ARMA manuscript No.
(will be inserted by the editor)

On Asymptotic Variational Wave Equations

Alberto Bressan, Ping Zhang, Yuxi Zheng

Abstract

We investigate the equation $(u_t + (f(u))_x)_x = f''(u)(u_x)^2/2$ where $f(u)$ is a given smooth function. Typically $f(u) = u^2/2$ or $u^3/3$. This equation models unidirectional and weakly nonlinear waves for the variational wave equation $u_{tt} - c(u)(c(u)u_x)_x = 0$ which models some liquid crystals with a natural sinusoidal c. The equation itself is also the Euler-Lagrange equation of a variational problem. Two natural classes of solutions can be associated with this equation. A conservative solution will preserve its energy in time, while a dissipative weak solution loses energy at the time when singularities appear. Conservative solutions are globally defined, forward and backward in time, and preserve interesting geometric features, such as the Hamiltonian structure. On the other hand, dissipative solutions appear to be more natural from the physical point of view.

We establish the well-posedness of the Cauchy problem within the class of conservative solutions, for initial data having finite energy and assuming that the flux function f has Lipschitz continuous second-order derivative. In the case where f is convex, the Cauchy problem is well-posed also within the class of dissipative solutions. However, when f is not convex, we show that the dissipative solutions do not depend continuously on the initial data.

1. Introduction

A nonlinear variational wave equation whose wave speed is a sinusoidal function of the wave amplitude arises in the study of nematic liquid crystals. It is given by

$$\partial_t^2\psi - c(\psi)\partial_x(c(\psi)\partial_x\psi) = 0, \tag{1}$$

with

$$c^2(\psi) = \alpha\sin^2(\psi) + \beta\cos^2(\psi), \tag{2}$$

where α and β are positive physical constants. We refer the reader to [11], [12], and [14] for background information on the equation. Glassey, Hunter, and Zheng [10] have shown that singularities can form from smooth data for equation (1)-(2). Assuming that the wave speed $c(\cdot)$ is a monotone increasing function, the global existence of (dissipative) weak solutions has been established in [19], [20], [21], and [23]. The general problem of the global existence and uniqueness of conservative solutions to the Cauchy problem of equation (1) will be addressed in a forthcoming paper [3].

The study of solutions to (1)-(2) consisting of a small-amplitude and high-frequency perturbation of a constant state has greatly contributed to the understanding of this equation [10], [19], [20], [21], and [23]. Hunter and Saxton first studied these waves in [12]. Given a constant state a, these perturbed solutions take the form

$$\psi(t,x) = a + \epsilon u(\epsilon t, x - c(a)t) + O(\epsilon^2).$$

Hunter and Saxton found that $u(\cdot,\cdot)$ satisfies

$$(u_t + u^n u_x)_x = \frac{1}{2} n u^{n-1} u_x^2 \tag{3}$$

up to a scaling and reflection of the independent variables, assuming that a is such that $c^{(k)}(a) = 0, k = 1, 2, \ldots n-1$, but $c^{(n)}(a) \neq 0$, for an integer $n \geq 1$. In connection with our sinusoidal function c modeling nematic liquid crystals in (2), the relevant approximations in (3) are those with $n = 1, 2$. The case $n = 1$ yields the *first-order asymptotic equation*

$$(u_t + u u_x)_x = \frac{1}{2} u_x^2, \tag{4}$$

for which existence and uniqueness of admissible conservative and dissipative weak solutions have both been established, see [13], [16], [17], and [18]. This equation is also an asymptotic equation of the Camassa-Holm equation [4], describing the motion of solitary waves in shallow water. For recent literature on the Camassa-Holm equation, we refer the reader to [5], [6], [7], [8], [9], [15], and in particular [2].

The case $n = 2$ yields the *second-order asymptotic equation*

$$(u_t + u^2 u_x)_x = u u_x^2 \,. \tag{5}$$

In [22] Zhang and Zheng established that dissipative solutions exist for (5) with BV data. In the analysis of (1), a major difficulty is concentration of energy at points where $c' = 0$, as in the example on p. 70 of [10]. We hope that investigation of singularities of the same type for the second-order asymptotic equation will be helpful toward the understanding of the original equation (1).

Rather than (3), in the present paper we study a somewhat more general class of equations, having the form

$$(u_t + f(u)_x)_x = \frac{1}{2} f''(u) u_x^2 \,. \tag{6}$$

Here $u = u(t,x)$ is a scalar function defined for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+$ where $\mathbb{R}_+ \doteq [0,\infty[$, and f is a C^2 function. More restrictions on f will be specified later. As initial and boundary data we take

$$u(0,x) = \bar{u}(x)\,, \qquad\qquad u(t,0) = 0\,. \tag{7}$$

Integrating equation (6) w.r.t. x, we obtain

$$u_t + f(u)_x = \frac{1}{2} \int_0^x f''(u) u_x^2 \, dx\,. \tag{8}$$

It is now clear that, to make sense of this equation, we should require that the function $u(t,\cdot)$ be absolutely continuous with derivative $u_x(t,\cdot)$ locally square integrable, for every fixed time t. Moreover, to satisfy the boundary condition at $x = 0$, one needs the nonnegativity of the characteristic speed at $u = 0$, namely

$$f'(0) \geq 0. \tag{9}$$

The local integrability assumption $u_x(t,\cdot) \in \mathbf{L}^2_{loc}(\mathbb{R}_+)$ imposes a certain degree of regularity on the function u. Therefore, there is no need to consider weak solutions in distributional sense and a stronger concept of solution can be adopted.

Definition 1.1 *A function $u = u(t,x))$ defined on $[0,T] \times \mathbb{R}_+$ is a* **solution** *of the initial-boundary value problem (7) –(9) if the following holds.*

(i) The function u is locally Hölder continuous w.r.t. both variables t, x. The initial and boundary conditions (7) hold pointwise. For each time t, the map $x \mapsto u(t,x)$ is absolutely continuous with $u_x(t,\cdot) \in \mathbf{L}^2_{loc}(\mathbb{R}_+)$.

(ii) For any $M > 0$, consider the restriction of u to the interval $x \in [0,M]$. Then the map $t \mapsto u(t,\cdot) \in \mathbf{L}^2([0,M])$ is absolutely continuous and satisfies the equation

$$\frac{d}{dt}u(t,\cdot) = -f'(u)u_x + \frac{1}{2}\int_0^* f''(u)\, u_x^2\, dx \tag{10}$$

for a.e. $t \in [0,T]$. Here equality is understood in the sense of functions in $\mathbf{L}^2([0,M])$.

In spite of the regularity assumptions, the requirements contained in the above definition are still not enough to single out a unique solution. Let us consider a simple example.

Example 1. Consider the flux $f(u) = u^2$ and choose the initial data

$$u(0,x) = \begin{cases} -x, & 0 \le x \le 1, \\ -1, & x > 1. \end{cases}$$

For $t \in [0, 1[$, a solution to (8) can be constructed by the method of characteristics, namely

$$u(t,x) = \begin{cases} -x/(1-t), & 0 \leq x \leq (1-t)^2, \\ -(1-t), & x \geq (1-t)^2. \end{cases}$$

Notice that the norm of the gradient $\|u_x(t)\|_{L^\infty}$ blows up as $t \to 1$. For $t = 1$ we have $u(1,x) = 0$ for all $x \geq 0$. At this stage, there are infinitely many ways to further prolong the solution. For example, we could set

$$u(t,x) \equiv 0\,, \qquad\qquad t \geq 1\,, \quad x \geq 0\,. \tag{11}$$

Or else we could choose an arbitrary point $b \geq 0$, an arbitrary amount of energy $k > 0$ and a time $\tau \geq 1$ and define

$$u(t,x) = 0 \qquad \text{for} \;\; 1 \leq t \leq \tau\,,$$

while for $t > \tau$,

$$u(t,x) = \begin{cases} 0\,, & 0 \leq x \leq b, \\ (x-b)/(t-\tau)\,, & b \leq x \leq k(t-\tau)^2 + b, \\ k(t-\tau)\,, & x > b + k(t-\tau)^2. \end{cases}$$

Among all these solutions, two in particular can be singled out. If we insist that the future configurations $u(t,\cdot)$ for $t > 1$ should be entirely determined only by the present configuration $u(1,\cdot)$, then the only reasonable choice is (11). On the other hand, if we look for solutions that satisfy the additional conservation equation

$$(u_x^2)_t + (2u\,u_x^2)_x = 0\,,$$

the natural choice should be

$$u(t,x) = \begin{cases} x/(t-1), & 0 \le x \le (t-1)^2\,, \\ t-1, & x \ge (t-1)^2\,, \end{cases} \qquad t > 1\,.$$

To express the fact that at time $t = 1$ this solution is different from the null solution, in some way we should think its derivative u_x as being not the zero function but the square root of a Dirac distribution at the origin. □

In the following, we say that a solution $u = u(t,x)$ is **dissipative** if the family of absolutely continuous measures $\{\mu_{(t)}\ ;\quad t \ge 0\}$ defined by $d\mu_{(t)} = u_x^2(t)\,dx$ provides a measure-valued solution to

$$w_t + (f'(u)w)_x \le 0. \tag{12}$$

More precisely, we require that

$$\int \phi(t,\cdot)\,d\mu_{(t)}\Big|_{t_1}^{t_2} \;\le\; \int_{t_1}^{t_2} \left[\int [\phi_t(t,\cdot) + \phi_x(t,\cdot)\,f'(u(t,\cdot))]\,d\mu_{(t)}\right] dt \tag{13}$$

for *every* $t_2 > t_1 \ge 0$ and any function $\phi \in \mathcal{C}_c^1$, $\phi \ge 0$.

On the other hand, to define a semigroup of conservative solutions we need to consider a domain $\mathcal{D}$ of couples (u,μ) where $u : \mathbb{R}_+ \mapsto \mathbb{R}$ is an absolutely continuous function with square integrable derivative and μ is a nonnegative measure on $\mathbb{R}_+$. Decomposing $\mu = \mu^a + \mu^s$ as a sum of an absolutely continuous and a singular part (w.r.t. Lebesgue measure), we shall require that $d\mu^a = u_x^2\,dx$. We say that a map $t \mapsto (u(t),\, \mu_{(t)}) \in \mathcal{D}$ is a **conservative solution** of (7)-(9) if u is a solution according to Definition 1.1 and (13) is satisfied as an equality for *all* $t_2 > t_1 \ge 0$ and $\phi \in \mathcal{C}_c^1$.

As mentioned earlier, Zhang and Zheng have established in [22] the finite-time singularity formation in smooth solutions and the global existence of a dissipative weak solution to (7)–(9) with initial data $\bar{u}(x)$ whose derivative is in BV, for $f(u) = u^3/3$.

In the present paper, we consider a general flux f with Lipschitz continuous second-order derivative such that $f'(0) \geq 0$. The initial data are chosen in the set of absolutely continuous functions $\bar{u}$ with $\bar{u}(0) = 0$ and $\bar{u}_x \in \mathbf{L}^2(\mathbb{R}_+)$. Our main results can be summarized as follows.

1. A flow of conservative solutions can be globally defined, forward and backward in time (Theorem 3.1). The conservative solution of the initial-boundary value problem (7)–(9) is unique, provided that a suitable non-degeneracy condition is satisfied (Theorem 4.1).

2. Assuming, in addition, that the flux f is convex, then there also exists a continuous semigroup of dissipative solutions. The dissipative solution of the initial-boundary value problem (7)–(9) is unique (Theorem 5.1).

3. If the flux f is not convex, the dissipative solutions do not depend continuously on the initial data, in general (see Example 2 in Section 6).

Before proving the main results in Section 3, we briefly discuss the action principle and some admissibility conditions, whose aim is to identify a unique physically relevant solution to equations (7)–(9).

2. Remarks on admissibility conditions

The decay estimate

$$u_x(t,x) \leq 2/t$$

was used as an admissibility criterion for dissipative solutions of the first-order asymptotic equation in [13], [16], [17], and [18]. We remark, however, that this is not appropriate in connection with dissipative solutions of (6). Indeed, for a solution of the second-order asymptotic equation, the gradient u_x can approach $+\infty$ as well as $-\infty$.

Another common criteria for selecting physically admissible solutions is by vanishing viscosity. One might conjecture that dissipative solutions are precisely the limits of vanishing viscosity approximations. We believe this is indeed the case when the flux function f is convex, see some proofs in [13] and [15] for $f = u^2/2$. On the other hand, when f is not convex, the dissipative solutions do not depend continuously on the initial data (see Section 6). We observe that the set of vanishing viscosity limits is closed, connected, and depends on the initial data in an upper semicontinuous way. Therefore, by a topological argument, the vanishing viscosity criterion cannot single out a unique limit, in general.

Concerning the vanishing dispersion limit, numerical experiments performed with a convex f seem to indicate that vanishing dispersion selects the conservative solutions, see [13].

Next, we discuss the admissibility of solutions in terms of a variational principle. For all asymptotic equations (3) we have the action functionals

$$\mathcal{A}_n \doteq \int_{t_1}^{t_2} \int [u_x u_t + u^n (u_x)^2]\, dx\, dt\ . \tag{14}$$

In other words, the Euler-Lagrange equations satisfied by functions u that render stationary the action $\mathcal{A}_n$ are precisely the asymptotic equations (3). These can be derived from the nonlinear variational wave equation (1)

$$\psi_{tt} - c(\psi)(c(\psi)\psi_x)_x = 0 \tag{15}$$

by a perturbation argument. Notice that (15) is the Euler-Lagrange equation corresponding to the Lagrangean

$$\mathcal{L} = \psi_t^2 - c^2(\psi)\psi_x^2. \tag{16}$$

This arises often in physical models. For weakly nonlinear waves of the form

$$\psi = \psi_0 + \epsilon u(\tau, \theta) + \epsilon^2 v(\tau, \theta) + O(\epsilon^3)$$

with

$$\tau = \epsilon t\,, \qquad \theta = x - c_0 t\,, \qquad c_0 \doteq c(\psi_0)\,,$$

assuming that $c_0' \doteq c'(\psi_0) \neq 0$ we have

$$\psi_{tt} - c(\psi)(c(\psi)\psi_x)_x = -2c_0\epsilon^2 \Big\{ (u_\tau + c_0' u u_\theta)_\theta - \frac{1}{2} c_0' u_\theta^2 \Big\} + O(\epsilon^3)\,.$$

Moreover

$$\psi_t^2 - c^2(\psi)\psi_x^2 = -2c_0\epsilon^3 [u_\tau u_\theta + c_0' u u_\theta^2] + O(\epsilon^4).$$

Therefore, u satisfies the first order asymptotic equation. The corresponding Lagrangean, approximated to order $O(\epsilon^3)$, is $-(u_\tau u_\theta + c_0' u u_\theta^2)$.

At first sight, one might hope that the physically relevant solutions to the equations (3) are those which maximize (or minimize) the action in (14). Unfortunately this is not the case, because the action $\mathcal{A}_n$ is not coercive. For any smooth solution u of (3) one can find compactly supported perturbations $u+\epsilon v$ which increase the value of $\mathcal{A}_n$, and others which decrease it. The extremality of the action thus cannot be used as a selective criterion.

3. Conservative solutions

We consider the evolution problem described by the equation

$$u_t + f(u)_x = \frac{1}{2}\int_0^x f''(u)u_x^2\,dx \qquad \text{for all} \quad t \geq 0\,, \quad x \geq 0, \tag{17}$$

together with the boundary conditions

$$u(t,0) = 0 \qquad \text{for all} \quad t \geq 0\,. \tag{18}$$

We assume that $f \in C^2(\mathbb{R})$ and

$$f'(0) \geq 0, \qquad |f''(u) - f''(v)| \leq L|u-v|, \quad \forall u, v \in \mathbb{R} \tag{19}$$

for a constant L.

One easily checks that every smooth solution satisfies the additional conservation law for the "energy" u_x^2, namely

$$(u_x^2)_t + \big[f'(u)(u_x^2)\big]_x = 0\,. \tag{20}$$

It is therefore natural to seek a continuous flow associated with (17)-(18) which preserves the energy $\int_0^\infty u_x^2(t,x)\,dx$. However, Example 1 in the Introduction already pointed out a basic difficulty which one encounters while

constructing a semigroup in the space $H^1_{\rm loc}$. Indeed, when the gradient u_x blows up, all the energy is concentrated at a single point, so that the measure $u_x^2\,dx$ approaches a Dirac mass.

Motivated by this example, to the equations (17)-(18) we will associate an evolution semigroup on a domain $\mathcal{D}$ defined as follows. An element of $\mathcal{D}$ is a couple (u,μ), where $u:\mathbb{R}_+\mapsto\mathbb{R}$ is a continuous function with $u(0)=0$ and whose distributional derivative u_x lies in $\mathbf{L}^2$, while $\mu=\mu^a+\mu^s$ is a bounded nonnegative Radon measure on $\mathbb{R}_+$, whose absolutely continuous part (w.r.t. Lebesgue measure) satisfies

$$d\mu^a = u_x^2\,dx\,. \tag{21}$$

In the following, on the family of Radon measures on $\mathbb{R}_+$ we consider the distance

$$d(\mu,\tilde\mu)\doteq\sup_{\varphi}\left|\int\varphi\,d\mu-\int\varphi\,d\tilde\mu\right|, \tag{22}$$

where the supremum is taken over all smooth functions φ with $|\varphi|\leq 1$, $|\varphi_x|\leq 1$.

We recall that a semigroup S on a domain $\mathcal{D}$ is a map $S:\mathcal{D}\times[0,\infty[\mapsto\mathcal{D}$ such that $S_0w=w$ and $S_s(S_tw)=S_{s+t}w$ for every $s,t\geq 0$ and $w\in\mathcal{D}$.

Theorem 3.1 *Assume that the flux function f satisfies condition (19). Then there exists a semigroup $S:\mathcal{D}\times[0,\infty[\mapsto\mathcal{D}$ with the following properties. Calling $t\mapsto S_t(\bar u,\bar\mu)=(u(t),\,\mu_{(t)})$ the trajectory corresponding to an initial data $(\bar u,\bar\mu)\in\mathcal{D}$, one has:*

(i) The function $u = u(t,x)$ is locally Hölder continuous in $\mathbb{R}_+ \times \mathbb{R}_+$. It provides a solution of (17)-(18) in the sense of Definition 1.1 with initial condition

$$u(0,x) = \bar{u}(x)\,. \tag{23}$$

(ii) The assignment $t \mapsto \mu_{(t)}$ provides a measure valued solution to the linear transport equation

$$w_t + \big[f'(u)w\big]_x = 0\,, \qquad\qquad w(0) = \bar{\mu}\,. \tag{24}$$

Moreover, the singular part of the measure $f''(u(t)) \cdot \mu_{(t)}$ vanishes at almost every time $t \geq 0$:

$$f''(u(t))\mu^s_{(t)} = 0, \qquad \text{a. e. } t. \tag{25}$$

(iii) (Temporal continuity) For every $M > 0$, the above solution u and the corresponding measure μ satisfy the Lipschitz continuity property:

$$\int_0^M |u(t,x) - u(s,x)|\ dx \leq C|t-s|, \tag{26}$$

$$d(\mu_{(t)}\,,\ \mu_{(s)}) \leq C|t-s|,$$

where the constant C depends only on M, the flux f, and the total energy $\bar{\mu}(\mathbb{R}_+) < \infty$.

(iv) (Continuous dependence on the initial data) Finally, consider a sequence of initial conditions $(\bar{u}^n, \bar{\mu}^n) \in \mathcal{D}$ with $\bar{u}^n \to \bar{u}$ uniformly on bounded sets and $d(\bar{\mu}^n, \bar{\mu}) \to 0$ as $n \to \infty$, for some $(\bar{u}, \bar{\mu}) \in \mathcal{D}$. Then the corresponding solutions satisfy

$$u^n(t,x) \to u(t,x) \tag{27}$$

uniformly for t, x in bounded sets, while

$$d(\mu^n_{(t)}, \mu_{(t)}) \to 0 \tag{28}$$

for every $t > 0$.

Proof. We treat here the case where $\bar\mu$ has compact support, say contained in the interval $[0, R]$, so that $\bar u$ is constant for $x > R$. The general case follows by an easy approximation argument. The proof will be given in several steps.

1. Construction of the solution. Let an initial data $(\bar u, \bar\mu) \in \mathcal{D}$ be given. Set $\bar\xi \doteq \bar\mu(\mathbb{R}_+) < \infty$. On the semi-infinite strip $\big\{t \geq 0\,,\ \xi \in [0, \bar\xi]\,\big\}$ we construct a function $U = U(t, \xi)$ by first setting

$$U(0, \xi) = \overline{U}(\xi) \doteq \bar u\,(\bar y(\xi))\,, \tag{29}$$

where

$$\bar y(\xi) \doteq \inf\Big\{x \geq 0\ ;\ \ \bar\mu\,([0, x]) \geq \xi\Big\} \tag{30}$$

for $0 < \xi \leq \bar\xi$, while

$$\bar y(0) = \sup\{x\ ;\ \ \bar\mu\,([0, x]) = 0\}\,. \tag{31}$$

Observe that the map $\xi \mapsto \bar y(\xi)$ is nondecreasing and left continuous, but it may well have upward jumps. The provision (31) makes it continuous at the point $\xi = 0$. In any case, the composed function $\xi \mapsto \bar u(\bar y(\xi))$ is always continuous. For positive times, the function U is then defined to be the solution of

$$\frac{\partial U}{\partial t}(t, \xi) = \frac{1}{2}\int_0^\xi f''(U(t, \eta))\,d\eta \tag{32}$$

with initial data (29). By the assumption of Lipschitz continuity of f'', the function U can be obtained as the unique fixed point of a contractive transformation. Details will be given at the next step.

Having constructed $U(t,\xi)$, the characteristic curves are obtained by solving the equation

$$y(0,\xi) = \bar{y}(\xi)\,, \qquad \frac{\partial y}{\partial t}(t,\xi) = f'(U(t,\xi))\,. \tag{33}$$

Explicitly:

$$y(t,\xi) = \bar{y}(\xi) + \int_0^t f'(U(\tau,\xi))\,d\tau. \tag{34}$$

Notice that $t \mapsto U(t,\xi)$ yields the values of our solution u along the characteristic curve $t \mapsto y(t,\xi)$ starting at $\bar{y}(\xi)$. A remarkable feature of equation (17) is that, if the energy is conserved, then these values can be determined in advance, before computing the actual position of the characteristic curve. The image of the mapping

$$\xi \to (y(t,\xi), U(t,\xi))$$

is now contained inside the graph of the desired solution $u(t,\cdot)$. More precisely, for any given (t,x) we define

$$u(t,x) = U(t,\xi(t,x)), \tag{35}$$

where

$$\xi(t,x) \doteq \sup\{\xi\,;\quad y(t,\xi) \leq x\}\,.$$

Finally, at time t the corresponding measure $\mu_{(t)}$ is defined as the push-forward of the Lebesgue measure on $[0,\bar{\xi}\,]$ through the mapping $\xi \mapsto y(t,\xi)$.

For each Borel set $J \subset \mathbb{R}_+$ we thus define

$$\mu_{(t)}(J) \doteq \text{meas}\left\{\xi \in [0, \bar{\xi}\,] \,;\quad y(t,\xi) \in J\right\}. \tag{36}$$

2. A contractive transformation. Consider the space of continuous functions $\mathcal{C}([0,\infty[\,\times[0,\bar{\xi}\,])$, with the equivalent weighted norm

$$\|U\|_* \doteq \sup_{t\geq 0,\ \xi\in[0,\bar{\xi}\,]} e^{-L\bar{\xi}t}\,|U(t,\xi)|\,, \tag{37}$$

where L is a Lipschitz constant for the function f''. The transformation $U \mapsto \mathcal{T}U$ is defined as

$$\mathcal{T}U\,(t,\xi) \doteq \bar{u}(\bar{y}(\xi)) + \frac{1}{2}\int_0^t \int_0^\xi f''(U(s,\eta))\,d\eta\,ds\,. \tag{38}$$

If now $\|U-V\|_* = \delta$, then

$$\Big|f''(U(\tau,\eta)) - f''(V(\tau,\eta))\Big| \leq L|U(\tau,\eta)-V(\tau,\eta)| \leq L\,e^{L\bar{\xi}\tau}\delta\,.$$

For every $t \geq 0$ and $\xi \in [0,\bar{\xi}\,]$ we thus have

$$|(\mathcal{T}U-\mathcal{T}V)(t,\xi)| \leq \frac{1}{2}\int_0^t \left[\int_0^\xi L\,e^{L\bar{\xi}\tau}\,\delta\,d\eta\right] d\tau \leq \frac{1}{2}\int_0^t L\xi\,e^{L\bar{\xi}\tau}\,\delta\,d\tau < \frac{1}{2}e^{L\bar{\xi}t}\,\delta\,.$$

By the above inequality we conclude

$$\|\mathcal{T}U-\mathcal{T}V\|_* \leq \frac{1}{2}\|U-V\|_*\,,$$

proving the contractivity of the map $\mathcal{T}$. By the contraction mapping theorem, it admits a unique fixed point $U = U(t,\xi)$, defined on $\mathbb{R}_+ \times [0,\bar{\xi}\,]$. In turn, the function $u = u(t,x)$ and the measures $\mu_{(t)}$ are well defined by (35)-(36).

3. Absolute continuity. We prove here that the map $\xi \mapsto U(t,\xi)$ is absolutely continuous, for each $t \geq 0$. Indeed, consider first the case $t = 0$. Let $[\xi_k, \xi_k']$, with $k = 1, \ldots, N$, be disjoint intervals contained in $[0, \bar{\xi}\,]$. Assume that

$$\sum_k |\xi_k' - \xi_k| \leq \varepsilon\,.$$

Call I_1 the set of indices k such that

$$\frac{|U(\xi_k') - U(\xi_k)|}{y(\xi_k') - y(\xi_k)} \leq \sqrt{\varepsilon}$$

and let I_2 be the set of indices where the above quantity is $> \sqrt{\varepsilon}$. Then

$$\sum_{k\in I_1} |U(\xi_k') - U(\xi_k)| \leq \sqrt{\varepsilon} \cdot \sum_{k\in I_1} |y(\xi_k') - y(\xi_k)| \leq \sqrt{\varepsilon}\, R\,,$$

while

$$\sum_{k\in I_2} |U(\xi_k') - U(\xi_k)| \leq \frac{1}{\sqrt{\varepsilon}} \sum_{k\in I_2} \frac{|U(\xi_k') - U(\xi_k)|^2}{y(\xi_k') - y(\xi_k)} \leq \frac{1}{\sqrt{\varepsilon}} \cdot \sum_{k\in I_2} \int_{y(\xi_k)}^{y(\xi_k')} u_x^2\, dx$$

$$\leq \frac{1}{\sqrt{\varepsilon}} \sum_{k\in I_2} |\xi_k' - \xi_k| \leq \sqrt{\varepsilon}\,.$$

Together, the two above inequalities yield

$$\sum_{k=1}^{N} |U(\xi_k') - U(\xi_k)| \leq (1+R)\,\sqrt{\varepsilon}\,,$$

proving the absolute continuity of the map $\xi \mapsto U(0,\xi)$.

For $t > 0$, the absolute continuity of $U(t,\cdot)$ follows from the absolute continuity of $U(0,\cdot)$ together with (32). Indeed,

$$|U(t,\xi') - U(t,\xi)| \leq |U(0,\xi') - U(0,\xi)| + |\xi' - \xi| \cdot \frac{t}{2} \sup_u |f''(u)|\,.$$

As a consequence, the partial derivative U_ξ exists at a.e. (t,ξ). By (32), it satisfies the evolution equation

$$\frac{\partial}{\partial t}U_\xi(t,\xi) = \frac{1}{2}f''(U(t,\xi))\,. \tag{39}$$

On the other hand, the map $\xi \mapsto y(t,\xi)$ can be discontinuous. However, if

$$\lim_{\xi\to\xi^*-} y(t,\xi) = y_1 < y_2 = \lim_{\xi\to\xi^*+} y(t,\xi)\,,$$

then the function $u(t,\cdot)$ must be constant on the interval $[y_1,y_2]$.

4. Measure transformations. To proceed, we first need to analyse the regular and the singular part of the push-forward of Lebesgue measure, under a continuous non-decreasing transformation.

Lemma 1. *Let $U:[0,\bar\xi]\mapsto\mathbb{R}$ be absolutely continuous with square integrable derivative. Let $\xi\mapsto y(\xi)$ be such that*

$$y(\xi) = y(0) + \int_0^\xi U_\xi^2(\zeta)\,d\zeta\,. \tag{40}$$

For $x\in[y(0),y(\bar\xi)]$ define the function $u=u(x)$ implicitly by

$$u(y(\xi)) \doteq U(\xi)\,. \tag{41}$$

Let μ be the push-forward of Lebesgue measure on $[0,\bar\xi]$ through the map y, i.e.

$$\mu(J) \doteq meas\,\{\xi\in[0,\bar\xi]\,;\quad y(\xi)\in J\}\,. \tag{42}$$

Then the absolutely continuous and the singular part of μ w.r.t. Lebesgue measure are respectively given by

$$\mu^a(A) = meas\,\{\xi\in[0,\bar\xi]\,;\quad y(\xi)\in A\,,\quad U_\xi(\xi)\neq 0\}\,. \tag{43}$$

$$\mu^s(A) = meas\,\{\xi \in [0,\bar\xi\,];\quad y(\xi)\in A\,,\quad U_\xi(\xi)=0\}\,. \tag{44}$$

In addition, on the set $[y(0),\, y(\bar\xi)]$ *one has*

$$d\mu^a = u_x^2\,dx\,. \tag{45}$$

Viceversa, if both U *and the map* y *are absolutely continuous and (41), (42), (45) are valid, then (40) must hold.*

Proof. By (40), the image of a set $I \subseteq [0,\bar\xi]$

$$I_\varepsilon \doteq \{\xi\in[0,\bar\xi\,];\quad |U_\xi(\xi)|\leq\varepsilon\}$$

under the mapping $\xi\mapsto y(\xi)$ has Lebesgue measure

$$\text{meas}\,(y(I)) = \int_I U_\xi^2(\xi)\,d\xi\,.$$

It is thus clear that the singular part of μ is the push-forward of Lebesgue measure on the set

$$I_0 \doteq \{\xi\in[0,\bar\xi\,];\quad U_\xi(\xi)=0\}$$

Next, for any fixed $\varepsilon>0$ take a measurable set $J\subset[0,\bar\xi\,]$ such that

$$U_\xi^2(\xi)\geq\varepsilon \qquad\qquad \text{for all } \xi\in J\,.$$

Then

$$\int_{y(J)} u_x^2(x)\,dx = \int_J \left[U_\xi\,\frac{d\xi}{dy}\right]^2\,\frac{dy}{d\xi}\cdot d\xi = \int_J \left[U_\xi\,U_\xi^{-2}\right]^2\,U_\xi^2\cdot d\xi = \text{meas}\,(J)\,.$$

Since $\varepsilon>0$ was arbitrary, this proves (43) and (45). To prove the last statement, assume (41), (42) and (45). Call

$$J_\varepsilon \doteq \{\xi\in[0,\bar\xi\,];\quad y_\xi(\xi)\geq\varepsilon\}\,.$$

Observe that, for $\xi \in J_\varepsilon$. the chain rule yields

$$u_x(y(\xi))\, y_\xi(\xi) = U_\xi(\xi)\,. \tag{46}$$

For $0 < a < b < \bar\xi$ we now obtain

$$\int_{[y(a),y(b)]\cap y(J_\varepsilon)} u_x^2(x)\, dx = \int_{[a,b]\cap J_\varepsilon} u_x^2(y(\xi))\, y_\xi(\xi)\, d\xi = \text{meas}\,([a,b]\cap J_\varepsilon) \tag{47}$$

Since a, b were arbitrary, this implies

$$y_\xi(\xi) = [u_x^2(y(\xi))]^{-1} \tag{48}$$

for $\xi \in J_\varepsilon$. Together with (46) this yields

$$u_x(y(\xi)) = U_\xi^{-1}(\xi)\,, \qquad\qquad y_\xi(\xi) = U_\xi^2(\xi) \tag{49}$$

for all $\xi \in J_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude

$$y(\xi) = y(0) + \int_0^\xi y_\xi(\zeta)\, d\zeta = y(0) + \lim_{\varepsilon\to 0} \int_{[0,\xi]\cap J_\varepsilon} y_\xi(\zeta)\, d\zeta = y(0) + \int_0^\xi U_\xi^2(\zeta)\, d\zeta\,,$$

proving (40).

5. A class of regular solutions. Having constructed the trajectory $t \mapsto (u(t,\cdot),\, \mu_{(t)})$, we still need to prove that it satisfies equation (17), coupled with the initial and boundary conditions (23) and (18). We carry out the analysis first assuming that the map $\xi \mapsto \bar y(\xi)$ is absolutely continuous. At the end, this assumption will be removed.

For each $t \geq 0$ and $\xi \in [0, \bar\xi\,]$ define

$$y(t,0) = \bar y(0) + t\, f'(0)\,, \qquad\qquad y(t,\xi) = y(t,0) + \int_0^\xi U_\xi^2(t,\zeta)\, d\zeta\,. \tag{50}$$

By (39) this implies

$$\frac{\partial}{\partial t} y_\xi(t,\xi) = \frac{\partial}{\partial t} U_\xi^2(t,\xi) = f''(U(t,\xi))\, U_\xi(t,\xi)\,. \tag{51}$$

We now check that the function $y = y(t,\xi)$ defined at (50) coincides with the one defined at (34). Indeed, by the second part of Lemma 1, their derivatives y_ξ coincide at time $t = 0$ and satisfy the same equation (51). In particular, from (50) it is clear that the map $t \mapsto y(t,\xi)$ is non-decreasing. In particular, characteristics never cross each other.

We begin by observing that the boundary condition (18) is clearly satisfied, because

$$u(t,0) = U(t,0) = U(0,0) + \int_0^t U_t(\tau,0)\, d\tau = 0\,.$$

Moreover, the initial condition (23) holds because of the definitions (29)-(30).

To check that the limit function u satisfies (17), fix a time $t > 0$. Since $u(t,x) \equiv 0$ for $x \in [0, y(t,0)]$, in this region the equation (17) trivially holds. For almost every $x \in [y(t,0),\ y(t,\bar\xi)]$, there exists a unique $\xi \in [0,\bar\xi\,]$ such that $x = y(t,\xi)$. In this case, our construction yields

$$u_t + f'(u)u_x = U_t(t,\xi) = \frac{1}{2}\int_0^\xi f''(U(t,\zeta))\, d\zeta = \frac{1}{2}\int_0^{y(t,\xi)} f''(u(t,\cdot))\, d\mu_{(t)}\,.$$

This implies (17), provided that we can show the identity of measures

$$f''(u)\, u_x^2\, dx = f''(u)\, d\mu_{(t)} \tag{52}$$

for almost every time $t \geq 0$. We shall now work toward a proof of (52).

Since the function u is continuous, by covering the open region

$$\Big\{(t,x)\in\mathbb{R}_+\times\mathbb{R}_+\,;\quad f''(u(t,x))\neq 0\Big\}$$

with countably many sets of the form

$$\Gamma\doteq\Big\{(t,x)\,;\quad t\in[t_1,t_2]\,,\quad x\in[y(t,a),\ y(t,b)]\Big\}$$

it suffices to prove the following statement.

Assume that

$$f''(u(t,x))>\delta>0\,,\qquad\qquad (t,x)\in\Gamma\,.$$

Then, for a.e. $t\in[t_1,t_2]$, the restriction of the measure $\mu_{(t)}$ to the interval $[y(t,a),\ y(t,b)]$ is absolutely continuous w.r.t. Lebesgue measure and satisfies $d\mu_{(t)}=u_x^2\,dx$.

By construction, as long as U ranges in a region where $f''>\delta$ we have

$$\frac{\partial}{\partial t}U_\xi(t,\xi)>\frac{\delta}{2}\,. \tag{53}$$

Hence, for any $\varepsilon>0$,

$$\text{meas}\Big(\{(t,\xi)\in\Gamma\,;\quad |U_\xi(t,\xi)|<\varepsilon\}\Big)<\frac{4\bar\xi}{\delta}\,\varepsilon\,.$$

Since $\varepsilon>0$ here is arbitrary, we conclude that there exists a set of times $\mathcal{N}$ of measure zero such that

$$\text{meas}\Big(\{\xi\in[a,b]\,;\quad U_\xi(t,\xi)=0\}\Big)=0$$

for all times $t\notin\mathcal{N}$. By Lemma 1, $t\notin\mathcal{N}$thus implies that the restriction of $\mu_{(t)}$ to the interval $[y(t,a),\ y(t,b)]$ is absolutely continuous w.r.t. Lebesgue

measure. Furthermore, by (50), Lemma 1 shows that its density is $d\mu_{(t)} = u_x^2(t)\,dx$. This concludes the proof of (i) and (ii) in Theorem 3.1, at least in the case where the function $\xi \mapsto \bar{y}(\xi)$ is absolutely continuous.

6. General initial data. We now consider a more general initial data $(\bar{u}, \bar{\mu}) \in \mathcal{D}$. In this case, the map $\xi \mapsto \bar{y}(\xi)$ is non-decreasing, left continuous but not necessarily continuous. Its distributional derivative is thus a measure, say $D_\xi \bar{y} = \sigma = \sigma^a + \sigma^s$. By the assumptions, the absolutely continuous part satisfies

$$d\sigma^a = \overline{U}_\xi^2\, d\xi\,,$$

so that

$$\bar{y}(\xi) = \bar{y}(0) + \int_0^\xi \overline{U}_\xi^2(\zeta)\, d\zeta + \sigma^s([0,\xi[\,)\,.$$

Consider a new initial condition $(\bar{u}^*, \bar{\mu}^*)$ defined by setting

$$\bar{y}^*(\xi) = \bar{y}(0) + \int_0^\xi \overline{U}_\xi^2(\zeta)\, d\zeta\,, \qquad\qquad \bar{u}^*(\bar{y}^*(\xi)) = \overline{U}(\xi)$$

$$\bar{\mu}^*(J) = \text{meas}\,\{\xi\,;\quad \bar{y}^*(\xi) \in J\}\,.$$

By construction, for this new initial data the function $\xi \mapsto y^*(0,\xi) = \bar{y}^*(\xi)$ is absolutely continuous. Hence, by the previous analysis, the corresponding function $u^*(t,x)$ provides a solution to the initial-boundary value problem (17)-(18) with initial data $(\bar{u}^*, \bar{\mu}^*)$. It is now easy to check that the function constructed in (32)–(35) for the original initial data $\bar{u}$ satisfies

$$u\Big(t,\ y(t,\xi) + \sigma^s([0,\xi[)\Big) = U(t,\xi)\,.$$

More precisely,

$$u(t,x) = U(t,\xi) \qquad \text{where} \quad \xi = \inf\{\zeta\,;\quad y(t,\zeta) + \sigma^s([0,\zeta]) \geq x\}\,.$$

By the previous analysis, u^* provides a solution. Hence the same is true of u.

7. Continuity properties. Recall that $\bar\xi = \bar\mu(\mathbb{R}_+) < \infty$ is the total mass of each of the measures $\mu_{(t)}$. We have

$$\text{Tot.Var.}\{u(t,\cdot)\,;\quad [0,M]\,\} \leq \sqrt{\bar\xi M}\,.$$

Since $u(t,0) = 0$, for any $x \in [0,M]$ we have

$$|u(t,x)| \leq \sqrt{\bar\xi M}\,.$$

This implies the Lipschitz continuity property w.r.t. time:

$$\int_0^M |u(t,x) - u(s,x)|\,dx \leq |t-s| \cdot \left\{ \sup_\omega |f'(\omega)| \cdot \sqrt{\bar\xi M} + \frac{\bar\xi M}{2} \cdot \sup_\omega |f''(\omega)| \right\} \tag{54}$$

where both sup are taken over $|\omega| \leq \sqrt{\bar\xi M}$.

Next, consider a convergent sequence of initial data $(\bar u^m, \bar\mu^m)_{m\geq 0}$. The assumption of Theorem 3.1 implies that the corresponding functions $\bar U^n$ satisfy

$$\overline{U}^m(\xi) \to \bar U(\xi)$$

uniformly on $[0,\bar\xi]$. Therefore $U^m(t,\xi) \to U(t,\xi)$ uniformly on the domain $[0,T]\times[0,\bar\xi\,]$, for any $T > 0$. In turn, this implies the convergence (27)-(28).

8. Hölder continuity. We show that $u(t,x)$ is Hölder continuous locally in (t,x). First we know by Sobolev embedding that u is Hölder continuous in x for each fixed time t with exponent $\alpha = 1/2$. In the time direction, we know that the derivative of u along a characteristic is bounded, thus u is Lipschitz continuous in time along a characteristic. The characteristic speed is u which is locally bounded, thus the distance traveled in the x direction is order one of time. Combining the two parts, we conclude that u is Hölder continuous locally in both space and time.

This completes the proof of Theorem 3.1.

Remark. The previous construction of solutions to (17)-(18) works equally well for negative times. The semigroup S can thus be extended to a group $\Psi : \mathcal{D} \times \mathbb{R} \mapsto \mathcal{D}$.

4. Characterization of semigroup trajectories

In the previous section, a solution u to the initial-boundary value problem (17)-(18), (23), was obtained as the fixed point of a contractive transformation. Hence, any other solution which provides a fixed point to the same transformation necessarily coincides with u. A straightforward uniqueness result can be stated as follows.

Theorem 4.1 *Assume that f satisfies (19). Consider a function $u = u(t,x)$ and a family of measures $\mu_{(t)}$ satisfying (i) and (ii) in Theorem 3.1.*

Moreover, calling

$$y(t,\xi) \doteq \inf\Big\{x \geq 0\ ;\quad \mu_{(t)}\left([0,x]\right) \geq \xi\Big\}\,, \tag{55}$$

$$U(t,\xi) \doteq u(t,\, y(t,\xi)), \tag{56}$$

assume that for a.e. ξ the map $t \mapsto U(t,\xi)$ is absolutely continuous and satisfies the differential equation (32). Then one has the identity

$$(u(t), \mu_{(t)}) = S_t(\bar{u}, \bar{\mu}). \tag{57}$$

In particular, the solution which satisfies the above conditions is unique.

We conjecture that a uniqueness result remains valid even without the assumption (32) on the corresponding function U. The basic ingredient toward a uniqueness result is the assumption

$$f''(u)\, d\mu^a(t) = f''(u)\, u_x^2(t)\, dx\,. \tag{58}$$

for a.e. t. We now show that this is indeed the case under the additional condition $f'' > 0$.

Theorem 4.2 *In addition to assumption (19), let $f''(\cdot) > 0$. Consider a function $u = u(t,x)$ and a family of measures $\mu_{(t)}$ satisfying (i) and (ii) in Theorem 3.1. Then identity (57) holds.*

Indeed, observe that the flow on $\mathbf{L}^1([0,\bar{\xi}])$ generated by the evolution equation (32) is Lipschitz continuous w.r.t. time and to the initial data. Adopting a semigroup notation, call $t \mapsto V(t) = \S_t\overline{V}$ the trajectory corresponding to the initial data $\overline{V} \in \mathbf{L}^1([0,\bar{\xi}])$. Since the couple $(u(t),\, \mu_{(t)})$

can be entirely recovered from the function $U(t,\cdot)$ and the initial mapping $\xi \mapsto \bar{y}(\xi)$, to prove uniqueness, it thus suffices to show that

$$\lim_{h\to 0+} \frac{1}{h} \int_0^{\bar{\xi}} \Big| U(t+h,\xi) - (\S_h U(t))(\xi) \Big| \, d\xi = 0 \tag{59}$$

for almost every time $t > 0$ (see Theorem 2.9 in [1]). Since $f'' > 0$, our assumption implies that the singular part of $\mu_{(t)}$ vanishes at a.e. t. Choose a time t where $\mu^s_{(t)} = 0$. Then

$$U_\xi(t,\xi) \neq 0 \qquad \text{for a.e.} \quad \xi \in [0, \bar{\xi}\,]\,. \tag{60}$$

Consider the map $\xi \mapsto y(t,\xi)$. Since

$$u_x^2 \, dy = d\xi\,, \qquad\qquad U_\xi = u_x \cdot \frac{dy}{d\xi} = \frac{1}{u_x}\,,$$

by (60) the pre-image of a set of measure zero through the map $\xi \mapsto y(t,\xi)$ has measure zero.

If now $u = u(t,x)$ is differentiable at the point $(t,\, y(t,\xi))$, we have the identity

$$\begin{aligned} \tfrac{\partial}{\partial t} U(t,\xi) &= \Big[u_t + f'(u) u_x \Big](t,\, y(t,\xi)) \\ &= \tfrac{1}{2} \textstyle\int_0^{y(t,\xi)} f''(u)\, u_x^2(t,x)\, dx = \tfrac{1}{2} \int_0^{\xi} f''(U(t,\eta))\, d\eta\,. \end{aligned} \tag{61}$$

Observing that $u(t,\cdot)$ is differentiable at a.e. x, we conclude that (61) holds at a.e. $\xi \in [0, \bar{\xi}\,]$. In turn, this implies (59), proving the theorem.

Notice how the condition on the vanishing of the singular part is essential to ensure uniqueness. Otherwise, in Example 1 the solution $u(t,x) \equiv 0$ for $t \geq 1$, with $\mu_{(t)}$ containing a unit mass at the origin, would satisfy all the other requirements of the theorem.

5. A semigroup of dissipative solutions

Next, we examine dissipative solutions. A major difference with the conservative case is that here the Cauchy problem is well-posed if the flux function f is strictly convex, but ill posed otherwise, as shown in the next section.

In this section, our main concern will be the construction of a semigroup of dissipative solutions under the additional assumption that $f'' \geq 0$. As domain $\mathcal{D}$ of our semigroup we choose the space

$$\mathcal{D} \doteq \Big\{ u : \mathbb{R}_+ \mapsto \mathbb{R}\,, \quad u \text{ is absolutely continuous, } \; u(0) = 0\,, \quad u_x \in \mathbf{L}^2 \Big\}.$$

Theorem 5.1 *Assume that the flux function f satisfies (19) and $f'' \geq 0$. Then there exists a semigroup $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ with the following properties. Calling $t \mapsto u(t) = S_t \bar{u}$ the trajectory corresponding to an initial data $\bar{u} \in \mathcal{D}$, one has:*

(i) The function $u = u(t, x)$ is Hölder continuous. It provides a solution of (17)-(18) with initial condition $u(0, x) = \bar{u}(x)$.

(ii) For every $M > 0$, the above solution u satisfies the Lipschitz continuity property in time:

$$\int_0^M |u(t, x) - u(s, x)| \; dx \leq C|t - s|, \tag{62}$$

(iii) Given a sequence of initial conditions $\bar{u}^n \in \mathcal{D}$, assume that

$$\|\bar{u}^n_x - \bar{u}_x\|_{\mathbf{L}^2([0,M])} \to 0$$

for every $M > 0$. Then the corresponding solutions satisfy

$$u^n(t,x) \to u(t,x) \tag{63}$$

uniformly for t, x in bounded sets.

Proof. Consider an initial condition $\bar{u} \in \mathcal{D}$. For simplicity, we again assume that $\bar{u}$ is constant outside a bounded interval, say $[0, R]$. The general case follows from an approximation argument.

To construct the corresponding trajectory we begin by setting

$$\bar{\xi} \doteq \int_0^R |u_x^2(x)|\, dx\,.$$

Then we define the initial data

$$\overline{U}(\xi) \doteq \bar{u}(\bar{y}(\xi))\,,$$

where

$$\bar{y}(\xi) \doteq \inf\left\{x \geq 0\,;\quad \int_0^x u_x^2(x)\, dx \geq \xi\right\}. \tag{64}$$

By the analysis in Section 3, the map $\xi \mapsto \overline{U}(\xi)$ is absolutely continuous, hence its derivative

$$\overline{Z}(\xi) = \frac{\partial}{\partial \xi}\overline{U}(\xi)$$

is a well defined function in $\mathbf{L}^1([0, \bar{\xi}\,])$.

Define the subset

$$J^- \doteq \left\{\xi \in [0, \bar{\xi}\,]\,;\ \ \overline{Z}(\xi) \leq 0\right\}.$$

Let L be a Lipschitz constant for f''. On the space of continuous functions $Y : \mathbb{R}_+ \mapsto L^1([0, \bar{\xi}\,])$ with weighted norm

$$\|Y\|_* \doteq \sup_t\ e^{-L\bar{\xi}t}\|Y(t)\|_{L^1}\,,$$

we now define a continuous transformation $Y \mapsto \mathcal{T}Y$ as follows.

$$\mathcal{T}Y(t,\xi) \doteq \overline{Z}(\xi) + \int_0^t \frac{1}{2} f''\left(\int_0^\xi \Phi(\eta, Y(s,\eta))\, d\eta\right) ds\,, \tag{65}$$

where

$$\Phi(\eta, Y) = \min\{Y, 0\} \qquad \text{if } \ \eta \in J^-\,,$$

$$\Phi(\eta, Y) = Y \qquad \text{if } \ \eta \in [0, \bar{\xi}\,] \setminus J^-\,.$$

To check that $\mathcal{T}$ is a strict contraction, assume that $\|Y - \widetilde{Y}\|_* = \kappa$, so that

$$\int_0^{\bar{\xi}} |Y(t,\xi) - \widetilde{Y}(t,\xi)|\, d\xi \leq \kappa\, e^{L\bar{\xi}t}$$

for all $t \geq 0$. Then for every $s \geq 0$

$$\int_0^\xi \Big|\Phi(\eta,\, Y(s,\eta)) - \Phi(\eta,\, \widetilde{Y}(s,\eta))\Big|\, d\eta \leq \kappa\, e^{L\bar{\xi}s},$$

and therefore

$$\int_0^{\bar{\xi}} |(\mathcal{T}Y - \mathcal{T}\widetilde{Y})(t,\xi)| d\xi \leq \int_0^{\bar{\xi}} \int_0^t \frac{L\kappa}{2}\, e^{L\bar{\xi}s}\, ds\, d\xi \leq \frac{\kappa}{2}\, e^{L\bar{\xi}t}.$$

By the definition of our weighted norm, this implies

$$\|\mathcal{T}Y - \mathcal{T}\widetilde{Y}\|_* \leq \frac{1}{2}\, \|Y - \widetilde{Y}\|_*\,.$$

Let now $Y = Y(t,\xi)$ be the unique fixed point of the transformation $\mathcal{T}$.

Then one easily checks that the function

$$Z(t,\xi) \doteq Y(t,\xi) \qquad \text{if } \ \xi \notin J^-\,,$$

$$Z(t,\xi) \doteq \min\{Y(t,\xi), \ \ 0\} \qquad \text{if } \ \xi \in J^-\,,$$

provides a solution to the equations

$$Z(0,\xi) = \overline{Z}(\xi)\,,$$

$$\frac{\partial Z}{\partial t}(t,\xi) = \frac{1}{2} f''\left(\int_0^\xi Z(t,\eta)\,d\eta\right) \qquad \text{if } \; Z(s,\xi) \neq 0 \;\; \text{for all } s \in [0,t]\,,$$

$$\frac{\partial Z}{\partial t}(t,\xi) = 0 \qquad \text{if } \; Z(s,\xi) = 0 \;\; \text{for some } s \in [0,t]\,.$$

In turn, we can now define

$$U(t,\xi) \doteq \int_0^\xi Z(t,\eta)\,d\eta$$

and the characteristic curves

$$y(t,\xi) \doteq \bar{y}(\xi) + \int_0^t f'(U(s,\xi))\,ds\,.$$

In a similar way as in Section 3, the dissipative solution u can now be obtained by setting

$$u(t,x) = U(t,\xi(t,x)),$$

where

$$\xi(t,x) \doteq \sup\left\{\xi\,;\;\; y(t,\xi) \leq x\right\}\,.$$

To see why this construction actually yields a solution to (17), consider first the case where the map $\xi \mapsto \bar{y}(\xi)$ is absolutely continuous. Then $y_\xi(0,\xi) = \overline{U}^2_\xi(0,\xi) = Z^2(0,\xi)$. Since

$$\frac{\partial}{\partial t} y_\xi = f''(U)\,U_\xi = f''(U)\,Z = \frac{\partial}{\partial t} Z^2$$

for all t, ξ we deduce the identity

$$y_\xi(t,\xi) = Z^2(t,\xi) = U^2_\xi(t,\xi).$$

Moreover, (50) again holds. As in the proof of Theorem 3.1, we obtain the relations

$$Z(t,\xi) = \frac{1}{u_x(t,y(t,\xi))}\,, \qquad\qquad [y_\xi(t,\xi)]^{-1} = u_x^2(t,y(t,\xi)). \tag{66}$$

For almost every $x \in [y(t,0)\,,\ y(t,\bar\xi)]$, if $x = y(t,\xi)$, then

$$[u_t + f'(u)u_x](t,x) = \frac{d}{dt}u(t,y(t,\xi)) = \frac{\partial}{\partial t}U(t,\xi) = \int_0^\xi \frac{\partial}{\partial t}Z(t,\eta)\,d\eta$$

$$= \frac{1}{2}\int_0^\xi f''(U(t,\eta))\,d\eta = \frac{1}{2}\int_0^x f''(u(t,y))\,u_x^2(t,y)\,dy\,.$$

The second identity in (66) was used here to change the variable of integration.

The extension to the case of general initial data, where the map $\xi \mapsto y(t,\xi)$ is not necessarily absolutely continuous, is carried out as in the earlier proof of Theorem 3.1. We skip the details.

6. Instability of dissipative solutions for non-convex flux

In this section, we show that if the convexity assumption $f'' \geq 0$ is dropped, then the Cauchy problem for the equation (17)-(18) is ill posed, in general.

Example 2. Consider the flux function $f(u) = u^3$. Let $U = U(t,\xi)$ be a solution of (32), with $\xi \in [0,3]$, such that at some time $t_0 > 0$ there holds

$$U(t_0,\xi) = \begin{cases} \xi, & \xi \in [0,1]\,, \\ 2-\xi, & \xi \in [1,2]\,, \\ 0, & \xi \in [2,3]\,. \end{cases}$$

Consider first the conservative solution $u = u(t,x)$. This is well defined forward and backward in time. At time $t = t_0$, its explicit values are

$$u(t_0,x) = \begin{cases} x\,, & x \in [0,1]\,, \\ 2-x\,, & x \in [1,2]\,, \\ 0\,, & x > 2 \end{cases}$$

while a unit mass is concentrated at the point $x = 2$. Assuming t_0 sufficiently small, we have

$$U_t(t,\xi) = \int_0^\xi 3U(t,\eta)\,d\eta > 0$$

for all $t \in [0,t_0]$ and $\xi \in]0,3]$. Hence

$$\frac{\partial}{\partial t}U_\xi(t,\xi) = 3U < 0, \qquad U_\xi(t,\xi) < 0 \qquad \text{for} \quad t \in [0,t_0[\,, \quad 2 < \xi < 3\,.$$

Next, consider a dissipative solution v coinciding with u at time $t = 0$. This means

$$v(0,x) = u(0,x) = U(0,\xi) \qquad \text{for} \quad x = y(0,\xi)\,. \tag{67}$$

We recall that

$$y(t,\xi) = \int_0^\xi U_\xi^2(t,\eta)\,d\eta\,.$$

Clearly, v will still coincide with u as long as its gradient remains bounded (equivalently, as long as U_ξ remains bounded away from zero). On the other hand, for $t > t_0$, the dissipative solution $v = v(t,x)$ coincides with the conservative one only on the interval where $x \le y(t,2)$, while v is constant for $x \ge y(t,2)$. In other words,

$$v(t,x) = u(t,x) \qquad \text{if} \quad t \in [0,t_0]\,,$$

$$v(t,x)=\begin{cases} u(t,x)\,, & 0\leq x\leq y(t,2)\,,\\ u(t,y(t,2))\,, & x>y(t,2)\,,\end{cases} \qquad \text{if } \ t\in[t_0,\,2t_0]\,.$$

Energy dissipation occurs at time $t=t_0$, namely

$$\int_0^\infty v_x^2(t,x)\,dx=\begin{cases}3\,, & t\in[0,t_0[,\\ 2\,, & t\geq t_0.\end{cases}$$

Next, consider a family of perturbed initial conditions, say

$$U^\varepsilon(0,\xi)=U(0,\xi)+\varepsilon\phi(\xi)\,,$$

where ϕ is a non-negative smooth function, whose support is contained in $[0,1]$. Since $U\mapsto f''(U)=6U$ is a monotone increasing function, by a comparison argument from (32) we deduce

$$U^\varepsilon(t,\xi)\geq U(t,\xi)$$

for all $\varepsilon,t>0$, $\xi\in[0,3]$. In fact, for a nontrivial ϕ we can assume a strict inequality:

$$U^\varepsilon(t,\xi)>U(t,\xi)\,, \qquad t>0\,, \quad \xi\in[2,3]\,.$$

For $2<\xi<3$ we now use the relations

$$\frac{\partial}{\partial t}U^\varepsilon_\xi(t,\xi)=3U^\varepsilon(t,\xi)>3U(t,\xi)=\frac{\partial}{\partial t}U_\xi(t,\xi)\,, \qquad U^\varepsilon_\xi(0,\xi)=U_\xi(0,\xi)\,,$$

and deduce

$$U^\varepsilon_\xi(t,\xi)>U_\xi(t,\xi)\geq 0\,, \qquad t\in[0,t_0]\,.$$

Moreover, for $t\geq t_0$ and $2<\xi<3$ one has

$$\frac{\partial}{\partial t}U^\varepsilon_\xi(t,\xi)=3U^\varepsilon(t,\xi)>3U(t,\xi)\geq 0\,.$$

Therefore, for each $\varepsilon > 0$, the quantity $U^\varepsilon_\xi(t,\xi)$ is still strictly positive at time $t = t_0$ and increases afterwards. It thus remains uniformly bounded away from zero.

Since $u_x = U_\xi^{-1}$, the above implies that, for any fixed $\varepsilon > 0$, the corresponding conservative solution $u^\varepsilon = u^\varepsilon(t,x)$ has a uniformly bounded gradient. The dissipative solution thus coincides with the conservative one. As $\varepsilon \to 0$, at time $t = 0$ our construction yields

$$\|u^\varepsilon(0) - u(0)\|_{\mathcal{C}^0} \to 0\,, \qquad\qquad \|u^\varepsilon_x(0) - u_x(0)\|_{\mathbf{L}^2} \to 0\,.$$

However, when $t > t_0$ and $x > y(t,2)$ the previous analysis yields

$$\lim_{\varepsilon\to 0+} u^\varepsilon(t,x) = u(t,x) \neq v(t,x)\,,$$

where u, v are respectively the conservative and the dissipative solutions of (17)-(18) with the same initial data (67). The example proves that dissipative solutions do not depend continuously on the initial data.

Remark. The previous example also shows that the family of dissipative solutions may not be closed. Since the set of solutions which are limits of vanishing viscosity approximations is closed and connected, we see that this set cannot coincide with the set of dissipative solutions.

Acknowledgments. We thank John Hunter for helpful conversations. This work is supported in part by NSF DMS-0305497 and NSF DMS-0305114 for Yuxi Zheng, NSF of China under Grants 10131050 and 10276036 (the innovation grants from Chinese Academy of Sciences) for Ping Zhang,

and by the Italian M.I.U.R. within the research project # 2002017219 "Equazioni iperboliche e paraboliche non lineari" for Alberto Bressan. This work was started when Ping Zhang visited Penn State University. He would like to thank the department for its hospitality.

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Department of Mathematics, Penn State University, UP, PA 16802;

E-mail: bressan@math.psu.edu

and

Academy of Mathematics and System Sciences, CAS, Beijing 100080;

E-mail: zp@mail.math.ac.cn

and

Department of Mathematics, Penn State University, UP, PA 16802;

E-mail: yzheng@math.psu.edu