Gradient Driven and Singular Flux Blowup of Smooth Solutions to Hyperbolic Conservation Laws

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Abstract

We consider two new classes of examples of sup-norm blowup in finite time for strictly hyperbolic systems of conservation laws. The explosive growth in amplitude is caused either by a gradient catastrophe or by a singularity in the flux function. The examples show that solutions of (uniformly) strictly hyperbolic systems can remain as smooth as the initial data until the time of blowup. Consequently, blowup in amplitude is not necessarily strictly preceded by shock formation.

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1 Introduction

Consider the Cauchy problem for a one-dimensional system of conservation laws of the form

$$W_t + F(W)_x = 0, \qquad x \in \mathbf{R}, \quad t \geq 0, \tag{1.1}$$

where $W(x,t) \in \mathbf{R}^n$. We will assume that the initial data $W_0(x)$ are smooth, which means either of class C^∞ or even analytic. The system will be strictly

hyperbolic for all values of W in some open set $\Omega \subset \mathbf{R}^n$, i.e. the eigenvalues λ_i, $i = 1, \ldots, n$, of the Jacobian DF are real and satisfy

$$\lambda_1(W) < \cdots < \lambda_n(W).$$

Also, each characteristic field will be either genuinely nonlinear or linearly degenerate, see [5].

The aim of this paper is to give two classes of examples of solutions of such systems, with the property that there is a time $t^* \in (0, \infty)$ such that

- the map $x \mapsto W(x, t)$ is as smooth as the initial data W_0 for each time $t < t^*$;

- $\lim_{t \uparrow t^*} \|W(\cdot, t)\|_\infty = +\infty$.

We will refer to this as *smooth L^∞ blowup*.

It has been conjectured that this type of behavior is nongeneric. More precisely, it is believed that finite time blowup in L^∞ is generically *strictly* preceded by blowup of the gradient. While our examples are dependent on certain symmetries in the data, and hence not generic, they do show that shock formation need not occur before the amplitude reaches infinity. Moreover, the large number of choices available suggests that such blowups are not necessarily rare events. Our examples serve as a starting point for addressing the interesting question of typical behavior for "large" solutions.

To motivate the first class of examples we recall recent work of the authors [7, 4, 1, 8] where it has been shown that solutions of (1.1) (with three or more equations) may blow up in finite time. However, in these examples the solutions were always discontinuous, containing either shocks or contact discontinuities in the initial data. The solutions were constructed by letting one of the equations solve a decoupled scalar equation with two approaching shocks forming a wedge in the (x, t)-plane. The solution of this equation then served as a source that forced the remaining two variables to blow up in amplitude. Using a similar technique, but using a centered compression wave instead of two approaching shocks, we will give examples of smooth blowup in L^∞.

This type of example of singular behavior may be called "gradient driven blowup", where it is the well known phenomenon of gradient catastrophe in one of the components that causes the other components to explode. We remark that the solution blows up (albeit at different times) for any x, so there is no possibility of extending the solution beyond the blowup time.

The second class of examples, again for systems of three or more equations, concerns the case where the flux F has a singularity in Ω. We give

2

a simple example of blowup where the solution remains as smooth as the data for all times $t < t^*$. It will be clear from the construction that a large number of similar examples can be built. While not surprising, this class of example highlights the fact that there is no general mechanism that prevents a solution of (1.1) from approaching the boundary of the domain of definition of the flux F.

We also stress the fact that the solutions we consider remain *uniformly* strictly hyperbolic. That is, the range of eigenvalues taken on over the entire solution lie in strictly separated intervals. We include an example which indicates that it may be impossible for this to happen for pairs of equations. On the other hand, there are recent examples of solutions to strictly hyperbolic systems of two (or more) equations that do blow up in L^∞. However, in these cases two of the eigenvalues coalesce as blowup is approached. We reproduce an example of this phenomenon from the recent monograph of Sever [6]. It is noteworthy that this "non-strictly hyperbolic blowup" may occur for systems with a strictly convex entropy; see [6] for a detailed discussion and for further references. Also, it is interesting to observe that this same example appears as a special case of the Born-Infeld system discussed in [2].

Finally, in all of the examples we consider the data are dependent on the particular fluxes F; once F is chosen, the data $W_0(x)$ is carefully selected to produce blowup. In particular, the total variation of the initial data exceeds the critical value determined by the flux, and these examples do not contradict Glimm's theorem [3] on time-global existence of weak solutions when the data have sufficiently small total variation. See [9] for an example of blowup with initial data having arbitrarily small total variation. This again does not violate Glimm's theorem because there is no *connected* neighborhood $\mathcal{U}$ of the range of the initial data throughout which the system is strictly hyperbolic.

The paper is organized as follows. In Section 2 we briefly consider the role of strict hyperbolicity vs. uniform strict hyperbolicity for 2×2 systems. In Section 3 we turn to blowup for 3×3 systems. We first construct a class of examples where a continuous solution blows up. We then extend this to arbitrarily smooth profiles. Finally, in Section 4 we consider the case where the flux has a singularity along a hypersurface in W-space. The main results are recorded in Theorem 1 in Section 3 and in Theorem 2 in Section 4.

2   Preliminary discussion

A basic feature of nonlinear hyperbolic equations is *gradient blowup*. In searching for ways in which a solution of a nonlinear hyperbolic system might become large, it is natural to consider the possibility of gradient blowup driving sup-norm blowup.

   We reproduce an example of this phenomenon for a 2×2 system, taken from the recent monograph [6], and which may also be obtained in a one-dimensional model problem for the Born-Infeld equations [2].

Example 2.1. *Consider the system*

$$\left(\frac{1}{v}\right)_t + \left(\frac{u}{v}\right)_x = 0 \tag{2.1}$$

$$\left(\frac{u}{v}\right)_t + \left(\frac{u^2 - v^2}{v}\right)_x = 0, \tag{2.2}$$

with $v > 0$. Introducing the "Lagrangian" coordinate y by $dy = dx/v - (u/v)dt$, we obtain the linear wave equation

$$v_t - u_y = 0 \tag{2.3}$$

$$u_t - v_y = 0. \tag{2.4}$$

One can clearly give data for u and v such that $v \to 0$ in finite time for the system (2.3)-(2.4), and this corresponds to a finite time blowup of amplitude in (2.1)-(2.2). Notice however that the eigenvalues for (2.1)-(2.2) are $u \pm v$, which coalesce exactly at $v = 0$.

   We have not been able to construct an example of a pair of equations exhibiting blowup of amplitude and where at the same time the system remains (uniformly) strictly hyperbolic. To highlight the issue consider the following example where it is seen how hyperbolicity quantifies the degree of blowup.

Example 2.2. *Consider the triangular 2×2-system,*

$$u_t + uu_x = 0 \tag{2.5}$$

$$v_t + u_x = 0. \tag{2.6}$$

We give initial data of the form

$$u_0(x) = \begin{cases} 1 & \text{for } x < 0, \\ 1 - x & \text{for } 0 \le x \le 1, \\ 0 & \text{for } x > 1, \end{cases}$$

4

and $v_0(x) \equiv 0$. The characteristic speeds are $\lambda_1 \equiv 0$ and $\lambda_2 = u$ so that strict hyperbolicity fails at $u = 0$. For any time $t \in [0, 1)$ we have,

$$u(x,t) = \begin{cases} 1 & \text{for } x < t, \\ \frac{1-x}{1-t} & \text{for } t \leq x \leq 1, \\ 0 & \text{for } x > 1. \end{cases} \tag{2.7}$$

Substituting into (2.6) shows that,

$$v(x,t) = \begin{cases} 0 & \text{for } x < 0, \text{ and for } x > 1 \\ -\ln(1-x) & \text{for } 0 \leq x \leq t, \\ -\ln(1-t) & \text{for } t \leq x \leq 1. \end{cases} \tag{2.8}$$

Thus, $v(\cdot, t)$ blows up in L^∞ at time $t^* = 1$. Observe that if we change the initial data of u to be

$$u_0(x) = \begin{cases} 1 & \text{for } x < 0, \\ 1 - x & \text{for } 0 \leq x \leq 1 - \alpha, \\ \alpha & \text{for } x > 1 - \alpha, \end{cases} \tag{2.9}$$

then the largest value $|v|$ now takes is $-\ln \alpha$. We can regard α in this case as a measure of the (strict) hyperbolicity of the system, and we see that less hyperbolicity allows for larger values in the solution.

The behavior exhibited here seems to be generic for 2×2 systems: that is, it appears that if we assume that gradient steepening of one family drives growth of the other, then shock formation generates blowup only if the eigenvalues coincide.

We proceed by considering systems of three (or more) equations in which case we can ensure uniform strict hyperbolicity along solutions that explode.

3 Gradient driven blowup

A strictly hyperbolic 3×3 system Our goal next is to construct a 3×3 system with a *continuous* solution which blows up in sup-norm. The idea is to replace the shocks of earlier constructions [7, 4, 1, 8] with a single large compression. We start with a Burgers' compression focusing at the point $(0, t^*)$. That is, with $W^{tr} = (u, v, z)$ we let the middle component solve

$$v_t + \left(\frac{v^2}{2}\right)_x = 0, \tag{3.1}$$

with decreasing data $v_0(x)$ which we choose to be antisymmetric about the origin, i.e. $v_0(-x) = -v_0(x)$.

Following [7] we augment equation (3.1) with a system of two conservation laws whose coefficients depend on the solution v. The two new fields will be the ones which blow up and, for simplicity, we will take these to be linearly degenerate. Because we want the system in conservation form, we take the two equations to be

$$\begin{pmatrix} u \\ z \end{pmatrix}_t + \left[A(v) \begin{pmatrix} u \\ z \end{pmatrix} \right]_x = 0, \tag{3.2}$$

where the smooth matrix function

$$A(v) = \begin{pmatrix} a_{11}(v) & a_{12}(v) \\ a_{21}(v) & a_{22}(v) \end{pmatrix}$$

is to be chosen later. As v is explicitly given, (3.2) is a linear 2×2 system with variable coefficients that are as smooth as v. This guarantees that the u- and z-solutions will remain as smooth as v.

The resulting 3×3 system thus has flux

$$F(W) = F \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} a_{11}(v)u + a_{12}(v)w \\ v^2/2 \\ a_{21}(v)u + a_{22}(v)w \end{pmatrix},$$

whose Jacobian has eigenvalues λ_1, λ_3 equal to those of $A(v)$, together with $\lambda_2 = v$.

Our solution will consist of the fixed compression (2-wave) which is focusing at $(0, t^*)$, plus 1- and 3-waves (carrying changes in u and z) which are blowing up. They do this as follows: forward 3-waves interact with 2-waves and reflect 1-waves. These 1-waves in turn interact with 2-waves and reflect 3-waves, *etc*. Because the compression converges at $(0, t^*)$, these interactions happen more and more quickly. We choose the data so that the 1- and 3-wave strengths continually increase, eventually resulting in blowup.

Note that these interactions are taking place inside the focusing compression. It will therefore be simpler to study the solutions by working with the natural rescaled equations.

Rescaled system Since the gradient of v blows up like $(1 - t)^{-1}$ it is natural to apply the coordinate transformation

$$y = \frac{x}{1 - t} \quad \text{and} \quad \tau = -\log(1 - t). \tag{3.3}$$

6

Defining

$$\begin{pmatrix} U(y,\tau) \\ V(y,\tau) \\ Z(y,\tau) \end{pmatrix} = \begin{pmatrix} u(x,t) \\ v(x,t) \\ z(x,t) \end{pmatrix},$$

the 2×2 system (3.2) takes the form

$$\begin{pmatrix} U \\ Z \end{pmatrix}_\tau + y \begin{pmatrix} U \\ Z \end{pmatrix}_y + \left[A(V) \begin{pmatrix} U \\ Z \end{pmatrix} \right]_y = 0, \qquad (3.4)$$

and

$$V_\tau + (y + V)V_y = 0. \qquad (3.5)$$

The idea is now to generate blowup of (u, z) by finding solutions (U, Z) of (3.4) whose sup-norm tend to ∞ as $\tau \to +\infty$. As this system is linear it is natural to seek time-exponential solutions.

3.1 Continuous blowup in a wedge

Consider the case where we give the following continuous data for v,

$$v_0(x) = \begin{cases} V_0 & x \le -V_0, \\ -x & -V_0 \le x \le V_0, \\ -V_0 & V_0 \le x, \end{cases} \qquad (3.6)$$

where $V_0 > 0$ will be specified later. In this case $t^* = 1$ and the solution of (3.1)-(3.6) is given explicitly, for times $t \in [0, 1)$, by

$$v(x,t) = \begin{cases} V_0 & \frac{x}{1-t} \le -V_0, \\ \frac{-x}{1-t} & -V_0 \le \frac{x}{1-t} \le V_0, \\ -V_0 & V_0 \le \frac{x}{1-t}. \end{cases} \qquad (3.7)$$

The transformed system (3.4) takes the following form in the three regions $y \le -V_0$, $-V_0 \le y \le V_0$, and $V_0 \le y$, respectively.

$$\begin{pmatrix} U \\ Z \end{pmatrix}_\tau + (y + A(V_0)) \begin{pmatrix} U \\ Z \end{pmatrix}_y = 0, \qquad \text{for } y \le -V_0, \qquad (3.8)$$

$$\begin{pmatrix} U \\ Z \end{pmatrix}_\tau + y \begin{pmatrix} U \\ Z \end{pmatrix}_y + \left[A(-y) \begin{pmatrix} U \\ Z \end{pmatrix} \right]_y = 0 \qquad \text{for } -V_0 \le y \le V_0, \quad (3.9)$$

$$\begin{pmatrix} U \\ Z \end{pmatrix}_\tau + (y + A(-V_0)) \begin{pmatrix} U \\ Z \end{pmatrix}_y = 0 \qquad \text{for } V_0 \le y. \qquad (3.10)$$

7

Boundary conditions. The solution $\mathcal{U} := (U, Z)^{tr}$ of (3.8)-(3.10) can be decomposed into backward (left-moving) and forward (right-moving) waves. We will keep the problem as simple as possible by insisting that there be no forward waves present in the left region $y \leq -V_0$, and no backward waves present in the right region $V_0 \leq y$. That is, there should be only outgoing waves from the central region $-V_0 \leq y \leq V_0$. Since we insist that our solution be continuous, these constraints provide boundary conditions for (3.9). To formulate these we introduce the following notation. Let $\lambda_-(V) < 0 < \lambda_+(V)$ denote the eigenvalues of $A(V)$ and let $\ell_-(V)$ $(r_-(V))$, $\ell_+(V)$ $(r_+(V))$ denote the corresponding left (right) row (column) eigenvectors. We thus have that

$$A = (r_- \ r_+) \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \begin{pmatrix} \ell_- \\ \ell_+ \end{pmatrix}.$$

We assume the normalizations $|\ell_\pm| = 1$ and $\ell_\pm \cdot r_\pm \equiv 1$, and we decompose the solution $\mathcal{U} = (U, Z)^{tr}$ along the basis of right eigenvectors

$$\mathcal{U} = \mathcal{U}^- r_- + \mathcal{U}^+ r_+,$$

such that

$$\mathcal{U}^\pm = \ell_\pm \cdot \mathcal{U}.$$

The no-wave requirements described above thus take the form

$$\mathcal{U}^+ = \ell_+ \cdot \begin{pmatrix} U \\ Z \end{pmatrix} \equiv 0 \qquad \text{for } y \leq -V_0, \tag{3.11}$$

and

$$\mathcal{U}^- = \ell_- \cdot \begin{pmatrix} U \\ Z \end{pmatrix} \equiv 0 \qquad \text{for } y \geq V_0. \tag{3.12}$$

The appropriate boundary conditions for (3.9) follow from (3.11)-(3.12), and the requirement that the solution should be continuous. That is, for the solution $\mathcal{U}$ of (3.9), we should have the boundary conditions

$$\ell_+(V_0) \cdot \mathcal{U}(-V_0) = 0, \qquad \ell_-(-V_0) \cdot \mathcal{U}(V_0) = 0. \tag{3.13}$$

Explicit solution. Rewrite (3.9) in conservation form as

$$\begin{pmatrix} U \\ Z \end{pmatrix}_\tau + \left[(A(-y) + y) \begin{pmatrix} U \\ Z \end{pmatrix} \right]_y = \begin{pmatrix} U \\ Z \end{pmatrix}, \tag{3.14}$$

8

and substitute the ansatz

$$\begin{pmatrix} U(y,\tau) \\ Z(y,\tau) \end{pmatrix} = e^{\tau} \begin{pmatrix} \hat{u}(y) \\ \hat{z}(y) \end{pmatrix}.$$

This yields the trivial system

$$\left[(A(-y) + y) \begin{pmatrix} \hat{u} \\ \hat{z} \end{pmatrix} \right]_y = 0,$$

with solution

$$\begin{pmatrix} \hat{u} \\ \hat{z} \end{pmatrix} = [A(-y) + y\,I]^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{3.15}$$

The constants α and β need to be chosen so that the boundary conditions (3.13) are satisfied. Because the eigenvectors of $(A + y\,I)^{-1}$ are those of A, we can choose such constants α and β provided that the vectors $\ell_-(-V_0)$ and $\ell_+(V_0)$ are linearly dependent.

We proceed to choose such left eigenvectors for the 2×2 matrix $A(v)$. Since $\ell_-(v)$ must rotate relative to $\ell_+(V_0)$, it is convenient to let $\ell_{\pm}(v)$ rotate as v varies, so that after v changes enough, the projected eigenvectors are linearly dependent. We thus let

$$\ell_-(v) = (-\cos v, \sin v), \qquad \ell_+(v) = (\sin v, \cos v).$$

Again, to simplify, we choose constant eigenvalues $\pm\lambda$ for the matrix $A(v)$. That is, $\lambda_{\pm} = \pm\lambda$, where we must have $\lambda > V_0$ to ensure strict hyperbolicity of the resulting 3×3 system. Thus our choice of the matrix $A(v)$ is

$$A(v) = \begin{pmatrix} \ell_- \\ \ell_+ \end{pmatrix}^{-1} \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \ell_- \\ \ell_+ \end{pmatrix} = \lambda \begin{pmatrix} -\cos 2v & \sin 2v \\ \sin 2v & \cos 2v \end{pmatrix}. \tag{3.16}$$

Choosing $V_0 = \pi/4$, we get

$$\ell_-(-V_0) = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} = -\ell_+(V_0),$$

and we can take $\alpha = -\beta = 1$ in (3.15). An easy calculation now shows that the solution of (3.9) and (3.13) is explicitly given by

$$\begin{pmatrix} U(y,\tau) \\ Z(y,\tau) \end{pmatrix} = \frac{e^{\tau}}{\lambda^2 - y^2} \begin{pmatrix} \lambda[\sin 2y - \cos 2y] - y \\ y - \lambda[\sin 2y + \cos 2y] \end{pmatrix}, \tag{3.17}$$

9

for $-\pi/4 \le y \le \pi/4$. Transforming back to the original coordinates we get that $(u, z)^{tr}$ is given explicitly by

$$
\begin{pmatrix} u(x,t) \\ z(x,t) \end{pmatrix} = e^\tau \begin{pmatrix} U(y,\tau) \\ Z(y,\tau) \end{pmatrix}
$$

$$
= \frac{1}{\lambda^2(1-t)^2 - x^2} \begin{pmatrix} \lambda(1-t)\left[\sin\left(\frac{2x}{1-t}\right) - \cos\left(\frac{2x}{1-t}\right)\right] - x \\ x - \lambda(1-t)\left[\sin\left(\frac{2x}{1-t}\right) + \cos\left(\frac{2x}{1-t}\right)\right] \end{pmatrix}, \quad (3.18)
$$

in the central wedge $|x|/(1-t) \le \pi/4$, $t \in [0,1)$. As a check we compute that

$$
\begin{pmatrix} u(0,t) \\ z(0,t) \end{pmatrix} = \frac{1}{\lambda(1-t)} \begin{pmatrix} 1 \\ 1 \end{pmatrix},
$$

which shows that the solution blows up in L^∞ norm at time $t^* = 1$.

To finish, we describe the solution outside the central wedge in the original variables. On the right of the wedge, we have $v(x,t) \equiv -\pi/4$, and from (3.10), the system becomes

$$
\begin{pmatrix} u \\ z \end{pmatrix}_t - \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} u \\ z \end{pmatrix}_x = 0.
$$

From the boundary condition (3.13) we have $u + z = 0$ in this region, and the state from the boundary $x_0 = (1 - t_0)\pi/4$ simply propagates forward along the lines $x - x_0 = \lambda(t - t_0)$, i.e. from (3.18),

$$
u(x,t) = -z(x,t) = u(x_0, t_0) = \frac{1}{1 - t_0} \frac{1}{\lambda + \pi/4},
$$

and where

$$
t_0 = \max\left\{ \frac{\pi/4 - (x - \lambda t)}{\lambda + \pi/4}, 0 \right\},
$$

which yields initial data that is compactly supported.

Similarly, on the left of the wedge, $v(x,t) \equiv \pi/4$, the system is

$$
\begin{pmatrix} u \\ z \end{pmatrix}_t + \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} u \\ z \end{pmatrix}_x = 0,
$$

and the solution is given by

$$
z(x,t) = -u(x,t) = z(x_0, t_0) = \frac{1}{1 - t_0} \frac{1}{\lambda + \pi/4},
$$

10

where
$$t_0 = \max\left\{\frac{\pi/4 + (x + \lambda\,t)}{\lambda + \pi/4}, 0\right\}.$$

This concludes the description of a continuous (C^0) solution that blows up in amplitude in finite time.

3.2 C^k blowup in a wedge

We now modify the construction above to obtain an example where the data are C^k-smooth and where the solution remains C^k-smooth for each time prior to blowup time. We begin with a Burgers equation (3.1) which we augment with the "linear" system (3.2). Applying the change of coordinates (3.3) we again obtain the system (3.4)-(3.5).

In regions in which $A(V)$ is described as a function of y only, we observe that (U, Z) solves a linear nonconstant coefficient system. We will choose V and $A(V)$ so that this is in fact the case, and then look for a solution of (3.4) which has the required blowup property. We must do these things in such a way that the solutions remain C^k in x.

We first choose V and the matrix $A(V)$ in such a way that $A(V)$ can be regarded as a function of y. To do so, we take $A(V)$ to be of class C^k and with constant values for $V > 1$ and $V < -1$:

$$\begin{aligned} A(V) &\equiv A(1) \quad \text{for} \quad V > 1, \quad \text{and} \\ A(V) &\equiv A(-1) \quad \text{for} \quad V < -1. \end{aligned} \tag{3.19}$$

Now choose non-increasing initial data $V_0 \in C^k(\mathbb{R})$ which are constant for large $|y|$, and such that $V_0(y) = -y$ for $|y| \leq 1$, and $V_0'(y) \geq -1$ for all y.

As a consequence of our choice of V_0, the v-wave breaks exactly at time $t = 1$ as before. We can then solve (3.5) for $V(y, t)$ for $t < 1$, to get

$$V(y, t) = -y \quad \text{for} \quad -1 \leq y \leq 1,$$

while $|V(y, t)| > 1$ for all $|y| > 1$.

With this choice of V, and with A chosen to satisfy (3.19), we indeed have that $A(V(y, t))$ is a function of y alone for all $t < 1$. Denoting the resulting matrix by $\mathcal{A}(y)$, the transformation (3.3) thus yields the linear 2×2 system

$$\begin{pmatrix} U \\ Z \end{pmatrix}_\tau + y \begin{pmatrix} U \\ Z \end{pmatrix}_y + \left[\mathcal{A}(y) \begin{pmatrix} U \\ Z \end{pmatrix}\right]_y = 0, \tag{3.20}$$

which is valid for *all* values of $y \in \mathbb{R}$.

It follows that if we can find a C^k solution (U, Z) of (3.20) which is unbounded as $\tau \to \infty$, then we will have C^k blowup in finite time for the original system.

It is convenient to choose the matrix $A(V)$ similar to our previous choice, but now of class C^k. Therefore, let

$$A(V) = \lambda(V) \begin{pmatrix} -\cos 2\theta(V) & \sin 2\theta(V) \\ \sin 2\theta(V) & \cos 2\theta(V) \end{pmatrix}, \qquad (3.21)$$

where we need to choose the C^k functions $\lambda(V)$ and $\theta(V)$. Since $A(V)$ has eigenvalues $\pm\lambda(V)$ we ensure strict hyperbolicity of the resulting 3×3 system by taking $\lambda(V)$ as any constant λ larger than $\sup|V_0|$. Also, we take $\theta = \theta(V)$ to be any monotone C^k function satisfying

$$\theta(V) = \frac{\pi}{4} \quad \text{for} \quad V \geq 1, \quad \text{and} \quad \theta(V) = -\frac{\pi}{4} \quad \text{for} \quad V \leq -1. \qquad (3.22)$$

Clearly, this guarantees that (3.19) is satisfied so that (3.20) holds with

$$\mathcal{A}(y) = \lambda \begin{pmatrix} -\cos 2\theta(-y) & \sin 2\theta(-y) \\ \sin 2\theta(-y) & \cos 2\theta(-y) \end{pmatrix},$$

As in (3.15), equation (3.20) has a particular solution of the form

$$\begin{pmatrix} U \\ Z \end{pmatrix} = e^\tau \left[\mathcal{A}(y) + y\,I\right]^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad (3.23)$$

where α and β are constants. In contrast to the above construction of a continuous solution, the expression (3.23) now gives a solution for *all* $y \in \mathbb{R}$. By choosing $\alpha = -\beta = 1$ we ensure that the solution satisfies

$$\ell_{\mp}(y) \cdot \begin{pmatrix} U(y,t) \\ Z(y,t) \end{pmatrix} \equiv 0 \qquad \text{for } y \gtrless \pm 1.$$

That is, there are only forward (linear) waves on the right and backward waves on the left of the central region $|y| \leq 1$. With these substitutions, the solution becomes

$$\begin{pmatrix} U(y,\tau) \\ Z(y,\tau) \end{pmatrix} = \frac{e^\tau}{y^2 - \lambda^2} \begin{pmatrix} \lambda[\sin 2\theta(-y) + \cos 2\theta(-y)] + y \\ \lambda[\cos 2\theta(-y) - \sin 2\theta(-y)] - y \end{pmatrix}, \qquad (3.24)$$

where the function θ is chosen as above. Notice that the solution (U, Z) remains C^k at $y = \pm\lambda$ due to the particular properties of the function $\theta(V)$.

Transforming back to the original variables (x, t), we now have a solution (u, v, z) which is C^k for times $t < 1$, and such that u and z blow up in sup-norm at time $t = 1$, as does the derivative v_x. This solution has initial data given by $v_0(x) = V_0(x)$ as described above, and

$$\begin{pmatrix} u_0(x) \\ z_0(x) \end{pmatrix} = \frac{1}{x^2 - \lambda^2} \begin{pmatrix} \lambda[\sin 2\theta(-x) + \cos 2\theta(-x)] + x \\ \lambda[\cos 2\theta(-x) - \sin 2\theta(-x)] - x \end{pmatrix}, \qquad (3.25)$$

which is bounded for all $x \in \mathbb{R}$. Of course, due to finite speed of propagation we may choose our data to have compact support.

We collect the results in the following theorem.

Theorem 1. *For any $k = 0, 1, 2, \dots$ there exist one-dimensional hyperbolic systems of conservation laws with three (or more) equations for which there are (classical) solutions with the following properties:*

(i) *the system is uniformly strictly hyperbolic on the range of the solution;*

(ii) *the supremum-norm of the solution approaches infinity at a finite time t^*;*

(iii) *the data are bounded, compactly supported and of class C^k, and the solution remains of class C^k at any time strictly preceding t^*.*

4   Smooth blowup for systems with singular flux

In this section we consider the possibility of blowup when the flux F in (1.1) has a singularity at some point. The following type of example is motivated by the fact that for systems of *three* or more equations one can let two of the components, u and z say, satisfy decoupled equations, while the evolution of the third component depends on a quantity (defined in terms of u and z) which blows up in finite time.

Consider again a system of conservation laws (1.1) which we now write out as

$$\begin{aligned} u_t + f(u, v, z)_x &= 0, && (4.1) \\ v_t + g(u, v, z)_x &= 0, && (4.2) \\ z_t + h(u, v, z)_x &= 0. && (4.3) \end{aligned}$$

As an example let two initial profiles $u_0(x)$ and $z_0(x)$ for the first and last components be given as in Figure 1. For simplicity let these be smooth,

13

monotone increasing and decreasing, respectively, and take constant values
0 and 1 outside $(-1, 1)$. Also let $f(u, v, z) = -u$ and $h(u, v, z) = z$. That is,
the solutions of the first and third equations are simply these initial profiles
shifted to the left and right with constant speeds $\lambda_1 \equiv -1$ and $\lambda_3 \equiv +1$. An
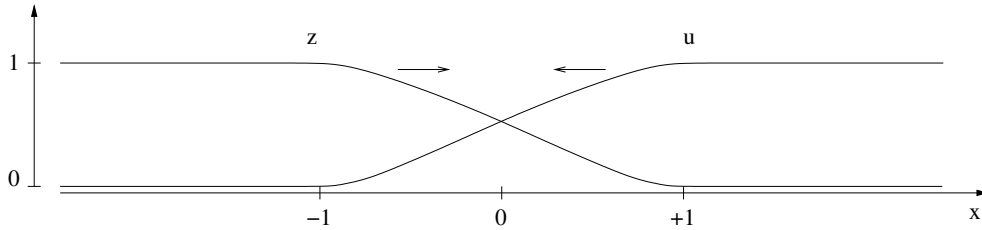

Figure 1

easy way to construct blowup in v is to let the flux in the second equation
be of the form

$$g(u, v, z) = g(u, z) = \frac{u - z}{\alpha - (u + z)},$$

for a constant $\alpha > 0$ to be determined. Note that since g depends only on
u and z, the second eigenvalue of the Jacobian of $F = (f, g, h)^{tr}$ is $\lambda_2 \equiv 0$.
The Jacobian of F thus has eigenvalues 0 and ± 1, which shows that the
system is strictly hyperbolic and linearly degenerate in all three fields.

With this choice for g the second equation takes the form

$$v_t = \frac{(2z - \alpha)u_x + (\alpha - 2u)z_x}{(\alpha - (u + z))^2}. \tag{4.4}$$

Assume for simplicity that $z_0(x) = 1 - u_0(x)$, so that

$$u(x, t) = u_0(x + t), \quad \text{and} \quad z(x, t) = z_0(x - t) = 1 - u_0(x - t).$$

Also, let the initial value of v be some positive constant,

$$v(x, 0) \equiv C_0 > 0.$$

To simplify further we assume that $u_0(x) + u_0(-x) \equiv 1$ (in particular $u_0(0) = 1/2$), and we also let $u_0''(x) < 0$ for $x \in (0, 1)$. Note that the function
$(u + z)(x, t)$ has first derivative $u_0'(x + t) - u_0'(x - t)$, whose only zero in
$(-1, 1)$ is at $x = 0$ (when $t > 0$). The second derivative of $(u + z)(x, t)$ at
$x = 0$ is $2u_0''(t)$ which is negative for $t \in (0, 1)$. This shows that $(u + z)(x, t)$
attains its unique maximum value at $x = 0$ when $t > 0$. Also, this maximum
value is strictly less than α for small $t > 0$ provided $\alpha \in (1, 2)$. As the

14

numerator of the right-hand side of (4.4) is a bounded function, it follows that if v blows up in sup-norm, then this will first happen at $x = 0$.

With these choices, equation (4.4) takes the form

$$v_t = \frac{[2 - 2u_0(x - t) - \alpha]u_0'(x + t) - [\alpha - 2u_0(x + t)]u_0'(x - t)}{[\alpha - 1 - u_0(x + t) + u_0(x - t)]^2}, \qquad (4.5)$$

Setting $V(t) := v(0, t)$ it follows that

$$\dot{V}(t) = -\frac{u_0'(t)}{\alpha/2 - u_0(t)}.$$

We now fix $\alpha \in (1, 2)$, and define x^* to be the unique number in $(0, 1)$ for which $u_0(x^*) = \alpha/2$. Since u_0 is convex down on $(0, 1)$ the graph of u_0 lies above the chord joining the points $(0, 1/2)$ and $(x^*, \alpha/2)$, see Figure 2.

Figure 2

Thus

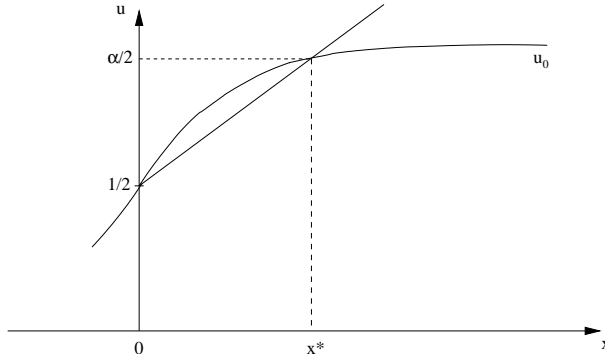$$0 < \frac{\alpha}{2} - u_0(t) < \frac{\alpha - 1}{2x^*}(x^* - t) \qquad \text{and} \qquad u_0'(t) > u_0'(x^*) \qquad \text{for } t \in (0, x^*).$$

It follows that

$$\dot{V}(t) < -\frac{C}{x^* - t},$$

where $C = 2x^* u_0'(x^*)/(\alpha - 1)$. Thus,

$$V(t) < C_0 + C \ln\left(1 - \frac{t}{x^*}\right),$$

which shows that $v(x, t)$ blows up in sup-norm at some finite time $t^* \leq x^*$. It follows by integrating (4.5) in time that $v(x, t)$ has the same degree of

15

smoothness (jointly in (x, t)) in $\mathbf{R} \times (0, 1)$ as $u_0'(x)$ has. This provides an example of (arbitrarily) smooth blowup in L^∞ for strictly hyperbolic systems of three conservation laws. It is clear from the construction that one can construct many similar examples of the same type. However, some care must be taken in choosing the singular component of the flux. For example, at first sight it may seem simpler to use the function

$$g(u, v, z) = \frac{1}{\alpha - (u + z)}.$$

However, the choice of the second flux function $g(u, v, z)$ is really dictated by the form of u_0 and z_0. So, if u_0' is to be symmetric about $x = 0$ and $z_0 = 1 - u_0$, then $u + z$ takes its maximum at $x = 0$, and $\dot{V}(t) = v_t(0, t)$ would be zero with this choice of flux. The factor $u - z$ is needed to "break the symmetry" in this case.

We finally note that due to finite speed of propagation we could just as well let the data u_0 and z_0 have compact support. We record these findings in the following theorem.

Theorem 2. *For any $k = 0, 1, 2, \ldots$ there exist one-dimensional hyperbolic systems of conservation laws of the form (1.1) where the flux F is smooth except on a smooth hypersurface $\mathcal{S}$ where it has an algebraic singularity, and for which there are solutions with the following properties:*

(i) *the system is uniformly strictly hyperbolic along the solution,*

(ii) *the supremum-norm of the solution approaches infinity at a finite time t^*,*

(iii) *the data are bounded, uniformly bounded away from $\mathcal{S}$, and of class C^k,*

(iv) *at any time strictly preceding t^* the solution is of class C^k.*

We also note that the system considered in this last section is linearly degenerate in all three characteristic fields. Theorem 2 thus highlights the fact that such systems do allow gradient blowup, albeit as a consequence of a singularity in the flux.

Remark. In both types of blowup we have considered the blowup first occurs at a *single point*. However, this is not necessarily the case for sup-norm blowup in general. See [8] for an example where the solution blows up at each point in a full spatial interval at a fixed time.

Acknowledgement

References

[1] P. Baiti, H. K. Jenssen, Blowup in L^∞ for a class of genuinely nonlinear hyperbolic systems of conservation laws. *Discrete Contin. Dynam. Systems* **7** (2001), 837–853.

[2] Y. Brenier, A note on deformations of 2D fluid motions using 3D Born-Infeld equations, to appear in *Monatsh. Math.*

[3] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations. *Comm. Pure Appl. Math.* **18** (1965) 697–715.

[4] H. K. Jenssen, Blowup for systems of conservation laws. *SIAM J. Math. Anal.* **31** (2000), 894–908

[5] P. D. Lax, Hyperbolic systems of conservation laws. II, *Comm. Pure Appl. Math.* **10** (1957), 537–566.

[6] M. Sever, Distribution solutions of nonlinear systems of conservation laws. Preprint, 2003.

[7] R. Young, Exact solutions to degenerate conservation laws. *SIAM J. Math. Anal.* **30** (1999), 537–558.

[8] R. Young, Blowup of solutions and boundary instabilities in nonlinear hyperbolic equations. Commun. Math. Sci. **1** (2003), 269–292.

[9] R. Young, W. Szeliga, Blowup with small BV data in hyperbolic conservation laws, *submitted*.