C^1 Measure Respecting Maps Preserve BV Iff they have Bounded Derivative

F. Colombini, T. Luo, J. Rauch*

Abstract

If $\Omega_j \in \mathbb{R}^d$ are bounded open subsets and $\Phi \in C^1(\Omega_1\,;\,\Omega_2)$ respects Lebesgue measure and satisfies $F \circ \Phi \in BV(\Omega_1)$ for all $F \in BV(\Omega_2)$ then Φ is uniformly Lipshitzean.

The problem addressed in this note was motivated by the study of the propagation of regularity in the transport by vector fields with bounded divergence,

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{d} a_j(x,t)\frac{\partial u}{\partial x_j} \;=\; 0,\;\; x \in \mathbb{R}^d, d \geq 2, t > 0, \tag{1}$$

where $x = (x_1, x_2, \cdots, x_d)$ and,

$$\mathrm{div}_x \mathbf{a} \;=\; \sum_{j=1}^{d} \partial_{x_j} a_j(x,t) \;\in\; L^\infty\big([0,T] \times \mathbf{R}^d\big) \tag{2}$$

in the sense of distribution. To guarantee the uniqueness of L^∞ solutions of Cauchy problem it suffices to assume that (cf. [Am])

$$\mathbf{a} = (a_1,\; a_2,\; \cdots, a_d) \in L^1([0,T], BV_{\mathrm{loc}}(\mathbf{R}^d)) \cap L^1([0,T],\; L^\infty(\mathbf{R}^d))\,.$$

Then for arbitrary initial data $u_0(x) \in L^\infty(\mathbf{R}^d)$ there is a unique solution $u(x,t) \in L^\infty\big([0,T] \times \mathbf{R}^d\big)$ with $u|_{t=0} = u_0$. With the same hypotheses, there is a well defined flow Φ_t and the solution is given by $u(t) = u_0 \circ \Phi_{-t}$. The flow respects Lebesgue measure in the sense of (3) below.

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We have given examples [CLR2] which show that such transport equations do not in general propagate either Holder or BV regularity. The counterexamples had flows which were mostly smooth with small singular sets. Thus there were large open sets on which the flow was a C^1 maps. On those sets, the next result shows that BV preservation implies that the flow must of necessity be uniformly Lipschitzean. In the examples it is easily verified that the derivative is not bounded.

The example (shown to us by L. Ambrosio) of the measure preserving map $\Phi :]0,2[\to]-1,1[$

$$\Phi(x) = x \quad \text{for} \quad 0 < x < 1\,, \qquad \Phi(x) = x - 2 \quad \text{for} \quad 1 < x < 2\,,$$

shows that measure preserving maps which are smooth except for jumps, can preserve BV without being Lipschitzean.

Theorem 1. *If Ω_j are bounded open subsets of $\mathbf{R}^d$ and Φ is a continuously differentiable map from Ω_1 to Ω_2 with the following two properties*

$$\exists \gamma > 0\,, \quad \forall \text{ Borel subsets } A \subset \Omega_1, \qquad \gamma\,\big|\Phi(A)\big| \;<\; \big|A\big| \;<\; \frac{1}{\gamma}\,\big|\Phi(A)\big|\,, \tag{3}$$

where $\big|\cdot\big|$ denotes Lebesgue measure and

$$\forall F \in BV(\Omega_2), \qquad F\circ\Phi \;\in\; BV(\Omega_1)\,. \tag{4}$$

Then, $\Phi \in W^{1,\infty}(\Omega_1)$.

The proof of the Theorem consists of two lemmas.

Lemma 2. *If $\Phi \in C^1$ but not in $W^{1,\infty}$ then for any positive number M, there exists an $F \in C_0^\infty(\Omega_2)$ such that*

$$||\big(F\circ\Phi\big)'||_{L^1(\Omega_1)} \;\geq\; M\,||F'||_{L^1(\Omega_2)}. \tag{5}$$

Proof. The chain rule implies that for any $F \in C_0^1$ and $1 \leq i \leq d$,

$$\int_{\tilde\Omega} \Big|\frac{\partial(F\circ\Phi)}{\partial x_i}\Big| dx = \int_{\Omega} \Big|\sum_{j=1}^{d} \frac{\partial F\big(\Phi(x)\big)}{\partial y_j}\,\frac{\partial \Phi_j}{\partial x_i}\Big|\; dx. \tag{6}$$

Since Φ' is not bounded, there is for any $M > 0$, an $\bar x \in \tilde\Omega$ such that

$$\max_{1\leq i,\ j\leq d}\ \Big|\frac{\partial \Phi_i}{\partial x_j}(\bar x)\Big| \;\geq\; 8M/\gamma. \tag{7}$$

Without loss of generality, we may assume that

$$\Big|\frac{\partial \Phi_1}{\partial x_1}(\bar{x})\Big| \;=\; \max_{1\le i,\; j\le d} \Big|\frac{\partial \Phi_i}{\partial x_j}(\bar{x})\Big| \;\ge\; 8M/\gamma. \tag{8}$$

Let $\bar{y} = (\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_d) =: \Phi(\bar{x})$. Choose $0 < \epsilon < \frac{1}{16(d-1)^2}$ such that

$$N_\epsilon(\bar{y}) = \Big\{ y \in \mathbf{R}^d \,:\, |y_1 - \bar{y}_1| < \epsilon, \quad |y_j - \bar{y}_j| < \sqrt{\epsilon} \;\text{ for }\; 2 \le j \Big\} \;\subset \Omega_2, \tag{9}$$

and for $x \in \Phi\left(N_\epsilon(\bar{y})\right)$,

$$\Big|\frac{\partial \Phi_1}{\partial x_1}(x)\Big| \ge \frac{1}{2}\Big|\frac{\partial \Phi_1}{\partial x_1}(\bar{x})\Big|, \;\text{ and for }\; j \ge 2, \;\; \Big|\frac{\partial \Phi_1}{\partial x_j}(x)\Big| \le 2\Big|\frac{\partial \Phi_1}{\partial x_1}(\bar{x})\Big|. \tag{10}$$

Choose $\phi \in C_0^\infty(\mathbf{R}^1)$ satisfying

$$\int_{-\infty}^{\infty} |\phi(z)| dz = 1, \quad \operatorname{supp}\phi \;\subset\; [-1.\ 1]. \tag{11}$$

Define

$$F \;=:\; \phi\left(\frac{y_1 - \bar{y}_1}{\epsilon}\right) \prod_{j=2}^{d} \phi\left(\frac{y_j - \bar{y}_j}{\sqrt{\epsilon}}\right).$$

Then,

$$\begin{aligned} ||F'\,||_{L^1(\Omega_2)} \;&=:\; \int_\Omega \sum_{j=1}^{d} \Big|\partial_{y_j} F(y)\Big|\, dy \;=\; \int_{N_\epsilon(\bar{y})} \sum_{j=1}^{d} \left|\partial_{y_j} F(Y)\right|\, dy \\ &= \epsilon^{(d-1)/2}(1 + (d-1)\sqrt{\epsilon}) \int_{-\infty}^{\infty} |\phi'(z)| dz. \end{aligned} \tag{12}$$

Since $\epsilon < \frac{1}{16(d-1)^2}$, we have

$$||F'\,||_{L^1(\Omega_2)} \;\le\; 2\,\epsilon^{(d-1)/2} \int_{-\infty}^{\infty} |\phi'(z)| dz. \tag{13}$$

In view of (6), (9) and (10), we have

$$\begin{aligned}
\int_{\tilde\Omega} \Big|\frac{\partial F\circ\Phi(x)}{\partial x_1}\Big|\, dx \;&=\; \int_{\Omega} \Big|\sum_{j=1}^{d} \frac{\partial F}{\partial y_j}\,\frac{\partial \Phi_j}{\partial x_1}\Big|\, dx \\
&\geq\; \int_{\Omega} \Big|\frac{\partial F}{\partial y_1}\,\frac{\partial \Phi_1}{\partial x_1}\Big|\, dX \;-\; \int_{\Omega} \sum_{j=2}^{d} \Big|\frac{\partial F}{\partial y_j}\,\frac{\partial \Phi_j}{\partial x_1}\Big|\, dx \\
&=\; \int_{N_\epsilon(\bar y)} \Big|\frac{\partial F}{\partial y_1}\,\frac{\partial \Phi_1}{\partial x_1}\Big|\, dX \;-\; \int_{N_\epsilon(\bar y)} \sum_{j=2}^{d} \Big|\frac{\partial F}{\partial y_j}\,\frac{\partial \Phi_j}{\partial x_1}\Big|\, dx \\
&\geq\; \frac{1}{2}\Big|\frac{\partial \Phi_1}{\partial x_1}(\bar x)\;\Big| \int_{N_\epsilon(\bar y)} \Big|\frac{\partial F}{\partial y_1}\Big|\, dX \;-\; 2\Big|\frac{\partial \Phi_1}{\partial x_1}(\bar x)\;\Big| \int_{N_\epsilon(\bar y)} \sum_{j=2}^{d} \Big|\frac{\partial F}{\partial y_j}\Big| dX \\
&=\; \Big|\frac{\partial \Phi_1}{\partial x_1}(\bar x)\Big| \left(\frac{1}{2} - 2\epsilon(d-1)\right) \epsilon^{(d-1)/2} \int_{-\infty}^{\infty} |\phi'(z)|dz\,.
\end{aligned}$$

Since $\epsilon < \frac{1}{16(d-1)^2} < \frac{1}{8(d-1)}$,

$$\int_{\tilde\Omega} \Big|\frac{\partial F\circ\Phi}{\partial x_1}\Big|\, dx \;\geq\; \frac{1}{4}\,\Big|\frac{\partial \Phi_1}{\partial x_1}(\bar x)\Big|\; \epsilon^{(d-1)/2} \int_{-\infty}^{\infty} |\phi'(z)|dz. \tag{14}$$

Estimates (13) and (14) and the fact that Φ respects Lebesgue measure in the sense of (3) imply

$$\int_{\tilde\Omega} \Big|\frac{\partial F\circ\Phi}{\partial x_1}\Big|\, dx \;\geq\; \frac{1}{4\gamma}\,\Big|\frac{\partial \Phi_1}{\partial x_1}(\bar x)\Big|\; \big||F'\big||_{L^1(\Omega_2)}. \tag{15}$$

(5) follows from (7) and (15). ∎

The next lemma completes the proof.

Lemma 3. *If $\Phi \in C^1(\Omega_2\,;\Omega_2)$ satisfies hypotheses (3) and (4) of Theorem 1, then there is a constant $C>0$ so that for all $F\in BV(\Omega_2)$*

$$\big\|(F\circ\Phi)'\,\big\|_{\mathrm{Var}} \;\leq\; C\,\|F'\,\|_{\mathrm{Var}}\,.$$

Proof. The space of $BV(\Omega_j)$ maps H modulo the constants is a Banach space normed by $\|H'\,\|_{\mathrm{Var}}$. It suffices to verify that the map from $BV(\Omega_2)$ to $BV(\Omega_1)$ which sends F to $F\circ\Phi$ has closed graph.

To that end, suppose that

$$F_n \ \to \ F \quad \text{in} \quad BV(\Omega_2)\,,$$

and

$$F_n \circ \Phi \to \ G \quad \text{in} \quad BV(\Omega_1)\,.$$

It suffices to show that $G' = (F \circ \Phi)'$.

Choose the representative $\tilde{F}_n$ of F_n and $\tilde{F}$ of F so that

$$\int_{\Omega_2} \tilde{F}_n \, dy \ = \ 0\,, \qquad \int_{\Omega_2} \tilde{F} \, dy \ = \ 0\,.$$

Then, passing to a subsequence, there is a Lebesgue null set $E_2 \subset \Omega_2$ so that

$$\tilde{F}_{n_k} \ \to \ F \quad \text{pointwise on} \ \ \Omega_2 \setminus E_2\,.$$

Then,

$$\tilde{F}_{n_k} \circ \Phi \ \to \ F \circ \Phi \quad \text{pointwise on} \ \ \Omega_1 \setminus \big(E_1 \ \cup \ \Phi^{-1}(E_2)\big)\,.$$

The exceptional set has measure zero since Φ respects the measure. Therefore, $F_{n_k} \circ \Phi \to F \circ \Phi$ a.e. on Ω_1 and therefore in the sense of distributions. It follows that $F \circ \Phi = G$ almost everywhere and the proof is complete. ∎

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1 References

[Am] L. Ambrosio, Transport equation and Cauchy problem for BV vector fields, preprint Scuola Norm. Sup. Pisa, March 2003.

[Bo] F. Bouchut, Renormalized solutions to the Vlasov equation with coefficients of bounded variation, Arch. Rational Mech. Anal. **157** (2001), 75-90.

[BJ] F. Bouchut, F. James, One-dimensional transport equations with discontinuous coefficients, Nonlinear Anal. **32** (1998), 891-933.

[CL1] F. Colombini, N. Lerner, Uniqueness of continuous solutions for BV vector fields, Duke Math. J. **111** (2002), 357-384.
[CL2] F. Colombini, N. Lerner, Sur les champs de vecteurs peu réguliers, Séminaire E.D.P., Ecole Polytechnique, 2000-2001, XIV 1-15.
[CL3] F. Colombini, N. Lerner, Uniqueness of L^∞ solutions for a class of conormal BV vector fields, Preprint Univ. Rennes 1, February 2003.
[CLR1] F. Colombini, T. Luo, J. Rauch, Uniqueness and Nonuniqueness for Nonsmooth Divergence Free Transport, Seminaire Equations aux Derivées Partielles, Ecole Polytechnique 2002-2003.
[CLR21] F. Colombini, T. Luo, J. Rauch, Nearly Lipschitzean divergence free transport propagates neither continuity nor BV regularity, preprint.
[DL] R.J. Di Perna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. **98** (1989), 511-547.
[Li] P.-L. Lions, Sur les équations différentielles ordinaires et les équations de transport, C.R.Acad.Sci.Paris Sér.I Math. **326** (1998), 833-838.
[PP] G. Petrova, B. Popov, Linear transport equations with discontinuous coefficients, Comm. P.D.E. **24** (1999), 1849-1873.
[PR] F. Poupaud, M. Rascle, Measure solutions to the linear multi-dimensional transport equation with non-smooth coefficients, Comm. P.D.E. **22** (1997), 337-358.

Ferruccio COLOMBINI
Dipartemento di Matematica
Università di Pisa
Pisa, Italia
colombini@dm.unipi.it

Tao LUO
Department of Mathematics
Georgetown University
Washington DC, 20057, USA
tl48@georgetown.edu

Jeffrey RAUCH
Department of Mathematics
University of Michigan
Ann Arbor 48104 MI, USA
rauch@umich.edu