mathbb{R}$. The spaces $\mathcal{H}(\Omega)$ and $\mathbb{H}(\Omega)$ are equipped with the norms:

$$\begin{aligned}
\|f\|^2_{\mathcal{H}(\Omega)} &:= \|f\|^2_{L^2(\Omega)} + \|\partial_{x_0} f\|^2_{L^2(\Omega)} + \dots + \|\partial_{x_{d-1}} f\|^2_{L^2(\Omega)}\,,\\
\|f\|^2_{\mathbb{H}(\Omega)} &:= \|f\|^2_{\mathcal{H}(\Omega)} + \|\partial_{x_0} f\|^2_{\mathcal{H}(\Omega)} + \dots + \|\partial_{x_d} f\|^2_{\mathcal{H}(\Omega)}\,.
\end{aligned}$$

The following density result is standard, and is proved by a truncation/regularization argument (see [2] for details):

Proposition A.1. *The space $\mathcal{C}_0^\infty(\mathbb{R}^{d+1})$ is dense in both $\mathcal{H}(\mathbb{R}^{d+1})$ and $\mathbb{H}(\mathbb{R}^{d+1})$.*
The space $\mathcal{C}_0^\infty(\overline{\Omega})$ is dense in both $\mathcal{H}(\Omega)$ and $\mathbb{H}(\Omega)$.

The following result is also very classical:

Proposition A.2. *There exist two continuous linear mappings*

$$E : \mathcal{H}(\Omega) \longrightarrow \mathcal{H}(\mathbb{R}^{d+1}) \qquad \text{and} \qquad \mathbb{E} : \mathbb{H}(\Omega) \longrightarrow \mathbb{H}(\mathbb{R}^{d+1})$$

such that for all $u \in \mathcal{H}(\Omega)$ (resp. $u \in \mathbb{H}(\Omega)$), $Eu = u$ (resp. $\mathbb{E}u = u$) almost everywhere in Ω.

Observe that for the extension operator E, it is sufficient to consider a continuation by 0 outside of Ω (this is because there is no "normal" derivative ∂_{x_d} in the definition of $\mathcal{H}(\Omega)$).

We end this short appendix with the following result:

Theorem A.1. *The mapping $\Gamma : u \in \mathcal{C}_0^\infty(\overline{\Omega}) \mapsto u(x', 0) \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ can be continued in a unique way as a continuous linear mapping $\Gamma : \mathbb{H}(\Omega) \to H^{3/2}(\mathbb{R}^d)$.*

[8] Here we take $\gamma = 1$ for the sake of simplicity, but it is clear that introducing the parameter γ in the definition of the norms does not change the results stated below.

Proof. With the help of Proposition A.1 and Proposition A.2, it is sufficient to show that the mapping

$$\begin{aligned} \Gamma \, : \, \mathcal{C}_0^\infty(\mathbb{R}^{d+1}) &\longrightarrow \mathcal{C}_0^\infty(\mathbb{R}^d) \, , \\ u &\longmapsto u(x',0) \, , \end{aligned}$$

satisfies the estimate

$$\|\Gamma u\|_{H^{3/2}(\mathbb{R}^d)} \leq C \, \|u\|_{\mathbb{H}(\mathbb{R}^{d+1})} \, ,$$

for a suitable constant C. The following formula is classical:

$$\widehat{\Gamma u}(\xi') = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{u}(\xi', \xi_d) \, d\xi_d \, .$$

Using Cauchy-Schwarz' inequality, we thus obtain

$$\begin{aligned} |\widehat{\Gamma u}(\xi')| &\leq C \left(\int_{\mathbb{R}} \frac{d\xi_d}{1+|\xi'|^2+\xi_d^2} \right)^{1/2} \left(\int_{\mathbb{R}} (1+|\xi|^2) |\widehat{u}(\xi)|^2 \, d\xi_d \right)^{1/2} \\ &\leq \frac{C}{(1+|\xi'|^2)^{1/4}} \left(\int_{\mathbb{R}} (1+|\xi|^2) |\widehat{u}(\xi)|^2 \, d\xi_d \right)^{1/2} \, . \end{aligned}$$

This bound immediately yields the estimate

$$\int_{\mathbb{R}^d} (1+|\xi'|^2)^{3/2} |\widehat{\Gamma u}(\xi')|^2 \, d\xi' \leq C \int_{\mathbb{R}^d} (1+|\xi'|^2) \int_{\mathbb{R}} (1+|\xi|^2) |\widehat{u}(\xi)|^2 \, d\xi_d \, d\xi' = C \, \|u\|_{\mathbb{H}(\mathbb{R}^{d+1})} \, .$$

The result follows. $\square$

B Estimates for commutators

In this appendix, we give the proof of Lemma 4.5. First, let $a \in W^{1,\infty}(\Omega)$. Decompose the commutator as

$$[a, \Theta_\varepsilon^\gamma] \, v = [a - T_a^\gamma, \Theta_\varepsilon^\gamma] \, v + [T_a^\gamma, \Theta_\varepsilon^\gamma] \, v \, .$$

Using Theorem 2.3 and Lemma 4.4, we have

$$\begin{aligned} \|\!|[a - T_a^\gamma, \Theta_\varepsilon^\gamma] \, v\|\!|_{1,\gamma} &\leq \|\!|(a - T_a^\gamma) \Theta_\varepsilon^\gamma v\|\!|_{1,\gamma} + \|\!|\Theta_\varepsilon^\gamma (a - T_a^\gamma) v\|\!|_{1,\gamma} \\ &\leq C \|a\|_{W^{1,\infty}(\Omega)} \, \|\!|\Theta_\varepsilon^\gamma v\|\!|_0 + \frac{1}{\gamma^2} \, \|\!|(a - T_a^\gamma) v\|\!|_{1,\gamma} \leq \frac{C}{\gamma^2} \|\!|v\|\!|_0 \, . \end{aligned}$$

The symbols ϑ_ε are bounded in $\Gamma_1^0(\Omega)$, hence the commutators $[T_a^\gamma, \Theta_\varepsilon^\gamma]$ are a bounded family of order ≤ -1, that is,

$$\|\!|[T_a^\gamma, \Theta_\varepsilon^\gamma] \, v\|\!|_{1,\gamma} \leq C \, \|\!|v\|\!|_0 \, ,$$

for a constant C that does not depend on ε. The uniform bound in Lemma 4.5 is thus clear. When $v \in \mathcal{C}_0^\infty(\Omega)$, it is clear that

$$a(\Theta_\varepsilon^\gamma v) - \Theta_\varepsilon^\gamma (av) \quad \text{and} \quad \partial_{x_j} \left(a(\Theta_\varepsilon^\gamma v) - \Theta_\varepsilon^\gamma (av) \right)$$

tend toward zero in $L^2(\Omega)$. This yields the convergence toward zero for all functions $v \in L^2(\Omega)$, using the density of $\mathcal{C}_0^\infty(\Omega)$ in $L^2(\Omega)$.

Consider now $a \in W^{2,\infty}(\Omega)$. Then we have

$$\|\!|(a\partial_{x_j} - T_a^\gamma \partial_{x_j}) \Theta_\varepsilon^\gamma v\|\!|_{1,\gamma} \leq C \, \|a\|_{W^{2,\infty}(\Omega)} \, \|\!|\Theta_\varepsilon^\gamma v\|\!|_0 \leq \frac{C}{\gamma^2} \, \|a\|_{W^{2,\infty}(\Omega)} \, \|\!|v\|\!|_0 \, ,$$

and similarly, we have

$$\|\Theta_\varepsilon^\gamma (a\partial_{x_j} - T_a^\gamma \partial_{x_j})v\|_{1,\gamma} \leq \frac{C}{\gamma^2} \, \|a\|_{W^{2,\infty}(\Omega)} \, \|v\|_0 \, .$$

The uniform bound is proved. When $v \in \mathcal{C}_0^\infty(\Omega)$, one shows that

$$[a\partial_{x_j} - T_a^\gamma \partial_{x_j}, \Theta_\varepsilon^\gamma]v \quad \text{and} \quad \partial_{x_k}[a\partial_{x_j} - T_a^\gamma \partial_{x_j}, \Theta_\varepsilon^\gamma]v$$

tend toward zero in $L^2(\Omega)$. The density of $\mathcal{C}_0^\infty(\Omega)$ in $L^2(\Omega)$ ends the proof.

C A trace lemma in $H^{-1}(\Omega)$

Recall the notation

$$Lu = \partial_{x_0} u + \sum_{j=1}^d A_j \, \partial_{x_j} u + D \, u \, ,$$

where $A_j \in W^{2,\infty}(\Omega)$ and $D \in W^{1,\infty}(\Omega)$. Let $\mathcal{E}$ denote the vector space $\{u \in H^{-1}(\Omega) \text{ s.t. } Lu \in H^{-1}(\Omega)\}$. It is equipped with the norm

$$\|u\|_{\mathcal{E}} := \Big(\|u\|_{H^{-1}(\Omega)}^2 + \|Lu\|_{H^{-1}(\Omega)}^2 \Big)^{1/2} \, .$$

The result is the following:

Lemma C.1. *The space $\mathcal{C}_0^\infty(\overline{\Omega})$ is dense in $\mathcal{E}$, and the mapping $\Gamma : u \in \mathcal{C}_0^\infty(\overline{\Omega}) \mapsto u(x',0) \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ can be uniquely continued as a continuous linear mapping $\Gamma : \mathcal{E} \to H^{-3/2}(\mathbb{R}^d)$. Moreover, the following Green's formula holds for all $u \in \mathcal{E}$ and all $v \in \mathcal{C}_0^\infty(\overline{\Omega})$:*

$$\langle\!\langle u, L^* v \rangle\!\rangle_{H^{-1}(\Omega),H^1(\Omega)} = \langle\!\langle Lu, v \rangle\!\rangle_{H^{-1}(\Omega),H^1(\Omega)} + \big\langle A_d u_{|x_d=0}, v_{|x_d=0} \big\rangle_{H^{-3/2}(\mathbb{R}^d),H^{3/2}(\mathbb{R}^d)} \, .$$

Proof. The Green's formula is clear when u is in $\mathcal{C}_0^\infty(\overline{\Omega})$, and it is therefore directly obtained by a continuity/density argument, provided that the first statement of the Lemma holds.

Let $u \in \mathcal{C}_0^\infty(\overline{\Omega})$, and let $\check{u}$ denote the continuation of u by 0 for $x_d < 0$. Then we have

$$A_d^{-1} L(\check{u}) = \widecheck{A_d^{-1}} Lu + \Gamma u \otimes \delta_{x_d=0} \, .$$

We thus have

$$\begin{aligned} \|\Gamma u \otimes \delta_{x_d=0}\|_{H^{-2}(\mathbb{R}^{d+1})} &\leq \|A_d^{-1} L(\check{u})\|_{H^{-2}(\mathbb{R}^{d+1})} + \|\widecheck{A_d^{-1}} Lu\|_{H^{-2}(\mathbb{R}^{d+1})} \\ &\leq C \, \|\check{u}\|_{H^{-1}(\mathbb{R}^{d+1})} + C \, \|A_d^{-1} Lu\|_{H^{-1}(\Omega)} \leq C \, \|u\|_{\mathcal{E}} \, . \end{aligned}$$

We also know that there exists a constant $c > 0$ such that

$$\|\Gamma u \otimes \delta_{x_d=0}\|_{H^{-2}(\mathbb{R}^{d+1})} = c \, \|\Gamma u\|_{H^{-3/2}(\mathbb{R}^d)} \, ,$$

see e.g. [3, chapter 2]. Consequently, it is now sufficient to prove the density of $\mathcal{C}_0^\infty(\overline{\Omega})$ in E and the Lemma will follow. The proof is done, as usual, by truncation and regularization. We refer to [3, chapter 7] for the details. The only difference with [3] is that, here, we use the property

$$\|[L, \varrho^\varepsilon] \, v\|_{H^{-1}(\mathbb{R}^{d+1})} \longrightarrow 0 \, ,$$

for all $v \in H^{-1}(\mathbb{R}^{d+1})$ (ϱ^ε denotes a mollifier with all the usual properties). □

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