DIVERGENCE-MEASURE FIELDS, SETS OF FINITE PERIMETER, AND CONSERVATION LAWS

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ABSTRACT. Divergence-measure fields in L^∞ over sets of finite perimeter are analyzed. A notion of normal traces over boundaries of sets of finite perimeter is introduced, and the Gauss-Green formula over sets of finite perimeter is established for divergence-measure fields in L^∞. The normal trace introduced here over a surface of finite perimeter is shown to be the weak-star limit of the normal traces introduced in Chen-Frid [6] over the Lipschitz deformation surfaces of the surface, which implies their consistency. As a corollary, an extension theorem of divergence-measure fields in L^∞ over sets of finite perimeter is also established. Then we apply the theory to the initial-boundary value problem of nonlinear hyperbolic conservation laws over sets of finite perimeter.

1. INTRODUCTION

We are concerned with divergence-measure ($\mathcal{DM}$) fields over sets of finite perimeter, especially their normal traces on the boundaries and the Gauss-Green formula. The $\mathcal{DM}$-fields arise naturally in the study of entropy solutions of nonlinear hyperbolic conservation laws, which take the form

$$\partial_t u + \nabla_x \cdot f(u) = 0, \qquad u \in \mathbb{R}^m, \ x \in \mathbb{R}^d, \tag{1.1}$$

where $f : \mathbb{R}^m \to (\mathbb{R}^m)^d$ is a nonlinear map.

The main feature of nonlinear hyperbolic conservation laws is that, no matter how smooth the initial data is, the solution may develop singularities and form shock waves in finite time. One must understand the solution in a weak sense, motivated by the Clausius-Duhem inequality involving entropy in fluid mechanics. In general, a function $\eta : \mathbb{R}^m \to \mathbb{R}$ is called a mathematical *entropy* of (1.1) if there exists $q : \mathbb{R}^m \to \mathbb{R}^d$ such that

$$\nabla q_k(u) = \nabla\eta(u)\nabla f_k(u), \qquad k = 1, 2, \cdots, d. \tag{1.2}$$

The function $q(u)$ is then called the entropy flux associated with the entropy $\eta(u)$, and the pair $(\eta(u), q(u))$ is called an entropy pair. The entropy pair $(\eta(u), q(u))$ is called a convex entropy pair on the domain $K \subset \mathbb{R}^m$ if the Hessian matrix $\nabla^2\eta(u) \geq 0$ for $u \in K$ and a strictly convex entropy pair on the domain K if $\nabla^2\eta(u) > 0$ for $u \in K$. Friedrichs-Lax [14] observed that most of the systems of conservation laws that result from continuum mechanics are endowed with a globally defined, strictly convex entropy.

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A function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ is called an entropy solution of (1.1) if $u = u(t,x)$ satisfies the Lax entropy inequality:

$$\partial_t \eta(u(t,x)) + \nabla_x \cdot q(u(t,x)) \leq 0 \tag{1.3}$$

in the sense of distributions, for any convex entropy pair $(\eta, q) : \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^d$. Taking $\eta = \pm u$, then we see that any entropy solution must be a weak solution.

One of the main issues in conservation laws is to study the behavior of solutions in this class to explore all possible information of solutions, including large-time behavior, uniqueness, stability, and traces of solutions, among others. The Schwartz lemma indicates from (1.3) that the distribution

$$\partial_t \eta(u(t,x)) + \nabla_x \cdot q(u(t,x))$$

is in fact a Radon measure, that is,

$$\operatorname{div}_{(t,x)}(\eta(u(t,x)), q(u(t,x))) \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^d). \tag{1.4}$$

Furthermore, since $u \in L^\infty$, (1.4) is also true for any C^2 entropy pair (η, q) (i.e. η is not necessarily convex) if system (1.1) is endowed with a strictly convex entropy, as first observed in Chen [4]. This implies that, for any C^2 entropy pair (η, q), the field $(\eta(u(t,x)), q(u(t,x)))$ is a $\mathcal{DM}$-field. Another motivation for studying the $\mathcal{DM}$-fields is to construct the flux functions by the Cauchy fluxes from the formulation of the balance law in the general framework of sets of finite perimeter [10, 17, 18, 27]. Divergence-measure fields also arise in various other nonlinear problems such as mass transfer problems and free boundary problems (see [2, 3]).

From the discussion above, it is clear that understanding more properties of $\mathcal{DM}$-fields can advance our understanding of the behavior of entropy solutions for hyperbolic conservation laws and other related nonlinear problems. In general, divergence-measure fields in L^∞ are vector fields in L^∞ whose divergences are Radon measures. More precisely, we have

Definition 1.1. *Let $D \subset \mathbb{R}^N$ be open. We say that F is a divergence-measure field in L^∞ over D, i.e., $F \in \mathcal{DM}^\infty(D)$, if*

$$\|F\|_{\mathcal{DM}^\infty(D)} := \|F\|_{L^\infty(D;\mathbb{R}^N)} + |\operatorname{div} F|(D) < \infty, \tag{1.5}$$

where

$$|\operatorname{div} F|(D) = \sup\{\int_D F \cdot \nabla\phi \, dx \; : \; \phi \in C_0^1(D;\mathbb{R}), \|\phi\|_C \leq 1\} \tag{1.6}$$

that is the total variation of the measure $\operatorname{div} F$ *over D. If $F \in \mathcal{DM}^\infty(\Omega)$ for any open set Ω with $\Omega \Subset D \subset \mathbb{R}^N$, then we say $F \in \mathcal{DM}^\infty_{loc}(D)$, where $\Omega \Subset D$ denotes that the closure $\bar{\Omega}$ of Ω is a compact subset of D.*

This space under norm (1.5) is a Banach space. This space is larger than the space of vector fields of bounded variation. The establishment of the Gauss-Green theorem, traces, and other properties of BV functions in the middle of last century (see Federer [13]) has advanced significantly our understanding of solutions of nonlinear partial differential equations and related problems in calculus of variations, differential geometry, and other areas (see [1, 12, 15, 24, 25, 28]).

A natural question is whether the $\mathcal{DM}^\infty$-fields have similar properties, especially the traces and Gauss-Green formula, as for the BV fields. The answer is negative in general since one cannot define the trace for each component of a $\mathcal{DM}^\infty$ field even over a Lipschitz surface in general, as opposed to the case of BV fields.

In Chen-Frid [6, 7], a theory of divergence-measure fields over sets of Lipschitz deformable boundaries was established, motivated by various nonlinear problems in conservation laws. In particular, a natural notion of normal traces over Lipschitz deformable surfaces was introduced by the neighborhood information via Lipschitz deformation under which the Gauss-Green formula is shown to hold for $\mathcal{DM}^\infty$ fields, and an explicit way to calculate the normal trace over any Lipschitz deformable surface was developed, suitable for applications, by using the neighborhood information of the field near the surface and the level set function of the Lipschitz deformation surfaces. This theory has been applied to solving several different problems; see [5, 8, 9] and the references cited therein.

In this paper, we extend the theory of divergence-measure fields in L^∞ for sets of Lipschitz deformable boundaries to that for sets of finite perimeter, which naturally arise in various areas such as conservation laws, free boundary problems, and mass transfer problems. We introduce a notion of normal traces over boundaries of sets of finite perimeter and establish the Gauss-Green formula for $\mathcal{DM}^\infty$ fields over sets of finite perimeter. It is shown that the normal trace over a surface of finite perimeter is the weak-star limit of the normal traces introduced in Chen-Frid [6] over the Lipschitz deformation surfaces of the surface. As a corollary, an extension theorem of divergence-measure fields in L^∞ over sets of finite perimeter is also established. Then we apply the theory to the initial-boundary value problem of nonlinear hyperbolic conservation laws over sets of finite perimeter.

In Section 2, we first analyze some basic properties of sets of finite perimeter and $\mathcal{DM}^\infty$ fields for subsequent development. In Section 3, we introduce the notion of normal traces over boundaries of sets of finite perimeter and establish the Gauss-Green formula over sets of finite perimeter for $\mathcal{DM}^\infty$ fields. As a corollary, we also establish an extension theorem for $\mathcal{DM}^\infty$ fields over sets of finite perimeter. Then, in Section 4, we further analyze the normal traces introduced in Section 3 and show that the normal trace over an $(N-1)$-dimensional surface of finite perimeter can be understood as the weak-star limit of the normal traces introduced in Chen-Frid [6] over the Lipschitz deformation surfaces of the surface, which implies their consistency. In Section 5, as a direct application, we apply the theory to the initial-boundary value problem of nonlinear hyperbolic conservation laws over sets of finite perimeter. It would be interesting to explore more applications of this theory in various problems involving L^∞ vector fields whose divergences are measures; see [5, 8, 9] and the references cited therein for applications of this theory over sets of Lipschitz deformable boundaries.

2. Sets of Finite Perimeter and Divergence-Measure Fields

In this section, we first recall some notions and analyze some basic properties of sets of finite perimeter and $\mathcal{DM}^\infty$ fields for subsequent development in Sections 3–5. For a detailed exposition on the theory of sets of finite perimeter and functions of bounded variation, see [1, 12, 15, 28].

We start with some basic notations and definitions. First $\mathcal{H}^N$ denotes the N-dimensional Hausdorff measure in $\mathbb{R}^N$, and $\mathcal{L}^N$ denotes the Lebesgue measure in $\mathbb{R}^N$. We recall that $\mathcal{L}^N = \mathcal{H}^N$. At times we shall denote $|E|$ as the $\mathcal{L}^N$-Lebesgue measure of the set E. In this paper, D and Ω denote open subsets of $\mathbb{R}^N$.

Definition 2.1. *Let E be an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$. For any open set D, we say that E is a set of finite perimeter in D if the characteristic function of E, χ_E, belongs to $BV(D)$. We shall refer to a set of finite perimeter in $\mathbb{R}^N$ simply as a* set of finite perimeter.

Remark 2.1. *If E is a set of finite perimeter in D, then $\nabla\chi_E$ (the gradient of χ_E in the sense of distributions) is a vector-valued Radon measure in D. We denote the total variation of $\nabla\chi_E$ as $|\nabla\chi_E|$. It can be shown (cf.* [1, 12]*) that*

$$\nabla\chi_E = \nu_E|\nabla\chi_E|,$$

where ν_E is the measure theoretic unit inward normal to E.

Definition 2.2. *Let E be a set of finite perimeter in D. The* **reduced boundary** *of E, denoted as $\partial^* E$, is the set of all points $x \in \operatorname{supp}(|\nabla\chi_E|) \cap D$ such that*

(i) $\int_{B(x,r)} |\nabla\chi_E| > 0$ *for all* $r > 0$;
(ii) $\lim_{r\to 0} \frac{\int_{B(x,r)} \nabla\chi_E}{\int_{B(x,r)} |\nabla\chi_E|} = \nu_E(x)$;
(iii) $|\nu_E(x)| = 1$.

We recall that the space of functions of bounded variation BV in fact represents an equivalence class of functions so that changing the value of a function in this class on a set of $\mathcal{L}^N$-measure zero does not change the function. From Definition 2.1 it follows that the same is true for sets of finite perimeter. Since we are concerned only with equivalence classes of sets, *we assume throughout this paper that a set of finite perimeter E is the representative given by the following proposition*, which can be found in [15].

Proposition 2.1. *If $E \subset \mathbb{R}^N$ is a Borel set, then there exists a Borel set $\tilde{E}$ equivalent to E (i.e., differs only by a set of $\mathcal{L}^N$- measure zero) such that*

$$0 < |\tilde{E} \cap B(x,r)| < \omega_N r^N \tag{2.1}$$

for all $x \in \partial\tilde{E}$ and all $r > 0$, where ω_N is the measure of the unit ball in $\mathbb{R}^N$.

With the above convention, there is no ambiguity when speaking of the topological boundary ∂E of a set E of finite perimeter.

Definition 2.3. *For every $\alpha \in [0,1]$ and every $\mathcal{L}^N$-measurable set $E \subset \mathbb{R}^N$, define*

$$E^\alpha := \{x \in \mathbb{R}^N \,:\, \lim_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = \alpha\}, \tag{2.2}$$

the set of all points with density α. We now define the **essential boundary** *of E, $\partial^s E$, as*

$$\partial^s E = \mathbb{R}^N \setminus (E^0 \cup E^1). \tag{2.3}$$

The sets E^0 and E^1 may be considered as the measure theoretic exterior and interior of E, which motivate the definition of essential boundaries.

Remark 2.2. *If E is a set of finite perimeter in D (cf.* [1]*), then*

$$\partial^* E \subset E^{\frac{1}{2}} \subset \partial^s E, \tag{2.4}$$

$$\mathcal{H}^{N-1}(\partial^s E \setminus \partial^* E) = 0, \tag{2.5}$$

and

$$|\nabla\chi_E| = \mathcal{H}^{N-1}\lfloor\partial^* E. \tag{2.6}$$

Definition 2.4. *Let $f \in L^1(D)$ and $a \in \mathbb{R}^N$. We say that $f_a(x_0)$ is the approximate limit of f at $x_0 \in D$ restricted to $\Pi_a := \{x \in \mathbb{R}^N : x \cdot a \geq 0\}$ if, for any $\epsilon > 0$,*

$$\lim_{r \to 0} \frac{|\{x \in \mathbb{R}^N : |f(x) - f_a(x_0)| < \epsilon\} \cap B(x_0, r) \cap \Pi_a|}{|B(x_0, r) \cap \Pi_a|} = 1. \tag{2.7}$$

Definition 2.5. *We say that $x_0 \in D$ is a regular point of a function $f \in BV(D)$ if there exists a vector $a \in \mathbb{R}^N$ such that the approximate limits $f_a(x_0)$ and $f_{-a}(x_0)$ exist. The vector a is called* **a defining vector**.

If x_0 is a regular point of $f \in BV(D)$, then there are two possibilities: either $f_a(x_0) = f_{-a}(x_0)$ or $f_a(x_0) \neq f_{-a}(x_0)$. It can be proved (cf. [25]) that, in the first case, any $b \in \mathbb{R}^N$ is a defining vector and $f_b(x_0) = f_a(x_0)$; in the second case, a is unique up to sign, i.e., the only defining vectors are a and $-a$.

Remark 2.3. *A classical result in the BV theory says that $\mathcal{H}^{N-1}$-almost every $x \in D$ is a regular point of $f \in BV(D)$; see* [1, 12, 25].

Definition 2.6. *Given $f \in L^1_{loc}(D)$, we define*

$$\overline{f}(x) := \lim_{\delta \to 0} f_\delta(x), \tag{2.8}$$

where $f_\delta := f * \omega_\delta$ with $\omega_\delta(x) = \delta^{-N}\omega(\frac{x}{\delta})$ for a positive symmetric mollifier ω.

Remark 2.4. *It can be proved that, if $f \in BV(D)$, then $\overline{f}$ is defined at each regular point. Moreover, if x_0 is a regular point of f, then*

$$\overline{f}(x_0) = \frac{1}{2}\left(f_a(x_0) + f_{-a}(x_0)\right),$$

where a is a defining vector (cf. [25]*).*

If E is a set of finite perimeter in D, we have from Remark 2.4 that $\overline{\chi}_E$ is defined $\mathcal{H}^{N-1}$-almost everywhere. In fact, we have

$$\overline{\chi}_E(x) = \begin{cases} \frac{1}{2} & \text{if } x \in \partial^* E, \\ 1 & \text{if } x \in E^1, \\ 0 & \text{if } x \in E^0. \end{cases} \tag{2.9}$$

We recall here that $\mathcal{H}^{N-1}(\partial^s E \setminus \partial^* E) = 0$.

As Proposition 2.2 below indicates, $\text{div}F << \mathcal{H}^{N-1}$. Thus, we do not have to be concerned with the values of $\overline{\chi}_E$ on the set $\partial^s E \setminus \partial^* E$. This fact is essential in the proof of the Gauss-Green formula presented in Section 3.

Proposition 2.2. *Let $F \in \mathcal{DM}^\infty(D)$. Then the Radon measure* $\text{div}F$ *in D is absolutely continuous with respect to the $(N-1)$-Hausdorff measure $\mathcal{H}^{N-1}$.*

A proof of Proposition 2.2 was given in Chen-Frid [6] by using the Gauss-Green formula for $\mathcal{DM}^\infty$ fields over sets of Lipschitz boundaries.

The next proposition is a direct consequence from the definitions above and can be proved as in Proposition 1.15 in [15].

Proposition 2.3. *Let $F \in \mathcal{DM}^\infty(D)$. If F has compact support in D, then*

$$|\mathrm{div} F_\delta|(D) \to |\mathrm{div} F|(D),$$

where $F_\delta := F * \omega_\delta$.

The following proposition is essentially contained in [6] and, for self-contained, we give a detailed proof.

Proposition 2.4. *Let $F \in \mathcal{DM}^\infty(D)$. If g is a Lipschitz function, then*

$$\mathrm{div}(g\,F) = g\,\mathrm{div} F + F \cdot \nabla g. \tag{2.10}$$

If $E \Subset D$ is a set of finite perimeter in D, then

$$\mathrm{div}(\chi_E F) = \overline{\chi}_E\,\mathrm{div} F + \overline{F \cdot \nabla \chi_E} \tag{2.11}$$

where $\overline{F \cdot \nabla \chi_E} = w - lim_{\delta\to 0} F \cdot \nabla (\chi_E)_\delta$ for $(\chi_E)_\delta = \chi_E * \omega_\delta$. Furthermore, the measure $\overline{F \cdot \nabla \chi_E}$ is absolutely continuous with respect to the measure $|\nabla \chi_E|$.

Proof. Identity (2.10) is actually (3.1) in Theorem 3.1 in [6], which can be checked directly from the definitions. Now we show (2.11).

Since $\chi_\delta := (\chi_E)_\delta$ is smooth and bounded, then (2.10) implies that $\chi_\delta F \in \mathcal{DM}^\infty(D)$ and

$$\mathrm{div}(\chi_\delta F) = \chi_\delta\,\mathrm{div} F + F \cdot \nabla \chi_\delta. \tag{2.12}$$

Using $\mathrm{div} F << \mathcal{H}^{N-1}$, we obtain from the Dominated Convergence Theorem and Remarks 2.3 and 2.4 that

$$\chi_\delta\,\mathrm{div} F \rightharpoonup \overline{\chi}\,\mathrm{div} F \qquad \text{in } \mathcal{M}(D).$$

Since $\{\mathrm{div}(\chi_\delta F)\}$ is uniformly bounded in $\mathcal{M}(D)$, then $\{\mathrm{div}(\chi_\delta F)\}$ converges weakly in $\mathcal{M}(D)$. On the other hand, this sequence converges to $\mathrm{div}(\chi_E F)$ in the sense of distributions over D. Therefore, the uniqueness of weak limits yields

$$\mathrm{div}(\chi_\delta F) \rightharpoonup \mathrm{div}(\chi_E F) \qquad \text{in } \mathcal{M}(D).$$

Hence, from (2.12), it follows that there exists a measure $\mu := \overline{F \cdot \nabla \chi_E} \in \mathcal{M}(D)$ such that

$$F \cdot \nabla \chi_\delta \rightharpoonup \overline{F \cdot \nabla \chi_E} \qquad \text{in } \mathcal{M}(D).$$

We now prove that $\overline{F \cdot \nabla \chi_E} << |\nabla \chi_E|$. Since μ is a Radon measure, it suffices to show that

$$\mu(A) = 0$$

for any compact set A with $|\nabla \chi_E|(A) = 0$. Given $\epsilon > 0$, we can cover A by a finite number J of balls with centers x_i and radii $r_i < \epsilon, 1 \le i \le J < \infty$, such that

$$A \subset \cup_{i=1}^J B(x_i, r_i) \quad \text{and} \quad |\nabla \chi_E|(\cup_{i=1}^J B(x_i, r_i)) < \epsilon.$$

We may assume without loss of generality that $|\nabla \chi_E|(\partial B(x_i, r_i)) = 0$, $i = 1, \dots, J$. Then, for any $\phi \in C_0(\cup_{i=1}^J B(x_i, r_i))$, we have

$$\begin{aligned} \int_{\cup_{i=1}^J B(x_i,r_i)} \phi\, d\mu &= \lim_{\delta\to 0} \int_{\cup_{i=1}^J B(x_i,r_i)} \phi(x)\, F(x) \cdot \nabla \chi_\delta(x) dx \\ &\le \|\phi\|_\infty \|F\|_\infty \limsup_{\delta\to 0} |\nabla \chi_\delta|(\cup_{i=1}^J B(x_i, r_i)) \\ &= \|\phi\|_\infty \|F\|_\infty |\nabla \chi_E|(\cup_{i=1}^J B(x_i, r_i)) \le \epsilon\, \|\phi\|_\infty \|F\|_\infty \end{aligned}$$

from the fact that

$$|\nabla \chi_\delta|(B) \to |\nabla \chi_E|(B)$$

for any open set $B \Subset D$ with $|\nabla\chi_E|(\partial B) = 0$. We can now choose $0 \le \phi \le 1$ such that

$$\phi \equiv 1 \text{ on } A \qquad \text{and} \qquad \Big|\int_{\cup_{i=1}^J B(x_i,r_i)\setminus A} \phi\, d\mu\Big| < \epsilon\, \|F\|_\infty.$$

Then

$$\begin{aligned}\mu(A) &= \int_{\cup_{i=1}^J B(x_i,r_i)} \phi\, d\mu - \int_{\cup_{i=1}^J B(x_i,r_i)\setminus A} \phi\, d\mu \\ &\le 2\epsilon\, \|F\|_\infty.\end{aligned}$$

Since ϵ is arbitrary, we conclude that $\mu(A) = 0$. This completes the proof. □

3. Gauss-Green Formula

In this section we establish the Gauss-Green formula, i.e., the integration by parts formula, on sets of finite perimeter for $\mathcal{DM}^\infty$ fields. This formula is a corollary of Theorem 3.1 below. We also establish an extension theorem for $\mathcal{DM}^\infty$ fields over sets of finite perimeter.

Proposition 3.1. *If $F \in \mathcal{DM}^\infty(D)$ has compact support in D, then*

$$\int_D \operatorname{div} F = 0. \tag{3.1}$$

Proof. Denote Ω an open set with smooth boundary such that $\operatorname{supp}(F) \Subset \Omega \Subset D$. If $\phi \in C_0^\infty(D)$ such that $\phi \equiv 1$ on Ω, then, for sufficiently small δ,

$$\int_D \operatorname{div} F_\delta = \int_D \phi \operatorname{div} F_\delta = -\int_D F_\delta \cdot \nabla\phi = 0.$$

Taking $\delta \to 0$ and using Proposition 2.3 yields the result. □

We will also use the following proposition, which can be found in [1] and [25].

Proposition 3.2. *Let $u \in BV(D)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $f(0) = 0$. Then*

$$v = f \circ u \in BV(D)$$

and

$$|\nabla v|(D) \le \operatorname{Lip}(f)|\nabla u|(D).$$

Furthermore, if $x \in D$ is a regular point of u, that is, there exists $a \in \mathbb{R}^N$ such that the approximate limits $u_{\pm a}(x)$ exist, then $x \in D$ is also a regular point of v with a as its defining vector such that

$$v_{\pm a}(x) = f(u_{\pm a}(x))$$

and

$$\bar{v}(x) = \frac{1}{2}\left(f(u_a(x)) + f(u_{-a}(x))\right).$$

We now prove the main theorem of this paper.

Theorem 3.1. *Let $F \in \mathcal{DM}^\infty(D)$. If $E \Subset D$ is a bounded set of finite perimeter, then there exists an $\mathcal{H}^{N-1}$-integrable function (denoted as) $F \cdot \nu$ on $\partial^s E$ such that*

$$\int_{E^1} \operatorname{div} F = -\int_{\partial^s E} \overline{F \cdot \nabla\chi} = -\int_{\partial^s E} F \cdot \nu\, d\mathcal{H}^{N-1}. \tag{3.2}$$

Proof. For simplicity, we denote χ_E by χ in the proof.

1. From Proposition 2.4, it follows that

$$\begin{aligned}\operatorname{div}(\chi^2 F) &= \operatorname{div}(\chi\,(\chi F)) \\ &= \overline{\chi}\operatorname{div}(\chi F) + \overline{\chi F\cdot\nabla\chi} \\ &= \overline{\chi}(\overline{\chi}\operatorname{div}F + \overline{F\cdot\nabla\chi}) + \overline{\chi F\cdot\nabla\chi} \\ &= (\overline{\chi})^2\operatorname{div}F + \overline{\chi}\,\overline{F\cdot\nabla\chi} + \overline{\chi F\cdot\nabla\chi}.\end{aligned} \tag{3.3}$$

On the other hand,

$$\operatorname{div}(\chi^2 F) = \operatorname{div}(\chi F) = \overline{\chi}\operatorname{div}F + \overline{F\cdot\nabla\chi}. \tag{3.4}$$

Combining (3.3) with (3.4) yields

$$((\overline{\chi})^2 - \overline{\chi})\operatorname{div}F + \overline{\chi}\,\overline{F\cdot\nabla\chi} + \overline{\chi F\cdot\nabla\chi} - \overline{F\cdot\nabla\chi} = 0. \tag{3.5}$$

Since $\overline{\chi}\equiv\frac{1}{2}$ on $\partial^* E$ and $\operatorname{div}F << \mathcal{H}^{N-1}$ (Proposition 2.2), it follows that

$$((\overline{\chi})^2 - \overline{\chi})\operatorname{div}F = -\frac{1}{4}\chi_{\partial^s E}\operatorname{div}F. \tag{3.6}$$

Therefore, from Proposition 2.4 and identities (3.5) and (3.6), we have

$$\begin{aligned}\frac{1}{2}\operatorname{div}(\chi F) &= \frac{1}{2}\overline{\chi}\operatorname{div}F + \frac{1}{2}\overline{F\cdot\nabla\chi} \\ &= \frac{1}{2}\overline{\chi}\operatorname{div}F + \frac{1}{2}\overline{F\cdot\nabla\chi} - \frac{1}{4}\chi_{\partial^s E}\operatorname{div}F + \overline{\chi}\,\overline{F\cdot\nabla\chi} + \overline{\chi F\cdot\nabla\chi} - \overline{F\cdot\nabla\chi} \\ &= \frac{1}{2}(\overline{\chi} - \frac{1}{2}\chi_{\partial^s E})\operatorname{div}F + \overline{\chi}\,\overline{F\cdot\nabla\chi} + \overline{\chi F\cdot\nabla\chi} - \frac{1}{2}\overline{F\cdot\nabla\chi}.\end{aligned} \tag{3.7}$$

2. Integrating both sides in (3.7), using Proposition 3.1, and noting that

$$(\overline{\chi} - \frac{1}{2}\chi_{\partial^s E})\operatorname{div}F = \chi_{E^1}\operatorname{div}F,$$

we obtain

$$0 = \frac{1}{2}\int_{E^1}\operatorname{div}F + \int_D \overline{\chi}\,\overline{F\cdot\nabla\chi} + \int_D \overline{\chi F\cdot\nabla\chi} - \int_D \frac{1}{2}\overline{F\cdot\nabla\chi}. \tag{3.8}$$

Now, we have

$$\overline{F\cdot\nabla\chi} << |\nabla\chi|, \qquad \overline{\chi F\cdot\nabla\chi} << |\nabla\chi|$$

from Proposition 2.4. Since

$$|\nabla\chi| = \mathcal{H}^{N-1}\lfloor\partial^* E,$$

it follows that

$$\overline{F\cdot\nabla\chi} \quad \text{and} \quad \overline{\chi F\cdot\nabla\chi} \quad \text{are supported on } \partial^* E.$$

Therefore, identity (3.8) implies

$$\frac{1}{2}\int_{E^1}\operatorname{div}F = -\int_{\partial^* E}\overline{\chi}\overline{F\cdot\nabla\chi} - \int_{\partial^* E}\overline{\chi F\cdot\nabla\chi} + \int_{\partial^* E}\frac{1}{2}\overline{F\cdot\nabla\chi}. \tag{3.9}$$

Since $\overline{\chi}\equiv\frac{1}{2}$ on $\partial^* E$, identity (3.9) reduces to

$$\frac{1}{2}\int_{E^1}\operatorname{div}F = -\int_{\partial^* E}\overline{\chi F\cdot\nabla\chi}. \tag{3.10}$$

3. We claim that $\overline{\chi F\cdot\nabla\chi}=\overline{\chi}\,\overline{F\cdot\nabla\chi}$. In fact, for any $\phi\in C_0(D)$, we first have

$$\left|\int_D \phi\,\overline{\chi}\,\overline{F\cdot\nabla\chi}-\int_D\phi\,(\chi F)\cdot\nabla\chi_\delta\right|\le I_1^{\tilde\delta}+I_2^{\delta,\tilde\delta}+I_3^{\delta,\tilde\delta}, \tag{3.11}$$

where

$$I_1^{\tilde\delta}=\left|\int_D\phi\,\overline{\chi}\,\overline{F\cdot\nabla\chi}-\int_D\phi\,\chi_{\tilde\delta}\,\overline{F\cdot\nabla\chi}\right|,$$
$$I_2^{\delta,\tilde\delta}=\left|\int_D\phi\,\chi_{\tilde\delta}\,\overline{F\cdot\nabla\chi}-\int_D(\phi\,\chi_{\tilde\delta})F\cdot\nabla\chi_\delta\right|,$$

and

$$I_3^{\delta,\tilde\delta}=\left|\int_D\chi_{\tilde\delta}\,(\phi F)\cdot\nabla\chi_\delta-\int_D\overline{\chi}\,(\phi F)\cdot\nabla\chi_\delta\right|.$$

We first fix $\tilde\delta$ and take the limit as $\delta\to0$ in (3.11). Then $I_2^{\delta,\tilde\delta}$ converges to zero as $\delta\to0$ since $F\cdot\nabla\chi_\delta\rightharpoonup\overline{F\cdot\nabla\chi}$. We note that $I_3^{\delta,\tilde\delta}$ can be rewritten as

$$I_3^{\delta,\tilde\delta}=\left|\int_D(\chi_{\tilde\delta}-\overline{\chi})\,(\phi F)\cdot\nabla\chi_\delta\right|. \tag{3.12}$$

Since

$$\nabla\chi_\delta(x)=\int_D\omega_\delta(x-y)\nabla\chi(y)=\int_{\partial^*E}\omega_\delta(x-y)\,\nu\,d\mathcal{H}^{N-1},$$

we obtain

$$\left|\int_D(\chi_{\tilde\delta}-\overline{\chi})(\phi F)\cdot\nabla\chi_\delta\right|=\left|\int_D(\chi_{\tilde\delta}-\overline{\chi})\,(\phi F)\cdot\left(\int_{\partial^*E}\omega_\delta(x-y)\,\nu\,d\mathcal{H}^{N-1}\right)dx\right|. \tag{3.13}$$

Using the boundedness of F and interchanging the limits of integration in (3.13) yields

$$\left|\int_D(\chi_{\tilde\delta}-\overline{\chi})(\phi F)\cdot\nabla\chi_\delta\right|\le C\int_{\partial^*E}\left(\int_D|\chi_{\tilde\delta}(x)-\overline{\chi}(x)|\,\omega_\delta(x-y)dx\right)d\mathcal{H}^{N-1}(y). \tag{3.14}$$

Then, from (3.11)–(3.14), we have

$$\begin{aligned}&\overline{\lim_{\delta\to0}}\left|\int_D\phi\,\overline{\chi}\,\overline{F\cdot\nabla\chi}-\int_D\phi\,(\chi F)\cdot\nabla\chi_\delta\right|\\&\le\left|\int_D\phi\,\overline{\chi}\,\overline{F\cdot\nabla\chi}-\int_D\phi\,\chi_{\tilde\delta}\,\overline{F\cdot\nabla\chi}\right|\\&\quad+C\,\overline{\lim_{\delta\to0}}\int_{\partial^*E}\int_D|\chi_{\tilde\delta}(x)-\overline{\chi}(x)|\omega_\delta(x-y)dx\,d\mathcal{H}^{N-1}(y).\end{aligned} \tag{3.15}$$

Define the function

$$f_{\tilde\delta}(x)=|\overline{\chi}(x)-\chi_{\tilde\delta}(x)|.$$

It follows from Proposition 3.2 that $f_{\tilde\delta}\in BV(D)$. Therefore,

$$\lim_{\delta\to0}\int_D|f_{\tilde\delta}(x)|\omega_\delta(x-y)dx=\overline{f_{\tilde\delta}}(y)\qquad\text{for }\mathcal{H}^{N-1}-\text{a.e. }y.$$

Using the Dominated Convergence Theorem in the last term on the right hand side of (3.15), we obtain

$$
\begin{aligned}
\overline{\lim_{\delta\to 0}}\left|\int_D \phi\overline{\chi}\,\overline{F\cdot\nabla\chi}-\int_D \phi\,(\chi F)\cdot\nabla\chi_\delta\right| \\
\le \left|\int_D \phi\overline{\chi}\,\overline{F\cdot\nabla\chi}-\int_D \phi\,\chi_{\tilde\delta}\,\overline{F\cdot\nabla\chi}\right|+C\int_{\partial^* E}\overline{f_{\tilde\delta}}(y)\,d\mathcal{H}^{N-1}.
\end{aligned}
\tag{3.16}
$$

Notice that the measure $\overline{F\cdot\nabla\chi}$ is supported on ∂^*E, since it is absolutely continuous with respect to $|\nabla\chi|=\mathcal{H}^{N-1}\lfloor\partial^*E$, and

$$
\chi_{\tilde\delta}(y)\xrightarrow{\tilde\delta\to 0}\overline{\chi}(y)\qquad\text{for every } y\in\partial^*E.
$$

Then the Dominated Convergence Theorem implies that the first term on the right hand side of (3.16) converges to zero as $\tilde\delta\to 0$.

We now prove that, for $\mathcal{H}^{N-1}$-a.e. $y\in\partial^*E$,

$$
\overline{f_{\tilde\delta}}(y)\to 0\qquad\text{as }\tilde\delta\to 0. \tag{3.17}
$$

We note that the function $f_{\tilde\delta}(y)$ can be written as the composition

$$
f_{\tilde\delta}=(g\circ h_{\tilde\delta})(y),
$$

where $h_{\tilde\delta}(y)=\overline{\chi}(y)-\chi_{\tilde\delta}(y)$ and $g(w)=|w|$.

We choose $y\in\partial^*E$ such that y is a regular point of $h_{\tilde\delta}(y)$. Then there exists $a(\tilde\delta)\in\mathbb{R}^N$ such that the approximate limits $(h_{\tilde\delta})_{\pm a(\tilde\delta)}(y)$ exist.

Using Proposition 3.2, we obtain

$$
\begin{aligned}
\overline{f_{\tilde\delta}}(y) &= \frac{1}{2}\left(g(h_{a(\tilde\delta)}(y))+g(h_{-a(\tilde\delta)}(y))\right)\\
&= \frac{1}{2}\left(|h_{a(\tilde\delta)}(y)|+|h_{-a(\tilde\delta)}(y)|\right).
\end{aligned}
$$

It suffices for (3.17) to show that

$$
\lim_{\tilde\delta\to 0}h_{\pm a(\tilde\delta)}(y)=0. \tag{3.18}
$$

In order to prove (3.18), we need to check that, for any fixed $\epsilon>0$,

$$
\lim_{\tilde\delta\to 0}\lim_{r\to 0}\frac{|\{h_{\tilde\delta}\le\epsilon\}\cap B(y,r)\cap\Pi_{a(\tilde\delta)}|}{|B(y,r)\cap\Pi_{a(\tilde\delta)}|}=1. \tag{3.19}
$$

Notice that $h_{\tilde\delta}\to 0$ for $\mathcal{L}^N$-a.e. y. For any fixed $\theta>0$, by Egorov's theorem, there exists a closed set $U\subset B(y,1)$ such that $|B(y,1)\setminus U|<\theta$ and $h_{\tilde\delta}\to 0$ uniformly on U. Hence, there exists $\tilde\delta_0$ such that, when $\tilde\delta<\tilde\delta_0$,

$$
h_{\tilde\delta}(z)<\epsilon\qquad\text{for any } z\in U.
$$

This implies that, for any $r,\tilde\delta<\tilde\delta_0$,

$$
|B(y,r)\setminus\{h_{\tilde\delta}\le\epsilon\}|<\theta.
$$

Since θ is arbitrary, then we arrive at the limit (3.19). In the same way, we can prove

$$
\lim_{\tilde\delta\to 0}h_{-a(\tilde\delta)}=0.
$$

Then using the Dominated Convergence Theorem yields

$$\int_{\partial^* E} \bar{f}_{\tilde{\delta}} d\mathcal{H}^{N-1} \to 0 \qquad \text{as } \tilde{\delta} \to 0.$$

Thus, we have

$$\int_D \phi\, (\chi F) \cdot \nabla \chi_\delta \to \int_D \phi\, \overline{\chi}\, \overline{F \cdot \nabla \chi}$$

for any $\phi \in C_0(D)$ as $\delta \to 0$, which implies that

$$(\chi F) \cdot \nabla \chi_\delta \rightharpoonup \overline{\chi}\, \overline{F \cdot \nabla \chi}.$$

On the other hand, from Proposition 2.4, we know that

$$(\chi F) \cdot \nabla \chi_\delta \rightharpoonup \overline{\chi F \cdot \nabla \chi}.$$

Therefore,

$$\overline{\chi}\, \overline{F \cdot \nabla \chi} = \overline{\chi F \cdot \nabla \chi}. \tag{3.20}$$

4. From (3.10), we obtain

$$\frac{1}{2} \int_{E^1} \operatorname{div} F = - \int_{\partial^* E} \overline{\chi}\, \overline{F \cdot \nabla \chi}. \tag{3.21}$$

Now, since $\overline{\chi} \equiv \frac{1}{2}$ on $\partial^* E$, we conclude

$$\int_{E^1} \operatorname{div} F = - \int_{\partial^* E} \overline{F \cdot \nabla \chi}. \tag{3.22}$$

Since $\overline{F \cdot \nabla \chi} << |\nabla \chi|$, the differentiation theorem for Radon measures (cf. [12]) implies that there exists a $|\nabla \chi|$-measurable function (denoted as) $F \cdot \nu$ such that

$$\overline{F \cdot \nabla \chi}(A) = \int_A F \cdot \nu\, |\nabla \chi| = \int_{A \cap \partial^* E} F \cdot \nu\, d\mathcal{H}^{N-1} \tag{3.23}$$

for any $|\nabla \chi|$-measurable set $A \subset D$. From (3.22) and (3.23), we conclude

$$\int_{E^1} \operatorname{div} F = - \int_{\partial^* E} F \cdot \nu\, d\mathcal{H}^{N-1} = - \int_{\partial^s E} F \cdot \nu\, d\mathcal{H}^{N-1}, \tag{3.24}$$

since $\mathcal{H}^{N-1}(\partial^s E \setminus \partial^* E) = 0$. $\square$

Remark 3.1. *It is proved in* [6] *that, if ∂E is Lipschitz continuous and $|\operatorname{div} F|(\partial E) = 0$, then the density function $F \cdot \nu$ coincides with the usual meaning of $F \cdot \nu$ for $\mathcal{H}^{N-1}$-a.e. $x \in \partial E$ with ν as the inward unit normal on ∂E.*

Then we have the following Gauss-Green formula.

Theorem 3.2. *(Gauss-Green formula). Let $F \in \mathcal{DM}^\infty(D)$. Let $E \Subset D$ be a bounded set of finite perimeter. Then there exists an $\mathcal{H}^{N-1}$-integrable function $F \cdot \nu$ on $\partial^s E$ such that, for any $\phi \in C_0^1(\mathbb{R}^N)$,*

$$\int_{E^1} \phi \operatorname{div} F = - \int_{\partial^s E} \phi F \cdot \nu d\mathcal{H}^{N-1} - \int_{E^1} F \cdot \nabla \phi$$

Proof. Using Theorem 3.1, we obtain that, for any $\phi \in C_0^1(\mathbb{R}^N)$,

$$\int_{E^1} \operatorname{div}(\phi F) = -\int_{\partial^s E} \overline{\phi F \cdot \nabla \chi}.$$

Since

$$\overline{\phi F \cdot \nabla \chi} = \phi\, \overline{F \cdot \nabla \chi}$$

and

$$\overline{F \cdot \nabla \chi} = F \cdot \nu \, d\mathcal{H}^{N-1},$$

we have

$$\int_{E^1} \operatorname{div}(\phi F) = -\int_{\partial^s E} \phi\, F \cdot \nu d\mathcal{H}^{N-1}. \tag{3.25}$$

On the other hand, since $\phi \in C_0^1(\mathbb{R}^N)$, Proposition 2.4 yields

$$\operatorname{div}(\phi F) = \phi \operatorname{div} F + F \cdot \nabla \phi,$$

which implies

$$\int_{E^1} \phi \operatorname{div} F = -\int_{E^1} F \cdot \nabla \phi + \int_{E^1} \operatorname{div}(\phi F). \tag{3.26}$$

We complete the proof by combining (3.25) with (3.26). $\square$

Finally, as a corollary of the Gauss-Green formula in $\mathcal{DM}^\infty$, we have the following extension theorem.

Theorem 3.3. *Let $\Omega \Subset E \Subset D$ be bounded open sets where E is a set of finite perimeter in $\mathbb{R}^N$. Let $F_1 \in \mathcal{DM}^\infty(D)$ and $F_2 \in \mathcal{DM}^\infty(\mathbb{R}^N - \bar{\Omega})$. Then*

$$F(y) = \begin{cases} F_1(y), & y \in E, \\ F_2(y), & y \in \mathbb{R}^N - \bar{E} \end{cases} \tag{3.27}$$

belongs to $\mathcal{DM}^\infty(\mathbb{R}^N)$, and

$$\|F\|_{\mathcal{DM}^\infty(\mathbb{R}^N)} \le \|F_1\|_{\mathcal{DM}^\infty(E)} + \|F_2\|_{\mathcal{DM}^\infty(\mathbb{R}^N - \bar{E})} + \|F_1 \cdot \nu - F_2 \cdot \nu\|_{L^1(\partial^s E; \mathcal{H}^{N-1})}.$$

Proof. Obviously, $F \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ and

$$\|F\|_{L^\infty(\mathbb{R}^N)} \le \|F_1\|_{L^\infty(E)} + \|F_2\|_{L^\infty(\mathbb{R}^N - \bar{E})}.$$

Now, choosing $\phi \in C_0^1(\mathbb{R}^N)$ such that $|\phi| \le 1$ and using the Gauss-Green formula yields

$$\begin{aligned} \int_{\mathbb{R}^N} F \cdot \nabla \phi \, dy &= \int_E F_1 \cdot \nabla \phi \, dy + \int_{\mathbb{R}^N - \bar{E}} F_2 \cdot \nabla \phi \, dy \\ &= -< \operatorname{div} F_1|_E, \phi > - < \operatorname{div} F_2|_{\mathbb{R}^N - \bar{E}}, \phi > + \int_{\partial^s E} \{F_1 \cdot \nu - F_2 \cdot \nu\} \, \phi \, d\mathcal{H}^{N-1} \\ &\le |\operatorname{div} F_1|(E) + |\operatorname{div} F_2|(\mathbb{R}^N - \bar{E}) + \|F_1 \cdot \nu - F_2 \cdot \nu\|_{L^1(\partial^s E; \mathcal{H}^{N-1})}. \end{aligned}$$

Hence, by the definition of the $\mathcal{DM}^\infty$ norm in (1.5) and that of $|\operatorname{div} F|(\mathbb{R}^N)$ in (1.6), we conclude the desired result. $\square$

4. Normal Traces and Lipschitz Deformations

In this section, we further analyze the normal traces introduced in Section 3 to show that the normal trace over an $(N-1)$-dimensional surface of finite perimeter can be understood as the weak-star limit of the normal traces introduced in Chen-Frid [6] over the Lipschitz deformation surfaces of the surface, which implies their consistency.

First we introduce the following Lemma.

Lemma 4.1. *Let E be a bounded set of finite perimeter. Then, for any small $\epsilon > 0$, there exists a closed set $Q^\epsilon \subset \partial^* E$ and a smooth vector field $\nu^\epsilon : \mathbb{R}^N \to \mathbb{R}^N$ that satisfy*

(i) $\mathcal{H}^{N-1}(\partial^* E \setminus Q^\epsilon) < \epsilon$,

(ii) $\nu^\epsilon|_{Q^\epsilon}$ *points toward the interior of E.*

Proof. Fix small $\epsilon > 0$. For each point $x \in \partial^* E$, there exists a measure-theoretic interior unit normal $\nu(x)$ and a measure-theoretic tangent plane $T(x)$ to the set E at x (cf. [15, 1]). Since the function $\nu : \partial^* E \to S^{N-1}$ is $\mathcal{H}^{N-1}$-measurable, we can apply Lusin's Theorem (cf. [12]) to ensure the existence of a closed set $Q^\epsilon \subset \partial^* E$ such that

$$\mathcal{H}^{N-1}(\partial^* E \setminus Q^\epsilon) < \epsilon$$

and

$$\nu \text{ restricted to } Q^\epsilon \text{ is continuous.}$$

We now apply Tietze's extension theorem (cf. [26]) to find a continuous extension of ν to all $\mathbb{R}^N$, say W. If we define $W_\delta := W * \omega_\delta$ (where $\omega_\delta(x)$ is defined in Definition 2.6), then $W_\delta \to W$ uniformly in a ball B such that $E \Subset B$. Therefore, we can choose δ small enough, say δ_0, such that the angle between $W_{\delta_0}(x)$ and the normal $\nu(x)$ is less than $\frac{\pi}{4}$, for all $x \in Q^\epsilon$. Since each $x \in Q^\epsilon$ belongs to the reduced boundary, we conclude that $W_{\delta_0}(x)$ points toward the interior of E. Finally, we define $\nu^\epsilon := W_{\delta_0}$. ☐

Definition 4.1. *For any $\epsilon > 0$, we define the function $\Psi^\epsilon : \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$ as*

$$\Psi^\epsilon(x,\tau) = x + \tau \nu^\epsilon, \tag{4.1}$$

where ν^ϵ is defined as in Lemma 4.1. *For any $0 < \tau < 1$, we define $\Psi^\epsilon_\tau : \mathbb{R}^N \to \mathbb{R}^N$ as $\Psi^\epsilon_\tau(x) = \Psi^\epsilon(x,\tau)$.*

Remark 4.1. *Let E be a bounded set of finite perimeter. Then, for τ small enough (depending only on the Lipschitz continuity of ν^ϵ on $\bar{E}$), the function $\Psi^\epsilon_\tau|_{\overline{E}}$ is a 1-1 map.*

Furthermore, we have

Lemma 4.2. *Let $\epsilon > 0$. Let E be a bounded set of finite perimeter, and let Q^ϵ be the set given in* Lemma 4.1. *Then there exists $K^\epsilon \subset Q^\epsilon$, $\mathcal{H}^{N-1}(Q^\epsilon \setminus K^\epsilon) < \epsilon$, and τ_0 small enough such that $\Psi^\epsilon_\tau(x) \in Int(E)$ for all $\tau \in (0,\tau_0)$ and $x \in K^\epsilon$, where $Int(E)$ denotes the topological interior of E.*

Proof. We define

$$A_k = \{x \in Q^\epsilon \,:\, \Psi^\epsilon_\tau(x) \in \mathrm{Int}(E), \tau \leq \frac{1}{k}\} \qquad \text{for } k = 1,2,3,\dots.$$

Then $A_k \subset A_{k+1}, k = 1,2,3,\dots$, and $Q^\epsilon = \cup_{k=1}^\infty A_k$. Since $A_k \subset A_{k+1}, k = 1,2,3,\dots$, it follows that

$$\lim_{k\to\infty} \mathcal{H}^{N-1}(A_k) = \mathcal{H}^{N-1}(Q^\epsilon). \tag{4.2}$$

From (4.2), it follows that there exists $k_0 = k_0(\epsilon)$ such that

$$\mathcal{H}^{N-1}(Q^\epsilon \setminus A_{k_0}) < \epsilon.$$

We define

$$K^\epsilon := A_{k_0}.$$

Then we have

$$\mathcal{H}^{N-1}(Q^\epsilon \setminus K^\epsilon) < \epsilon.$$

Finally, we set $\tau_0 := \frac{1}{k_0}$ to arrive at the result. □

Definition 4.2. *For any $\epsilon > 0$, we define $K^\epsilon_\tau = \Psi^\epsilon_\tau(K^\epsilon)$ and $E_\tau = \Psi^\epsilon_\tau(E)$, where K^ϵ is given in* Lemma 4.2.

Then we have the following theorem.

Theorem 4.1. *For any $\epsilon > 0$, the normal trace $F \cdot \nu$ on K^ϵ is the weak-star limit of the normal traces on K^ϵ_τ introduced in* [6] *as $\tau \to 0$. That is, for any $\phi \in L^1(K^\epsilon)$,*

$$\begin{aligned}(4.3)\qquad \int_{K^\epsilon} (F\cdot\nu)(w)\phi(w)d\mathcal{H}^{N-1} &= \lim_{\tau\to 0}\int_{K^\epsilon_\tau} (F\cdot\nu_\tau)(w)(\phi\circ(\Psi^\epsilon_\tau)^{-1})(w)d\mathcal{H}^{N-1} \\ &= \lim_{\tau\to 0}\int_{K^\epsilon} (F\cdot\nu_\tau)\circ\Psi^\epsilon_\tau(w)\phi(w)d\mathcal{H}^{N-1}.\end{aligned}$$

Proof. Let $P \subset K^\epsilon$ be a compact subset. We choose a test function $\phi \in C^1_0(D)$ such that ϕ vanishes outside a neighborhood of P with $\phi|_P \neq 0$ and $\phi|_{\partial^s E - K^\epsilon} = 0$. From Theorem 3.2, we have

$$(4.4)\qquad \int_{E^1_\tau} \phi\,\mathrm{div}F = -\int_{E^1_\tau} F\cdot\nabla\phi - \int_{\partial^s E_\tau} \phi F\cdot\nu_\tau\, d\mathcal{H}^{N-1}.$$

On the other hand, Theorem 3.2 also yields

$$(4.5)\qquad \int_{E^1} \phi\,\mathrm{div}F = -\int_{E^1} F\cdot\nabla\phi - \int_{\partial^s E} \phi F\cdot\nu d\mathcal{H}^{N-1}.$$

Taking the limit in (4.4) as $\tau \to 0$ and using the Dominated Convergence Theorem, we obtain

$$\int_{E^1_\tau} \phi\,\mathrm{div}F \to \int_{E^1} \phi\,\mathrm{div}F,$$

and

$$\int_{E^1_\tau} F\cdot\nabla\phi \to \int_{E^1} F\cdot\nabla\phi.$$

This implies that

$$\int_{\partial^s E_\tau} \phi F\cdot\nu_\tau d\mathcal{H}^{N-1} \to \int_{\partial^s E} \phi F\cdot\nu d\mathcal{H}^{N-1}.$$

Our choice of ϕ implies

$$(4.6)\qquad \int_{K^\epsilon_\tau} \phi F\cdot\nu_\tau d\mathcal{H}^{N-1} \to \int_{K^\epsilon} \phi F\cdot\nu d\mathcal{H}^{N-1} \qquad \text{as } \tau\to 0.$$

Now, since $\phi|_{K^\epsilon_\tau}$ can be substituted by $\phi|_{K^\epsilon}\circ(\Psi^\epsilon_\tau)^{-1}$ with an error that goes to zero when $\tau \to 0$, we obtain

$$(4.7)\qquad \lim_{\tau\to 0}\int_{K^\epsilon_\tau} \phi((\Psi^\epsilon_\tau)^{-1}(w))(F\cdot\nu_\tau)(w)d\mathcal{H}^{N-1}(w) = \int_{K^\epsilon} \phi(w)(F\cdot\nu)(w)d\mathcal{H}^{N-1}(w).$$

We can approximate any $\phi \in L^1(K^\epsilon)$ with a sequence of C^1 functions $\{\phi_j\}$ on a neighborhood of ∂E such that each ϕ_j is a function that vanishes outside a neighborhood of $P_j \Subset K^\epsilon$ with $P_j \to K^\epsilon$ as $j \to \infty$. We find that, for τ small enough,

$$
\begin{aligned}
(4.8)\qquad & \Big|\int_{K^\epsilon_\tau} \phi((\Psi^\epsilon_\tau)^{-1}(w))(F\cdot\nu_\tau)(w)d\mathcal{H}^{N-1}(w) - \int_{K^\epsilon} \phi(w)(F\cdot\nu)(w)d\mathcal{H}^{N-1}(w)\Big| \\
&\le \int_{K^\epsilon_\tau} |\phi((\Psi^\epsilon_\tau)^{-1}(w))(F\cdot\nu_\tau)(w) - \phi_j((\Psi^\epsilon_\tau)^{-1}(w))(F\cdot\nu_\tau)(w)|d\mathcal{H}^{N-1} \\
&\quad + \Big|\int_{K^\epsilon_\tau} \phi_j((\Psi^\epsilon_\tau)^{-1}(w))(F\cdot\nu_\tau)(w)d\mathcal{H}^{N-1} - \int_{K^\epsilon} \phi_j(w)(F\cdot\nu)(w)d\mathcal{H}^{N-1}\Big| \\
&\quad + \int_{K^\epsilon} |\phi_j(w)(F\cdot\nu)(w) - \phi(w)(F\cdot\nu)(w)|d\mathcal{H}^{N-1} \\
&= I_1^\tau + I_2^\tau + I_3^\tau.
\end{aligned}
$$

Using the area formula,

$$I_1^\tau \le \int_{K^\epsilon} |J\Psi^\epsilon_\tau|\,|(\phi-\phi_j)(w)(F\cdot\nu_\tau)(\Psi^\epsilon_\tau(w))|d\mathcal{H}^{N-1}$$

with $|J\Psi^\epsilon_\tau| \le C$ and $|F\cdot\nu_\tau| \le C$ for all small $\tau > 0$. We first fix j and let $\tau \to 0$, which yields that the term I_2^τ converges to zero since each ϕ_j vanishes outside a neighborhood of $P_j \Subset K^\epsilon$. Then we let $j \to \infty$ to yield that I_1^τ and I_3^τ converge to zero, since $\phi_j \to \phi$ in $L^1(K^\epsilon)$.

Furthermore, we apply the area formula to the right-hand side of the first identity of (4.3) and use that the deformation is regular, i.e, $\lim_{\tau\to 0} J\Psi^\epsilon_\tau = 1$, to obtain the second identity of (4.3). ☐

5. Applications to nonlinear conservation laws

Let E be an open bounded set of finite perimeter in $\mathbb{R}^d$. We define

$$Q := (0,\infty)\times E \subset \mathbb{R}^{d+1}_+, \qquad \Gamma = (0,\infty)\times\partial^s E.$$

For each $T > 0$, we define

$$Q_T = (0,T)\times E, \qquad \Gamma_T = (0,T)\times\partial^s E.$$

We consider the following initial-boundary value problem:

$$\partial_t u + \nabla_x\cdot f(u) = 0 \qquad \text{in } Q, \tag{5.1}$$

$$u|_{E\times\{0\}} = u_0, \tag{5.2}$$

$$u|_{\partial E\times(0,\infty)} = u_b, \tag{5.3}$$

where $u : Q \to U \subset \mathbb{R}^m, f \in C^1(U;\mathbb{R}^{m\times d}), u_0 \in L^\infty(E;\mathbb{R}^m)$, and $u_b \in L^\infty(\Gamma;\mathbb{R}^m)$.

We say that a convex function $\eta \in C^1(\mathbb{R}^m;\mathbb{R})$ is an entropy for (5.1), with associated entropy flux $q \in C^1(\mathbb{R}^m;\mathbb{R}^d)$, if

$$\nabla q_j(u) = \nabla\eta(u)\nabla f_j(u), \qquad j = 1,...,d.$$

Then the pair $(\eta(u), q(u))$ is called a convex entropy pair. As in [22], the function $\mathcal{F}(u,v) = (\alpha(u,v),\beta(u,v))$ is a boundary entropy pair if and only if, for each $v \in \mathbb{R}^m$, $(\alpha(u,v),\beta(u,v))$ is a convex entropy pair satisfying

$$\alpha(u,u) = \beta(v,v) = \partial_u\alpha(v,v) = 0. \tag{5.4}$$

Definition 5.1. *We say that $u(t,x) \in L^\infty(Q_T;\mathbb{R}^m)$ is an entropy solution for* (5.1)–(5.3) *if*

- $\partial_t u + \nabla_x \cdot f(u) = 0$ *holds in the sense of distributions in Q;*
- *for any nonnegative $\phi \in C_0^\infty((-\infty,T)\times\mathbb{R}^d)$,*

$$\begin{aligned}\int\int_{Q_T} &(\alpha(u(t,x),v)\phi_t + \beta(u(t,x),v)\cdot\nabla_x\phi)dxdt \\ &+\int_E \alpha(u_0(x),v)\phi(0,x)dx \\ &+\int_{\Gamma_T} \beta(u_b(r),v)\cdot\nu(r)\,\phi(r)d\mathcal{H}^d(r) \geq 0,\end{aligned} \tag{5.5}$$

where the unit vector $\nu(r)$ is the measure theoretic inward normal to Γ, which is defined $\mathcal{H}^d$-almost everywhere in Γ.

The existence of entropy solutions to problem (5.1)–(5.3) may be obtained by using the vanishing viscosity method. In fact, for each $\delta > 0$, we consider the problem:

$$\partial_t u^\delta + \nabla_x \cdot f(u^\delta) = \delta\Delta u^\delta \qquad \text{in } Q_T, \tag{5.6}$$

$$u^\delta|_{E\times\{0\}} = u_0^\delta, \tag{5.7}$$

$$u^\delta|_{\partial E\times(0,T)} = u_b^\delta, \tag{5.8}$$

where u_0^δ and u_b^δ are smooth functions satisfying $\lim_{\delta\to 0} u_0^\delta = u_0$ in $L^1(E)$ and $\lim_{\delta\to 0} u_b^\delta = u_b$ in $L^1(\partial E\times(0,T))$.

The solvability of problem (5.6)–(5.8) reduces to the problem of the solvability of the Dirichlet problem for the Laplace's equation on the open set of finite perimeter E. That is, we ask for an harmonic function w on E such that $w|_{\partial E} = f \in C(\partial E)$. From Lemma A.1, we obtain the desired w, which satisfies $\lim_{y\to x} w(y) = f(x)$ for every $x \in \partial^* E$. For our application, this boundary regularity will be enough since $(E^0 \cup E^1)\cap\partial E$ do not appear in the Gauss-Green formula (see Theorem 3.1).

Let $u^\delta = u^\delta(t,x)$ be the solutions of problem (5.6)–(5.8), uniformly bounded in L^∞. Assume that $u^\delta(t,x)$ converges a.e. to an L^∞ function $u = u(t,x)$ as $\delta \to 0$.

Proceeding as in the standard way, for fixed $v \in \mathbb{R}^d$, we multiply equation (5.6) by $\partial_u\alpha(u^\delta(t,x),v)$ and use that $u \mapsto (\alpha(u,v),\beta(u,v))$ is a convex entropy pair to obtain

$$\partial_t\alpha(u^\delta(t,x),v) + \nabla_x\cdot\beta(u^\delta(t,x),v) \leq \delta\Delta\alpha(u^\delta(t,x),v). \tag{5.9}$$

Then we multiply equation (5.9) by a nonnegative test function $\phi \in C_0^\infty((-\infty,T)\times\mathbb{R}^d)$ and integrate by parts to obtain

$$\begin{aligned}&-\int_{Q_T}\alpha(u^\delta(t,x),v)\partial_t\phi\,dxdt - \int_E \alpha(u^\delta(0,x),v)\phi(0,x)dx \\ &-\int_{Q_T}\beta(u^\delta(t,x),v)\cdot\nabla_x\phi\,dxdt - \int_0^T\int_{\partial^s E}\beta(u^\delta(t,x),v)\cdot\nu\,\phi\,d\mathcal{H}^d(x)dt \\ &\leq \delta\int_{Q_T}\Delta_x\alpha(u^\delta(t,x),v)dxdt.\end{aligned}$$

Therefore, we have

$$\int_{Q_T} \alpha(u^\delta(t,x),v)\partial_t\phi\,dxdt + \int_E \alpha(u_0^\delta,v)\phi(0,x)dx + \int_{Q_T} \beta(u^\delta(t,x),v)\cdot\nabla_x\phi\,dxdt$$
$$+\int_{\Gamma_T} \beta(u_b^\delta(r),v)\cdot\nu\,\phi\,d\mathcal{H}^d(r) + \delta\int_{Q_T} \Delta(\alpha(u^\delta(t,x),v))dxdt \geq 0.$$

Then, letting $\delta\to 0$ in the previous inequality, we conclude that $u=u(t,x)$ is an entropy solution of (5.1)–(5.3).

Since Q_T is a set of finite perimeter in $\mathbb{R}^{d+1}$, we can apply the results from Section 4 to this set. Therefore, for any $\epsilon>0$, we obtain the existence of a set $\Gamma_T^\epsilon\subset\Gamma_T$, $\mathcal{H}^d((\Gamma_T\cap\partial^*Q_T)\setminus\Gamma_T^\epsilon)<\epsilon$ and a smooth vector field ν^ϵ defined on $\mathbb{R}^{d+1}$ such that, when restricted to Γ_T^ϵ, $\nu^\epsilon|_{\Gamma_T^\epsilon}$ points toward the interior of Q_T; we also obtain that there exists $\tau_0=\tau_0(\epsilon)$ small enough such that, for $\tau\leq\tau_0$,

$$\Psi^\epsilon(t,x,\tau)=(t,x)+\tau\nu^\epsilon, \qquad (t,x)\in\Gamma_T^\epsilon,$$

defines a 1-1 deformation of Γ_T^ϵ. We define $\Gamma_{T,\tau}^\epsilon=\Psi_\tau^\epsilon(\Gamma_T^\epsilon)$. Using this deformation, we can prove the following proposition.

Proposition 5.1. *For any $\gamma\in L^1(\Gamma_T^\epsilon)$,*

$$\begin{aligned}\int_{\Gamma_T^\epsilon}(F(u)\cdot\nu)(w)\gamma(w)d\mathcal{H}^d &= \lim_{\tau\to 0}\int_{\Gamma_{T,\tau}^\epsilon}(F(u)\cdot\nu_\tau)(w)(\gamma\circ(\Psi_\tau^\epsilon)^{-1})(w)d\mathcal{H}^d\\ &= \lim_{\tau\to 0}\int_{\Gamma_T^\epsilon}(F(u)\cdot\nu_\tau)\circ\Psi_\tau^\epsilon(w)\gamma(w)d\mathcal{H}^d,\end{aligned}$$

where $F(u)=(u,f(u))$ or $(\eta(u),q(u))$ for any convex entropy pair (η,q), and Ψ_τ^ϵ is any regular deformation.

This proposition follows from the convergence (4.3) and from the fact that the fields $(u,f(u))$ and $(\eta(u),q(u))$ are divergence measure fields.

Remark 5.1. *If system* (5.1) *is endowed with a strict convex entropy η_* on the subset in the state space $\{u:|u|\leq\|u(\cdot,\cdot)\|_{L^\infty}\}$, then Proposition* 5.1 *holds for any C^2 entropy pair (i.e. η is not necessary to be convex) (see Chen* [4]*).*

Furthermore, we have

Proposition 5.2. *For any $\gamma\in L^1(\Gamma_T^\epsilon)$, $\gamma\geq 0$ $\mathcal{H}^d$-a.e., and for any boundary entropy flux $\mathcal{F}=(\alpha,\beta)$,*

$$ess\lim_{\tau\to 0}\int_{\Gamma_T^\epsilon}\mathcal{F}(u\circ\Psi_\tau^\epsilon(r),u_b(r))\cdot\nu_\tau(\Psi_\tau^\epsilon(r))\gamma(r)d\mathcal{H}^d(r)\leq 0,$$

where ν_τ is the measure theoretic inward normal to $\Gamma_{T,\tau}^\epsilon$.

Proof. Let $h:\mathbb{R}^{d+1}\to[0,1]$ be defined by setting $h(t,x)=\tau$, if $(t,x)\in\Gamma_{T,\tau}^\epsilon$ for $\tau\leq\tau_0$ with τ_0 small enough; $h(t,x)=0$, if $(t,x)\notin Q$; and $h(t,x)=\tau_0$, otherwise. In (5.5), we choose

$$\phi(t,x)=\gamma((\Psi_{h(t,x)}^\epsilon)^{-1}(t,x))\zeta(h(t,x)) \qquad \text{for } (t,x)\in\mathrm{Image}(\Psi_\tau^\epsilon) \text{ and } \tau\leq\tau_0, \tag{5.10}$$

where $\gamma\in\mathrm{Lip}(\Gamma)$, $\mathrm{spt}(\gamma)\subset\Gamma_T^\epsilon$, $\gamma\geq 0$, $\zeta\in C_0^\infty(-\infty,\tau_0)$, and set $\phi(t,x)=0$ for $(t,x)\in Q\setminus\mathrm{Image}(\Psi_\tau^\epsilon),\tau\leq\tau_0$. We extend ϕ to all $\mathbb{R}^{d+1}$ as a Lipschitz function with compact support contained in $(0,T)\times\mathbb{R}^d$. With this choice of ϕ in (5.5) and using the coarea formula, we obtain

$$-\int_0^{\tau_0}\left(\int_{\Gamma^\epsilon_{T,\tau}}\mathcal{F}(u(r),v)\cdot\nu_\tau(r)\,\gamma((\Psi^\epsilon_\tau)^{-1}(r))d\mathcal{H}^d(r)\right)\zeta'(s)ds$$
$$+C\int_0^{\tau_0}\zeta(s)ds+\int_{\Gamma^\epsilon_T}\beta(u_b(r),v)\cdot\nu(r)\,\gamma(r)d\mathcal{H}^d(r)\zeta(0)\geq 0.$$

Choosing $\zeta(s)=\chi_{[-\delta,\delta]}$, $0<\delta<\tau_0$ (mollifying and passing to the limit), and then making $\delta\to 0$, we get

$$\operatorname{ess}\lim_{\tau\to 0}\int_{\Gamma^\epsilon_{T,\tau}}\mathcal{F}(u(r),v)\cdot\nu_\tau(r)\,\gamma((\Psi^\epsilon_\tau)^{-1}(r))d\mathcal{H}^d(r)$$
$$\leq\int_{\Gamma^\epsilon_T}\beta(u_b(r),v)\cdot\nu(r)\,\gamma(r)\,d\mathcal{H}^d(r),$$

where we used Proposition 5.1 to ensure the existence of the limit on the left-hand side. By approximation, we conclude that the above inequality holds for any $\gamma\in L^1(\Gamma^\epsilon_T)$, $\gamma\geq 0$, $\mathcal{H}^d$-a.e. Using the area formula, one obtains

$$\operatorname{ess}\lim_{\tau\to 0}\int_{\Gamma^\epsilon_T}\mathcal{F}(u\circ\Psi^\epsilon_\tau(r),v)\cdot\nu_\tau(\Psi^\epsilon_\tau(r))\gamma(r)d\mathcal{H}^d(r)$$
$$\leq\int_{\Gamma^\epsilon_T}\beta(u_b(r),v)\cdot\nu(r)\,\gamma(r)d\mathcal{H}^d(r).$$

Now, considering first simple step functions $v_b(r)$ and using a standard approximation argument again, one deduces from the last inequality that

$$\operatorname{ess}\lim_{\tau\to 0}\int_{\Gamma^\epsilon_T}\mathcal{F}(u\circ\Psi^\epsilon_\tau(r),v_b(r))\cdot\nu_\tau(\Psi^\epsilon_\tau(r))\gamma(r)d\mathcal{H}^d(r)$$
$$\leq\int_{\Gamma^\epsilon_T}\beta(u_b(r),v_b(r))\cdot\nu(r)\,\gamma(r)d\mathcal{H}^d(r)$$

for any $v_b\in L^1(\Gamma^\epsilon_T)$. Taking $v_b=u_b$ and using the fact that $\beta(u,u)=0$ yields the desired result □. □

For $m=1$, we define

$$\Gamma_{act}=\{r\in\Gamma_T\,:\,u\mapsto(u,f(u))\cdot\nu(r)\quad\text{is increasing}\}.$$

Choosing $\mathcal{F}(u,v)=(|u-v|,\operatorname{sign}(u-v)(f(u)-f(v)))$ and proceeding as in Proposition 4.1 in [6] with the aid of Proposition 5.2, we obtain

Theorem 5.1. *If $u\in L^\infty(Q_T)$ and $u_b\in L^\infty(\Gamma_T)$ satisfy* (5.5) *for any boundary entropy pair associated with* (5.1)*, then*

$$ess\lim_{\tau\to 0}\int_{\Gamma_{act}\cap\Gamma^\epsilon_T}|u\circ\Psi^\epsilon_\tau(r)-u_b(r)|d\mathcal{H}^d(r)=0.$$

Appendix A

Lemma A.1. *Let* $E \subset \mathbb{R}^N$ *be an open bounded set of finite perimeter and* $f \in C(\partial E)$*. Then, there exists a function* w *with continuous second derivatives satisfying*

(i) $\Delta w = 0$ *in* E*,*

(ii) $\lim_{y \to x, y \in E} w(y) = f(x)$ *for every* x *in* $\partial^* E$*.*

Proof. We refer to [19] (Chapter 8) for the existence of the desired w by using the *Perron-Wiener-Brelot method*. The boundary regularity behavior of the solution w follows by the *Wiener criterion* (see [21], Chapter 2), which implies that $\lim_{y \to x, y \in E} w(y) = f(x)$ if and only if $\mathbb{R}^N \setminus E$ **is not thin** at x. We refer to [21] for the rigorous definition of **thinness** of a set at a point, which involves the concept of *capacity* of the set. Using Corollary 2.51 in [21], we obtain that, if $\mathbb{R}^N \setminus E$ is thin at x, then $x \in E^1$, that is, x is a point of density 1 for E. Since $\partial^* E \subset E^{\frac{1}{2}}$, we conclude that $\mathbb{R}^N \setminus E$ is not thin at every $x \in \partial^* E$ and hence (ii) holds for every $x \in \partial^* E$. $\square$

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