The genuinely nonlinear non-isotropic ultra-parabolic equation

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Abstract

The work is devoted to a study of the degenerate quasilinear parabolic-hyperbolic equation

$$\partial_t u + \mathrm{div}_x \boldsymbol{a}(\boldsymbol{x}, t, u) - \mathrm{div}_x (A(\boldsymbol{x}, t)\nabla_x b(u)) = 0$$

such that $b(u)$ is strictly increasing in u, the rank of the nonnegative diffusion $d \times d$-matrix A may vary in $\boldsymbol{x}$ and t, the convection coefficients $\boldsymbol{a} = (a_1, \ldots, a_d)$ may be non-smooth in $\boldsymbol{x}$ and t, and the genuine nonlinearity condition holds in the sense that the Lebesgue measure of the intersection of the sets $\{(\boldsymbol{x}, t, \lambda) \,|\, A(\boldsymbol{x}, t)\boldsymbol{y} \cdot \boldsymbol{y} = 0\}$ and $\{(\boldsymbol{x}, t, \lambda) \,|\, \tau + (\boldsymbol{a}'_\lambda(\boldsymbol{x}, t, \lambda) - b'(\lambda)\mathrm{div}_x A(x, t)) \cdot \boldsymbol{y} = 0\}$ is equal to zero for any fixed $(\boldsymbol{y}, \tau) \in \mathbb{R}^d \times \mathbb{R}$ such that $|\boldsymbol{y}|^2 + \tau^2 = 1$. The main results of the work consist in justification that any bounded in L^∞ set of entropy solutions of the equation is relatively compact in L^1_{loc} and that the Cauchy problem with bounded initial data has an entropy solution. The proofs are based on the Chen–Perthame-type kinetic formulation of the equation and on Panov's theorem on a version of Tartar H-measures.

1 Introduction

We consider the Cauchy problem for a quasilinear diffusion-convection equation of the form

$$u_t + \partial_{x_i} a_i(\boldsymbol{x}, t, u) - \partial_{x_i}(a_{ij}(\boldsymbol{x}, t)\partial_{x_j} b(u)) = 0, \quad \boldsymbol{x} \in \mathbb{R}^d, t \in (0, T), \tag{1.1a}$$

endowed with initial data belonging to $L^\infty(\mathbb{R}^d)$,

$$u|_{t=0} = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{1.1b}$$

In (1.1), u is the unknown function, the flux $\boldsymbol{a} := (a_i)$, the diffusion matrix $A := (a_{ij})$, and the diffusion function b are given such that

$$a_i, D_{x_i} a_i \in L^2_{loc}(\mathbb{R}^d_x \times (0,T); C^1_{loc}(\mathbb{R}_u)),\ a_{ij} \in C^2_{loc}(\mathbb{R}^d_x \times [0,T]), \tag{1.2}$$
$$a_{ij} = a_{ji}, \quad a_{ij}(\boldsymbol{x},t)\xi_i\xi_j \geq 0, \quad \forall \boldsymbol{\xi}, \boldsymbol{x} \in \mathbb{R}^d,\ t \in [0,T], \tag{1.3}$$
$$b \in C^2_{loc}(\mathbb{R}), \quad b'(u) > 0, \quad \forall\, u \in \mathbb{R}, \tag{1.4}$$

and such that the maximum principle for (1.1) is a priori guaranteed, for example [1, chapter I, theorem 2.9] such that the inequality

$$uD_{x_i} a_i(\boldsymbol{x},t,u) \geq -c_1 u^2 - c_2, \quad \text{for a.e. } \boldsymbol{x} \in \mathbb{R}^d,\ t \in [0,T],\ \forall\, u \in \mathbb{R}, \tag{1.5}$$

holds with some positive constants c_1 and c_2.

In (1.1)–(1.5) and further in the paper, the conventional summation rule over repeating indexes is in use. The partial derivative D_{x_i} is defined by the formula $D_{x_i} g(\boldsymbol{x},t,u) = (\partial_{x_i} g(\boldsymbol{x},t,\lambda))|_{\lambda=u(x,t)}$, $\forall\, g \in C^1(\mathbb{R}^d_x \times (0,T) \times \mathbb{R}_\lambda)$. In particular, the partial derivatives ∂_{x_i} and D_{x_i} relate via the equality $\partial_{x_i} g(\boldsymbol{x},t,u) = D_{x_i} g(\boldsymbol{x},t,u) + \partial_u g(\boldsymbol{x},t,u)\partial_{x_i} u$. Also, we denote $\Pi := \mathbb{R}^d_x \times (0,T)$ throughout the paper.

We assume that, in general, the rank of matrix A varies in $\boldsymbol{x}$ and t. Thus, (1.1a) is an ultra-parabolic equation. Equations of such type arise in fluid dynamics, combustion theory and financial mathematics [2]. They describe, in particular, non-stationary transport of matter or temperature, in cases, when effects of diffusion in some spatial directions in some subdomains of $\mathbb{R}^d_x$ are negligible, as compared to convection.

Let us notice that, since A is symmetric and nonnegative, there exists a unique square root $A^{1/2} = \{\alpha_{ij}\}_{i,j=1,\ldots,d}$ which is a symmetric nonnegative matrix. We are now in a position to define an entropy solution of (1.1).

Definition 1. *Function $u = u(\boldsymbol{x},t)$ is called an entropy solution of* (1.1)*, if it satisfies initial data* (1.1b)*, the conditions*

$$u \in L^\infty(\Pi), \quad \alpha_{ij}\partial_{x_j} u \in L^2_{loc}(\Pi) \tag{1.6}$$

and the entropy inequality

$$\begin{aligned}
&\varphi(u)_t + \partial_{x_i} q_i(\boldsymbol{x},t,u) + \varphi'(u) D_{x_i} a_i(\boldsymbol{x},t,u) - D_{x_i} q_i(\boldsymbol{x},t,u)\\
&- \partial_{x_i}\left(a_{ij}(\boldsymbol{x},t)\partial_{x_j} w(u)\right) + \varphi''(u) b'(u)(\alpha_{il}(\boldsymbol{x},t)\partial_{x_i} u)(\alpha_{jl}(\boldsymbol{x},t)\partial_{x_j} u) \leq 0
\end{aligned} \tag{1.7}$$

for all functions φ, q_i, and w such that

$$\begin{aligned}
&\varphi \in C^2_{loc}(\mathbb{R}), \quad \varphi''(u) \geq 0,\\
&\partial_u q_i(\boldsymbol{x},t,u) = \varphi'(u)\partial_u a_i(\boldsymbol{x},t,u), \quad w'(u) = \varphi'(u) b'(u).
\end{aligned} \tag{1.8}$$

Inequality (1.7) and initial data (1.1b) are understood in the sense of distributions and therefore can be equivalently collected in the integral formulation

$$\int_{\Pi} (\zeta_t \varphi(u) + \zeta_{x_i} q_i(\boldsymbol{x},t,u) - \zeta\varphi'(u) D_{x_i} a_i(\boldsymbol{x},t,u) + \zeta D_{x_i} q_i(\boldsymbol{x},t,u)$$
$$+ w(u)\partial_{x_i}(a_{ij}(\boldsymbol{x},t)\partial_{x_j}\zeta) - \zeta\varphi''(u)b'(u)(\alpha_{il}(\boldsymbol{x},t)\partial_{x_i}u)(\alpha_{lj}(\boldsymbol{x},t)\partial_{x_j}u)\big)\, d\boldsymbol{x}dt$$
$$+ \int_{\mathbb{R}^d} \varphi(u_0)\zeta(\boldsymbol{x},0)d\boldsymbol{x} \geq 0, \quad (1.9)$$

where $\zeta \in C^2(\Pi)$ is an arbitrary nonnegative function vanishing near the plane $\{t = T\}$ and for large $|\boldsymbol{x}|$.

Definition 2. *We say that u is an entropy solution of* (1.1a) *if it satisfies* (1.7) *in the sense of distributions, i.e. if it satisfies* (1.9) *for all* $\zeta \in C_0^2(\Pi)$.

Additionally to (1.2)–(1.5), we impose the following demand on a_i, a_{ij} and b.

Condition G. (The genuine nonlinearity condition). *The functions* a_i*,* a_{ij} *and* b *are such that the Lebesgue measure of the intersection of the sets*

$$\mathbb{I}_1 := \{(\boldsymbol{x},t,\lambda) \in \Pi \times \mathbb{R}_\lambda \quad | \quad a_{ij}(\boldsymbol{x},t)\xi_i\xi_j = 0\}$$

and

$$\mathbb{I}_2 := \{(\boldsymbol{x},t,\lambda) \in \Pi \times \mathbb{R}_\lambda \quad | \quad \tau + \big(a_{i\lambda}(\boldsymbol{x},t,\lambda) - b'(\lambda)a_{ijx_j}(\boldsymbol{x},t)\big)\,\xi_i = 0\}$$

is equal to zero for any fixed $(\boldsymbol{\xi},\tau) \in \mathbb{S}^d$.

Here and further in the paper $\mathbb{S}^d$ is a unit sphere in $\mathbb{R}^{d+1}$, $\mathbb{S}^d = \{(\boldsymbol{\xi},\tau) \in \mathbb{R}^{d+1} \quad | \quad |\boldsymbol{\xi}|^2 + \tau^2 = 1\}$.

The following theorems are the main results of the article.

Theorem 1. *Let equation* (1.1a) *be genuinely nonlinear in the sense of Condition G. Then the Cauchy problem* (1.1) *has an entropy solution for any initial data* $u_0 \in L^\infty(\mathbb{R}^d)$.

Theorem 2. *Any bounded in* $L^\infty(\Pi)$ *family of entropy solutions of genuinely nonlinear equation* (1.1a) *is relatively compact in* $L^1_{loc}(\Pi)$.

Genuinely nonlinear equations of the forms similar to (1.1a) are in focus of many studies. One of the first results for such equations was obtained by P. D. Lax [3], who proved in 1957 that the Cauchy problem for the equation $u_t + a(u)_x = 0$ has an entropy solution in the case when $a(u)$ is either convex or concave. Since then, the theory of entropy solutions of genuinely nonlinear PDEs stepped much forward and it is worth to recall the results by L. Tartar [4], P. L. Lions, B. Perthame and E. Tadmor [5], and E. Yu. Panov [6], because the considerations of the present article may be referred to as a continuation of the works [4, 5, 6]. In [4], it is shown that any bounded set of entropy solutions of equation $u_t + \partial_{x_i} a_i(\boldsymbol{x},t,u) = 0$, $\boldsymbol{x} \in \mathbb{R}^2$ is relatively compact in $L^1_{loc}(\mathbb{R}^2_x \times \mathbb{R}^+)$.

In [6] this result is extended to any space dimension d. The article [5] is devoted to the equations $u_t + \partial_{x_i} a_i(u) = 0$ and $u_t + \partial_{x_i} a_i(u) - \partial^2_{x_i x_j} a_{ij}(u) = 0$, where the rank of a nonnegative matrix (a'_{ij}) may vary in u, and the compactness results in L^1_{loc} are established for the both equations. In [4, 5, 6], the genuine nonlinearity conditions (that are also called *the nondegeneracy* conditions) are analogous to Condition G.

The theory of existence and uniqueness of entropy solutions to the above mentioned equations from [4, 5, 6] was constructed by S. Kruzhkov [7] and G.-Q. Chen and B. Perthame [8] without imposing demands like genuine nonlinearity. On the other hand, the observations in [7, 8] do not cover the cases when the rank of the matrix (a_{ij}) varies in $\boldsymbol{x}$ and t and when the flux $\boldsymbol{a}$ is non-smooth. Therefore Theorem 1 brings a novelty to the existence theory.

The proofs of the present article rely on the method of kinetic equation that allows to reduce quasilinear equations and systems to linear scalar equations on 'distribution' functions involving additional 'kinetic' variables. This method was created and applied recently to study a wide range of topics, for example isentropic gas dynamics and p-systems [9], and scalar conservation laws [5, 8, 10, 11, 12]. Along with this method we apply the theory of H-measures that was constructed originally by L. Tartar [13] and P. Gérard [14] and later developed by E. Yu. Panov in [6].

2 The kinetic formulation of (1.1a)

We introduce the notion of the kinetic formulation of (1.1a) in the form similar to [8, Definition 2.2].

Problem K. Find a kinetic function $f(\boldsymbol{x}, t, \lambda)$ and nonnegative Borel measures $m, n \in \mathbb{M}(\Pi \times \mathbb{R}_\lambda)$ satisfying the equation

$$f_t + a_{i\lambda}(\boldsymbol{x}, t, \lambda) f_{x_i} - a_{i x_i}(\boldsymbol{x}, t, \lambda) f_\lambda - b'(\lambda)\partial_{x_i}(a_{ij}(\boldsymbol{x}, t)\partial_{x_j} f) + (m + b'(\lambda) n)_\lambda = 0, \tag{2.1a}$$

and the constraints

$$f(\boldsymbol{x}, t, \lambda) = \begin{cases} 1, & \text{for} \quad \lambda \geq u(\boldsymbol{x}, t), \\ 0, & \text{for} \quad \lambda < u(\boldsymbol{x}, t), \end{cases} \tag{2.1b}$$

$$\operatorname{spt} m \subset \{(\boldsymbol{x}, t, \lambda) \in \Pi \times \mathbb{R}_\lambda \, : \, |\lambda| \leq \|u\|_{L^\infty}\}, \tag{2.1c}$$

and

$$dn(\boldsymbol{x}, t, \lambda) = |A^{1/2} \nabla_x u(\boldsymbol{x}, t)|^2 d\gamma_{u(x,t)}(\lambda) d\boldsymbol{x} dt \tag{2.1d}$$

with some function $u \in L^\infty(\Pi)$ such that $A^{1/2}\nabla_x u \in L^2_{loc}(\Pi)$.

Here and further in the paper, $\mathbb{M}(X)$ denotes the Banach space of bounded Radon measures on a set X. In (2.1d), $\gamma_{u(x,t)}$ denotes the Dirac measure on $\mathbb{R}_\lambda$ concentrated at the point $\lambda = u(\boldsymbol{x}, t)$.

The kinetic equation (2.1a) is understood in the sense of distributions and therefore is equivalent to the integral formulation

$$\int_{\Pi\times\mathbb{R}_\lambda} (\zeta_t + a_{i\lambda}(\boldsymbol{x},t,\lambda)\zeta_{x_i} - a_{ix_i}(\boldsymbol{x},t,\lambda)\partial_\lambda\zeta \\ +b'(\lambda)\partial_{x_i}(a_{ij}(\boldsymbol{x},t)\partial_{x_j}\zeta)\big) f d\boldsymbol{x}dtd\lambda \\ + \int_{\Pi\times\mathbb{R}_\lambda} \zeta_\lambda b'(\lambda)dn(\boldsymbol{x},t,\lambda) + \int_{\Pi\times\mathbb{R}_\lambda} \zeta_\lambda dm(\boldsymbol{x},t,\lambda) = 0, \quad (2.2)$$

where $\zeta \in C_0^2(\Pi\times\mathbb{R}_\lambda)$ is an arbitrary test function.

Remark 1. *In view of the simple representation*

$$\varphi(u(\boldsymbol{x},t)) = -\int_{\mathbb{R}} \varphi'(\lambda) f(\boldsymbol{x},t,\lambda)d\lambda, \quad \forall\,\varphi\in C_0^1(\mathbb{R}), \tag{2.3}$$

it is easy to see that the triple (f,m,n) *is a solution of Problem K if and only if the function* u *that appears in* (2.1b)–(2.1d) *is an entropy solution of* (1.1a).

The rest of the paper is organized as follows. In Section 3, we introduce the family of H-measures corresponding to a weakly convergent sequence of solutions of Problem K. In Section 4, we formulate and in Sections 5–7 prove Theorem 3 on the localization principle for the H-measures. In Section 8, we apply this localization principle and deduce the assertion of Theorem 2. In Section 9, we adapt the proof of Theorem 2 for verification of Theorem 1.

3 Notion of H-measures

Let (f^k, m^k, n^k), $k\in\mathbb{N}$, be a sequence of solutions of (2.1a)–(2.1d) such that all the set of functions f^k is uniformly supported in some interval $[-u_*, u_*]$, $u_* = \text{const} > 0$. This means that the corresponding sequence of entropy solutions $\{u^k\}$ of (1.1a) is uniformly bounded in $L^\infty(\Pi)$ and $\|u^k\|_{L^\infty(\Pi)} \le u_*$. Extracting a proper subsequence from $k\in\mathbb{R}$ we define weakly* convergent subsequences $\{f^k\}$ and $\{u^k\}$ and limiting functions $f\in L^\infty(\Pi\times\mathbb{R}_\lambda)$ and $u\in L^\infty(\Pi)$ such that

$$f^k \to f \quad \text{weakly* in} \quad L^\infty(\Pi\times\mathbb{R}_\lambda), \text{ as } k\nearrow\infty, \tag{3.1}$$
$$u^k \to u \quad \text{weakly* in} \quad L^\infty(\Pi), \text{ as } k\nearrow\infty. \tag{3.2}$$

It is easy to see that $f=0$ for all $\lambda < -u_*$ and $f=1$ for all $\lambda\ge u_*$. The following lemma yields also that f is monotonous nondecreasing and right-continuous in λ. Such the structure of f allows us to make use of Panov's theorem on a version of H-measures [6, Theorem 3] and introduce a family of H-measures corresponding to $f^k - f$.

Lemma 1. *The limiting function f in* (3.1) *is the distribution function of the Young measure $\nu_{x,t} \in Prob(\mathbb{R}_\lambda)$ associated with the subsequence $\{u^k\}$, i.e.*

$$f(\boldsymbol{x}, t, \lambda) = \int_{\mathbb{R}_s} 1_{\lambda \geq s} d\nu_{x,t}(s). \tag{3.3}$$

$Prob(\mathbb{R}_\lambda)$ is the subset of $\mathbb{M}(\mathbb{R}_\lambda)$ consisting of all nonnegative measures with the norms equal to one. The notion of the Young measure will be recalled within the proof.

Proof. Let $\zeta \in C_0(\Pi; C_0^1(\mathbb{R}_\lambda))$ be an arbitrary function. From (3.1) it follows that

$$\int_{\Pi\times\mathbb{R}_\lambda} f^k \zeta_\lambda d\boldsymbol{x}dtd\lambda \underset{k\nearrow\infty}{\longrightarrow} \int_{\Pi\times\mathbb{R}_\lambda} f\zeta_\lambda d\boldsymbol{x}dtd\lambda. \tag{3.4}$$

Representation (2.3) gives

$$\int_{\Pi\times\mathbb{R}_\lambda} f^k(\boldsymbol{x}, t, \lambda)\zeta_\lambda(\boldsymbol{x}, t, \lambda) d\boldsymbol{x}dtd\lambda = -\int_\Pi \zeta(\boldsymbol{x}, t, u^k(\boldsymbol{x}, t))d\boldsymbol{x}dt. \tag{3.5}$$

On the strength of the Tartar theorem on Young measures [4], [15, Ch.3, Theorem 2.3], there exists a bounded weakly measurable mapping $(\boldsymbol{x}, t) \mapsto \nu_{x,t}$ from Π into $Prob(\mathbb{R}_\lambda)$ such that

$$\operatorname{spt}\nu_{x,t} \subset \{\lambda \,:\, |\lambda| \leq u_*\}, \tag{3.6}$$

and for any ζ the limiting relation

$$\lim_{k\nearrow\infty} \int_\Pi \zeta(\boldsymbol{x}, t, u^k(\boldsymbol{x}, t))d\boldsymbol{x}dt = \int_\Pi \left(\int_{\mathbb{R}_\lambda} \zeta(\boldsymbol{x}, t, \lambda) d\nu_{x,t}(\lambda) \right) d\boldsymbol{x}dt \tag{3.7}$$

holds. Using the notion of the Stieltjes integral generated by the distribution function $g(\boldsymbol{x}, t, \lambda) := \int_{\mathbb{R}_s} 1_{\lambda\geq s} d\nu_{x,t}(\lambda)$ of the measure $\nu_{x,t}$ we can represent the right hand side of (3.7) in the form

$$\int_\Pi \left(\int_{\mathbb{R}_\lambda} \zeta(\boldsymbol{x}, t, \lambda) d\nu_{x,t}(\lambda) \right) d\boldsymbol{x}dt = \int_\Pi \left(\int_{\mathbb{R}_\lambda} \zeta(\boldsymbol{x}, t, \lambda) d_\lambda g(\boldsymbol{x}, t, \lambda) \right) d\boldsymbol{x}dt, \tag{3.8}$$

where $d_\lambda g(\boldsymbol{x}, t, \cdot)$ is the parametrized Stieltjes measure on $\mathbb{R}_\lambda$. On the strength of the theory of the Stieltjes integral, for a.e. $(\boldsymbol{x}, t) \in \Pi$ and for an arbitrary $\psi \in C_0(\mathbb{R}_\lambda)$ the equality

$$\int_{\mathbb{R}_\lambda} \psi(\lambda) d_\lambda g(\boldsymbol{x}, t, \lambda) = -\int_{\mathbb{R}_\lambda} \psi'(\lambda) g(\boldsymbol{x}, t, \lambda) d\lambda,$$

is valid. Using it we rewrite the right hand side of (3.8) in the form

$$\int_\Pi \left(\int_{\mathbb{R}_\lambda} \zeta d_\lambda g \right) d\boldsymbol{x}dt = -\int_{\Pi\times\mathbb{R}_\lambda} \zeta_\lambda g d\boldsymbol{x}dtd\lambda. \tag{3.9}$$

Aggregating (3.5), (3.7)–(3.9) and comparing with (3.4) we conclude that the functions f and g coincide for a.e. $(\boldsymbol{x}, t, \lambda) \in \Pi \times \mathbb{R}_\lambda$, which finishes the proof of the lemma. $\square$

Let us introduce the set

$$\mathcal{E} := \{\lambda_0 \in \mathbb{R} \,|\, f(\lambda) \to f(\lambda_0) \text{ strongly in } L^1_{loc}(\Pi), \text{ as } \lambda \to \lambda_0\}.$$

Lemma [6, Lemma 4] and Panov's theorem on a version of Tartar H-measures [6, Theorem 3] yield the following.

Lemma 2. *The complement of $\mathcal{E}$ in $\mathbb{R}$ is at most countable, and for any $\lambda \in \mathcal{E}$ the limiting relation $f^k(\cdot,\cdot,\lambda) \underset{k\nearrow\infty}{\longrightarrow} f(\cdot,\cdot,\lambda)$ weakly* in $L^\infty(\Pi)$ holds.*

Theorem H. (Existence of H-measures). *There exists a family of locally finite Radon measures $\{\mu^{pq}\}_{p,q\in\mathcal{E}}$ on $\Pi \times \mathbb{S}^d$ and a subsequence from $\{f^k(\lambda) - f(\lambda)\}$, $\lambda \in \mathcal{E}$, such that for all $\Phi_1, \Phi_2 \in C_0(\mathbb{R}^d \times [0,T])$ and $\psi \in C(\mathbb{S}^d)$ the equality*

$$\int_{\Pi\times\mathbb{S}^d} \Phi_1(\boldsymbol{x},t)\overline{\Phi_2(\boldsymbol{x},t)}\psi(\boldsymbol{y})d\mu^{pq}(\boldsymbol{x},t,\boldsymbol{y}) =$$
$$\lim_{k\nearrow\infty} \int_{\mathbb{R}^{d+1}} \mathcal{F}[\Phi_1\{f^k(p) - f(p)\}](\boldsymbol{\xi})\overline{\mathcal{F}[\Phi_2\{f^k(q) - f(q)\}](\boldsymbol{\xi})}\psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi},$$
$$\forall\, p, q \in \mathcal{E}, \quad (3.10)$$

holds.

In the formulation of Theorem 3 and further in the paper, $\bar{\varphi}$ means the complex conjugate of φ. $\mathcal{F}$ is the Fourier transform with respect to $\boldsymbol{x}$ and t,

$$\mathcal{F}[\varphi](\boldsymbol{\xi}) = \int_{\mathbb{R}^{d+1}} \varphi(\boldsymbol{x},t)e^{2\pi i(\xi_0 t+\xi_1 x_1+\ldots+\xi_d x_d)}d\boldsymbol{x}dt$$

for any integrable φ. We assume that any function defined merely for $t \in [0,T]$ is extended outside $[0,T]$ by zero. Further, sometimes we also denote $x_0 := t$.

Definition 3. *The family of measures $\{\mu^{pq}\}_{p,q\in\mathcal{E}}$ is called the H-measure associated with the extracted subsequence $\{f^k - f\}$.*

The general theory of H-measures states the following.

Lemma 3.

(1) *For any finite set $E := \{p_1, \ldots, p_n\} \subset \mathcal{E}$ the measures $(\mu^{p_i p_j})_{i,j=1,\ldots,n}$ are hermitian nonnegative, i.e.*

$$\mu^{p_i p_j} = \mu^{p_j p_i}, \quad \langle \mu^{p_i p_j}, \Phi_i \overline{\Phi}_j \psi\rangle \geq 0 \qquad (3.11)$$

for all $\Phi_1, \ldots, \Phi_n \in C_0(\Pi)$ and $\psi \in C(\mathbb{S}^d)$, $\psi \geq 0$ [13, Corollary 1.2].

(2) *The mapping $(p,q) \mapsto \mu^{pq}$ is continuous from $\mathcal{E} \times \mathcal{E}$ into $\mathbb{M}(\Pi \times \mathbb{S}^d)$ [6, Theorem 3].*

(3) For any $p, q \in \mathcal{E}$, measure μ^{pq} is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{d+1}$. As a functional on $C(\Omega \times \mathbb{S}^d)$, it admits a natural expansion onto $L^2(\Omega, C(\mathbb{S}^d))$ and therefore the decomposition $d\mu^{pq}(\boldsymbol{x}, t, \boldsymbol{y}) = d\sigma^{pq}_{x,t}(\boldsymbol{y}) d\boldsymbol{x} dt$ takes place, where the mapping $(\boldsymbol{x}, t) \mapsto \sigma^{pq}_{x,t}$ belongs to $L^2_w(\Pi, \mathbb{M}(S^d))$ and is uniquely defined by μ^{pq} [16, Section 1.2].

(4) $f^k(\cdot, \cdot, \lambda) \to f(\cdot, \cdot, \lambda)$ strongly in $L^1_{loc}(\Pi)$ for all $\lambda \in \mathcal{E}$, as $k \nearrow \infty$, if and only if $\mu^{\lambda\lambda} \equiv 0$ for all $\lambda \in \mathcal{E}$ [13].

In item (3) of this lemma and further in the paper, $L^2_w(\Pi, \mathbb{M}(\mathbb{S}^d))$ is the space of weakly measurable with respect to Lebesgue measure on Π mappings $\boldsymbol{x} \mapsto \sigma_x$ from Π into $\mathbb{M}(\mathbb{S}^d)$ equipped with the norm

$$\|\sigma\|_{L^2_w(\Pi, \mathbb{M}(\mathbb{S}^d))} = \left(\int_\Pi \|\sigma_{x,t}\|^2_{\mathbb{M}(\mathbb{S}^d)} d\boldsymbol{x} dt \right)^{1/2}, \quad \forall \sigma \in L^2_w(\Pi, \mathbb{M}(\mathbb{S}^d)).$$

4 The formulation of the localization principle for the H-measures

Theorem 3. *H-measure $\mu^{\lambda\lambda}$ associated with the extracted subsequence $\{f^k - f\}$ satisfies the integral equalities*

$$\int_{\mathbb{R}_\lambda} \left(\int_{\Pi \times \mathbb{S}^d} a_{ij}(\boldsymbol{x}, t) y_i y_j \zeta(\boldsymbol{x}, t, \lambda, \boldsymbol{y}) d\mu^{\lambda\lambda}(\boldsymbol{x}, t, \boldsymbol{y}) \right) d\lambda = 0 \tag{4.1}$$

and

$$\int_{\mathbb{R}_\lambda} \Big(\int_{\Pi \times \mathbb{S}^d} \big(y_0 + \big(a_{i\lambda}(\boldsymbol{x}, t, \lambda) - b'(\lambda) a_{ij x_j}(\boldsymbol{x}, t, \lambda)\big) y_i\big) \beta(\boldsymbol{x}, t, \lambda, \boldsymbol{y}) d\mu^{\lambda\lambda}(\boldsymbol{x}, t, \boldsymbol{y}) \Big) d\lambda = 0 \tag{4.2}$$

for all $\beta, \zeta \in C_0(\Pi \times \mathbb{R}_\lambda \times \mathbb{S}^d_y)$.

Remark 2. *Theorem 3 amounts to the assertion that the support of the H-measure $\mu^{\lambda\lambda}$ for a.e. $\lambda \in \mathbb{R}$ lays entirely in the intersection of the sets*

$$\{(\boldsymbol{x}, t, \boldsymbol{y}) \in \Pi \times \mathbb{S}^d \mid a_{ij}(\boldsymbol{x}, t) y_i y_j = 0\}$$

and

$$\{(\boldsymbol{x}, t, \boldsymbol{y}) \in \Pi \times \mathbb{S}^d \mid y_0 + \big(a_{i\lambda}(\boldsymbol{x}, t, \lambda) - b'(\lambda) a_{ij x_j}(\boldsymbol{x}, t, \lambda)\big) y_i = 0\}.$$

5 Proof of Theorem 3. Part I: preliminaries

We start the proof by establishing the following auxiliary lemma.

Lemma 4. *There exists a Borel measure $H \in \mathbb{M}(\Pi \times \mathbb{R}_\lambda)$ supported in the set $\mathbb{I}_* = \{(\boldsymbol{x}, t, \lambda) \in \Pi \times \mathbb{R}_\lambda \, : \, |\lambda| \leq u_*\}$ such that the limiting relation*

$$m^k + b'(\lambda) n^k \to H \text{ weakly* in } \mathbb{M}(\Pi \times \mathbb{R}_\lambda), \text{ as } k \nearrow \infty, \tag{5.1}$$

holds true.

Proof. On the strength of (2.1a), (3.1) and (3.2), the uniform bound

$$\|m^k + b' n^k\|_{(C^2(\Pi \times \mathbb{R}_\lambda))^*} \leq c_* \tag{5.2}$$

holds true with a constant c_* that does not depend on $k \in \mathbb{N}$. Since $m^k + b'n^k$ is a nonnegative measure for any $k \in \mathbb{N}$, we conclude by the standard arguments that there exists a unique natural continuation of $m^k + b'n^k$ onto $\mathbb{M}(\Pi \times \mathbb{R}_\lambda)$ and that the family $\{m^k + b'n^k\}_{k \in \mathbb{N}}$ is bounded uniformly in $\mathbb{M}(\Pi \times \mathbb{R}_\lambda)$ by the constant c_* [17, Chapter III, §1, Proposition 2]. This bound yields that for some subsequence from $k \in \mathbb{N}$ the limiting relation (5.1) holds true.

Finally, we observe that the support of H lays entirely in $\mathbb{I}_*$ because both the supports of m^k and n^k lay in $\mathbb{I}_*$ for all $k \in \mathbb{N}$. □

Besides, in order to prove Theorem 3, we will repeatedly use the notions of the Riesz potentials and the pseudo-differential operators (henceforth, p.d.o.'s) of zero order, in particular, the Riesz transforms. Let us recall [18, Chapter 5, §1] that the Riesz potential $\mathcal{I}_\alpha$ $(0 < \alpha < d+1)$ is defined by the formula

$$\mathcal{F}[\mathcal{I}_\alpha[\varphi]](\boldsymbol{\xi}) = (2\pi|\boldsymbol{\xi}|)^{-\alpha} \mathcal{F}[\varphi](\boldsymbol{\xi})$$

for any function $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$. The Hardy–Littlewood–Sobolev theorem [18, Chapter 5, §1] states that the Riesz potential is well-defined on $L^p(\mathbb{R}^{d+1})$ for all $p \in (1, +\infty)$ and that it is bounded from $L^p(\mathbb{R}^{d+1})$ into $L^q(\mathbb{R}^{d+1})$ with $q^{-1} = p^{-1} - \alpha(d+1)^{-1}$, i.e.

$$\|\mathcal{I}_\alpha[\varphi]\|_{L^q(\mathbb{R}^{d+1})} \leq c_{p,q} \|\varphi\|_{L^p(\mathbb{R}^{d+1})}, \quad \forall \varphi \in L^p(\mathbb{R}^{d+1}). \tag{5.3}$$

The p.d.o. of zero order $\mathcal{A}$ with the principal symbol $\psi \in C(\mathbb{S}^d)$ is defined by the formula

$$\mathcal{F}[\mathcal{A}[\varphi]](\boldsymbol{\xi}) = \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|) \mathcal{F}[\varphi](\boldsymbol{\xi})$$

for any $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$. The p.d.o. of zero order $\mathcal{R}_j$ with the principal symbol $-i\xi_j/|\boldsymbol{\xi}|$ is called the Riesz transform $\mathcal{R}_j$ $(j = 0, \dots, d)$ [18, Chapter 3]. The p.d.o.'s of zero order are well-defined and bounded on $L^p(\mathbb{R}^{d+1})$ for all $p \in (1, +\infty)$, and the bound

$$\|\mathcal{A}[\varphi]\|_{L^p(\mathbb{R}^{d+1})} \leq c_p \|\varphi\|_{L^p(\mathbb{R}^{d+1})} \quad \forall \varphi \in L^p(\mathbb{R}^{d+1}) \tag{5.4}$$

holds true [18, Chapter 3, Theorem 3].

On the strength of the calculus of p.d.o.'s, the Riesz potentials and the p.d.o.'s of zero order commutate with each other and with the partial differentiating, are self-adjoint in $L^2(\mathbb{R}^{d+1})$, and the following identities hold for all admissible test functions φ (for example, for $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$):

$$(\mathcal{I}_\alpha \circ \mathcal{I}_\beta)[\varphi] = \mathcal{I}_{\alpha+\beta}[\varphi] \quad \forall\, \alpha, \beta, \alpha+\beta \in (1, d+1), \tag{5.5}$$

$$\mathcal{I}_1[\partial_{x_j}\varphi] = \mathcal{R}_j[\varphi], \quad j = 0, \dots, d, \ x_0 := t. \tag{5.6}$$

The Sobolev embedding theorem and the above stated properties of the Riesz potentials imply the following [18, Chapter 5, Theorem 2].

Lemma 5. *If $p > d+1$ then the Riesz potential $\mathcal{I}_1$ is relatively compact from $L^p_{loc}(\mathbb{R}^{d+1})$ into $C_{loc}(\mathbb{R}^{d+1})$. If $1 < p \le d+1$, then the Riesz potential $\mathcal{I}_1$ is relatively compact from $L^p_{loc}(\mathbb{R}^{d+1})$ into $L^q_{loc}(\mathbb{R}^{d+1})$ for any $q \in [1, p(d+1)(d+1-p)^{-1})$.*

Finishing this review of the calculus of p.d.o.'s, we remark that, applying Parseval's theorem to the equality (3.10), we can equivalently define H-measures by the formula

$$\int_{\Pi\times\mathbb{S}^d} \Phi_1 \bar{\Phi}_2 \psi d\mu^{pq}(\boldsymbol{x}, t, \boldsymbol{y}) = \lim_{k\nearrow\infty} \int_\Pi \Phi_1(f^k(p) - f(p))\overline{\mathcal{A}[\Phi_2(f^k(q) - f(q))]} d\boldsymbol{x}dt, \tag{5.7}$$

where $\mathcal{A}$ is the p.d.o. of zero order in $\mathbb{R}^{d+1}$ with the principal symbol ψ.

6 Proof of Theorem 3. Part II: derivation of equality (4.1)

We denote $U_k^\lambda(\boldsymbol{x}, t) := f^k(\boldsymbol{x}, t, \lambda) - f(\boldsymbol{x}, t, \lambda)$ for simplicity of notations. On the strength of the limiting relations (3.1), (3.2) and (5.1), we deduce from (2.2) that

$$\int_{\Pi\times\mathbb{R}_\lambda} \left(\zeta_t + a_{i\lambda}(\boldsymbol{x}, t, \lambda)\zeta_{x_i} - a_{ix_i}(\boldsymbol{x}, t, \lambda)\zeta_\lambda + b'(\lambda)\partial_{x_i}(a_{ij}(\boldsymbol{x}, t)\zeta_{x_j})\right) U_k^\lambda d\boldsymbol{x}dtd\lambda + \int_{\Pi\times\mathbb{R}_\lambda} \zeta_\lambda dH_k = 0, \tag{6.1}$$

where $H_k := m^k + b'(\lambda)n^k - H$ and ζ is a test function defined in (2.2).

Let us multiply this integral equality by the factor $\int_{\mathbb{R}_p} \zeta_0(p)dp$, where $\zeta_0 \in C_0^2(\mathbb{R})$ is arbitrary. Since the linear span of the set $\{\zeta(\boldsymbol{x}, t, \lambda)\zeta_0(p)\}$ is dense in

$C_0^2(\Pi \times \mathbb{R}^2_{\lambda,p})$, equality (6.1) yields that

$$\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} (\zeta_t + a_{i\lambda}(\boldsymbol{x},t,\lambda)\zeta_{x_i} - a_{ix_i}(\boldsymbol{x},t,\lambda)\zeta_\lambda + b'(\lambda)\partial_{x_i}(a_{ij}(\boldsymbol{x},t)\zeta_{x_j})\big) U_k^\lambda d\boldsymbol{x}dtd\lambda dp + \int_{\mathbb{R}_p}\int_{\Pi\times\mathbb{R}_\lambda} \zeta_\lambda dH_k dp = 0, \quad (6.2)$$

where $\zeta = \zeta(\boldsymbol{x},t,\lambda,p)$ is smooth and finite.

The rest of the proof of validity of (4.1) is based on a special choice of the test function in (6.2) and on the limiting transition, as $k \nearrow +\infty$.

We take ζ in the form

$$\zeta(\boldsymbol{x},t,\lambda,p) = \zeta_1(\boldsymbol{x},t)\zeta_2(\lambda)(\mathcal{I}_2\circ\mathcal{A})[\zeta_3(\cdot,\cdot,p)U_k^p](\boldsymbol{x},t), \quad (6.3)$$

where $\zeta_1 \in C_0^2(\Pi)$, $\zeta_2 \in C_0^2(\mathbb{R})$, and $\zeta_3 \in C_0^2(\Pi\times\mathbb{R}_p)$ are arbitrary, and $\mathcal{A}$ is the p.d.o. of zero order with an arbitrary principal symbol $\psi \in C^1(\mathbb{S}^d)$. In view of the properties of pseudo-differential operators, that were stated in Section 5, this choice of the test function is legal because all the integrals in (6.2) are well defined.

Applying formulas (5.5) and (5.6) we obtain

$$\begin{aligned}\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} &(\zeta_{1t}\zeta_2(\mathcal{I}_2\circ\mathcal{A})[\zeta_3U_k^p] + \zeta_1\zeta_2(\mathcal{I}_1\circ\mathcal{A}\circ\mathcal{R}_0)[\zeta_3U_k^p]\\
&+ a_{i\lambda}(\boldsymbol{x},t,\lambda)\zeta_{1x_i}\zeta_2(\mathcal{I}_2\circ\mathcal{A})[\zeta_3U_k^p] + a_{i\lambda}(\boldsymbol{x},t,\lambda)\zeta_1\zeta_2(\mathcal{I}_1\circ\mathcal{A}\circ\mathcal{R}_i)[\zeta_3U_k^p]\\
&- a_{ix_i}(\boldsymbol{x},t,\lambda)\zeta_1\zeta_{2\lambda}(\mathcal{I}_2\circ\mathcal{A})[\zeta_3U_k^p] + b'(\lambda)a_{ijx_i}(\boldsymbol{x},t)\zeta_{1x_j}\zeta_2(\mathcal{I}_2\circ\mathcal{A})[\zeta_3U_k^p]\\
&+ b'(\lambda)a_{ijx_i}(\boldsymbol{x},t)\zeta_1\zeta_2(\mathcal{I}_1\circ\mathcal{A}\circ\mathcal{R}_j)[\zeta_3U_k^p] + b'(\lambda)a_{ij}(\boldsymbol{x},t)\zeta_{1x_ix_j}\zeta_2(\mathcal{I}_2\circ\mathcal{A})[\zeta_3U_k^p]\\
&+ 2b'(\lambda)a_{ij}(\boldsymbol{x},t)\zeta_{1x_i}\zeta_2(\mathcal{I}_1\circ\mathcal{A}\circ\mathcal{R}_j)[\zeta_3U_k^p]\\
&+ b'(\lambda)a_{ij}(\boldsymbol{x},t)\zeta_1\zeta_2(\mathcal{A}\circ\mathcal{R}_i\circ\mathcal{R}_j)[\zeta_3U_k^p]\big) U_k^\lambda d\boldsymbol{x}dtd\lambda dp\\
&+ \int_{\mathbb{R}_p}\int_{\Pi\times\mathbb{R}_\lambda} \zeta_1\zeta_{2\lambda}(\mathcal{I}_2\circ\mathcal{A})[\zeta_3U_k^p]dH_k(\boldsymbol{x},t,\lambda)dp = 0. \quad (6.4)\end{aligned}$$

On the strength of Lemma 2, $U_k^p \to 0$ weakly* in $L^\infty(\Pi)$ for any $p \in \mathcal{E}$ and for a.e. $p \in \mathbb{R}$, as $k \nearrow +\infty$. Using this limiting relation, applying Lebesgue's dominated convergence theorem, Lemma 4 and Lemma 5, and extracting a proper subsequence from $\{k\} \subset \mathbb{N}$, if necessary, we arrive at the equality

$$\int_{\mathbb{R}^2_{\lambda,p}} \lim_{k\nearrow+\infty}\int_\Pi b'(\lambda)a_{ij}(\boldsymbol{x},t)\zeta_1\zeta_2(\mathcal{A}\circ\mathcal{R}_i\circ\mathcal{R}_j)[\zeta_3U_k^p]U_k^\lambda d\boldsymbol{x}dtd\lambda dp = 0. \quad (6.5)$$

Using Theorem H and the fact that $\mathcal{A}\circ\mathcal{R}_i\circ\mathcal{R}_j$ is the p.d.o. of zero order with the principal symbol $-\psi(\boldsymbol{y})y_iy_j$, where $\boldsymbol{y} \in \mathbb{S}^d$, we derive from (6.5) that the equality

$$\int_{\mathbb{R}^2_{\lambda,p}}\int_{\Pi\times\mathbb{S}^d} b'(\lambda)a_{ij}(\boldsymbol{x},t)\zeta_1(\boldsymbol{x},t)\zeta_2(\lambda)\zeta_3(\boldsymbol{x},t,p)\psi(\boldsymbol{y})y_iy_jd\mu^{p\lambda}(\boldsymbol{x},t,\boldsymbol{y})d\lambda dp = 0 \quad (6.6)$$

holds for all functions ζ_1, ζ_2, ζ_3, and ψ that were defined in (6.3).

Now, we notice that the linear span of the set $\{\zeta_2(\lambda)\zeta_3(\boldsymbol{x},t,p)\}$ is dense in the space $\{\zeta_4 \in C_0^2(\Pi \times \mathbb{R}^2_{\lambda,p})\}$ and we take Kruzhkov's test function [7] for ζ_4,

$$\zeta_4^\varepsilon(\boldsymbol{x},t,\lambda,p) := \frac{1}{\varepsilon}\zeta_5(\boldsymbol{x},t)\zeta_6\left(\frac{\lambda-p}{\varepsilon}\right)\zeta_7\left(\frac{\lambda+p}{2}\right), \quad \varepsilon > 0, \tag{6.7}$$

where ζ_5 is smooth and finite in Π, ζ_6 is nonnegative even infinitely smooth and has a compact support in $[-1,1]$ and the mean value equal to one, i.e. $\int \zeta_6(\lambda)d\lambda = 1$, and ζ_7 is smooth and finite in $\mathbb{R}$.

Changing the variable p by $\kappa = \frac{p-\lambda}{\varepsilon}$ we deduce from (6.6) that

$$\int_{\mathbb{R}^2_{\lambda,\kappa}}\int_{\Pi\times\mathbb{S}^d} b'(\lambda)a_{ij}\zeta_1\zeta_5\zeta_6(\kappa)\zeta_7\left(\frac{2\lambda+\kappa\varepsilon}{2}\right)\psi(\boldsymbol{y})y_iy_j d\mu^{\lambda(\lambda+\kappa\varepsilon)}(\boldsymbol{x},t,\boldsymbol{y})d\lambda d\kappa = 0. \tag{6.8}$$

On the strength of the properties of test functions ζ_6 and ζ_7, Lemma 2, item 2 of Lemma 3, and Lebesgue's dominated convergence theorem, passing to the limit as $\varepsilon \searrow 0$ we derive from (6.8) that

$$\int_{\mathbb{R}_\lambda}\int_{\Pi\times\mathbb{S}^d} b'(\lambda)a_{ij}(\boldsymbol{x},t)\zeta_1(\boldsymbol{x},t)\zeta_5(\boldsymbol{x},t)\zeta_7(\lambda)\psi(\boldsymbol{y})y_iy_j d\mu^{\lambda\lambda}(\boldsymbol{x},t,\boldsymbol{y})d\lambda = 0,$$

which easily yields (4.1) due to arbitrariness of ζ_1, ζ_5, ζ_7, and ψ.

7 Proof of Theorem 3. Part III: derivation of equality (4.2)

Let us introduce the mollifying kernel $\omega \in C_0^\infty(\mathbb{R})$ which has the same properties, as the function ζ_6 that was defined in the previous section. We denote

$$\omega_h(\boldsymbol{x}) := \frac{1}{h^d}\omega\left(\frac{x_1}{h}\right)\dots\omega\left(\frac{x_d}{h}\right), \quad (\dots)_h := (\dots) * \omega_h,$$

$$U^p_{k,h}(\boldsymbol{x},t) := (U^p_k * \omega_h)(\boldsymbol{x},t) = \int_{\mathbb{R}^d}\omega_h(\boldsymbol{x}-\tilde{\boldsymbol{x}})U^p_k(\tilde{\boldsymbol{x}},t)d\tilde{\boldsymbol{x}},$$

and set

$$\zeta(\boldsymbol{x},t,\lambda,p) = b'(p)\big(\zeta_1(\mathcal{I}_1\circ\mathcal{A})[\zeta_2 U^p_{k,h}]\big) * \omega_h, \tag{7.1}$$

where $\zeta_1 = \zeta_1(\boldsymbol{x},t,p,\lambda)$ and $\zeta_2 = \zeta_2(\boldsymbol{x},t)$ are arbitrary finite smooth functions such that ζ_1 is symmetric in λ and p, i.e. $\zeta_1(\boldsymbol{x},t,\lambda,p) = \zeta_1(\boldsymbol{x},t,p,\lambda)$ for all λ and p, and $\mathcal{A}$ is the p.d.o. of zero order with an arbitrary principal symbol $\psi \in C^1(\mathbb{S}^d)$.

Since $U^p_{k,h}$ is infinitely smooth in $\boldsymbol{x}$, the defined by (7.1) function ζ is a legal test function for (6.2). Substituting this function into (6.2) and using (5.5), (5.6) and the well known property of mollifying kernels $\langle\varphi_{1h},\varphi_2\rangle=\langle\varphi_1,\varphi_{2h}\rangle$ we deduce from (6.2) that

$$\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} b'(p)\Big(U^\lambda_{k,h}\zeta_{1t}(\mathcal{I}_1\circ\mathcal{A})[\zeta_2U^p_{k,h}]+U^\lambda_{k,h}\zeta_1(\mathcal{A}\circ\mathcal{R}_0)[\zeta_2U^p_{k,h}]$$
$$+(U^\lambda_k a_{i\lambda})_h\zeta_{1x_i}(\mathcal{I}_1\circ\mathcal{A})[\zeta_2U^p_{k,h}]+(U^\lambda_k a_{i\lambda})_h\zeta_1(\mathcal{A}\circ\mathcal{R}_i)[\zeta_2U^p_{k,h}]$$
$$-(U^\lambda_k a_{ix_i})_h\zeta_{1\lambda}(\mathcal{I}_1\circ\mathcal{A})[\zeta_2U^p_k]+b'(\lambda)(U^\lambda_k a_{ijx_i})_h\zeta_{1x_j}(\mathcal{I}_1\circ\mathcal{A})[\zeta_2U^p_{k,h}]$$
$$+b'(\lambda)(U^\lambda_k a_{ijx_i})_h\zeta_1(\mathcal{A}\circ\mathcal{R}_j)[\zeta_2U^p_{k,h}]$$
$$+2b'(\lambda)(U^\lambda_k a_{ij})_h\zeta_1\partial_{x_i}(\mathcal{A}\circ\mathcal{R}_j)[\zeta_2U^p_{k,h}]$$
$$+2b'(\lambda)(U^\lambda_k a_{ij})_h\zeta_{1x_i}(\mathcal{A}\circ\mathcal{R}_j)[\zeta_2U^p_{k,h}]$$
$$+b'(\lambda)(U^\lambda_k a_{ij})_h\zeta_{1x_ix_j}(\mathcal{I}_1\circ\mathcal{A})[\zeta_2U^p_{k,h}]\Big)\,d\boldsymbol{x}dtd\lambda dp$$
$$+\int_{\mathbb{R}_p}\int_{\Pi\times\mathbb{R}_\lambda} b'(p)\Big(\zeta_{1\lambda}(\mathcal{I}_1\circ\mathcal{A})[\zeta_2U^p_{k,h}]\Big)_h\,dH_k(\boldsymbol{x},t,\lambda)dp=0. \quad (7.2)$$

Except for the terms involving the derivatives $\partial_{x_i}(\mathcal{A}\circ\mathcal{R}_j)[\zeta_2U^p_{k,h}]$, all the integrals in (7.2) are well defined, if we substitute $U^p_{k,h}$ by U^p_k and $(\ldots)_h$ by $(\ldots)$ in them. Since $U^p_{k,h}(\cdot,t)\underset{h\searrow 0}{\to}U^p_k(\cdot,t)$ strongly in $L^1_{loc}(\mathbb{R}^d)$, we conclude that all these integrals converge to the integrals of the same forms with U^p_k on the places of $U^p_{k,h}$ and $(\ldots)$ on the places of $(\ldots)_h$. The limiting transition as $h\searrow 0$ (and, after that, as $k\nearrow+\infty$) in the integrals involving the derivatives $\partial_{x_i}(\mathcal{A}\circ\mathcal{R}_j)[\zeta_2U^p_{k,h}]$ are based on three following lemmas.

Lemma 6. *Let $\mathcal{A}$ be the p.d.o. of zero order with a principle symbol $\psi\in C^1(\mathbb{S}^d)$, let $\mathcal{B}$: $L^2(\mathbb{R}^{d+1})\mapsto L^2(\mathbb{R}^{d+1})$ be the operator of multiplication on a function $B\in C^2_0(\mathbb{R}^{d+1}_{x,t})$, i.e. $\mathcal{B}[\varphi](\boldsymbol{x},t)=B(\boldsymbol{x},t)\varphi(\boldsymbol{x},t)$, $\forall\,\varphi\in L^2(\mathbb{R}^{d+1})$.*

Then, the commutator $[\mathcal{A},\mathcal{B}]:=\mathcal{A}\circ\mathcal{B}-\mathcal{B}\circ\mathcal{A}$ is a continuous operator from $L^2(\mathbb{R}^{d+1}_{x,t})$ into $W^1_2(\mathbb{R}^{d+1}_{x,t})$, and the operator $\varphi\mapsto\partial_{x_i}[\mathcal{A},\mathcal{B}][\varphi]$ ($i=0,\ldots,d$) has the structure

$$\partial_{x_i}[\mathcal{A},\mathcal{B}][\varphi]=(\mathcal{A}_{ij}\circ\mathcal{B}_j)[\varphi]+\mathcal{C}_i[\varphi]\quad\forall\,\varphi\in L^2(\mathbb{R}^{d+1}), \quad (7.3)$$

where the summing over j is fulfilled from $j=0$ to $j=d$, $\mathcal{A}_{ij}$ is the p.d.o. of zero order with the principal symbol $\psi_{ij}\in C(\mathbb{S}^d)$, which is defined by the principal symbol of $\mathcal{A}$ via the formula

$$\psi_{ij}(\boldsymbol{\xi}/|\boldsymbol{\xi}|)=\xi_i\frac{\partial\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)}{\partial\xi_j},\quad \boldsymbol{\xi}\in\mathbb{R}^{d+1}, \quad (7.4)$$

$\mathcal{B}_j$ is the operator of multiplication on the function $\partial_{x_j}B$ ($x_0:=t$), and $\mathcal{C}_i$: $L^2(\mathbb{R}^{d+1})\mapsto L^2(\mathbb{R}^{d+1})$ is a compact operator.

Remark 3. *In terms of variables $y_i:=(\xi_i/|\boldsymbol{\xi}|)\in\mathbb{S}^d$, the formula (7.4) takes the shape $\psi_{ij}(\boldsymbol{y})=y_i(\delta_{jl}+y_jy_l)\partial_{y_l}\psi(\boldsymbol{y})$, were the summing over l is fulfilled from $l=0$ to $l=d$,* Lemma 6 was proved in [13].

Lemma 7. *The identity*

$$
\begin{aligned}
2\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} & b'(\lambda)b'(p)U^\lambda_{k,h}a_{ij}\zeta_1\partial_{x_i}(\mathcal{A}\circ\mathcal{R}_j)[\zeta_2U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&= \int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h}b'(\lambda)\Big(a_{ij}\zeta_1\partial_{x_i}[\mathcal{A}\circ\mathcal{R}_j,\mathcal{Z}_2][b'(p)\chi U^p_{k,h}] \\
&+ a_{ij}\zeta_1\zeta_{2x_i}(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U^p_{k,h}] - (a_{ij}\zeta_1)_{x_i}\zeta_2(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U^p_{k,h}] \\
&\qquad -\zeta_2\partial_{x_i}[\mathcal{A}\circ\mathcal{R}_j,\mathcal{Z}_{1ij}][b'(p)\chi U^p_{k,h}]\Big)\, d\boldsymbol{x}dtd\lambda dp \qquad (7.5)
\end{aligned}
$$

holds true for all ζ_1, ζ_2 *and* $\mathcal{A}$ *introduced in* (7.1).

In the formulation of Lemma 7, $\mathcal{Z}_{1ij}$ and $\mathcal{Z}_2$ are the operators of multiplication on the functions $a_{ij}\zeta_1$ and ζ_2, respectively, and $\chi(\boldsymbol{x},t) = \chi^{\lambda,p}(\boldsymbol{x},t) = 1_{\mathrm{supp}\,\zeta_1\cap\mathrm{supp}\,\zeta_2}(\boldsymbol{x},t)$. Notice that $\chi\zeta_1 = \zeta_1$ and $\chi\zeta_2 = \zeta_2$.

Proof. The following chain of equalities holds true due to the properties of p.d.o.'s from Section 5.

$$
\begin{aligned}
&\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} b'(\lambda)b'(p)U^\lambda_{k,h}a_{ij}\zeta_1\partial_{x_i}(\mathcal{A}\circ\mathcal{R}_j)[\zeta_2U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \qquad (7.6) \\
&= -\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}(U^\lambda_{k,h}b'(\lambda)a_{ij}\zeta_1)(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\zeta_2U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&= -\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}(U^\lambda_{k,h}b'(\lambda)a_{ij}\zeta_1)[\mathcal{A}\circ\mathcal{R}_j,\mathcal{Z}_2][b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&-\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}(U^\lambda_{k,h}b'(\lambda)a_{ij}\zeta_1)\zeta_2(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&= \int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h}b'(\lambda)a_{ij}\zeta_1\partial_{x_i}[\mathcal{A}\circ\mathcal{R}_j,\mathcal{Z}_2][b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&-\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}(U^\lambda_{k,h}b'(\lambda)a_{ij}\zeta_1\zeta_2)(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&+\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h}b'(\lambda)a_{ij}\zeta_1\zeta_{2x_i}(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h}b'(\lambda)a_{ij}\zeta_1\partial_{x_i}[\mathcal{A}\circ\mathcal{R}_j,\mathcal{Z}_2][b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&+\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h}b'(\lambda)a_{ij}\zeta_1\zeta_{2x_i}(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&-\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h}b'(\lambda)a_{ijx_i}\zeta_1\zeta_2(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h} b'(\lambda) a_{ij}\zeta_{1x_i}\zeta_2(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&+ \int_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}(U^\lambda_{k,h} b'(\lambda)\zeta_2)[\mathcal{A}\circ\mathcal{R}_j, \mathcal{Z}_{1ij}][b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&- \int_{\Pi\times\mathbb{R}^2_{\lambda,p}} \partial_{x_i}(U^\lambda_{k,h} b'(\lambda)\zeta_2)(\mathcal{A}\circ\mathcal{R}_j)[b'(p)a_{ij}\zeta_1 U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&= \int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h} b'(\lambda) a_{ij}\zeta_1\partial_{x_i}[\mathcal{A}\circ\mathcal{R}_j, \mathcal{Z}_2][b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&+ \int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h} b'(\lambda) a_{ij}\zeta_1\zeta_{2x_i}(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&- \int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h} b'(\lambda) a_{ijx_i}\zeta_1\zeta_2(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&- \int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h} b'(\lambda) a_{ij}\zeta_{1x_i}\zeta_2(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^\lambda_{k,h} b'(\lambda)\zeta_2\partial_{x_i}[\mathcal{A}\circ\mathcal{R}_j, \mathcal{Z}_{1ij}][b'(p)\chi U^p_{k,h}]d\boldsymbol{x}dtd\lambda dp \\
&- \left(\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U^p_{k,h} b'(p) a_{ij}\zeta_1\partial_{x_i}(\mathcal{A}\circ\mathcal{R}_j)[b'(\lambda)\zeta_2 U^\lambda_{k,h}]d\boldsymbol{x}dtd\lambda dp\right). \quad (7.7)
\end{aligned}
$$

Notice that the integral in the brackets in (7.7) and the integral (7.6) are the same to the renaming of variables p and λ since ζ_1 is symmetric in p and λ. Thus, the above chain of equalities yields the identity (7.5). □

Lemma 8. *The limiting relation*

$$
\partial_{x_i}\left((U^\lambda_k a_{ij})_h - U^\lambda_{k,h} a_{ij}\right) \underset{h\searrow 0}{\longrightarrow} 0 \text{ strongly in } L^p_{loc}(\Pi) \quad (7.8)
$$

holds true for any $p < +\infty$.

Proof. This lemma immediately follows from [19, Lemma II.1]. □

Now, let us turn back to the limiting transition in (7.2), as $h \searrow 0$. We have

$$\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} 2b'(p)b'(\lambda)(U_k^\lambda a_{ij})_h \zeta_1 \partial_{x_i}(\mathcal{A}\circ\mathcal{R}_j)[\zeta_2 U_{k,h}^p] d\boldsymbol{x}dtd\lambda dp$$
$$= -\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} 2b'(p)b'(\lambda)\partial_{x_i}\left((U_k^\lambda a_{ij})_h - U_{k,h}^\lambda a_{ij}\right)\zeta_1(\mathcal{A}\circ\mathcal{R}_j)[\zeta_2 U_{k,h}^p] d\boldsymbol{x}dtd\lambda dp$$
$$+\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} 2b'(p)b'(\lambda)U_{k,h}^\lambda a_{ij}\zeta_1 \partial_{x_i}(\mathcal{A}\circ\mathcal{R}_j)[\zeta_2 U_{k,h}^p] d\boldsymbol{x}dtd\lambda dp \tag{7.9}$$
$$= -\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} 2b'(p)b'(\lambda)\partial_{x_i}\left((U_k^\lambda a_{ij})_h - U_{k,h}^\lambda a_{ij}\right)\zeta_1(\mathcal{A}\circ\mathcal{R}_j)[\zeta_2 U_{k,h}^p] d\boldsymbol{x}dtd\lambda dp \tag{7.10}$$
$$+\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_{k,h}^\lambda b'(\lambda)a_{ij}\zeta_1\partial_{x_i}[\mathcal{A}\circ\mathcal{R}_j,\mathcal{Z}_2][b'(p)\chi U_{k,h}^p] d\boldsymbol{x}dtd\lambda dp \tag{7.11}$$
$$+\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_{k,h}^\lambda b'(\lambda)a_{ij}\zeta_1\zeta_{2x_i}(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U_{k,h}^p] d\boldsymbol{x}dtd\lambda dp \tag{7.12}$$
$$-\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_{k,h}^\lambda b'(\lambda)(a_{ij}\zeta_1)_{x_i}\zeta_2(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U_{k,h}^p] d\boldsymbol{x}dtd\lambda dp \tag{7.13}$$
$$-\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_{k,h}^\lambda b'(\lambda)\zeta_2\partial_{x_i}[\mathcal{A}\circ\mathcal{R}_j,\mathcal{Z}_{1ij}][b'(p)\chi U_{k,h}^p] d\boldsymbol{x}dtd\lambda dp \tag{7.14}$$

$$\xrightarrow[h\searrow 0]{}$$
$$\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_k^\lambda b'(\lambda)a_{ij}\zeta_1\partial_{x_i}[\mathcal{A}\circ\mathcal{R}_j,\mathcal{Z}_2][b'(p)\chi U_k^p] d\boldsymbol{x}dtd\lambda dp \tag{7.15}$$
$$+\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_k^\lambda b'(\lambda)a_{ij}\zeta_1\zeta_{2x_i}(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U_k^p] d\boldsymbol{x}dtd\lambda dp \tag{7.16}$$
$$-\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_k^\lambda b'(\lambda)(a_{ij}\zeta_1)_{x_i}\zeta_2(\mathcal{A}\circ\mathcal{R}_j)[b'(p)\chi U_k^p] d\boldsymbol{x}dtd\lambda dp \tag{7.17}$$
$$-\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} U_k^\lambda b'(\lambda)\zeta_2\partial_{x_i}[\mathcal{A}\circ\mathcal{R}_j,\mathcal{Z}_{1ij}][b'(p)\chi U_k^p] d\boldsymbol{x}dtd\lambda dp \tag{7.18}$$

On the strength of Lemma 8, the integral (7.10) vanishes, as $h \searrow 0$. On the strength of Lemmas 6 and 7, (7.9) has the representation (7.11)–(7.14) and tends to the sum (7.15)–(7.18), as $h \searrow 0$. Aggregating all the above established limiting relations, we deduce from (7.2) that the following integral equality holds true, as $h \searrow 0$:

$$\int_{\Pi\times\mathbb{R}^2_{\lambda,p}} b'(p)\left(U_k^\lambda\zeta_{1t}(\mathcal{I}_1\circ\mathcal{A})[\zeta_2U_k^p] + U_k^\lambda\zeta_1(\mathcal{A}\circ\mathcal{R}_0)[\zeta_2U_k^p]\right.$$
$$+ U_k^\lambda a_{i\lambda}\zeta_{1x_i}(\mathcal{I}_1\circ\mathcal{A})[\zeta_2U_k^p] + U_k^\lambda a_{i\lambda}\zeta_1(\mathcal{A}\circ\mathcal{R}_i)[\zeta_2U_k^p]$$
$$- U_k^\lambda a_{ix_i}\zeta_{1\lambda}(\mathcal{I}_1\circ\mathcal{A})[\zeta_2U_k^p] + b'(\lambda)U_k^\lambda a_{ijx_i}\zeta_{1x_j}(\mathcal{I}_1\circ\mathcal{A})[\zeta_2U_k^p]$$
$$+ b'(\lambda)U_k^\lambda a_{ijx_i}\zeta_1(\mathcal{A}\circ\mathcal{R}_j)[\zeta_2U_k^p] + b'(\lambda)U_k^\lambda a_{ij}\zeta_1\partial_{x_i}[\mathcal{A}\circ\mathcal{R}_j,\mathcal{Z}_2][\chi U_k^p]$$
$$+ b'(\lambda)U_k^\lambda a_{ij}\zeta_1\zeta_{2x_i}(\mathcal{A}\circ\mathcal{R}_j)[\chi U_k^p] - b'(\lambda)U_k^\lambda(a_{ij}\zeta_1)_{x_i}\zeta_2(\mathcal{A}\circ\mathcal{R}_j)[\chi U_k^p]$$
$$- b'(\lambda)U_k^\lambda\zeta_2\partial_{x_i}[\mathcal{A}\circ\mathcal{R}_j,\mathcal{Z}_{1ij}][\chi U_k^p] + 2b'(\lambda)U_k^\lambda a_{ij}\zeta_{1x_i}(\mathcal{A}\circ\mathcal{R}_j)[\zeta_2U_k^p]$$
$$\left.+b'(\lambda)U_k^\lambda a_{ij}\zeta_{1x_ix_j}(\mathcal{I}_1\circ\mathcal{A})[\zeta_2U_k^p]\right)d\boldsymbol{x}dtd\lambda dp$$
$$+\int_{\mathbb{R}_p}\int_{\Pi\times\mathbb{R}_\lambda} b'(p)\zeta_{1\lambda}(\mathcal{I}_1\circ\mathcal{A})[\zeta_2U_k^p]dH_k(\boldsymbol{x},t,\lambda)dp = 0. \quad (7.19)$$

Using Theorem H, Lemmas 4, 5 and 6, and Lebesgue's theorem on dominated convergence, we pass to the limit in (7.19), as $k\nearrow+\infty$, (extracting a proper subsequence $\{k\}\subset\mathbb{N}$, if necessary):

$$\int_{\Pi\times\mathbb{S}^d\times\mathbb{R}^2_{\lambda,p}} b'(p)\left(\zeta_1(\boldsymbol{x},t,\lambda,p)\zeta_2(\boldsymbol{x},t)\psi(\boldsymbol{y})y_0\right.$$
$$+ a_{i\lambda}(\boldsymbol{x},t,\lambda)\zeta_1(\boldsymbol{x},t,\lambda,p)\zeta_2(\boldsymbol{x},t)\psi(\boldsymbol{y})y_i$$
$$+ b'(\lambda)a_{ij}(\boldsymbol{x},t)\zeta_1(\boldsymbol{x},t,\lambda,p)\left(\psi_{y_r}(\boldsymbol{y}) + y_ry_l\psi_{y_l}(\boldsymbol{y}) + y_r\psi(\boldsymbol{y})\right)y_iy_j\zeta_{2x_r}(\boldsymbol{x},t)$$
$$+ 2b'(\lambda)a_{ij}(\boldsymbol{x},t)\zeta_1(\boldsymbol{x},t,\lambda,p)\zeta_{2x_i}(\boldsymbol{x},t)\psi(\boldsymbol{y})y_j$$
$$- b'(\lambda)a_{ijx_r}(\boldsymbol{x},t)\zeta_1(\boldsymbol{x},t,\lambda,p)\left(\psi_{y_r}(\boldsymbol{y}) + y_ry_l\psi_{y_l}(\boldsymbol{y}) + y_r\psi(\boldsymbol{y})\right)y_iy_j\zeta_2(\boldsymbol{x},t)$$
$$- b'(\lambda)a_{ij}(\boldsymbol{x},t)\zeta_{1x_r}(\boldsymbol{x},t,\lambda,p)\left(\psi_{y_r}(\boldsymbol{y}) + y_ry_l\psi_{y_l}(\boldsymbol{y}) + y_r\psi(\boldsymbol{y})\right)y_iy_j\zeta_2(\boldsymbol{x},t)$$
$$\left.-b'(\lambda)a_{ijx_j}(\boldsymbol{x},t)\zeta_1(\boldsymbol{x},t,\lambda,p)\zeta_2(\boldsymbol{x},t)\psi(\boldsymbol{y})y_i\right)d\mu^{\lambda p}(\boldsymbol{x},t,\boldsymbol{y})d\lambda dp = 0. \quad (7.20)$$

Choosing the test function $\zeta_2 = \zeta_{2N}$ such that $\|\zeta_{2N}\|_{C^1(\Pi)} \le c$ and $\zeta_{2N}\to 1$ pointwise in Π, as $N\nearrow+\infty$, on the strength of Lebesgue's dominated convergence theorem, we easily conclude that $\zeta_2\equiv 1$ is a valid test function for (7.20). In turn, we take the test function ζ_1 in the form (6.7) which is a valid choice. Passing to the limit, as $\varepsilon\searrow 0$, and using the same arguments, as in Section 6, we derive from (7.20) that the equality

$$\int_{\Pi\times\mathbb{S}^d\times\mathbb{R}_\lambda}(b'(\lambda))^2\zeta_5\zeta_7(\lambda)\psi(\boldsymbol{y})\left(y_0 + (a_{i\lambda} - a_{ijx_j}y_i)\right)d\mu^{\lambda\lambda}(\boldsymbol{x},t,\boldsymbol{y})d\lambda$$
$$-\int_{\Pi\times\mathbb{S}^d\times\mathbb{R}_\lambda}(b'(\lambda))^2\zeta_{5x_r}\zeta_7(\lambda)a_{ij}\left(\psi_{y_r}(\boldsymbol{y})\right.$$
$$\left.+y_ry_l\psi_{y_l}(\boldsymbol{y}) + y_r\psi(\boldsymbol{y})\right)y_iy_jd\mu^{\lambda\lambda}(\boldsymbol{x},t,\boldsymbol{y})d\lambda$$
$$-\int_{\Pi\times\mathbb{S}^d\times\mathbb{R}_\lambda}(b'(\lambda))^2\zeta_5\zeta_7(\lambda)a_{ijx_r}\left(\psi_{y_r}(\boldsymbol{y})\right.$$
$$\left.+y_ry_l\psi_{y_l}(\boldsymbol{y}) + y_r\psi(\boldsymbol{y})\right)y_iy_jd\mu^{\lambda\lambda}(\boldsymbol{x},t,\boldsymbol{y})d\lambda = 0 \quad (7.21)$$

holds for arbitrary functions $\zeta_5 \in C_0^1(\Pi)$ and $\zeta_7 \in C_0(\mathbb{R})$.

The second integral in (7.21) vanishes due to equality (4.1). The third integral in (7.21) has the representation

$$\int_{\Pi\times\mathbb{S}^d\times\mathbb{R}_\lambda} (b'(\lambda))^2 \zeta_5(\boldsymbol{x},t)\zeta_7(\lambda) a_{ijx_r}(\boldsymbol{x},t) \left(\psi_{y_r}(\boldsymbol{y}) + y_r y_l \psi_{y_l}(\boldsymbol{y}) + y_r \psi(\boldsymbol{y})\right) y_i y_j \, d\mu^{\lambda\lambda}(\boldsymbol{x},t,\boldsymbol{y}) d\lambda$$
$$= \int_{\mathbb{R}_\lambda} (b'(\lambda))^2 \zeta_7(\lambda) \int_\Pi \zeta_5(\boldsymbol{x},t) \int_{\mathbb{S}^d} a_{ijx_r}(\boldsymbol{x},t) \left(\psi_{y_r}(\boldsymbol{y}) + y_r y_l \psi_{y_l}(\boldsymbol{y}) + y_r \psi(\boldsymbol{y})\right) y_i y_j \, d\sigma_{x,t}^{\lambda\lambda}(\boldsymbol{y}) d\boldsymbol{x} dt d\lambda \quad (7.22)$$

on the strength of item 4 of Lemma 3. On the strength of item 4 of Lemma 3 and equality (4.1), we have that the support of the measure $\sigma_{x,t}^{\lambda\lambda}$ lays entirely in the set $\{\boldsymbol{y} \in \mathbb{S}^d : a_{ij}(\boldsymbol{x},t) y_i y_j = 0\}$ for a.e. $(\boldsymbol{x},t) \in \Pi$. On the other hand, for any $\boldsymbol{y} \in \mathbb{S}^d$, the intersection of the sets $\{(\boldsymbol{x},t) \in \Pi : a_{ij}(\boldsymbol{x},t) y_i y_j = 0\}$ and $\{(\boldsymbol{x},t) \in \Pi : a_{ijx_r}(\boldsymbol{x},t) y_i y_j \neq 0\}$ has zero Lebesgue measure. This observation and the fact that $\sigma^{\lambda\lambda} \in L^2_w(\Pi, \mathbb{M}(\mathbb{S}^d))$ imply that the function

$$(\boldsymbol{x},t) \mapsto \int_{\mathbb{S}^d} a_{ijx_r}(\boldsymbol{x},t) \left(\psi_{y_r}(\boldsymbol{y}) + y_r y_l \psi_{y_l}(\boldsymbol{y}) + y_r \psi(\boldsymbol{y})\right) y_i y_j \, d\sigma_{x,t}^{\lambda\lambda}(\boldsymbol{y})$$

vanishes for a.e. $(\boldsymbol{x},t) \in \Pi$. Thus, the third integral in (7.21) vanishes. Hence, (7.21) yields that

$$\int_{\Pi\times\mathbb{S}^d\times\mathbb{R}_\lambda} (b'(\lambda))^2 \zeta_5 \zeta_7 \psi \left(y_0 + (a_{i\lambda}(\boldsymbol{x},t,\lambda) - a_{ijx_j}(\boldsymbol{x},t)) y_i\right) d\mu^{\lambda\lambda} d\lambda = 0 \quad (7.23)$$

which, in turn, easily yields (4.2) due to arbitrariness of ζ_5, ζ_7 and ψ.

8 Proof of Theorem 2

Condition G and Theorem 3 imply that the H-measure $\mu^{\lambda\lambda}$ is zero measure for a.e. $\lambda \in \mathbb{R}$. This fact and item 4 of Lemma 3 yield that $f^k(\cdot,\cdot,\lambda) \underset{k\nearrow+\infty}{\longrightarrow} f(\cdot,\cdot,\lambda)$ strongly in $L^1_{loc}(\Pi)$ for a.e. $\lambda \in \mathbb{R}$ and therefore pointwise in $\Pi \times \mathbb{R}_\lambda$. Since f^k takes only two values, 0 and 1, and f is monotonous nondecreasing and right continuous in λ for a.e. $(\boldsymbol{x},t)$, and $f \equiv 0$ for $\lambda < -u_*$ and $f \equiv 1$ for $\lambda \geq u_*$, this means that f has the form

$$f(\boldsymbol{x},t,\lambda) = \begin{cases} 1, & \text{for} \quad \lambda \geq \tilde{u}(\boldsymbol{x},t), \\ 0, & \text{for} \quad \lambda < \tilde{u}(\boldsymbol{x},t), \end{cases} \quad (8.1)$$

with some function $\tilde{u} \in L^\infty(\Pi)$, $\|\tilde{u}\|_{L^\infty} \leq u_*$. Formula (2.3) and the limiting relations (3.1) and (3.2) yield that $\tilde{u}$ coincide with $u = \text{w-}\lim\limits_{k\nearrow+\infty} u^k$ and that $\|u^k\|_{L^2(\mathcal{Q})} \underset{k\nearrow+\infty}{\longrightarrow} \|u\|_{L^2(\mathcal{Q})}$ for any bounded measurable set $\mathcal{Q} \in \Pi$. In turn, this yields that $u^k \underset{k\nearrow+\infty}{\longrightarrow} u$ strongly in $L^2_{loc}(\Pi)$ and thus in $L^1_{loc}(\Pi)$. Theorem 2 is proved.

9 Proof of Theorem 1

We introduce the parabolic approximation of (1.1)

$$u_t + \partial_{x_i} a_i(\boldsymbol{x}, t, u) - \partial_{x_i}(a_{ij}(\boldsymbol{x}, t)\partial_{x_j} b(u)) - \varepsilon\partial^2_{x_i x_i} u = 0, \quad \varepsilon > 0, \tag{9.1}$$

endowed with initial data (1.1b).

It follows from the general theory of the second order parabolic equations [1] that the problem (9.1), (1.1b) has a unique smooth solution u_ε for any fixed $\varepsilon > 0$. The maximum principle and the first energy estimate imply the inequalities

$$-u_* \le u_\varepsilon \le u_* \text{ a.e. in } \Pi, \tag{9.2}$$

$$\|\alpha_{ij}\nabla_x u_\varepsilon\|^2_{L^2(\mathcal{Q})} + \varepsilon\|\nabla_x u_\varepsilon\|^2_{L^2(\mathcal{Q})} \le c(\mathcal{Q}), \tag{9.3}$$

where $\mathcal{Q} \subset \Pi$ is a bounded domain with sufficiently smooth boundary and the constant $c(\mathcal{Q})$ does not depend on ε.

We observe that (9.1) admits the kinetic formulation (2.1) with

$$dm_\varepsilon(\boldsymbol{x}, t, \lambda) = \varepsilon\partial_{x_i} u_\varepsilon \partial_{x_i} u_\varepsilon d\gamma_{u_\varepsilon(x,t)}(\lambda) d\boldsymbol{x}dt \tag{9.4}$$

and that it is possible to choose a subsequence $\varepsilon = \varepsilon_k$ such that the weak limiting relations (3.1) and (3.2) hold true for u_{ε_k} and f_{ε_k}. Using the same arguments, as in Sections 4–8, we prove that

$$u_{\varepsilon_k} \underset{k\nearrow+\infty}{\longrightarrow} u \text{ strongly in } L^1_{loc}(\Pi) \tag{9.5}$$

after extracting one more subsequence ε_k (if necessary).

Multiplying the both sides of (9.1) on $\zeta\varphi'(u)$, where $\zeta \in C^2(\Pi)$ is an arbitrary nonnegative function vanishing near the plane $\{t = T\}$ and finite in $\boldsymbol{x}$ and $\varphi \in C^2_{loc}(\mathbb{R})$ is an arbitrary convex function, and integrating on Π, we arrive at the equality

$$\begin{aligned}\int_\Pi \big(\zeta_t\varphi(u_\varepsilon) &+ \zeta_{x_i} q_i(\boldsymbol{x}, t, u_\varepsilon) - \zeta\varphi'(u_\varepsilon) D_{x_i} a_i(\boldsymbol{x}, t, u_\varepsilon) \\ &+ \zeta D_{x_i} q_i(\boldsymbol{x}, t, u_\varepsilon) + w(u_\varepsilon)\partial_{x_i}(a_{ij}(\boldsymbol{x}, t)\partial_{x_j}\zeta) \\ &- \zeta\varphi''(u_\varepsilon) b'(u_\varepsilon)(\alpha_{il}(\boldsymbol{x}, t)\partial_{x_i} u_\varepsilon)(\alpha_{lj}(\boldsymbol{x}, t)\partial_{x_j} u_\varepsilon) \\ &+ \varepsilon\varphi(u_\varepsilon)\partial^2_{x_i x_i}\zeta - \varepsilon\zeta\varphi''(u_\varepsilon)\partial_{x_i} u_\varepsilon \partial_{x_i} u_\varepsilon\big)\, d\boldsymbol{x}dt \\ &+ \int_{\mathbb{R}^d} \varphi(u_0)\zeta(\boldsymbol{x}, 0) d\boldsymbol{x} \ge 0.\end{aligned} \tag{9.6}$$

On the strength of the limiting relation (9.5), the inequality

$$\int_\Pi \varepsilon\zeta\varphi''(u_\varepsilon)\partial_{x_i} u_\varepsilon \partial_{x_i} u_\varepsilon d\boldsymbol{x}dt \ge 0,$$

and the lower semicontinuity property (see, for example, [20, Chapter 1, §1.1.3, Definition; Chapter 2, §2.3, Proposition 2.3.2])

$$\liminf_{\varepsilon\searrow 0}\int_\Pi \zeta\varphi''(u_\varepsilon)b'(u_\varepsilon)(\alpha_{il}(\boldsymbol{x},t)\partial_{x_i}u_\varepsilon)(\alpha_{lj}(\boldsymbol{x},t)\partial_{x_j}u_\varepsilon)d\boldsymbol{x}dt$$
$$\geq \int_\Pi \zeta\varphi''(u)b'(u)(\alpha_{il}(\boldsymbol{x},t)\partial_{x_i}u)(\alpha_{lj}(\boldsymbol{x},t)\partial_{x_j}u)d\boldsymbol{x}dt,$$

as $\varepsilon_k \searrow 0$, we derive the inequality (1.9) from the equality (9.6), which completes the proof of Theorem 1.

A On the generalization to the case $b'(u) \geq 0$

The restriction to the case when $b'(u) > 0$ is not fundamental, i.e. arguments in the paper can be generalized (in a natural way) to the case when $b'(u)$ may vanish in some points or on some intervals, i.e. when the condition (1.4) is substituted by the condition

$$b \in C^2_{loc}(\mathbb{R}), \quad b'(u) \geq 0, \quad \forall\, u \in \mathbb{R}. \tag{A.1}$$

Consequently, Theorems 1 and 2 hold true for this case as well and the corresponding genuine nonlinearity condition has the following shape.

Condition G2. *The functions a_i, a_{ij} and b are such that the Lebesgue measure of the intersection of the sets*

$$\mathbb{I}_1 := \{(\boldsymbol{x},t,\lambda) \in \Pi\times\mathbb{R}_\lambda \quad | \quad b'(\lambda)a_{ij}(\boldsymbol{x},t)\xi_i\xi_j = 0\}$$

and

$$\mathbb{I}_2 := \{(\boldsymbol{x},t,\lambda) \in \Pi\times\mathbb{R}_\lambda \quad | \quad \tau + \big(a_{i\lambda}(\boldsymbol{x},t,\lambda) + b'(\lambda)a_{ijx_j}(\boldsymbol{x},t)\big)\,\xi_i = 0\}$$

is equal to zero for any fixed $(\boldsymbol{\xi},\tau) \in \mathbb{S}^d$.

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