A Half-space Problem for the Boltzmann Equation with Specular Reflection Boundary Condition

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Contents

Abstract

There are many open problems on the stability of nonlinear wave patterns to the Boltzmann equation even though the corresponding stability theory has been comparatively well-established for the gas dynamical systems. In this paper, we study the nonlinear stability of a rarefaction wave profile to the Boltzmann equation with the boundary effect imposed by specular reflection for both the hard sphere model and the hard potential model with angular cut-off. The analysis is based on the property of the solution and its derivatives which are either odd or even functions at the boundary coming from specular reflection, and the decomposition on both the solution and the Boltzmann equation introduced in [24, 26] for energy method.

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1 Introduction

The Boltzmann equation was introduced by Ludwig Boltzmann in 1872 through the study in statistics physics. It is a fundamental equation for rarefied gas in kinetic theory and provides many challenging mathematical problems. When the Knudsen number tends to zero, the Boltzmann equation yields the Euler equations as the first order in the Hilbert expansion, and the Navier-Stokes equations as the second order in the Chapman-Enskog expansion. Hence, the Boltzmann equation has close relation to the systems of gas dynamics. As we know, the solutions to the systems of gas dynamics have rich nonlinear wave phenomena and the stability of these nonlinear wave patterns has been extensively studied. It is natural to work on the corresponding stability problems to the Boltzmann equation due to their close relation. Some work has been done in this direction, especially recently, for the hard sphere model, such as the stability of the shock profile and rarefaction wave in [26] and [25] respectively, and the stability of shock profile with reflective boundary condition in [21].

In this paper, we consider the one dimensional Boltzmann equation in half space, i.e. $x \geq 0$ with specular boundary condition given at $x = 0$. Notice that the problem with specular reflection has been studied in many mathematical and physical settings, and it could be the first step to include the boundary effect with physical meaning. One of the reason comes from its simplicity in mathematical analysis around the boundary because of the non-appearance of boundary layer without source in the Navier-Stokes equations and the Boltzmann equation. When the initial data is assumed to be a small perturbation of the local Maxwellian given by a nonlinear wave pattern containing one rarefaction wave with positive speed to the Euler equations, we will show that the solution to the Boltzmann equation converges to this local Maxwellian as time tends to infinity. Thus, this yields the nonlinear time asymptotic stability of the rarefaction wave to the Boltzmann equation with specular boundary condition. Same as the case on the Cauchy problem with rarefaction wave profiles considered in [25], the strength of the rarefaction wave is not particularly small. In fact, the bound on the strength of the rarefaction wave is required by the variation of the linearized H-theorem which gives the dissipation on the non-fluid component.

In the analysis, the boundary condition of specular reflection is fully used which give the even or odd property on the solution and its derivatives at the boundary. These property is particularly useful for the case of the hard potential with angular cut-off where the convection has linear growth in ξ while the linearized operator has dissipative effect on non-fluid component only with the weight $|\xi|^\beta$, $0 < \beta < 1$. Another technique used in the proof is the micro-macro decomposition of the solution into local Maxwellian and the non-fluid component. This provides a way to re-write the Boltzmann equation into a system similar to the gas dynamics coupled with an equation for the non-fluid component, [24, 26]. Writing the Boltzmann equation in this form allows the use of energy method in a straightforward way.

Consider the one dimensional Boltzmann equation in half space

$$f_t + \xi_1 f_x = Q(f,f), \qquad (f,t,x,\xi) \in \mathbf{R} \times \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}^3, \tag{1.1}$$

with initial data

$$f(0,x,\xi) = f_0(x,\xi), \qquad (x,\xi) \in \mathbf{R}_+ \times \mathbf{R}^3, \tag{1.2}$$

and boundary condition

$$f(t,0,R\xi) = f(t,0,\xi), \qquad (t,\xi) \in \mathbf{R}_+ \times \mathbf{R}^3, \tag{1.3}$$

where $R\xi = R(\xi_1,\xi_2,\xi_3) = (-\xi_1,\xi_2,\xi_3)$, and $f(t,x,\xi)$ represents the distributional density of particles at time-space (t,x) with velocity ξ. Here, $Q(f,f)$ is a bilinear collision operator, cf.

[4], given by

$$Q(f,g)(\xi) \equiv \frac{1}{2}\int_{\mathbf{R}^3}\int_{\mathbf{S}_+^2} \Big(f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi') - f(\xi)g(\xi_*) - f(\xi_*)g(\xi)\Big)q(|V|,\theta)\, d\xi_* d\Omega,$$

where $V = \xi - \xi_*$, $\cos\theta = \frac{V\cdot\Omega}{|V|}$, and $\mathbf{S}_+^2 = \{\Omega \in \mathbf{S}^2 : \ (\xi - \xi_*)\cdot\Omega \geq 0\}$. By conservation of momentum and energy, the velocities (ξ, ξ_*) before and (ξ', ξ'_*) after collision have the following relation

$$\begin{cases} \xi' = \xi - [(\xi - \xi_*)\cdot\Omega]\,\Omega, \\ \xi'_* = \xi_* + [(\xi - \xi_*)\cdot\Omega]\,\Omega. \end{cases}$$

Throughout this paper, the collision kernel $q(|V|,\theta)$ is assumed to satisfy the following two conditions:

(A1): There is $0 \leq \delta_1 < 1$ such that

$$0 \leq q(V,\theta) \leq C_1\Big(|V| + |V|^{-\delta_1}\Big)|\cos\theta|.$$

(A2): There are constants $0 < \beta \leq 1$ such that

$$C_2(1+|\xi|)^\beta \leq \nu(\xi) \leq C_3(1+|\xi|)^\beta, \tag{1.4}$$

where $\nu(\xi)$ is the collision frequency defined in (1.14), and $C_i > 0$, $i = 1,2,3$ are positive constants.

Notice that both the hard sphere model and the hard potential model with angular cut-off satisfy the above two conditions (A1) and (A2).

Now we will introduce some notations to state our main theorem in the paper. First, it is well known that the equilibrium state, i.e. the Maxwellian $\mathbf{M} = \mathbf{M}_{[\rho,u,\theta]}$ depending on five parameters $(\rho,u,\theta) \in \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}_+$ representing the density, velocity and temperature, is the only function such that $Q(\mathbf{M},\mathbf{M}) = 0$. And there are five collision invariants corresponding to the five dimensional sub-space of the fluid components, denoted by $\psi_\alpha(\xi)$, cf. [4], as

$$\begin{cases} \psi_0(\xi) \equiv 1, \\ \psi_i(\xi) \equiv \xi^i, \qquad \text{for} \ \ i = 1,2,3, \\ \psi_4(\xi) \equiv \frac{1}{2}|\xi|^2, \end{cases} \tag{1.5}$$

satisfying

$$\int_{\mathbf{R}^3} \psi_j(\xi)Q(h,g)d\xi = 0, \qquad \text{for} \ \ j = 0,1,2,3,4.$$

For a solution $f(t,x,\xi)$ to the Boltzmann equation, we decompose it into the macroscopic (fluid) component, i.e., the local Mawellian $\mathbf{M} = \mathbf{M}(t,x,\xi) = \mathbf{M}_{[\rho,u,\theta]}(\xi)$; and the microscopic (non-fluid) component, i.e., $\mathbf{G} = \mathbf{G}(t,x,\xi)$ as follows, [24]:

$$f(t,x,\xi) = \mathbf{M}(t,x,\xi) + \mathbf{G}(t,x,\xi).$$

The local Maxwellian $\mathbf{M}$ is naturally defined by the five conserved quantities, that is, the mass density $\rho(t,x)$, momentum $m(t,x) = \rho(t,x)u(t,x)$, and energy $e(t,x) + \frac{1}{2}|u(t,x)|^2$:

$$\begin{cases} \rho(t,x) \equiv \displaystyle\int_{\mathbf{R}^3} f(t,x,\xi)d\xi, \\ m^i(t,x) \equiv \displaystyle\int_{\mathbf{R}^3} \psi_i(\xi)f(t,x,\xi)d\xi \text{ for } i = 1,2,3, \\ \left[\rho\left(e + \frac{1}{2}|u|^2\right)\right](t,x) \equiv \displaystyle\int_{\mathbf{R}^3} \psi_4(\xi)f(t,x,\xi)d\xi, \end{cases} \tag{1.6}$$

by

$$\mathbf{M} \equiv \mathbf{M}_{[\rho,u,\theta]}(t,x,\xi) \equiv \frac{\rho(t,x)}{\sqrt{(2\pi R\theta(t,x))^3}} \exp\left(-\frac{|\xi-u(t,x)|^2}{2R\theta(t,x)}\right). \tag{1.7}$$

Here $\theta(t,x)$ is the temperature which is related to the internal energy e by $e=\frac{3}{2}R\theta$ with R being the gas constant, and $u(t,x)$ is the fluid velocity.

The space of function for the solution of the initial boundary value problem (1.1)-(1.3) considered in this paper is $H^s_x\left(L^2_{\xi,\mathcal{M}}\right)=H^s_x\left(L^2_{\xi,\mathcal{M}}\left(\mathbf{R}_+\times\mathbf{R}^3\right)\right)$ for some global or local Maxwellian $\mathcal{M}$, the inner product is given by:

$$\langle h,g\rangle \equiv \int_{\mathbf{R}^3}\frac{1}{\mathcal{M}}h(\xi)g(\xi)d\xi,$$

for any functions $h,\ g$ of ξ such that the above integral is well-defined, and $\|h\|_{L^2_{\xi,\mathcal{M}}}=\langle h,h\rangle^{\frac{1}{2}}$. In particular, when $\mathcal{M}=\mathbf{M}$, the orthogonal basis for the space spanned by $\{\psi_\alpha,\alpha=0,1,\cdots,4\}$ with respect to the inner product is given by:

$$\begin{cases}\chi_0(\xi;\rho,u,\theta)\equiv\frac{1}{\sqrt{\rho}}\mathbf{M},\\ \chi_i(\xi;\rho,u,\theta)\equiv\frac{\xi^i-u^i}{\sqrt{R\theta\rho}}\mathbf{M}\quad\text{for}\quad i=1,2,3,\\ \chi_4(\xi;\rho,u,\theta)\equiv\frac{1}{\sqrt{6\rho}}\left(\frac{|\xi-u|^2}{R\theta}-3\right)\mathbf{M},\\ \qquad\langle\chi_i,\chi_j\rangle=\delta_{ij},\quad\text{for } i,j=0,1,2,3,4.\end{cases} \tag{1.8}$$

Therefore, the orthogonal projection $\mathbf{P}_0$ on the fluid space spanned by $\{\psi_\alpha,\alpha=0,\cdots,4\}$, and the corresponding orthogonal projection $\mathbf{P}_1$ on its orthogonal complement, i.e. the non-fluid component can be defined as:

$$\begin{cases}\mathbf{P}_0h\equiv\sum\limits_{j=0}^{4}\langle h,\chi_j\rangle\chi_j,\\ \mathbf{P}_1h\equiv h-\mathbf{P}_0h.\end{cases} \tag{1.9}$$

Notice that the operators $\mathbf{P}_0$ and $\mathbf{P}_1$ are projections satisfying

$$\mathbf{P}_0\mathbf{P}_0=\mathbf{P}_0,\qquad \mathbf{P}_1\mathbf{P}_1=\mathbf{P}_1,\qquad \mathbf{P}_0\mathbf{P}_1=\mathbf{P}_1\mathbf{P}_0=0.$$

Under this decomposition, the solution $f(t,x,\xi)$ of the Boltzmann equation satisfies

$$\mathbf{P}_0f=\mathbf{M},\qquad \mathbf{P}_1f=\mathbf{G}.$$

Then by replacing $f(t,x,\xi)$ by $\mathbf{M}(t,x,\xi)+\mathbf{G}(t,x,\xi)$, the Boltzmann equation becomes:

$$(\mathbf{M}+\mathbf{G})_t+\xi_1(\mathbf{M}+\mathbf{G})_x=\Big(2Q(\mathbf{G},\mathbf{M})+Q(\mathbf{G},\mathbf{G})\Big), \tag{1.10}$$

and the system of conservation laws is obtained by taking the inner product of the Boltzmann equation with the collision invariants $\psi_\alpha(\xi)$, $\alpha=0,1,\cdots,4$:

$$\begin{cases}\rho_t+(\rho u_1)_x=0,\\ (\rho u_1)_t+(\rho u_1^2+p)_x=-\left(\int_{\mathbf{R}^3}\xi_1^2\mathbf{G}d\xi\right)_x,\\ (\rho u_2)_t+(\rho u_1u_2)_x=-\left(\int_{\mathbf{R}^3}\xi_1\xi_2\mathbf{G}d\xi\right)_x,\\ (\rho u_3)_t+(\rho u_1u_3)_x=-\left(\int_{\mathbf{R}^3}\xi_1\xi_3\mathbf{G}d\xi\right)_x,\\ \left[\rho(\frac{1}{2}|u|^2+e)\right]_t+\left(u_1\left(\rho\left(\frac{1}{2}|u|^2+e\right)+p\right)\right)_x=-\frac{1}{2}\left(\int_{\mathbf{R}^3}\xi_1|\xi|^2\mathbf{G}d\xi\right)_x.\end{cases} \tag{1.11}$$

Here p is the pressure for the monatomic gas:

$$p = \frac{2}{3}\rho e = R\rho\theta.$$

Note that this system is not self-contained and we need one more equation for the non-fluid component $\mathbf{G}$ which can be obtained by applying the projection $\mathbf{P}_1$ on (1.10):

$$\mathbf{G}_t + \mathbf{P}_1(\xi_1\mathbf{G}_x + \xi_1\mathbf{M}_x) = L_{\mathbf{M}}\mathbf{G} + Q(\mathbf{G},\mathbf{G}), \tag{1.12}$$

i.e.,

$$\begin{aligned} \mathbf{G} &= L_{\mathbf{M}}^{-1}(\xi_1\mathbf{M}_x) + L_{\mathbf{M}}^{-1}\Big(\mathbf{G}_t + \mathbf{P}_1(\xi_1\mathbf{G}_x) - Q(\mathbf{G},\mathbf{G})\Big) \\ &:= L_{\mathbf{M}}^{-1}(\xi_1\mathbf{M}_x) + \Theta, \end{aligned} \tag{1.13}$$

where $L_{\mathbf{M}}$ is the usual linearized operator around the local Maxwellian $\mathbf{M}$ given by

$$L_{\mathbf{M}}g = L_{[\rho,u,\theta]}g \equiv Q\Big(\mathbf{M}+g, \mathbf{M}+g\Big) - Q(g,g).$$

Recall that the linearized collision operator $L_{\mathbf{M}}$ is symmetric:

$$\langle h, L_{\mathbf{M}}g\rangle = \langle L_{\mathbf{M}}h, g\rangle,$$

and the null space N of $L_{\mathbf{M}}$ contains only the fluid components spanned by:

$$\chi_j, \qquad j = 0, \cdots, 4.$$

$L_{\mathbf{M}}$ can also be written as, cf. [17, 15],

$$(L_{\mathbf{M}}h)(\xi) = -\nu(\xi;\rho,u,\theta)h(\xi) + \sqrt{\mathbf{M}(\xi)}K_{\mathbf{M}}\left(\left(\frac{h}{\sqrt{\mathbf{M}}}\right)(\xi)\right). \tag{1.14}$$

Here $K_{\mathbf{M}}(\cdot) = -K_{1\mathbf{M}}(\cdot) + K_{2\mathbf{M}}(\cdot)$ is a symmetric compact L^2-operator. And $\nu(\xi;\rho,u,\theta)$ and $K_{i\mathbf{M}}(\cdot)$ have the following estimates, cf. [10],

$$\nu(\xi;\rho,u,\theta) = \int_{\mathbf{R}^3} \mathbf{M}(\xi_*)q(|V|,\theta)d\xi_* d\Omega,$$

and

$$\begin{aligned} k_{1\mathbf{M}}(\xi,\xi_*) &= \mathbf{M}^{\frac12}(\xi)\int_{\mathbf{S}^2}\mathbf{M}^{\frac12}(\xi_*)q(V,\theta)\,d\Omega \\ &\le C\Big(|\xi-\xi_*| + |\xi-\xi_*|^{-\delta_1}\Big)\exp\Big(-\tfrac{|\xi-u|^2}{4R\theta} - \tfrac{|\xi_*-u|^2}{4R\theta}\Big), \\ k_{2\mathbf{M}}(\xi,\xi_*) &= a(\xi,\xi_*)\exp\left(-\frac{1}{8R\theta}\frac{(|\xi_*|^2-|\xi|^2)^2}{|\xi_*-\xi|^2} - \frac{|\xi_*-\xi|^2}{8R\theta}\right), \end{aligned}$$

with

$$a(\xi,\xi_*) \le C|\xi_*-\xi|^{-1},$$

where $k_{i\mathbf{M}}(\xi,\xi_*)$ is the kernel of the operator $K_{i\mathbf{M}}$, $i = 1, 2$, and $C > 0$ is a constant. Furthermore, the linearized H-theorem which reveals the dissipative effort of $L_{\mathbf{M}}$ on the non-fluid component implies that there exists $\sigma_0(\rho,u,\theta) > 0$ such that for any function $h(\xi) \in N$,

$$\langle h, L_{\mathbf{M}}h\rangle \le -\sigma_0(\rho,u,\theta)\langle h,h\rangle,$$

which yields cf. [15]

$$\langle h, L_{\mathbf{M}}h\rangle \le -\sigma(\rho,u,\theta)\langle \nu(\xi)h,h\rangle, \tag{1.15}$$

with some constant $\sigma(\rho, u, \theta) > 0$.

To have a clear representation related to fluid dynamics intuitively, by plugging (1.13) into (1.11), the system of conservation laws (1.11) becomes the Navier-Stokes equations plus some extra terms involving Θ which is of high order in some sense:

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = -\left(\int_{\mathbf{R}^3} \xi_1^2 L_{\mathbf{M}}^{-1}(\xi_1 \mathbf{M}_x) d\xi\right)_x - \left(\int_{\mathbf{R}^3} \xi_1^2 \Theta d\xi\right)_x, \\ (\rho u_2)_t + (\rho u_1 u_2)_x = -\left(\int_{\mathbf{R}^3} \xi_1 \xi_2 L_{\mathbf{M}}^{-1}(\xi_1 \mathbf{M}_x) d\xi\right)_x - \left(\int_{\mathbf{R}^3} \xi_1 \xi_2 \Theta d\xi\right)_x, \\ (\rho u_3)_t + (\rho u_1 u_3)_x = -\left(\int_{\mathbf{R}^3} \xi_1 \xi_3 L_{\mathbf{M}}^{-1}(\xi_1 \mathbf{M}_x) d\xi\right)_x - \left(\int_{\mathbf{R}^3} \xi_1 \xi_3 \Theta d\xi\right)_x, \\ \left[\rho(\frac{1}{2}|u|^2 + e)\right]_t + \left(u_1\left(\rho\left(\frac{1}{2}|u|^2 + e\right) + p\right)\right)_x \\ \qquad = -\frac{1}{2}\left(\int_{\mathbf{R}^3} \xi_1 |\xi|^2 L_{\mathbf{M}}^{-1}(\xi_1 \mathbf{M}_x) d\xi\right)_x - \frac{1}{2}\left(\int_{\mathbf{R}^3} \xi_1 |\xi|^2 \Theta d\xi\right)_x. \end{cases} \tag{1.16}$$

Notice that here the first terms on the right hand side of (1.16) not involving Θ evaluate as usual the viscosity and thermal conductivity. For the Navier-Stokes equations, the stability of rarefaction waves with or without boundary effect has been extensively studied, cf. [27, 22, 28, 29, 30, 35]. Moreover, the case for the Broadwell model of a discrete velocity gas was studied in [31]. And the problem we considered in this paper corresponds to those for Navier-Stokes equations with ideal gas law when $p = R\rho\theta$ with $\theta = \frac{3R}{2} e$ (For the corresponding study for the nonisentropic compressible Navier-Stokes equations, see [20, 23, 35] and the references therein. These results show that, even for general gas, the strength of the rarefaction waves can be arbitrarily large and for some special case, global stability results can also be obtained, cf. [35]. Note also that in our present paper for the Boltzmann equation, the strength of the rarefaction wave need not be small and the bound on the strength of the rarefaction wave is required by the variation of the linearized H-theorem which gives the dissipation on the non-fluid part). Hence, one can expect that the energy method which works well for the stability problems of Navier-Stokes equations works also here for the Boltzmann equation.

For later use, notice also that the projections $\mathbf{P}_0$ and $\mathbf{P}_1$ have the following basic properties:

$$\begin{cases} \mathbf{P}_0(\psi_j \mathbf{M}) = \psi_j \mathbf{M}, \ \mathbf{P}_1(\psi_j \mathbf{M}) = 0, \ j = 0, 1, 2, 3, 4, \\ L_{\mathbf{M}} \mathbf{P}_1 = \mathbf{P}_1 L_{\mathbf{M}} = L_{\mathbf{M}}, \ \mathbf{P}_1(Q(h, h)) = Q(h, h), \\ L_{\mathbf{M}} \mathbf{P}_0 = \mathbf{P}_0 L_{\mathbf{M}} = 0, \ \mathbf{P}_0(Q(h, h)) = 0, \\ \langle \psi_j \mathbf{M}, h \rangle = \langle \psi_j \mathbf{M}, \mathbf{P}_0 h \rangle, \ j = 0, 1, 2, 3, 4, \\ \langle h, L_{\mathbf{M}} g \rangle = \langle \mathbf{P}_1 h, L_{\mathbf{M}}(\mathbf{P}_1 g) \rangle, \\ \left\langle h, L_{\mathbf{M}}^{-1}(\mathbf{P}_1 g) \right\rangle = \left\langle L_{\mathbf{M}}^{-1}(\mathbf{P}_1 h), \mathbf{P}_1 g \right\rangle = \left\langle \mathbf{P}_1 h, L_{\mathbf{M}}^{-1}(\mathbf{P}_1 g) \right\rangle. \end{cases}$$

Now we turn to define the nonlinear time asymptotic rarefaction wave profile to the Botlzmann equation. For smooth solution, it is clear from the specular boundary condition that $u(t, 0) \equiv 0$ for any time $t \geq 0$. Assume

$$f_0(x, \xi) \to \mathbf{M}_r \equiv \mathbf{M}_{[\rho_r, u_r, \theta_r]} = \frac{\rho_r}{\sqrt{(2\pi R\theta_r)^3}} \exp\left(-\frac{|\xi - u_r|^2}{2R\theta_r}\right), \quad x \to +\infty. \tag{1.17}$$

Here $\rho_r, \theta_r > 0, u_r = (u_{1r}, 0, 0)$ are constants such that there exists a unique constant state

$(\rho_l, 0, \theta_l)$ with $\rho_l, \theta_l > 0$, such that the Riemann problem for the compressible Euler equations

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + \left(\rho u_1^2 + p\right)_x = 0, \\ (\rho u_2)_t + (\rho u_1 u_2)_x = 0, \\ (\rho u_3)_t + (\rho u_1 u_3)_x = 0, \\ \left[\rho(\frac{1}{2}|u|^2 + e)\right]_t + \left(u_1\left(\rho\left(\frac{1}{2}|u|^2 + e\right) + p\right)\right)_x = 0, \end{cases} \tag{1.18}$$

$$(\rho, u, \theta)(t, x)|_{t=0} = (\rho_0^r, u_0^r, \theta_0^r)(x) = \begin{cases} (\rho_l, 0, \theta_l), & x < 0, \\ (\rho_r, u_r, \theta_r), & x > 0, \end{cases} \tag{1.19}$$

admits a rarefaction wave solution of the third family, denoted by $\left(\rho^R(t, x), u^R(t, x), \theta^R(t, x)\right)$, i.e., $(\rho_r, u_r, \theta_r) \in R_3(\rho_l, 0, \theta_l)$. Here

$$R_3(\rho_l, 0, \theta_l) = \left\{ (\rho, u, \theta) \;\middle|\; \begin{array}{c} S = \overline{S},\ u_1 - \sqrt{15k}\rho^{\frac{1}{3}} \exp\left(\frac{S}{2}\right) = -\sqrt{15k}\rho_l^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right) \\ u_2 = u_3 = 0,\ u_1 > 0,\ \rho < \rho_l \end{array} \right\}, \tag{1.20}$$

where

$$\begin{cases} S = -\frac{2}{3}\ln\rho + \ln(2\pi R\theta) + 1 = -\frac{2}{3}\ln\rho_l + \ln(2\pi R\theta_l) + 1 \\ \quad = -\frac{2}{3}\ln\rho_r + \ln(2\pi R\theta_r) + 1 \equiv \overline{S}, \\ k = \frac{1}{2\pi e}. \end{cases}$$

Notice that this solution is only Lipschitz continuous at the edge of the wave. Similar to the corresponding work on the Navier-Stokes equations, cf. [30], we need to construct an approximate rarefaction wave which is in $H_x^s\left(L^2_{\xi,\mathcal{M}}\right)$ space. For this, let $w(t, x)$ be the unique global smooth solution to the following Cauchy problem of the Burgers' equation

$$\begin{cases} w_t + w w_x = 0, \\ w(t, x)|_{t=0} = w_0(x) = \frac{1}{2}(w_r + w_l) + \frac{1}{2}(w_r - w_l)\tanh(x), \end{cases} \tag{1.21}$$

where

$$\begin{cases} w_l = \lambda_3(\rho_l, 0, \theta_l) = \frac{\sqrt{15k}}{3}\rho_l^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right) > 0, \\ w_r = \lambda_3(\rho_r, u_r, \theta_r) = u_{1r} + \frac{\sqrt{15k}}{3}\rho_r^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right) > 0. \end{cases} \tag{1.22}$$

Then, we have the approximation of the rarefaction wave profile $\left(\overline{\rho}, \overline{u}, \overline{\theta}\right)(t, x)$ given by

$$\left(\overline{\rho}, \overline{u}, \overline{\theta}\right)(t, x) = \left(\rho^A, u^A, \theta^A\right)(t + t_0, x), \tag{1.23}$$

where t_0 is a suitably large but fixed positive constant and $\left(\rho^A, u^A, \theta^A\right)(t, x)$ satisfies

$$\begin{cases} u_1^A(t, x) + \frac{\sqrt{15k}}{3}\left(\rho^A(t, x)\right)^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right) = w(t, x), \\ u_1^A(t, x) - \sqrt{15k}\left(\rho^A(t, x)\right)^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right) = u_{1r} - \sqrt{15k}\rho_r^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right), \\ \theta^A(t, x) = \frac{3}{2}k\left(\rho^A(t, x)\right)^{\frac{2}{3}} \exp(\overline{S}),\ u_2^A = u_3^A = 0. \end{cases} \tag{1.24}$$

As for the nonlinear stability of rarefaction waves for the Boltzmann equation in the whole space, the strength of the rarefaction wave here need not be small. And both the energy estimates with respect to the global Maxwellian state $\mathbf{M}_-$ and the one with respect to the local Maxwellian state $\mathbf{M}$ are required.

With the above notations, we can now state the main result in this paper as follows.

Theorem 1.1 *Under the assumptions (A1) and (A2), let the approximate rarefaction wave* $\left(\overline{\rho},\overline{u},\overline{\theta}\right)(t,x)$ *be defined in (1.23). If*

$$\begin{cases} \delta=\max\left\{|\rho_r-\rho_l|+u_r+|\theta_r-\theta_l|\right\}<\eta_0, \\ \frac{1}{2}\sup\limits_{(t,x)\in\mathbf{R}_+\times\mathbf{R}}\overline{\theta}(t,x)<\inf\limits_{(t,x)\in\mathbf{R}_+\times\mathbf{R}}\overline{\theta}(t,x), \end{cases} \tag{1.25}$$

with $u_r>0$. *There exists a global Maxwellian* $\mathbf{M}_-$ *and sufficiently small positive constants* ε_0, t_0^{-1} *such that the following holds. Let the initial data* $f_0(x,\xi)$ *satisfy*

$$\int_{\mathbf{R}^3} f_0(0,\xi)\xi_1 d\xi=0,$$

and for $s\geq 2$,

$$\left\|f_0(x,\xi)-\mathbf{M}_{[\overline{\rho}(0,x),\overline{u}(0,x),\overline{\theta}(0,x)]}\right\|_{H^s_x\left(L^2_{\xi,\mathbf{M}_-}\right)}\leq\varepsilon_0, \tag{1.26}$$

then the initial boundary value problem (1.1), (1.2), (1.3) admits a unique global solution $f(t,x,\xi)$ *satisfying*

$$\left\|f(t,x,\xi)-\mathbf{M}_{[\overline{\rho},\overline{u},\overline{\theta}]}\right\|_{H^s_x\left(L^2_{\xi,\mathbf{M}_-}\right)}\leq C\left(\varepsilon_0+t_0^{-\frac{1}{8}}\right), \tag{1.27}$$

for some positive constant C, *and*

$$\lim_{t\to\infty}\left\|f(t,x,\xi)-\mathbf{M}_{[\rho^R,u^R,\theta^R]}\right\|_{L^\infty_x\left(L^2_{\xi,\mathbf{M}_-}\right)}=0. \tag{1.28}$$

Here the constant t_0 *comes from the definition of the approximate rarefaction wave.* $\mathbf{M}_-=\mathbf{M}_{[\rho_-,u_-,\theta_-]}$ *is a global Maxwellian satisfying* $\frac{1}{2}\theta(t,x)<\theta_-<\theta(t,x), u_{1-}=0$, *and* $|\rho(t,x)-\rho_-|+|u(t,x)-u_-|+|\theta(t,x)-\theta_-|<\eta_0$ *for all* $(t,x)\in\mathbf{R}_+\times\mathbf{R}$. *Here* $\eta_0>0$ *is a constant defined in Lemma 2.2 for the variation of the microscopic H-theorem.*

Remark 1.1 *Unlike the Cauchy problem, for the Boltzmann equation with specular reflection boundary condition, the global Maxwellian* $\mathbf{M}_-(\xi)$ *must be suitably chosen such that it is an even function of* ξ_1.

Besides the study on stability of nonlinear wave profiles for the Boltzmann equation, there have also been extensive study on the Boltzmann equation in other aspects related to fluid dynamics, such as the Knudsen layer, ghost effects, incompressible flow limit etc. Since they are beyond the scope of this paper, we will not refer them here. Before the energy method based on the decomposition (1.16) is used, the elegant and important analysis using the spectral properties of the linearized collision operator $L_{\mathbf{M}}$ has been used to obtained existence and large time behavior of solutions to the Boltzmann equation, see [19, 34, 37] and references therein.

This rest of this paper is arranged as follows: The microscopic and a macroscopic H-theorems with specular boundary condition will be given in Section 2 together with some properties on the smooth approximation of the rarefaction wave solution connecting to the boundary. The energy estimates will be given in Section 3: Section 3.1 is devoted to estimates on the boundary terms and the lower order energy estimate and the higher order energy estimates are presented in Section 3.2 and 3.3 respectively. Compared with that of [25], the main differences are two-fold: The first is due to the occurrance of the boundary terms and the other is that in our present paper, the assumptions we imposed on the collision kernel $q(|V|,\theta)$ is weaker than that of [25] which leads to some technical difficulties. The local existence in $H_x^s\left(L^2_{\xi,\mathcal{M}}\right)$ space and the proof of Theorem 1.1 will be given in Section 4 for the case when $s=2$. The case when $s>2$ can be discussed similarly. In the sequel, λ is used to denote a small positive constant.

2 Preliminaries

In this section, we give some known results concerning the properties on the smooth approximation of the rarefaction wave solution and the two versions of the $H-$theorem for the Boltzmann equation.

First, we list the properties of $\left(\overline{\rho},\overline{u}_1,\overline{\theta}\right)(t,x)$ constructed in (1.23) in the following lemma, cf. [27].

Lemma 2.1 *The approximate rarefaction wave $(\overline{\rho},\overline{u}_1,\overline{\theta})(t,x)$ constructed in (1.23) satisfies*

(i). $\overline{u}_{1x}(t,x)>0,\ \forall (t,x)\in\mathbf{R}_+\times\mathbf{R},\ i=1,3$;

(ii). For any $p(1\leq p\leq\infty)$, there exists a constant $C(p)>0$, depending only on p, such that

$$\begin{cases}\left\|\left(\overline{\rho},\overline{u}_1,\overline{\theta}\right)_x(t,x)\right\|_{L^p}\leq C(p)(t+t_0)^{-1+\frac{1}{p}},\\ \left\|\frac{\partial^j}{\partial x^j}\left(\overline{\rho},\overline{u}_1,\overline{\theta}\right)(t,x)\right\|_{L^p}\leq C(p)(t+t_0)^{-1},\quad j\geq 2;\end{cases}$$

(iii). $(\overline{\rho},\overline{u}_1,\overline{\theta})(t,x)$ solves

$$\begin{cases}\overline{\rho}_t+\left(\overline{\rho u}_1\right)_x=0,\\ \left(\overline{\rho u}_1\right)_t+\left[\overline{\rho}|\overline{u}_1|^2+\frac{2}{3}\overline{\rho\theta}\right]_x=0,\\ \left[\overline{\rho}\left(\frac{1}{2}|\overline{u}_1|^2+\overline{\theta}\right)\right]_t+\left[\frac{1}{2}\overline{\rho u}_1\left(|\overline{u}_1|^2+\frac{10}{3}\overline{\theta}\right)\right]_x=0.\end{cases}$$

Consequently

$$\left|\left(\overline{\rho}_t,\overline{u}_{1t},\overline{\theta}_t\right)(t,x)\right|\leq O(1)\left|\left(\overline{\rho}_x,\overline{u}_{1x},\overline{\theta}_x\right)(t,x)\right|;$$

(iv). $|\overline{u}_1(t,0)|\leq O(1)\exp\left(-d_1(t+t_0)\right)$;

(v). $\lim\limits_{t\to+\infty}\sup\limits_{x\in\mathbf{R}}\left|\left(\overline{\rho},\overline{u}_1,\overline{\theta}\right)(t,x)-\left(\rho^R,u_1^R,\theta^R\right)\left(\frac{x}{t}\right)\right|=0.$

Here and in what follows, $O(1)$ will be used to denote a generic positive constant independent of t and x and $d_1=\lambda_3\left(\rho_l,0,\theta_l\right)>0$.

Now we turn to the $H-$theorem for the Boltzmann equation. It is based on the special property of the bilinear structure of $Q(f,f)$ which satisfies

$$\int_{\mathbf{R}^3}Q(f,f)\ln f d\xi\leq 0,$$

and the equality holds only when the function $f(t,x,\xi)$ is a Maxwellian. According to the dissipative effects on the macroscopic and microscopic components, the H-theorem can be viewed from two aspects. The first one is mainly on the linearized collision operator $L_{\mathbf{M}}$ acting on the microscopic components stated in (1.15) called the microscopic $H-$theorem. The second one comes from the nonlinear collision operator which gives dissipation of entropy in the macroscopic level.

Since the perturbation of a nonlinear wave pattern considered may not be small, a combination of the energy estimates with respect to a global Maxwellian state $\mathbf{M}_-$ and the local Maxwellian state $\mathbf{M}$ will be used. For this reason, another form of the microscopic $H-$theorem is needed to relate the dissipation estimates with different weights. In fact, motivated by the proof of Lemma 3.2 in [25], we have the following estimate.

Lemma 2.2 *If $\frac{\theta}{2} < \theta_- < \theta$, then there exist two constants $\overline{\sigma} = \overline{\sigma}(\rho, u, \theta; \rho_-, u_-, \theta_-) > 0$ and $\eta_0 = \eta_0(\rho, u, \theta; \rho_-, u_-, \theta_-) > 0$ such that if $|\rho - \rho_-| + |u - u_-| + |\theta - \theta_-| < \eta_0$, we have for $h(\xi) \in N^{\perp}$,*

$$-\int_{\mathbf{R}^3} \frac{hL_{\mathbf{M}}h}{\mathbf{M}_-} d\xi \geq \overline{\sigma} \int_{\mathbf{R}^3} \frac{\nu(\xi)h^2}{\mathbf{M}_-} d\xi. \tag{2.1}$$

Lemma 2.2 is proved in [25] for the hard sphere case whose proof is straightforward by using the Cauchy inequality and an inequality on the collision operator from [13]. We note, however, that the proof given in [25] can be used to deduce Lemma 2.2 since the above mentioned inequality on the collision operator established in [13] holds also for the collision kernel $q(|V|, \theta)$ satisfying (A_1) and (A_2), i.e.

Lemma 2.3 *Suppose that $q(|V|, \theta)$ satisfies (A1) and (A2), then there exists a positive constant $C_4 > 0$ such that*

$$\int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} Q(f,g)^2}{\mathbf{M}} d\xi \leq \frac{C_4}{2} \left\{ \int_{\mathbf{R}^3} \frac{\nu(\xi) f^2}{\mathbf{M}} d\xi \cdot \int_{\mathbf{R}^3} \frac{g^2}{\mathbf{M}} d\xi + \int_{\mathbf{R}^3} \frac{f^2}{\mathbf{M}} d\xi \cdot \int_{\mathbf{R}^3} \frac{\nu(\xi) g^2}{\mathbf{M}} d\xi \right\},$$

where $\mathbf{M}$ can be any Maxwellian so that the above integrals are well-defined.

The following is a direct corollary of Lemma 2.2 and the Cauchy inequality.

Corollary 2.1 *Under the assumptions in Lemma 2.2, we have*

$$\begin{cases} \int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}} \left| L_{\mathbf{M}}^{-1} h \right|^2 d\xi \leq \sigma^{-2} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} h^2(\xi)}{\mathbf{M}} d\xi, \\ \int_{\mathbf{R}^3} \frac{\nu(\xi)}{\mathbf{M}_-} \left| L_{\mathbf{M}}^{-1} h \right|^2 d\xi \leq \overline{\sigma}^{-2} \int_{\mathbf{R}^3} \frac{\nu(\xi)^{-1} h^2(\xi)}{\mathbf{M}_-} d\xi \end{cases} \tag{2.2}$$

holds for each $h(\xi) \in N^{\perp}$.

For study the nonlinear wave behavior of the solutions, the following calculation on the macroscopic version of the $H-$theorem reveals the dissipation of entropy, cf. [24]. To be self-contained, we include it as follows. Set the macroscopic entropy S by

$$-\frac{3}{2}\rho S \equiv \int_{\mathbf{R}^3} \mathbf{M} \ln \mathbf{M} d\xi. \tag{2.3}$$

Direct calculation yields

$$-\frac{3}{2}(\rho S)_t - \frac{3}{2}(\rho u_1 S)_x + \left(\int_{\mathbf{R}^3} (\xi_1 \ln \mathbf{M}) \mathbf{G} d\xi \right)_x = \int_{\mathbf{R}^3} \frac{\mathbf{G} \xi_1 \partial_x \mathbf{M}}{\mathbf{M}} d\xi \tag{2.4}$$

and

$$\begin{cases} S = -\frac{2}{3} \ln \rho + \ln(2\pi R \theta) + 1, \\ p = \frac{2}{3} \rho \theta = k \rho^{\frac{5}{3}} \exp(S), \\ \mathbf{E} = \theta, \quad R = \frac{2}{3}. \end{cases} \tag{2.5}$$

Denote the conservation laws (1.11) by

$$\mathbf{m}_t + \mathbf{n}_x = -\begin{pmatrix} 0 \\ \int_{\mathbf{R}^3} \xi_1^2 \mathbf{G} d\xi \\ \int_{\mathbf{R}^3} \xi_1\xi_2 \mathbf{G} d\xi \\ \int_{\mathbf{R}^3} \xi_1\xi_3 \mathbf{G} d\xi \\ \frac{1}{2}\int_{\mathbf{R}^3} \xi_1|\xi|^2 \mathbf{G} d\xi \end{pmatrix}_x . \tag{2.6}$$

Here

$$\begin{cases} \mathbf{m} = (m_0, m_1, m_2, m_3, m_4)^t = \left(\rho, \rho u_1, \rho u_2, \rho u_3, \rho\left(\frac{1}{2}|u|^2 + \theta\right)\right)^t, \\ \mathbf{n} = (n_0, n_1, n_2, n_3, n_4)^t = \left(\rho u_1, \rho u_1^2 + \frac{2}{3}\rho\theta, \rho u_1 u_2, \rho u_1 u_3, \rho u_1\left(\frac{1}{2}|u|^2 + \frac{5}{3}\theta\right)\right)^t. \end{cases}$$

Then define an entropy-entropy flux pair (η, q) around a Maxwellian $\overline{\mathbf{M}} = \mathbf{M}_{[\overline{\rho},\overline{u},\overline{\theta}]}$ ($\overline{u}_i = 0, i = 2,3$) as

$$\begin{cases} \eta = \overline{\theta}\left\{-\frac{3}{2}\rho S + \frac{3}{2}\overline{\rho}\overline{S} + \frac{3}{2}\nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\overline{\mathbf{m}}}(\mathbf{m} - \overline{\mathbf{m}})\right\}, \\ q = \overline{\theta}\left\{-\frac{3}{2}\rho u_1 S + \frac{3}{2}\overline{\rho}\overline{u}_1\overline{S} + \frac{3}{2}\nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\overline{\mathbf{m}}}(\mathbf{n} - \overline{\mathbf{n}})\right\}. \end{cases} \tag{2.7}$$

It is easy to see that (cf. [25])

$$\begin{cases} \eta = \frac{3}{2}\left\{\rho\theta - \overline{\theta}\rho S + \rho\left[\left(\overline{S} - \frac{5}{3}\right)\overline{\theta} + \frac{|u-\overline{u}|^2}{2}\right] + \frac{2}{3}\overline{\rho}\overline{\theta}\right\}, \\ q = u_1\eta + (u_1 - \overline{u}_1)\left(\rho\theta - \overline{\rho}\overline{\theta}\right). \end{cases} \tag{2.8}$$

and for $\mathbf{m}$ in any closed bounded region in $\Sigma = \{\mathbf{m} : \rho > 0, \theta > 0\}$, there exists a positive constant C_5 such that

$$C_5^{-1}\left|\mathbf{m} - \overline{\mathbf{m}}\right|^2 \le \eta \le C_5\left|\mathbf{m} - \overline{\mathbf{m}}\right|^2. \tag{2.9}$$

Since

$$\begin{aligned} \eta_t + q_x = & \left[\nabla_{(\overline{\rho},\overline{u},\overline{S})}\eta \cdot \left(\overline{\rho}, \overline{u}, \overline{S}\right)_t + \nabla_{(\overline{\rho},\overline{u},\overline{S})} q \cdot \left(\overline{\rho}, \overline{u}, \overline{S}\right)_x\right] \\ & + \int_{\mathbf{R}^3}\left[\xi_1\partial_x\left(\overline{\theta}\ln\mathbf{M}\right) - \frac{3}{2}\overline{u}_{1x}|\xi_1|^2\right]\mathbf{G}d\xi \\ & - \left(\int_{\mathbf{R}^3}\left(\overline{\theta}\xi_1\ln\mathbf{M} - \frac{1}{2}\xi_1|\xi|^2 - \frac{3}{2}\overline{u}_1|\xi_1|^2\right)\mathbf{G}d\xi\right)_x, \end{aligned} \tag{2.10}$$

and there exists a positive constant $d_2 > 0$ such that (cf. [25])

$$\left[\nabla_{(\overline{\rho},\overline{u},\overline{S})}\eta \cdot \left(\overline{\rho}, \overline{u}, \overline{S}\right)_t + \nabla_{(\overline{\rho},\overline{u},\overline{S})} q \cdot \left(\overline{\rho}, \overline{u}, \overline{S}\right)_x\right] \le -d_2\overline{u}_{1x}\left|\left(\rho - \overline{\rho}, u_1 - \overline{u}_1, \theta - \overline{\theta}\right)\right|^2, \tag{2.11}$$

we have the entropy estimate:

$$\begin{aligned} \eta_t + q_x \le & -d_2\overline{u}_{1x}\left|\left(\rho - \overline{\rho}, u_1 - \overline{u}_1, \theta - \overline{\theta}\right)\right|^2 \\ & + \int_{\mathbf{R}^3}\left[\xi_1\partial_x\left(\overline{\theta}\ln\mathbf{M}\right) - \frac{3}{2}\overline{u}_{1x}|\xi_1|^2\right]\mathbf{G}d\xi \\ & - \left(\int_{\mathbf{R}^3}\left(\overline{\theta}\xi_1\ln\mathbf{M} - \frac{1}{2}\xi_1|\xi|^2 - \frac{3}{2}\overline{u}_1|\xi_1|^2\right)\mathbf{G}d\xi\right)_x. \end{aligned} \tag{2.12}$$

3 Energy Estimates

In this section, we perform the energy estimates. Our main purpose is to get the following result

Theorem 3.1 *Under the assumptions listed in Theorem 1.1, we have that the solution $f(t,x,\xi)$ to the initial boundary value problem (1.1)-(1.3) satisfies the following estimates*

$$\begin{aligned}&\int_0^\infty \eta(t)dx+\int_0^t\int_0^\infty\left(|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)|^2+\overline{u}_{1x}|(\tilde\rho,\tilde u_1,\tilde\theta)|^2+\sum_{|\alpha|=1}|\partial^\alpha(\rho_x,u_x,\theta_x)|^2\right)dxd\tau\\&+\int_0^\infty\int_{\mathbf{R}^3}\tfrac{1}{\mathbf{M}_-}\left(\tilde{\mathbf{G}}^2+\sum_{|\alpha|=1}\left(|\partial^\alpha\mathbf{M}|^2+|\partial^\alpha\mathbf{G}|^2\right)+\sum_{|\alpha|=2}|\partial^\alpha f|^2\right)d\xi dxd\tau\\&+\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\nu(\xi)}{\mathbf{M}_-}\left(\tilde{\mathbf{G}}^2+\sum_{|\alpha|=1}^2|\partial^\alpha\mathbf{G}|^2\right)d\xi dxd\tau\\&\leq O(1)\left(t_0^{-\frac14}+N(0)^2\right).\end{aligned}\tag{3.1}$$

Once we obtained (3.1), Theorem 1.1 follows immediately from it and the local existence results in $H^2_x\left(L^2_{\xi,\mathcal{M}}\right)$ given in the next section. Denote ∂^α the differential operator $\partial^\alpha=\partial^{(\alpha_0,\alpha_1)}=\partial_t^{\alpha_0}\partial_x^{\alpha_1}, |\alpha|=\alpha_0+\alpha_1$, where α_0 and α_1 are nonnegative integers. Set

$$\begin{cases}\tilde\rho(t,x)=\rho(t,x)-\overline{\rho}(t,x),\\ \tilde u(t,x)=u(t,x)-\overline{u}(t,x),\\ \tilde\theta(t,x)=\theta(t,x)-\overline{\theta}(t,x),\\ \tilde{\mathbf{G}}(t,x,\xi)=\mathbf{G}(t,x,\xi)-\overline{\mathbf{G}}(t,x,\xi)\end{cases}$$

with

$$\overline{\mathbf{G}}(t,x,\xi)=\frac{1}{R\theta(t,x)}L^{-1}_{\mathbf{M}_{[(\rho,u,\theta)(t,x)]}}\left\{\mathbf{P}_1\left[\xi_1\left(\frac{|\xi-u(t,x)|^2}{2\theta(t,x)}\overline{\theta}_x(t,x)+\xi_1\cdot\overline{u}_{1x}\right)\mathbf{M}(t,x)\right]\right\}.\tag{3.2}$$

Here we subtract $\overline{\mathbf{G}}(t,x,\xi)$ from $\mathbf{G}(t,x,\xi)$ because $\left\|\left(\overline{u}_x,\overline{\theta}_x\right)(t)\right\|^2_{L^2}$ is not integrable with respect to t. To get the desired energy estimates (3.1), all we need is to close the following a priori assumption

$$\begin{aligned}N(t)^2=&\sup_{0\leq\tau\leq t}\left\{\int_{\mathbf{R}_+}\eta(\tau)dx+\int_{\mathbf{R}_+}\int_{\mathbf{R}^3}\left(\tfrac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}+\sum_{|\alpha|=1}\tfrac{(\partial^\alpha\mathbf{M})^2+(\partial^\alpha\mathbf{G})^2}{\mathbf{M}_-}+\sum_{|\alpha|=2}\tfrac{(\partial^\alpha f)^2}{\mathbf{M}_-}\right)d\xi dx\right\}\\&<\delta_0^2.\end{aligned}\tag{3.3}$$

Here $\delta_0>0$ is a suitably chosen sufficiently small constant.

From (1.11), (3.3) yields the following $L^\infty_{(t,x)}$ estimates by Sobolev imbedding theorem.

$$\begin{aligned}\sup_{\tau\in[0,t],x\in\mathbf{R}_+}&\left\{\left|(\tilde\rho,\tilde u,\tilde\theta)(\tau,x)\right|+\sum_{0\leq|\alpha|\leq1}\left(|\partial^\alpha(\rho,u,\theta)(\tau,x)|+\left\|\tfrac{\partial^\alpha\mathbf{G}(\tau,x)}{\sqrt{\mathbf{M}_-(\tau,x)}}\right\|_{L^2_\xi}\right)\right\}\\&<O(1)\left(t_0^{-1}+\delta_0\right),\end{aligned}\tag{3.4}$$

where t_0 is the constant in the definition of the approximate rarefaction waves.

Under the a priori assumptions (3.3), by choosing δ_0 and t_0^{-1} to be sufficiently small, there exists a constant state $(\rho_-,u_-,\theta_-)(\rho_->0,\theta_->0)$ with $u_{1-}=0$ such that for all $(\tau,x)\in[0,t]\times\mathbf{R}_+$

$$\frac12\theta(\tau,x)<\theta_-<\theta(\tau,x),\quad|\theta(\tau,x)-\theta_-|+|u(\tau,x)-u_-|+|\rho(\tau,x)-\rho_-|<\eta_0.\tag{3.5}$$

Therefore, the microscopic $H-$theorem, i.e. (2.1) holds for the global Maxwellian $\mathbf{M}_- = \mathbf{M}_{[\rho_-,u_-,\theta_-]}$.

In the following three sub-sections, we will give estimates on the boundary terms; the energy estimates on the entropy and the derivatives with the weight of the local Maxwellian $\mathbf{M}$; and then the derivatives with the weight of the global Maxwellian $\mathbf{M}_-$ respectively.

3.1 Estimates on the Boundary Terms

This subsection is devoted to estimating the boundary terms. Our first result is to show that at the boundary $x=0$, the solution $f(t,x,\xi)$ of the initial boundary value problem (1.1)-(1.3) and its derivatives with respect to t and x are either odd and even functions of ξ_1.

Lemma 3.1 *Let $f(t,x,\xi)$ be a solution of the initial boundary value problem (1.1)-(1.3), then we have*

$$\partial_t^i\partial_x^j f(t,0,R\xi) = (-1)^j\partial_t^i\partial_x^j f(t,0,\xi) \tag{3.6}$$

and

$$u_1(t,0)=0. \tag{3.7}$$

Lemma 3.1 follows directly from the fact that $f(t,x,\xi)$ satisfies (1.1)-(1.3), we omit the details for brevity.

From (3.7) and the fact that $u_{1-}=0$, we know that $\mathbf{M}(t,0,\xi)$ and $\mathbf{M}_-(\xi)$ are even functions of ξ_1 and consequently $\mathbf{G}(t,0,\xi)$ is also an even function of ξ_1. This together with (3.6) give the following lemma.

Lemma 3.2 *Under the assumptions listed in Lemma 3.1, we have*

$$\int_0^t\int_{\mathbf{R}^3}\frac{\xi_1|\partial^\alpha f(\tau,0,\xi)|^2}{\mathbf{M}(\tau,0,\xi)}d\xi d\tau = \int_0^t\int_{\mathbf{R}^3}\frac{\xi_1|\partial^\alpha f(\tau,0,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi d\tau = 0 \tag{3.8}$$

and

$$\begin{aligned}\int_0^t\int_{\mathbf{R}^3}\Big[\overline{\theta}(\tau,0)\xi_1\ln\mathbf{M}(\tau,0,\xi)-\tfrac{1}{2}\xi_1|\xi|^2-\tfrac{3}{2}\overline{u}_1(\tau,0)|\xi_1|^2\Big]\mathbf{G}(\tau,0,\xi)d\xi d\tau\\ = -\frac{3}{2}\int_0^t\int_{\mathbf{R}^3}|\xi_1|^2\overline{u}_1(\tau,0)\mathbf{G}(\tau,0,\xi)d\xi d\tau.\end{aligned} \tag{3.9}$$

Now we turn to the estimates on the boundary terms arising from the later energy estimates on the solution.

Lemma 3.3 *Under the a priori assumption (3.3), we have*

$$
\begin{cases}
I_1=\displaystyle\int_0^t q(\tau,0)d\tau\le O(1)\exp\Big(-d_1t_0\Big),\\
I_2=\displaystyle\int_0^t\int_{\mathbf{R}^3}\overline{u}_{1x}(\tau,0)|\xi_1|^2|\mathbf{G}(\tau,0,\xi)|d\xi d\tau\\
\qquad\le O(1)t_0^{-\frac13}+O(1)\displaystyle\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi dxd\tau,\\
I_3=\displaystyle\int_0^t\int_{\mathbf{R}^3}\left\{\frac{1}{\theta}\xi_1\left(\xi\cdot\tilde u+\frac{\tilde\theta}{2\theta}|\xi-u|^2\right)L_{\mathbf{M}}^{-1}\left\{\mathbf{P}_1\left[\xi_1\left(\frac{|\xi-u|^2}{2\theta}\overline{\theta}_x+\xi_1\overline{u}_{1x}\right)\mathbf{M}\right]\right\}\right\}(\tau,0,\xi)d\xi d\tau\\
\quad\le O(1)t_0^{-\frac12}\left(1+\displaystyle\int_0^t\int_0^{+\infty}\left|(\tilde u_x,\tilde\theta_x)\right|^2dxd\tau\right),\\
I_4=\displaystyle\int_0^t\int_{\mathbf{R}^3}\left(\frac{\xi_1\tilde{\mathbf{G}}^2}{\mathbf{M}}\right)(\tau,0,\xi)d\xi d\tau\le O(1)t_0^{-\frac13}+O(1)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi dxd\tau,\\
I_5=\displaystyle\int_0^t\int_{\mathbf{R}^3}\left(\frac{\xi_1\tilde{\mathbf{G}}^2}{\mathbf{M}_-}\right)(\tau,0,\xi)d\xi d\tau\le O(1)t_0^{-\frac13}+O(1)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}_-}d\xi dxd\tau.
\end{cases}
\tag{3.10}
$$

Proof. From $(2.8)_2$, (3.4), (3.7), and (iv) of Lemma 2.1, we have

$$
|I_1|\le O(1)\int_0^t\exp\Big(-d_1(\tau+t_0)\Big)\Big|(\rho\theta-\overline{\rho}\overline{\theta})(\tau,0)\Big|d\tau\le O(1)\exp\Big(-d_1t_0\Big),
$$

and $(3.10)_1$ is proved.

As to I_2, we get from (ii) of Lemma 2.1 and (3.4) that

$$
\begin{aligned}
|I_2|&\le O(1)\int_0^t(\tau+t_0)^{-1}\left(\int_{\mathbf{R}^3}\frac{\mathbf{G}^2(\tau,0,\xi)}{\mathbf{M}_{[\rho_-,u_-,2\theta_-]}(\xi)}d\xi\right)^{\frac12}d\tau\\
&\le O(1)\int_0^t(\tau+t_0)^{-1}\left(\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}^2}{\mathbf{M}}d\xi dx\right)^{\frac14}\left(\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi dx\right)^{\frac14}d\tau\\
&\le O(1)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi dxd\tau+O(1)\int_0^t(\tau+t_0)^{-\frac43}\left(\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}^2}{\mathbf{M}}d\xi dx\right)^{\frac13}d\tau\\
&\le O(1)t_0^{-\frac13}+O(1)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi dxd\tau,
\end{aligned}
$$

from which $(3.10)_2$ follows.

Now we turn to estimate I_3. The properties of the operators $L_{\mathbf{M}}^{-1}$ and $\mathbf{P}_1$ yield

$$
\begin{aligned}
|I_3|&\le O(1)\int_0^t\left|(\tilde u,\tilde\theta)(\tau,0)\right|\left|(\overline{u}_{1x},\overline{\theta}_x)(\tau,0)\right|d\tau\\
&\le O(1)\int_0^t(\tau+t_0)^{-1}\left|(\tilde u,\tilde\theta)(\tau,0)\right|d\tau\\
&\le O(1)\int_0^t(\tau+t_0)^{-1}\left\|(\tilde u,\tilde\theta)(\tau)\right\|^{\frac12}\left\|(\tilde u_x,\tilde\theta_x)(\tau)\right\|^{\frac12}d\tau\\
&\le t_0^{-\frac12}\int_0^t\int_0^\infty\left|(\tilde u_x,\tilde\theta_x)\right|^2dxd\tau+O(1)\int_0^t(\tau+t_0)^{-\frac32}d\tau\\
&\le O(1)t_0^{-\frac12}\left(1+\int_0^t\int_0^\infty\left|(\tilde u_x,\tilde\theta_x)\right|^2dxd\tau\right),
\end{aligned}
$$

which is $(3.10)_3$.

Notice that I_5 can be treated similarly. As for I_4, noting that $\mathbf{M}(\tau,0,\xi)$ and $\mathbf{G}(\tau,0,\xi)$ are even function of ξ_1, we obtain from the fact $\frac{\theta}{2}<\theta_-<\theta$ that

$$
\begin{aligned}
I_4 &= \int_0^t \int_{\mathbf{R}^3} \left(\frac{\xi_1 \left[\overline{\mathbf{G}}^2 - 2\mathbf{G}\overline{\mathbf{G}} \right]}{\mathbf{M}} \right) (\tau,0,\xi) d\xi d\tau \\
&\leq O(1)\int_0^t \left[\left|(\overline{u}_{1x},\overline{\theta}_x)(\tau,0)\right|^2 + \left|(\overline{u}_{1x},\overline{\theta}_x)(\tau,0)\right| \left(\int_{\mathbf{R}^3} \frac{\mathbf{G}^2(\tau,0,\xi)}{\mathbf{M}_{[\rho_-,u_-,2\theta_-]}(\xi)} d\xi \right)^{\frac{1}{2}} \right] d\tau \\
&\leq O(1)\int_0^t \left[(\tau+t_0)^{-2} + (\tau+t_0)^{-1} \left(\int_0^\infty \int_{\mathbf{R}^3} \frac{\mathbf{G}^2}{\mathbf{M}} d\xi dx \right)^{\frac{1}{4}} \left(\int_0^\infty \int_{\mathbf{R}^3} \frac{\mathbf{G}_x^2}{\mathbf{M}} d\xi dx \right)^{\frac{1}{4}} \right] d\tau \\
&\leq O(1)t_0^{-1} + O(1)\int_0^t \int_0^\infty \int_{\mathbf{R}^3} \frac{\mathbf{G}_x^2}{\mathbf{M}} d\xi dx d\tau \\
&\quad + O(1)\int_0^t (\tau+t_0)^{-\frac{4}{3}} \left(\int_0^\infty \int_{\mathbf{R}^3} \frac{\mathbf{G}^2}{\mathbf{M}} d\xi dx \right) d\tau \\
&\leq O(1)t_0^{-\frac{1}{3}} + O(1)\int_0^t \int_0^\infty \int_{\mathbf{R}^3} \frac{\mathbf{G}_x^2}{\mathbf{M}} d\xi dx d\tau,
\end{aligned}
$$

This gives $(3.10)_4$ and completes the proof of the lemma.

For the boundary terms coming from higher order energy estimates, we have the following lemma.

Lemma 3.4 *Under the assumptions listed in Lemma 3.3, we have*

$$
\left\{
\begin{aligned}
I_6 &= \int_0^t \int_{\mathbf{R}^3} \left(\tfrac{\mathbf{M}_x \mathbf{M}_t}{\mathbf{M}}\right)(\tau,0,\xi)d\xi d\tau\\
&\leq \lambda \int_0^t \int_0^\infty \left[\left|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)\right|^2 + \left|(\rho_{xx}, u_{xx}, \theta_{xx})\right|^2 + \int_{\mathbf{R}^3} \tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}} d\xi\right] dxd\tau + O(1)t_0^{-\frac13}\\
&\quad + \tfrac{O(1)}{\lambda}\int_0^t\int_0^\infty\int_{\mathbf{R}^3} \tfrac{\mathbf{G}_{tx}^2+\mathbf{G}_{xx}^2}{\mathbf{M}} d\xi dx d\tau + O(1)\left(\delta_0 + t_0^{-1}\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3} \tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}_-} d\xi dx d\tau,\\
I_7 &= \int_0^t \int_{\mathbf{R}^3} \left(\tfrac{\xi_1(\mathbf{M}_x^2+\mathbf{M}_t^2)}{\mathbf{M}}\right)(\tau,0,\xi)d\xi d\tau\\
&\leq O(1)t_0^{-\frac13} + \lambda \int_0^t \int_0^\infty \left[\left|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)\right|^2 + \left|(\rho_{xx}, u_{xx}, \theta_{xx})\right|^2 + \int_{\mathbf{R}^3} \tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}} d\xi\right] dxd\tau\\
&\quad + \tfrac{O(1)}{\lambda}\int_0^t\int_0^\infty\int_{\mathbf{R}^3} \tfrac{\mathbf{G}_{tx}^2+\mathbf{G}_{xx}^2}{\mathbf{M}} d\xi dx d\tau + O(1)\left(\delta_0 + t_0^{-1}\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3} \tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}_-} d\xi dx d\tau,\\
I_8 &= \int_0^t \int_{\mathbf{R}^3} \left(\tfrac{\xi_1(\mathbf{G}_x^2+\mathbf{G}_t^2)}{\mathbf{M}}\right)(\tau,0,\xi)d\xi d\tau\\
&\leq O(1)t_0^{-\frac13} + \lambda \int_0^t \int_0^\infty \left[\left|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)\right|^2 + \left|(\rho_{xx}, u_{xx}, \theta_{xx})\right|^2 + \int_{\mathbf{R}^3} \tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}} d\xi\right] dxd\tau\\
&\quad + \tfrac{O(1)}{\lambda}\int_0^t\int_0^\infty\int_{\mathbf{R}^3} \tfrac{\mathbf{G}_{tx}^2+\mathbf{G}_{xx}^2}{\mathbf{M}} d\xi dx d\tau + O(1)\left(\delta_0 + t_0^{-1}\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3} \tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}_-} d\xi dx d\tau,\\
I_9 &= \left|\int_0^t \int_{\mathbf{R}^3} \left(\tfrac{\xi_1 \mathbf{M}_x \mathbf{G}_x}{\mathbf{M}}\right)(\tau,0,\xi)d\xi d\tau\right| + \left|\int_0^t \int_{\mathbf{R}^3} \left(\tfrac{\xi_1 \mathbf{M}_t \mathbf{G}_t}{\mathbf{M}}\right)(\tau,0,\xi)d\xi d\tau\right|\\
&\leq \left(\lambda + \delta_0^{\frac12} + t_0^{\frac12}\right)\int_0^t \int_0^\infty \left[\left|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)\right|^2 + \left|(\rho_{xx}, u_{xx}, \theta_{xx})\right|^2 + \int_{\mathbf{R}^3} \tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}} d\xi\right] dxd\tau\\
&\quad + O(1)t_0^{-\frac14} + \tfrac{O(1)}{\lambda}\int_0^t\int_0^\infty\int_{\mathbf{R}^3} \tfrac{\mathbf{G}_{tx}^2+\mathbf{G}_{xx}^2}{\mathbf{M}} d\xi dx d\tau\\
&\quad + O(1)\left(\delta_0^{\frac12} + t_0^{-\frac12}\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3} \tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}_-} d\xi dx d\tau,\\
I_{10} &= \int_0^t \int_{\mathbf{R}^3} \left(\tfrac{\xi_1(\mathbf{G}_x^2+\mathbf{G}_t^2)}{\mathbf{M}_-}\right)(\tau,0,\xi)d\xi d\tau\\
&\leq O(1)t_0^{-\frac13} + \lambda \int_0^t \int_0^\infty \left[\left|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)\right|^2 + \left|(\rho_{xx}, u_{xx}, \theta_{xx})\right|^2 + \int_{\mathbf{R}^3} \tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}_-} d\xi\right] dxd\tau\\
&\quad + \tfrac{O(1)}{\lambda}\int_0^t\int_0^\infty\int_{\mathbf{R}^3} \tfrac{\mathbf{G}_{tx}^2+\mathbf{G}_{xx}^2}{\mathbf{M}_-} d\xi dx d\tau.
\end{aligned}
\right. \tag{3.11}
$$

Here and in what follows, $\lambda > 0$ is used to denote a sufficiently small constant.

Proof. Since

$$
\left\{
\begin{aligned}
\mathbf{M}_x &= \tfrac{\rho_x}{\sqrt{\rho}}\chi_0 + \tfrac{\sqrt{6}\rho}{2\theta}\theta_x\chi_4 + \sqrt{\tfrac{\rho}{R\theta}}\sum_{i=1}^3 u_{ix}\chi_i,\\
\mathbf{M}_t &= \tfrac{\rho_t}{\sqrt{\rho}}\chi_0 + \tfrac{\sqrt{6}\rho}{2\theta}\theta_t\chi_4 + \sqrt{\tfrac{\rho}{R\theta}}\sum_{i=1}^3 u_{it}\chi_i,
\end{aligned}
\right. \tag{3.12}
$$

we have from (1.8) that

$$
\int_{\mathbf{R}^3}\left(\frac{\mathbf{M}_x\mathbf{M}_t}{\mathbf{M}}\right)(\tau,0,\xi)d\xi = \left[\frac{\rho_x\rho_t}{\rho} + \frac{3\rho}{2\theta^2}\theta_x\theta_t + \frac{\rho}{R\theta}\sum_{i=1}^3 u_{ix}u_{it}\right](\tau,0). \tag{3.13}
$$

Since $u_1(\tau,0)=0$, (1.11) gives

$$\begin{aligned}\Big[\tfrac{\rho_x\rho_t}{\rho}+\tfrac{3\rho}{2\theta^2}\theta_x\theta_t+\tfrac{\rho}{R\theta}\sum_{i=1}^{3}u_{ix}u_{it}\Big](\tau,0)&=-2\Big[\rho_xu_{1x}+\tfrac{\rho}{\theta}u_{1x}\theta_x\Big](\tau,0)\\&\quad+O(1)\Big|(u_x,\theta_x)(\tau,0)\Big|\int_{\mathbf{R}^3}|\xi|^2|\mathbf{G}_x(\tau,0,\xi)|d\xi.\end{aligned}\tag{3.14}$$

Hence

$$\begin{aligned}\int_{\mathbf{R}^3}\Big(\tfrac{\mathbf{M}_x\mathbf{M}_t}{\mathbf{M}}\Big)(\tau,0,\xi)d\xi&=-2\Big[\rho_xu_{1x}+\tfrac{\rho}{\theta}u_{1x}\theta_x\Big](\tau,0)\\&\quad+O(1)\Big|(u_x,\theta_x)(\tau,0)\Big|\int_{\mathbf{R}^3}|\xi|^2|\mathbf{G}_x(\tau,0,\xi)|d\xi.\end{aligned}\tag{3.15}$$

Similarly, we have

$$\int_{\mathbf{R}^3}\left(\frac{\xi_1\mathbf{M}_x^2}{\mathbf{M}}\right)(\tau,0,\xi)d\xi=2\left[\rho_xu_{1x}+\frac{\rho}{\theta}u_{1x}\theta_x\right](\tau,0),\tag{3.16}$$

$$\begin{aligned}\int_{\mathbf{R}^3}\Big(\tfrac{\xi_1\mathbf{M}_t^2}{\mathbf{M}}\Big)(\tau,0,\xi)d\xi&=\tfrac{20}{9}\Big[\tfrac{1}{\theta}\left(\rho_xu_{1x}+\tfrac{\rho}{\theta}u_{1x}\theta_x\right)\Big](\tau,0)\\&\quad+O(1)\Big|(\rho_x,u_x,\theta_x)(\tau,0)\Big|\int_{\mathbf{R}^3}|\xi|^2|\mathbf{G}_x(\tau,0,\xi)|d\xi.\end{aligned}\tag{3.17}$$

Since

$$\begin{aligned}&\int_0^t\left(\Big|(\rho_x,u_x,\theta_x)(\tau,0)\Big|\int_{\mathbf{R}^3}|\xi|^2|\mathbf{G}_x(\tau,0,\xi)|d\xi\right)d\tau\\
&\le O(1)\int_0^t\left(\Big|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)(\tau,0)\Big|+\Big|(\overline\rho_x,\overline u_x,\overline\theta_x)\Big|\right)\left(\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_x^2(\tau,0,\xi)}{\mathbf{M}_{[\rho_-,u_-,2\theta_-]}(\xi)}d\xi\right)^{\frac12}d\tau\\
&\le O(1)\int_0^t\Big[\|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\|^{\frac12}\|(\rho_{xx},u_{xx},\theta_{xx})\|^{\frac12}+O(1)(\tau+t_0)^{-1}\Big]\\
&\qquad\times\left(\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_x^2}{\mathbf{M}}d\xi dx\right)^{\frac14}\left(\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_{xx}^2}{\mathbf{M}}d\xi dx\right)^{\frac14}d\tau\\
&\le\lambda\int_0^t\int_0^\infty\left[\Big|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\Big|^2+\Big|(\rho_{xx},u_{xx},\theta_{xx})\Big|^2+\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}}d\xi\right]dxd\tau\\
&\qquad+O(1)t_0^{-\frac13}+\tfrac{O(1)}{\lambda}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_{xx}^2}{\mathbf{M}}d\xi dxd\tau,\end{aligned}\tag{3.18}$$

$(3.11)_2$ follows directly from $(3.11)_1$, (3.15)-(3.18).

For $(3.11)_1$, since $\mathbf{M}_x(\tau,0,\xi)=(f_x-\mathbf{G}_x)(\tau,0,\xi),\mathbf{M}_t(\tau,0,\xi)=(f_t-\mathbf{G}_t)(\tau,0,\xi)$, and $f_x(\tau,0,\xi)$ and $f_t(\tau,0,\xi)$ are odd and even functions of ξ_1 respectively, we have

$$\begin{aligned}I_6&=\int_0^t\int_{\mathbf{R}^3}\Big(\tfrac{(f_x-\mathbf{G}_x)(f_t-\mathbf{G}_t)}{\mathbf{M}}\Big)(\tau,0,\xi)d\xi d\tau\\
&=-\int_0^t\int_{\mathbf{R}^3}\Big(\tfrac{\mathbf{M}_x\mathbf{G}_t+\mathbf{M}_t\mathbf{G}_x+\mathbf{G}_x\mathbf{G}_t}{\mathbf{M}}\Big)(\tau,0,\xi)d\xi d\tau\\
&\le O(1)\int_0^t\left[|(\rho_x,u_x,\theta_x)(\tau,0)|\left(\int_{\mathbf{R}^3}\tfrac{(\mathbf{G}_x^2+\mathbf{G}_t^2)(\tau,0,\xi)}{\mathbf{M}_{[\rho_-,u_-,2\theta_-]}(\xi)}d\xi\right)^{\frac12}+\int_{\mathbf{R}^3}\Big(\tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}}\Big)(\tau,0,\xi)d\xi\right]d\tau\end{aligned}\tag{3.19}$$

Since

$$\begin{aligned}\int_{\mathbf{R}^3}\tfrac{(\mathbf{G}_x^2+\mathbf{G}_t^2)(\tau,0,\xi)}{\mathbf{M}_{[\rho_-,u_-,2\theta_-]}(\xi)}d\xi&\le O(1)\int_0^\infty\int_{\mathbf{R}^3}\left|\left(\frac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}_{[\rho_-,u_-,2\theta_-]}(\xi)}\right)_x\right|d\xi dx\\
&\le O(1)\left(\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}}d\xi dx\right)^{\frac12}\left(\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_{xx}^2+\mathbf{G}_{xt}^2}{\mathbf{M}}d\xi dx\right)^{\frac12},\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbf{R}^3}\left(\tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}}\right)(\tau,0,\xi)d\xi\leq O(1)\int_0^\infty\int_{\mathbf{R}^3}\left|\left(\tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}}\right)_x\right|d\xi dx\\
&\leq O(1)\int_0^\infty\int_{\mathbf{R}^3}\tfrac{|\mathbf{G}_t\mathbf{G}_{tx}|+|\mathbf{G}_x\mathbf{G}_{xx}|}{\mathbf{M}}d\xi dx+O(1)\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}^2}|\mathbf{M}_x|d\xi dx\\
&\leq O(1)\Big(\delta_0+t_0^{-1}\Big)\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}_-}d\xi dx\\
&\quad+O(1)\left(\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}}d\xi dx\right)^{\frac12}\left(\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_{xt}^2+\mathbf{G}_{xx}^2}{\mathbf{M}}d\xi dx\right)^{\frac12},
\end{aligned}$$

we get

$$\begin{aligned}
|I_6|&\leq O(1)\int_0^t\Big(\|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\|^{\frac12}\|(\rho_{xx},u_{xx},\theta_{xx})\|^{\frac12}+(\tau+t_0)^{-1}\Big)\\
&\quad\times\Bigg[\left(\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}}d\xi dx\right)^{\frac14}\left(\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_{xt}^2+\mathbf{G}_{xx}^2}{\mathbf{M}}d\xi dx\right)^{\frac14}\\
&\quad+\Big(\delta_0+t_0^{-1}\Big)\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}_-}d\xi dx\\
&\quad+\left(\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}}d\xi dx\right)^{\frac12}\left(\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_{xt}^2+\mathbf{G}_{xx}^2}{\mathbf{M}}d\xi dx\right)^{\frac12}\Bigg]d\tau\\
&\leq\lambda\int_0^t\int_0^\infty\left[\Big|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\Big|^2+\Big|(\rho_{xx},u_{xx},\theta_{xx})\Big|^2+\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}}d\xi\right]dxd\tau\\
&\quad+O(1)t_0^{-\frac13}+\tfrac{O(1)}{\lambda}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_{tx}^2+\mathbf{G}_{xx}^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\left(\delta_0+t_0^{-1}\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}_-}d\xi dxd\tau,
\end{aligned}\tag{3.20}$$

i.e., $(3.11)_1$.

For I_8, we have

$$\begin{aligned}
I_8&=\int_0^t\int_{\mathbf{R}^3}\left(\tfrac{\xi_1(\mathbf{G}_x^2+\mathbf{G}_t^2)}{\mathbf{M}}\right)(\tau,0,\xi)d\xi d\tau\\
&=\int_0^t\int_{\mathbf{R}^3_+}\tfrac{\xi_1[(\mathbf{G}_x^2+\mathbf{G}_t^2)(\tau,0,\xi)-(\mathbf{G}_x^2+\mathbf{G}_t^2)(\tau,0,R\xi)]}{\mathbf{M}(\tau,0,\xi)}d\xi d\tau
\end{aligned}\tag{3.21}$$

and

$$\begin{cases}\mathbf{G}_x(\tau,0,R\xi)+\mathbf{G}_x(\tau,0,\xi)=-\Big[\mathbf{M}_x(\tau,0,\xi)+\mathbf{M}_x(\tau,0,R\xi)\Big],\\ \mathbf{G}_t(\tau,0,R\xi)-\mathbf{G}_t(\tau,0,\xi)=\mathbf{M}_t(\tau,0,\xi)-\mathbf{M}_t(\tau,0,R\xi).\end{cases}$$

Hence

$$\begin{aligned}
|I_8|&\leq O(1)\int_0^t\int_{\mathbf{R}^3}\left(\tfrac{|\xi_1|(|\mathbf{M}_x|+|\mathbf{M}_t|)(|\mathbf{G}_x|+|\mathbf{G}_t|)}{\mathbf{M}}\right)(\tau,0,\xi)d\xi d\tau\\
&\leq O(1)\int_0^t\Bigg(|(\rho_x,u_x,\theta_x)(\tau,0)|+|(\rho_t,u_t,\theta_t)(\tau,0)|\Bigg)\left(\int_{\mathbf{R}^3}\left(\tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}}\right)(\tau,0,\xi)\right)^{\frac12}d\tau\\
&\leq O(1)\int_0^t\Bigg(|(\rho_x,u_x,\theta_x)(\tau,0)|+\left(\int_{\mathbf{R}^3}\left(\tfrac{\mathbf{G}_x^2}{\mathbf{M}}\right)(\tau,0,\xi)d\xi\right)^{\frac12}\Bigg)\\
&\quad\times\left(\int_{\mathbf{R}^3}\left(\tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}}\right)(\tau,0,\xi)\right)^{\frac12}d\tau\\
&\leq\lambda\int_0^t\int_0^\infty\left[\Big|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\Big|^2+\Big|(\rho_{xx},u_{xx},\theta_{xx})\Big|^2+\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}}d\xi\right]dxd\tau+O(1)t_0^{-\frac13}\\
&\quad+\tfrac{O(1)}{\lambda}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_{tx}^2+\mathbf{G}_{xx}^2}{\mathbf{M}}d\xi dxd\tau+O(1)\left(\delta_0+t_0^{-1}\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}_-}d\xi dxd\tau,
\end{aligned}\tag{3.22}$$

Which gives $(3.11)_3$. Note that $(3.11)_4$ and $(3.11)_5$ can be proved similarly. This completes the proof of lemma.

3.2 Lower Order Estimate

In this subsection, we will give the energy estimates on the entropy and the non-fluid component $\mathbf{G}(t,x,\xi)$ and we have

Lemma 3.5 *Under the a priori assumption (3.3), we have*

$$\begin{aligned}
&\int_0^\infty \eta(t)dx + \int_0^\infty \int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}^2}{\mathbf{M}} d\xi dx \Big|_0^t \\
&\quad + \int_0^t \int_0^\infty \left[\overline{u}_{1x} \Big|(\tilde\rho, \tilde u_1, \tilde\theta)\Big|^2 + \Big|(\tilde u_x, \tilde\theta_x)\Big|^2 + \int_{\mathbf{R}^3} \frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}} d\xi \right] dx d\tau \\
&\le O(1)\left(\int_0^\infty \eta(0)dx + t_0^{-\frac14}\right) + O(1)\left(\delta_0 + t_0^{-1}\right) \int_0^t \int_0^\infty \int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-} d\xi dx d\tau \\
&\quad + O(1) \int_0^t \int_0^\infty \int_{\mathbf{R}^3} \frac{\mathbf{G}_x^2 + \mathbf{G}_t^2}{\mathbf{M}} d\xi dx d\tau + O(1) t_0^{-1} \int_0^t \int_0^\infty \Big|\tilde\rho_x\Big|^2 dx d\tau
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
&\int_0^\infty \eta(t)dx + \int_0^\infty \int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-} d\xi dx \Big|_0^t \\
&\quad + \int_0^t \int_0^\infty \left[\overline{u}_{1x} \Big|(\tilde\rho, \tilde u_1, \tilde\theta)\Big|^2 + \Big|(\tilde u_x, \tilde\theta_x)\Big|^2 + \int_{\mathbf{R}^3} \frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}_-} d\xi \right] dx d\tau \\
&\le O(1)\left(\int_0^\infty \eta(0)dx + t_0^{-\frac14}\right) + O(1) t_0^{-1} \int_0^t \int_0^\infty \Big|\tilde\rho_x\Big|^2 dx d\tau \\
&\quad + O(1) \int_0^t \int_0^\infty \int_{\mathbf{R}^3} \frac{\mathbf{G}_x^2 + \mathbf{G}_t^2}{\mathbf{M}_-} d\xi dx d\tau.
\end{aligned} \tag{3.24}$$

Proof. First from (2.12), $(3.10)_1$-$(3.10)_3$, by similar argument on entropy to the one in [25], we have

$$\begin{aligned}
&\int_0^\infty \eta(t)dx + \int_0^t \int_0^\infty \left(\overline{u}_{1x} |(\tilde\rho, \tilde u_1, \tilde\theta)|^2 + |(\tilde u_x, \tilde\theta_x)|^2 \right) dx d\tau \\
&\le O(1)\left(\int_0^\infty \eta(0)dx + t_0^{-\frac14}\right) + O(1) \int_0^t \int_0^\infty \int_{\mathbf{R}^3} \frac{\mathbf{G}_x^2 + \mathbf{G}_t^2 + \nu(\xi)^{-1} Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}} d\xi dx d\tau.
\end{aligned} \tag{3.25}$$

As for the microscopic part $\tilde{\mathbf{G}}$ which solves

$$\tilde{\mathbf{G}}_t - L_{\mathbf{M}}\tilde{\mathbf{G}} = -\frac{1}{R\theta}\mathbf{P}_1\left[\xi_1\left(\frac{|\xi-u|^2}{2\theta}\tilde\theta_x + \xi\cdot\tilde u_x\right)\mathbf{M}\right] - \mathbf{P}_1(\xi_1\mathbf{G}_x) + Q(\mathbf{G},\mathbf{G}) - \overline{\mathbf{G}}_t, \tag{3.26}$$

by multiplying (3.26) by $\frac{\tilde{\mathbf{G}}}{\mathbf{M}}$ and integrating over $[0,t]\times\mathbf{R}_+\times\mathbf{R}^3$, we have from (1.15) that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}^2}{\mathbf{M}} d\xi dx \Big|_0^t + \sigma \int_0^t \int_0^\infty \int_{\mathbf{R}^3} \frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}} d\xi dx d\tau \\
&\le -\int_0^t \int_0^\infty \int_{\mathbf{R}^3} \frac{1}{R\theta} \frac{\tilde{\mathbf{G}}}{\mathbf{M}} \mathbf{P}_1\left[\xi_1\left(\frac{|\xi-u|^2}{2\theta}\tilde\theta_x + \xi\cdot\tilde u_x\right)\mathbf{M}\right] d\xi dx d\tau \\
&\quad - \int_0^t \int_0^\infty \int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}^2}{\mathbf{M}^2}\mathbf{M}_t d\xi dx d\tau - \int_0^t \int_0^\infty \int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}}{\mathbf{M}} \mathbf{P}_1\left(\xi_1\mathbf{G}_x\right) d\xi dx d\tau \\
&\quad - \int_0^t \int_0^\infty \int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}}{\mathbf{M}} \overline{\mathbf{G}}_t d\xi dx d\tau + \int_0^t \int_0^\infty \int_{\mathbf{R}^3} \frac{\tilde{\mathbf{G}}}{\mathbf{M}} Q(\mathbf{G},\mathbf{G}) d\xi dx d\tau \\
&= \sum_{j=11}^{15} I_j
\end{aligned} \tag{3.27}$$

Similar to the case of the Cauchy problem for hard sphere model considered in [25], $I_j (j = 11, 12, 14, 15)$ can be estimated as follows.

$$\left|I_{12}\right| \leq O(1)\Big(\delta_0 + t_0^{-1}\Big)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dx d\tau, \tag{3.28}$$

$$\left|I_{11}\right| \leq \frac{\sigma}{5}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dx d\tau + O(1)\int_0^t\int_0^\infty\left|(\tilde{u}_x, \tilde{\theta}_x)\right|^2 dx d\tau, \tag{3.29}$$

$$\left|I_{15}\right| \leq \frac{\sigma}{5}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dx d\tau + O(1)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}}d\xi dx d\tau, \tag{3.30}$$

$$\begin{aligned}\left|I_{14}\right| \leq{}& \tfrac{\sigma}{5}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dx d\tau + O(1)t_0^{-\frac{1}{4}}\\ &+O(1)t_0^{-1}\int_0^t\int_0^\infty\left(|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)|^2 + \int_{\mathbf{R}^3}\tfrac{\mathbf{G}_x^2}{\mathbf{M}}d\xi\right)dx d\tau.\end{aligned} \tag{3.31}$$

The estimate on I_{13} is different because we only assume that $q(V, \theta)$ satisfies (A1) and (A2). For this, first notice that

$$\mathbf{P}_1\Big(\xi_1\mathbf{G}_x\Big) = \xi_1\mathbf{G}_x - \sum_{\alpha=0}^{4}\langle\xi_1\mathbf{G}_x, \chi_\alpha\rangle\chi_\alpha = \xi_1\tilde{\mathbf{G}}_x - \sum_{\alpha=0}^{4}\langle\xi_1\mathbf{G}_x, \chi_\alpha\rangle\chi_\alpha + \xi_1\overline{\mathbf{G}}_x.$$

We have

$$\begin{aligned}I_{13} ={}& -\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\tilde{\mathbf{G}}}{\mathbf{M}}\left[\xi_1\tilde{\mathbf{G}}_x - \sum_{\alpha=0}^{4}\langle\xi_1\mathbf{G}_x, \chi_\alpha\rangle\chi_\alpha + \xi_1\overline{\mathbf{G}}_x\right]d\xi dx d\tau\\ ={}& \int_0^t\int_{\mathbf{R}^3}\left(\tfrac{\xi_1\tilde{\mathbf{G}}^2}{\mathbf{M}}\right)(\tau, 0, \xi)d\xi d\tau\\ &+\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\left[\left(\sum_{\alpha=0}^{4}\langle\xi_1\mathbf{G}_x, \chi_\alpha\rangle\chi_\alpha - \xi_1\overline{\mathbf{G}}_x\right)\tfrac{\tilde{\mathbf{G}}}{\mathbf{M}} - \tfrac{\xi_1\tilde{\mathbf{G}}^2}{\mathbf{M}^2}\mathbf{M}_t\right]d\xi dx d\tau\\ ={}& J_{13}^1 + J_{13}^2.\end{aligned} \tag{3.32}$$

Since

$$\langle\xi_1\mathbf{G}_x, \chi_\alpha\rangle \leq O(1)\left(\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi\right)^{\frac{1}{2}},$$

we get

$$\begin{aligned}\left|J_{13}^2\right| \leq{}& \tfrac{\sigma}{10}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dx d\tau + O(1)\Big(\delta_0 + t_0^{-1}\Big)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dx d\tau\\ &+O(1)t_0^{-\frac{1}{4}} + O(1)t_0^{-1}\int_0^t\int_0^\infty\left|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)\right|^2 dx d\tau\\ &+O(1)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_x^2}{\mathbf{M}}d\xi dx d\tau.\end{aligned} \tag{3.33}$$

Plugging $(3.10)_4$ and (3.33) into (3.32) yields

$$\begin{aligned}\left|I_{13}\right| \leq{}& \tfrac{\sigma}{10}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dx d\tau + O(1)\Big(\delta_0 + t_0^{-1}\Big)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dx d\tau\\ &+O(1)t_0^{-\frac{1}{4}} + O(1)t_0^{-1}\int_0^t\int_0^\infty\left|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)\right|^2 dx d\tau + O(1)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_x^2}{\mathbf{M}}d\xi dx d\tau.\end{aligned} \tag{3.34}$$

Substituting (3.28)-(3.31) and (3.34) into (3.27), we obtain

$$
\begin{aligned}
&\int_0^\infty\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dx\Big|_0^t+\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dxd\tau\\
&\le O(1)\Big(\delta_0+t_0^{-1}\Big)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dxd\tau+O(1)t_0^{-\frac14}\\
&\quad+O(1)t_0^{-1}\int_0^t\int_0^\infty\Big|\tilde\rho_x\Big|^2dxd\tau+O(1)\int_0^t\int_0^\infty\left[|(\tilde u_x,\tilde\theta_x)|^2+\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi\right]dxd\tau\\
&\quad+O(1)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}}d\xi dxd\tau.
\end{aligned}\tag{3.35}
$$

Similarly, using the weight $\mathbf{M}_-$ instead of $\mathbf{M}$, we have

$$
\begin{aligned}
&\int_0^\infty\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dx\Big|_0^t+\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\le O(1)\int_0^t\int_0^\infty\left[|(\tilde u_x,\tilde\theta_x)|^2+\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2+\nu(\xi)^{-1}Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}_-}d\xi\right]dxd\tau\\
&\quad+O(1)t_0^{-\frac14}+O(1)t_0^{-1}\int_0^t\int_0^\infty\Big|\tilde\rho_x\Big|^2dxd\tau.
\end{aligned}\tag{3.36}
$$

Since for $\mathbf{M}_i=\mathbf{M}_-$ or $\mathbf{M}$, Lemma 2.3 and (3.3) give

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}_i}d\xi\le O(1)\int_{\mathbf{R}^3}\frac{\nu(\xi)\mathbf{G}^2}{\mathbf{M}_i}d\xi\int_{\mathbf{R}^3}\frac{\mathbf{G}^2}{\mathbf{M}_i}d\xi\\
&\le O(1)\left[\int_{\mathbf{R}^3}\frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}_i}d\xi+\int_{\mathbf{R}^3}\frac{\nu(\xi)\overline{\mathbf{G}}^2}{\mathbf{M}_i}d\xi\right]\left[\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_i}d\xi+\int_{\mathbf{R}^3}\frac{\overline{\mathbf{G}}^2}{\mathbf{M}_i}d\xi\right]\\
&\le O(1)\int_{\mathbf{R}^3}\frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}_i}d\xi\left[\int_{\mathbf{R}^3}\frac{\mathbf{G}^2}{\mathbf{M}_i}d\xi+\int_{\mathbf{R}^3}\frac{\nu(\xi)\overline{\mathbf{G}}^2}{\mathbf{M}_i}d\xi\right]+O(1)\left[\int_{\mathbf{R}^3}\frac{\nu(\xi)\overline{\mathbf{G}}^2}{\mathbf{M}_i}d\xi\right]^2\\
&\le O(1)\Big(\delta_0+t_0^{-1}\Big)\int_{\mathbf{R}^3}\frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}_i}d\xi+O(1)\Big|(\overline u_{1x},\overline\theta_x)(\tau,x)\Big|^4,
\end{aligned}\tag{3.37}
$$

(3.23) and (3.24) follow immediately from (3.25), (3.35)-(3.37) and this completes the proof of Lemma 3.5.

Notice that in the above two estimates (3.23) and (3.24), the double integral of $\tilde\rho_x^2$ and $\tilde\rho_t^2$ are not included. In the following, we will show that they can be recovered from the system of conservation laws. For results in this direction, we have

Lemma 3.6 *Under the a priori assumption (3.3), we have*

$$
\begin{aligned}
\int_0^t\int_0^\infty|\tilde\rho_x|^2dxd\tau&\le O(1)\exp\Big(-d_1t_0\Big)+O(1)\left|\int_0^\infty\Big(\tilde\rho_x\tilde u_1\Big)(t)dx\Big|_0^t\right|\\
&\quad+O(1)\int_0^t\int_0^\infty\left[|(\tilde u_x,\tilde\theta_x)|^2+\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi\right]dxd\tau\\
&\quad+O(1)t_0^{-1}\int_0^t\int_0^\infty\overline u_{1x}|(\tilde\rho,\tilde u_1)|^2dxd\tau
\end{aligned}\tag{3.38}
$$

and

$$
\int_0^t\int_0^\infty|\tilde\rho_t|^2dxd\tau\le O(1)\int_0^t\int_0^\infty|(\tilde\rho_x,\tilde u_{1x})|^2dxd\tau+O(1)t_0^{-1}\int_0^t\int_0^\infty\bar u_{1x}|(\tilde\rho,\tilde u_1)|^2dxd\tau.\tag{3.39}
$$

Proof. From (1.11) and (iii) of Lemma 2.1, $\left(\tilde{\rho},\tilde{u},\tilde{\theta}\right)(t,x)$ solves

$$\begin{cases} \tilde{\rho}_t+(\tilde{\rho}\tilde{u}_1)_x=-H_1,\\ \tilde{u}_{1t}+\tilde{u}_1\tilde{u}_{1x}+\frac{2}{3}\tilde{\theta}_x+\frac{2\theta}{3\rho}\tilde{\rho}_x=-\int_{\mathbf{R}^3}\frac{|\xi_1|^2\mathbf{G}_x}{\rho}d\xi-H_2,\\ \tilde{u}_{2t}+\tilde{u}_1\tilde{u}_{2x}=-\int_{\mathbf{R}^3}\frac{\xi_1\xi_2\mathbf{G}_x}{\rho}d\xi-\overline{u}_1\tilde{u}_{2x},\\ \tilde{u}_{3t}+\tilde{u}_1\tilde{u}_{3x}=-\int_{\mathbf{R}^3}\frac{\xi_1\xi_3\mathbf{G}_x}{\rho}d\xi-\overline{u}_1\tilde{u}_{3x},\\ \tilde{\theta}_t+\tilde{u}_1\tilde{\theta}_x+\frac{2}{3}\tilde{\theta}\tilde{u}_{1x}=\int_{\mathbf{R}^3}\frac{\xi_1\left(\xi\cdot u-\frac{1}{2}|\xi|^2\right)}{\rho}\mathbf{G}_xd\xi-H_3. \end{cases}\tag{3.40}$$

Here

$$\begin{cases} H_1=\left(\overline{\rho}\tilde{u}_1+\overline{u}_1\tilde{\rho}\right)_x,\\ H_2=\tilde{u}_1\overline{u}_{1x}+\overline{u}_1\tilde{u}_{1x}+\frac{2}{3}\frac{\overline{\rho}\tilde{\theta}-\tilde{\rho}\overline{\theta}}{\rho\overline{\rho}}\overline{\rho}_x,\\ H_3=\frac{2}{3}\left(\overline{u}_{1x}\tilde{\theta}+\tilde{u}_{1x}\overline{\theta}\right)+\left(\tilde{u}_1\overline{\theta}_x+\overline{u}_1\tilde{\theta}_x\right). \end{cases}\tag{3.41}$$

Multiplying $(3.40)_2$ by $\tilde{\rho}_x$ and integrating with respect to t and x over $[0,t]\times\mathbf{R}_+$, we have

$$\begin{aligned} \int_0^t\int_0^\infty\frac{2\theta}{3\rho}\left|\tilde{\rho}_x\right|^2dxd\tau&=-\int_0^t\int_0^\infty\left[\tilde{u}_{1t}\tilde{\rho}_x+\tilde{\rho}_x\tilde{u}_1\tilde{u}_{1x}+\frac{2}{3}\tilde{\rho}_x\tilde{\theta}_x\right]dxd\tau\\ &\quad-\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\tilde{\rho}_x|\xi_1|^2\mathbf{G}_x}{\rho}d\xi dxd\tau-\int_0^t\int_0^\infty\tilde{\rho}_x\left[\tilde{u}_1\overline{u}_{1x}+\overline{u}_1\tilde{u}_{1x}+\frac{2}{3}\frac{\overline{\rho}\tilde{\theta}-\tilde{\rho}\overline{\theta}}{\rho\overline{\rho}}\overline{\rho}_x\right]dxdt\\ &=\sum_{i=16}^{18}I_i. \end{aligned}\tag{3.42}$$

For this, we have

$$\left|I_{17}\right|\leq\lambda\int_0^t\int_0^\infty\left|\tilde{\rho}_x\right|^2dxd\tau+O(1)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi dxd\tau,\tag{3.43}$$

$$\begin{aligned} \left|I_{18}\right|\leq\lambda\int_0^t\int_0^\infty\left|\tilde{\rho}_x\right|^2dxd\tau+O(1)\int_0^t\int_0^\infty\left|\tilde{u}_x\right|^2dxd\tau\\ +O(1)t_0^{-1}\int_0^t\int_0^\infty\overline{u}_{1x}\left|(\tilde{\rho},\tilde{u}_1,\tilde{\theta})\right|^2dxd\tau. \end{aligned}\tag{3.44}$$

As to I_{16}, since

$$\begin{aligned} \tilde{u}_{1t}\tilde{\rho}_x+\tilde{\rho}_x\tilde{u}_1\tilde{u}_{1x}+\frac{2}{3}\tilde{\rho}_x\tilde{\theta}_x&=\left[\tilde{u}_1\tilde{\rho}_x\right]_t+\left\{\overline{u}_1\left(\overline{\rho}\tilde{u}_1+\overline{u}_1\tilde{\rho}\right)_x-\tilde{u}_1\left(\tilde{\rho}\tilde{u}_1)_x\right\}_x\\ &\quad-\tilde{\rho}\left|\tilde{u}_{1x}\right|^2+\frac{2}{3}\tilde{\rho}_x\tilde{\theta}_x-\tilde{u}_{1x}\left(\overline{\rho}\tilde{u}_1+\tilde{\rho}\overline{u}_1\right)_x, \end{aligned}$$

we have

$$\begin{aligned} I_{16}&=-\int_0^\infty\left(\tilde{\rho}_x\tilde{u}_1\right)(t)dt\Big|_0^t+\int_0^t\left[\overline{u}_1\left(\overline{\rho}\tilde{u}_1+\tilde{\rho}\overline{u}_1\right)_x-\tilde{u}_1\left(\tilde{\rho}\tilde{u}_1\right)_x\right](\tau,0)d\tau\\ &\quad+\int_0^t\int_0^\infty\left\{\tilde{\rho}|\tilde{u}_{1x}|^2-\frac{2}{3}\tilde{\rho}_x\tilde{\theta}_x+\tilde{u}_{1x}\left(\overline{\rho}\tilde{u}_1+\tilde{\rho}\overline{u}_1\right)_x\right\}dxd\tau\\ &=\sum_{j=1}^3J_{16}^j. \end{aligned}\tag{3.45}$$

For J^3_{16}, we have

$$\begin{aligned}\left|J^3_{16}\right| \leq \lambda\int_0^t\int_0^\infty |\tilde\rho_x|^2 dxd\tau + O(1)\int_0^t\int_0^\infty |(\tilde u_x,\tilde\theta_x)|^2 dxd\tau \\ +O(1)t_0^{-1}\int_0^t\int_0^\infty \overline{u}_{1x}|(\tilde\rho,\tilde u_1)|^2 dxd\tau.\end{aligned} \tag{3.46}$$

As for J^2_{16}, since

$$\begin{aligned}\Big[\overline{u}_1\Big(\overline{\rho}\tilde u_1+\tilde\rho\overline{u}_1\Big)_x-\tilde u_1\Big(\tilde\rho\tilde u_1\Big)_x\Big](\tau,0) &= \Big[\overline{u}_1\Big(\rho u_{1x}-\overline{\rho}_x\overline{u}_1-\overline{\rho u}_{1x}\Big)\Big](\tau,0)\\ &\leq O(1)\exp\Big(-d_1(\tau+t_0)\Big),\end{aligned}$$

we set

$$\left|J^2_{16}\right| \leq O(1)\int_0^t \exp\Big(-d_1(\tau+t_0)\Big)d\tau \leq O(1)\exp\Big(-d_1t_0\Big). \tag{3.47}$$

Combining (3.45), (3.46) and (3.47) gives

$$\begin{aligned}\left|I_{16}\right| \leq O(1)\left|\int_0^\infty\Big(\tilde\rho_x\tilde u_1\Big)(t)dx\Big|_0^t\right| + \lambda\int_0^t\int_0^\infty|\tilde\rho_x|^2dxd\tau \\ +O(1)\exp\Big(-d_1t_0\Big)+O(1)t_0^{-1}\int_0^t\int_0^\infty\overline{u}_{1x}|(\tilde\rho,\tilde u_1)|^2dxd\tau \\ +O(1)\int_0^t\int_0^\infty|(\tilde u_x,\tilde\theta_x)|^2dxd\tau.\end{aligned} \tag{3.48}$$

By putting (3.43),(3.44) and (3.48) into (3.42), we can deduce (3.38) immediately and (3.39) can be proved similarly. This completes the proof of Lemma 3.6.

(3.23)-(3.24) and (3.38)-(3.39) give the complete lower order estimates.

3.3 Higher Order Estimates

In this subsection, we will consider the higher order energy estimates of $\partial^\alpha\mathbf{M},\partial^\alpha\mathbf{G}$, and $\partial^\beta f$ for $|\alpha|=1,|\beta|=2$ with respect to both the local Maxwellian $\mathbf{M}=\mathbf{M}_{[\rho(t,x),u(t,x),\theta(t,x)]}$ and the global Maxwellian $\mathbf{M}_-=\mathbf{M}_{[\rho_-,u_-,\theta_-]}$.

For estimates on $\partial^\alpha\mathbf{M}$ with $\alpha=(1,0)$ or $\alpha=(0,1)$, applying $\mathbf{P}_0$ to (1.10) gives

$$\mathbf{M}_t+\mathbf{P}_0\Big(\xi_1\mathbf{M}_x\Big)+\mathbf{P}_0\Big(\xi_1\mathbf{G}_x\Big)=0 \tag{3.49}$$

and we can get that

Lemma 3.7 *Under the a priori assumption (3.3), we have for $|\alpha|=1$ that*

$$\begin{aligned}&\int_0^\infty\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{M}|^2}{\mathbf{M}}d\xi dx\Big|_0^t+\int_0^t\int_0^\infty|\partial^\alpha(u_x,\theta_x)|^2dxd\tau\leq O(1)t_0^{-\frac14}\\ &+O(1)\left(\lambda+\delta_0^{\frac12}+t_0^{-\frac12}\right)\int_0^t\int_0^\infty\left[|\partial_x(\tilde\rho,\tilde u,\tilde\theta)|^2+|\partial_{xx}(\rho,u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)(\tilde{\mathbf{G}}^2+\mathbf{G}_x^2+\mathbf{G}_t^2)}{\mathbf{M}}d\xi\right]dxd\tau\\ &+O(1)\left(1+\frac1\lambda\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_{xx}^2+\mathbf{G}_{xt}^2}{\mathbf{M}}d\xi dxd\tau+O(1)\Big(\delta_0^{\frac12}+t_0^{-\frac12}\Big)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}_-}d\xi dxd\tau.\end{aligned} \tag{3.50}$$

Proof. Applying ∂^α to (3.49) and integrating its product with $\frac{\partial^\alpha \mathbf{M}}{\mathbf{M}}$ over $[0,t]\times\mathbf{R}_+\times\mathbf{R}^3$ give

$$\begin{aligned}
&\tfrac{1}{2}\int_0^\infty\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{M}|^2}{\mathbf{M}}d\xi dx\Big|_0^t = -\tfrac{1}{2}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{M}|^2}{\mathbf{M}^2}\mathbf{M}_t d\xi dx d\tau\\
&\quad -\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\partial^\alpha\mathbf{M}}{\mathbf{M}}\partial^\alpha\Big[\mathbf{P}_0(\xi_1\mathbf{M}_x)\Big]d\xi dx d\tau-\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\partial^\alpha\mathbf{M}}{\mathbf{M}}\partial^\alpha\Big[\mathbf{P}_0(\xi_1\mathbf{G}_x)\Big]d\xi dx d\tau\\
&=\sum_{j=19}^{21}I_j.
\end{aligned}\tag{3.51}$$

Now we estimate $I_j (j=19,20,21)$ term by term as follows. First, we have

$$\begin{aligned}
\Big|I_{19}\Big| &\le O(1)\int_0^t\int_0^\infty|\partial^\alpha(\rho,u,\theta)|^2|(\rho_t,u_t,\theta_t)|dxd\tau\\
&\le O(1)t_0^{-\frac14}+O(1)\Big(\delta_0+t_0^{-1}\Big)\int_0^t\int_0^\infty\left[\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi+|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)|^2\right]dxd\tau.
\end{aligned}\tag{3.52}$$

As to I_{20}, noticing that $\partial^\alpha\mathbf{M}\in N$ for $|\alpha|=1$, we have from $(3.11)_2$ and (3.3) that

$$\begin{aligned}
I_{20} &= -\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\partial^\alpha\mathbf{M}}{\mathbf{M}}\xi_1\partial^\alpha\mathbf{M}_x d\xi dx d\tau\\
&=\tfrac12\int_0^t\int_{\mathbf{R}^3}\left(\frac{\xi_1|\partial^\alpha\mathbf{M}|^2}{\mathbf{M}}\right)(\tau,0,\xi)d\xi d\tau-\tfrac12\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\xi_1|\partial^\alpha\mathbf{M}|^2}{\mathbf{M}^2}\mathbf{M}_x d\xi dx d\tau\\
&\le\Big(\lambda+\delta_0^{\frac12}+t_0^{-\frac12}\Big)\int_0^t\int_0^\infty\left[|\partial_x(\tilde\rho,\tilde u,\tilde\theta)|^2+|\partial_{xx}(\rho,u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}}d\xi\right]dxd\tau\\
&\quad+O(1)t_0^{-\frac14}+\frac{O(1)}{\lambda}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_{xx}^2+\mathbf{G}_{xt}^2}{\mathbf{M}}d\xi dx d\tau\\
&\quad+O(1)\Big(\delta_0^{\frac12}+t_0^{-\frac12}\Big)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}_-}d\xi dx d\tau.
\end{aligned}\tag{3.53}$$

Similarly

$$\begin{aligned}
I_{21} &= -\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\partial^\alpha\mathbf{M}}{\mathbf{M}}\xi_1\partial^\alpha\mathbf{G}_x d\xi dx d\tau\\
&=\int_0^t\int_{\mathbf{R}^3}\left(\frac{\xi_1\partial^\alpha\mathbf{M}\partial^\alpha\mathbf{G}}{\mathbf{M}}\right)(\tau,0,\xi)d\xi d\tau-\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\xi_1\partial^\alpha\mathbf{M}\partial^\alpha\mathbf{G}}{\mathbf{M}^2}\mathbf{M}_x d\xi dx d\tau\\
&\quad+\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\xi_1\partial^\alpha\mathbf{M}_x\partial^\alpha\mathbf{G}}{\mathbf{M}}d\xi dx d\tau\\
&=\sum_{i=1}^3 J_{21}^i
\end{aligned}\tag{3.54}$$

$(3.11)_4$ implies that

$$\begin{aligned}
\Big|J_{21}^1\Big| &\le\Big(\lambda+\delta_0^{\frac12}+t_0^{-\frac12}\Big)\int_0^t\int_0^\infty\left[|\partial_x(\tilde\rho,\tilde u,\tilde\theta)|^2+|\partial_{xx}(\rho,u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}}d\xi\right]dxd\tau\\
&\quad+O(1)t_0^{-\frac14}+\frac{O(1)}{\lambda}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_{xx}^2+\mathbf{G}_{xt}^2}{\mathbf{M}}d\xi dx d\tau\\
&\quad+O(1)\Big(\delta_0^{\frac12}+t_0^{-\frac12}\Big)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_t^2+\mathbf{G}_x^2}{\mathbf{M}_-}d\xi dx d\tau.
\end{aligned}\tag{3.55}$$

Moreover, we have

$$\begin{aligned}
\Big|J_{21}^2\Big| &\le O(1)\int_0^t\int_0^\infty|(\rho_x,u_x,\theta_x)||\partial^\alpha(\rho,u,\theta)|\left(\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi\right)^{\frac12}dxd\tau\\
&\le O(1)t_0^{-\frac14}+O(1)\Big(\delta_0+t_0^{-1}\Big)\int_0^t\int_0^\infty\left[|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)|^2+\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi\right]dxd\tau.
\end{aligned}\tag{3.56}$$

Now we turn to estimate J_{21}^3. Noticing (1.13), we have

$$
\begin{aligned}
J_{21}^3 &= \int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\xi_1\partial^\alpha\mathbf{M}_x}{\mathbf{M}}\partial^\alpha\Big[L_{\mathbf{M}}^{-1}(\xi_1\mathbf{M}_x)+L_{\mathbf{M}}^{-1}\left(\mathbf{G}_t+\mathbf{P}_1(\xi_1\mathbf{G}_x)-Q(\mathbf{G},\mathbf{G})\right)\Big]d\xi dxd\tau\\
&= \int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{P}_1(\xi_1\partial^\alpha\mathbf{M}_x)}{\mathbf{M}}\partial^\alpha\Big[L_{\mathbf{M}}^{-1}(\xi_1\mathbf{M}_x)+L_{\mathbf{M}}^{-1}\left(\mathbf{G}_t+\mathbf{P}_1(\xi_1\mathbf{G}_x)-Q(\mathbf{G},\mathbf{G})\right)\Big]d\xi dxd\tau\\
&\leq -\tfrac{2}{3}\sigma\int_0^t\int_0^\infty|\partial^\alpha(u_x,\theta_x)|^2dxd\tau+O(1)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_{xx}^2+\mathbf{G}_{xt}^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)t_0^{-\frac14}+O(1\Big(\delta_0+t_0^{-1}\Big)\int_0^t\int_0^\infty\Big[|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)|^2+\int_{\mathbf{R}^3}\tfrac{\tilde{\mathbf{G}}^2+\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}}d\xi\Big]dxd\tau.
\end{aligned}
\tag{3.57}
$$

Substituting (3.55)-(3.57) into (3.54) deduce

$$
\begin{aligned}
&I_{21}\\
&\leq -\tfrac{2}{3}\sigma\int_0^t\int_0^\infty|\partial^\alpha(u_x,\theta_x)|^2dxd\tau+O(1)t_0^{-\frac14}+O(1)\left(1+\tfrac{1}{\lambda}\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_{xx}^2+\mathbf{G}_{xt}^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+\Big(\lambda+\delta_0^{\frac12}+t_0^{-\frac12}\Big)\int_0^t\int_0^\infty\Big[|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)|^2+|(\rho_{xx},u_{xx},\theta_{xx})|^2+\int_{\mathbf{R}^3}\tfrac{\tilde{\mathbf{G}}^2+\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}}d\xi\Big]dxd\tau\\
&\quad+O(1)\Big(\delta_0^{\frac12}+t_0^{-\frac12}\Big)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}_-}d\xi dxd\tau.
\end{aligned}
\tag{3.58}
$$

Putting (3.51), (3.52), (3.53) and (3.58) together, we can get (3.50) immediately and this completes the proof of Lemma 3.7.

For the corresponding estimates on $\partial^\alpha\mathbf{G}$ with $\alpha=(1,0)$ or $(0,1)$, compared with the Cauchy problem for the hard sphere model considered in [25], the main difference is on the estimate on the following term coming from the assumption on $q(V,\theta)$.

$$
I_{22}=\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\partial^\alpha\mathbf{G}\partial^\alpha[\mathbf{P}_1(\xi_1\mathbf{G}_x)]}{\mathbf{M}}d\xi dxd\tau. \tag{3.59}
$$

To deal with (3.59), noticing

$$
\partial^\alpha\Big(\mathbf{P}_1h\Big)=\mathbf{P}_1(\partial^\alpha h)-\sum_{j=0}^4\Big[\langle h,\partial^\alpha\chi_j\rangle\chi_j+\langle h,\chi_j\rangle\partial^\alpha\chi_j\Big],
$$

we have from $(3.11)_3$ that

$$
\begin{aligned}
I_{23} &= \int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\partial^\alpha\mathbf{G}\Big\{\xi_1\partial^\alpha\mathbf{G}_x-\sum\limits_{j=0}^4\Big[\langle\xi_1G_x,\partial^\alpha\chi_j\rangle\chi_j+\langle\xi_1\mathbf{G}_x,\chi_j\rangle\partial^\alpha\chi_j\Big]\Big\}}{\mathbf{M}}d\xi dxd\tau\\
&= -\tfrac12\int_0^t\int_{\mathbf{R}^3}\Big(\tfrac{\xi_1|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}\Big)(\tau,0,\xi)d\xi d\tau+\tfrac12\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\xi_1|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}^2}\mathbf{M}_td\xi dxd\tau\\
&\quad-\sum_{j=0}^4\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\partial^\alpha\mathbf{G}\Big[\langle\xi_1\mathbf{G}_x,\partial^\alpha\chi_j\rangle\chi_j+\langle\xi_1\mathbf{G}_x,\chi_j\rangle\partial^\alpha\chi_j\Big]}{\mathbf{M}}d\xi dxd\tau\\
&\leq \lambda\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau+O(1)\left(1+\tfrac{1}{\lambda}\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_{xx}^2+\mathbf{G}_{tx}^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+\Big(\lambda+\delta_0^{\frac12}+t_0^{-\frac12}\Big)\int_0^t\int_0^\infty\Big[|\partial_x(\tilde\rho,\tilde u,\tilde\theta)|^2+|\partial_{xx}(\rho,u,\theta)|^2+\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}}d\xi\Big]dxd\tau\\
&\quad+O(1)\Big(\delta_0^{\frac12}+t_0^{-\frac12}\Big)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}_-}d\xi dxd\tau.
\end{aligned}
\tag{3.60}
$$

Thus, we can deduce that

Lemma 3.8 *Under the a priori assumption (3.3), we have for $|\alpha|=1$ that*

$$
\begin{aligned}
&\int_0^\infty\int_{\mathbf{R}^3}\tfrac{|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx\Big|_0^t+\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\nu(\xi)|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\le O(1)t_0^{-\frac14}\\
&+\Big(\lambda+\delta_0^{\frac12}+t_0^{-\frac12}\Big)\int_0^t\int_0^\infty\left[|\partial_x(\tilde\rho,\tilde u,\tilde\theta)|^2+|\partial_{xx}(\rho,u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)\Big(\tilde{\mathbf{G}}^2+\mathbf{G}_x^2+\mathbf{G}_t^2\Big)}{\mathbf{M}}d\xi\right]dxd\tau\\
&+O(1)\int_0^t\int_0^\infty\Big|\partial^\alpha(u_x,\theta_x)\Big|^2dxd\tau+O(1)\left(1+\tfrac1\lambda\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_{xx}^2+\mathbf{G}_{xt}^2}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)\Big(\delta_0^{\frac12}+t_0^{-\frac12}\Big)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_x^2+\mathbf{G}_t^2}{\mathbf{M}_-}d\xi dxd\tau.
\end{aligned}
\tag{3.61}
$$

To obtain the 2-th order derivatives with respect to x and/or t on $\mathbf{G}$, we need to work on the original Boltzmann equation (1.1) to avoid the appearance of the 3-th oeder derivatives. This can be summarized in the following lemma.

Lemma 3.9 *Under the a priori assumption (3.3), we have for each $|\beta|=2$ that*

$$
\begin{aligned}
&\int_0^\infty\int_{\mathbf{R}^3}\tfrac{|\partial^\beta f|^2}{\mathbf{M}}d\xi dx\Big|_0^t+\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\nu(\xi)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\le O(1)t_0^{-\frac14}+O(1)\Big(\delta_0+t_0^{-1}\Big)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{|\partial^2 f|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&+O(1)\Big(\delta_0+t_0^{-1}\Big)\int_0^t\int_0^\infty\Bigg[|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)|^2+\sum_{|\alpha|=1}|\partial^\alpha(\rho_x,u_x,\theta_x)|^2\\
&+\int_{\mathbf{R}^3}\frac{\nu(\xi)\left[\tilde{\mathbf{G}}^2+\sum\limits_{|\alpha|=1}^{2}|\partial^\alpha\mathbf{G}|^2\right]}{\mathbf{M}}d\xi\Bigg]dxd\tau.
\end{aligned}
\tag{3.62}
$$

Proof. Applying ∂^β with $|\beta|=2$ to (1.1) and integrating its product with $\frac{\partial^\beta f}{\mathbf{M}}$ over $[0,t]\times\mathbf{R}_+\times\mathbf{R}_0^3$, we have from (3.8) that

$$
\begin{aligned}
&\tfrac12\int_0^\infty\int_{\mathbf{R}^3}\tfrac{|\partial^\beta f|^2}{\mathbf{M}}d\xi dx\Big|_0^t=-\tfrac12\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{|\partial^\beta f|^2}{\mathbf{M}^2}\Big[\mathbf{M}_t+\xi_1\mathbf{M}_x\Big]d\xi dxd\tau\\
&+\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\partial^\beta f\partial^\beta\Big(L_{\mathbf{M}}\mathbf{G}\Big)}{\mathbf{M}}d\xi dxd\tau+\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\partial^\beta f\partial^\beta\Big[Q(\mathbf{G},\mathbf{G})\Big]}{\mathbf{M}}d\xi dxd\tau\\
&=\sum_{j=23}^{25}I_j.
\end{aligned}
\tag{3.63}
$$

First,

$$
\Big|I_{23}\Big|\le O(1)\Big(\delta_0+t_0^{-1}\Big)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{|\partial^\beta f|^2}{\mathbf{M}_-}d\xi dxd\tau.
\tag{3.64}
$$

As to I_{24}, since

$$
\begin{aligned}
I_{24}&=\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\partial^\beta\mathbf{M})\partial^\beta\Big(L_{\mathbf{M}}G\Big)}{\mathbf{M}}d\xi dxd\tau+\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\partial^\beta\mathbf{G}\partial^\beta\Big(L_{\mathbf{M}}\mathbf{G}\Big)}{\mathbf{M}}d\xi dxd\tau\\
&=J_{24}^1+J_{24}^2,
\end{aligned}
\tag{3.65}
$$

we have

$$
\begin{aligned}
J_{24}^1 \leq\ & O(1)t_0^{-\frac{1}{4}} + O(1)\Big(\delta_0 + t_0^{-1}\Big)\int_0^t\int_0^\infty\Bigg[|\partial_x(\tilde{\rho},\tilde{u},\tilde{\theta})|^2 + |\partial_{xx}(\rho,u,\theta)|^2 \\
& + \int_{\mathbf{R}^3}\frac{\nu(\xi)\Big[\tilde{\mathbf{G}}^2+\mathbf{G}_x^2+\mathbf{G}_t^2+\mathbf{G}_{xx}^2+\mathbf{G}_{xt}^2\Big]}{\mathbf{M}}d\xi\Bigg]dxd\tau,
\end{aligned}
\tag{3.66}
$$

and

$$
\begin{aligned}
J_{24}^2 =\ & \int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\partial^\beta\mathbf{G}L_{\mathbf{M}}\Big(\partial^\beta\mathbf{G}\Big)}{\mathbf{M}}d\xi dxd\tau \\
& + O(1)\sum_{|\alpha|=1}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{|\partial^\beta\mathbf{G}|\Big[|Q(\partial^\alpha\mathbf{G},\partial^{\beta-\alpha}\mathbf{M})|+|Q(\mathbf{G},\partial^\beta\mathbf{M})|\Big]}{\mathbf{M}}d\xi dxd\tau \\
\leq\ & -\frac{\sigma}{2}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau \\
& + O(1)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}\Bigg[\sum\limits_{|\alpha|=1}\big|Q(\partial^\alpha\mathbf{G},\partial^{\beta-\alpha}\mathbf{M})\big|^2+\big|Q(\mathbf{G},\partial^\beta\mathbf{M})\big|^2\Bigg]}{\mathbf{M}}d\xi dxd\tau \\
\leq\ & -\frac{\sigma}{2}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau + O(1)t_0^{-\frac{1}{4}} \\
& + O(1)\Big(\delta_0 + t_0^{-1}\Big)\int_0^t\int_0^\infty\Bigg[|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)|^2 + |(\rho_{xx},u_{xx},\theta_{xx})|^2 + |(\rho_{xt},u_{xt},u_{xt})|^2 \\
& + \int_{\mathbf{R}^3}\frac{\nu(\xi)\Bigg[|\tilde{\mathbf{G}}|^2+\sum\limits_{|\alpha|=1}^{2}|\partial^\alpha\mathbf{G}|^2\Bigg]}{\mathbf{M}}d\xi\Bigg]dxd\tau.
\end{aligned}
\tag{3.67}
$$

Consequently

$$
\begin{aligned}
I_{24} \leq\ & -\frac{\sigma}{2}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau + O(1)t_0^{-\frac{1}{4}} \\
& + O(1)\Big(\delta_0 + t_0^{-1}\Big)\int_0^t\int_0^\infty\Bigg[|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)|^2 + \sum_{|\alpha|=1}|\partial^\alpha(\rho_x,u_x,\theta_x)|^2 \\
& + \int_{\mathbf{R}^3}\frac{\nu(\xi)\Bigg[|\tilde{\mathbf{G}}|^2+\sum\limits_{|\alpha|=1}^{2}|\partial^\alpha\mathbf{G}|^2\Bigg]}{\mathbf{M}}d\xi\Bigg]dxd\tau.
\end{aligned}
\tag{3.68}
$$

Finally, we estimate I_{25} by first rewriting it as

$$
\begin{aligned}
I_{25} =\ & \int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{P}_1\Big(\partial^\beta\mathbf{M}\Big)\partial^\beta[Q(\mathbf{G},\mathbf{G})]}{\mathbf{M}}d\xi dxd\tau + \int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\partial^\beta\mathbf{G}\partial^\beta[Q(\mathbf{G},\mathbf{G})]}{\mathbf{M}}d\xi dxd\tau \\
=\ & J_{25}^1 + J_{25}^2.
\end{aligned}
\tag{3.69}
$$

Since

$$
\partial^\beta[Q(\mathbf{G},\mathbf{G})] = 2Q(\partial^\beta\mathbf{G},\mathbf{G}) + O(1)\sum_{\substack{|\alpha|=1\\ \alpha<\beta}} Q\Big(\partial^\alpha\mathbf{G},\partial^{\beta-\alpha}\mathbf{G}\Big),
$$

we have

$$\begin{aligned}
\left|J_{25}^{1}\right| \le O(1)&\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{|\mathbf{P}_1(\partial^\beta\mathbf{M})||Q(\partial^\beta\mathbf{G},\mathbf{G})|}{\mathbf{M}}d\xi dx d\tau\\
&+O(1)\sum_{0<\alpha<\beta}\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{|\mathbf{P}_1(\partial^\beta\mathbf{M})Q(\partial^\alpha\mathbf{G},\partial^{\beta-\alpha}\mathbf{G})|}{\mathbf{M}}d\xi dx d\tau\\
\le O(1)\int_0^t\int_0^\infty&\left(\sum_{|\alpha|=1}|\partial^\alpha(\rho,u,\theta)|^2\right)\left(\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}\left[|Q(\partial^\beta\mathbf{G},\mathbf{G})|^2+\sum\limits_{0<\alpha<\beta}|Q(\partial^\alpha\mathbf{G},\partial^{\beta-\alpha}\mathbf{G})|^2\right]}{\mathbf{M}}d\xi\right)^{\frac12}dxd\tau.
\end{aligned}\tag{3.70}$$

Since

$$\begin{aligned}
\int_{\mathbf{R}^3}\frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi &\le O(1)\int_0^\infty\int_{\mathbf{R}^3}\left|\left[\frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}}\right]_x\right|d\xi dx\\
&\le O(1)\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)|\tilde{\mathbf{G}}\tilde{\mathbf{G}}_x|}{\mathbf{M}}d\xi dx+O(1)\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)|\tilde{\mathbf{G}}^2}{\mathbf{M}^2}\left|\mathbf{M}_x\right|d\xi dx\\
&\le O(1)t_0^{-1}+O(1)\left(\delta_0+t_0^{-1}\right)\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)|\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dx+O(1)\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)[\tilde{\mathbf{G}}^2+\mathbf{G}_x^2]}{\mathbf{M}}d\xi dx
\end{aligned}$$

and

$$\sup_{x\in\mathbf{R}_+}\left(\sum_{|\alpha|=1}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi\right)\le O(1)\delta_0,$$

we deduce that

$$\begin{aligned}
&\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}|Q(\partial^\beta\mathbf{G},\mathbf{G})|^2}{\mathbf{M}}d\xi dx\\
&\le O(1)\int_0^\infty\left\{\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi\int_{\mathbf{R}^3}\frac{\mathbf{G}^2}{\mathbf{M}}d\xi+\int_{\mathbf{R}^3}\frac{\nu(\xi)\mathbf{G}^2}{\mathbf{M}}d\xi\int_{\mathbf{R}^3}\frac{|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi\right\}dx\\
&\le O(1)\delta_0\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx+O(1)\int_0^\infty\left\{\left[\int_{\mathbf{R}^3}\frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi+\int_{\mathbf{R}^3}\frac{\nu(\xi)\overline{\mathbf{G}}^2}{\mathbf{M}}d\xi\right]\int_{\mathbf{R}^3}\frac{|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi\right\}dx\\
&\le O(1)\left(\delta_0+t_0^{-1}\right)\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx+O(1)\sup_{x\in\mathbf{R}_+}\left[\int_{\mathbf{R}^3}\frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi\right]\int_0^\infty\int_{\mathbf{R}^3}\frac{|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx\\
&\le O(1)\left(\delta_0+t_0^{-1}\right)\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx+O(1)\delta_0\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)\left[\tilde{\mathbf{G}}^2+\mathbf{G}_x^2\right]}{\mathbf{M}}d\xi dx d\tau\\
&\le O(1)\left(\delta_0+t_0^{-1}\right)\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)\left[\tilde{\mathbf{G}}^2+\sum\limits_{|\alpha|=1}^{2}|\partial^\alpha\mathbf{G}|^2\right]}{\mathbf{M}}d\xi dx d\tau,
\end{aligned}\tag{3.71}$$

and

$$\begin{aligned}
&\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}|Q(\partial^\alpha\mathbf{G},\partial^{\beta-\alpha}\mathbf{G})|^2}{\mathbf{M}}d\xi dx\\
&\le O(1)\left(\delta_0+t_0^{-1}\right)\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)\left[\tilde{\mathbf{G}}^2+\sum\limits_{|\alpha|=1}^{2}|\partial^\alpha\mathbf{G}|^2\right]}{\mathbf{M}}d\xi dx.
\end{aligned}\tag{3.72}$$

Hence

$$
\begin{aligned}
\left|J_{25}^1\right| \le\, & O(1)\sum_{|\alpha|=1}\int_0^t\int_0^\infty |\partial^\alpha(\rho,u,\theta)|^4 dxd\tau \\
& +\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}\left[|Q(\partial^\beta\mathbf{G},\mathbf{G})|^2+\sum_{0<\alpha<\beta}|Q(\partial^\alpha\mathbf{G},\partial^{\beta-\alpha}\mathbf{G})|^2\right]}{\mathbf{M}}d\xi dxd\tau \\
\le\, & O(1)t_0^{-\frac{1}{4}}+O(1)\left(\delta_0+t_0^{-1}\right)\int_0^t\int_0^\infty\left[|\partial_x(\tilde\rho,\tilde u,\tilde\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)\left[\tilde{\mathbf{G}}^2+\sum_{|\alpha|=1}^{2}|\partial^\alpha\mathbf{G}|^2\right]}{\mathbf{M}}d\xi\right]dxd\tau.
\end{aligned}
\tag{3.73}
$$

Similarly, we have

$$
\begin{aligned}
\left|J_{25}^2\right| \le\, & \lambda\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}\left|\partial^\beta\mathbf{G}\right|^2}{\mathbf{M}}d\xi dxd\tau \\
& +O(1)\left(\delta_0+t_0^{-1}\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)\left[\tilde{\mathbf{G}}^2+\sum_{|\alpha|=1}^{2}\left|\partial^\alpha\mathbf{G}\right|^2\right]}{\mathbf{M}}d\xi dxd\tau.
\end{aligned}
\tag{3.74}
$$

Hence

$$
\begin{aligned}
\left|I_{25}\right| \le\, & O(1)t_0^{-\frac{1}{4}}+\lambda\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau \\
& +O(1)\left(\delta_0+t_0^{-1}\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)\left[\tilde{\mathbf{G}}^2+\sum_{|\alpha|=1}^{2}|\partial^\alpha\mathbf{G}|^2\right]}{\mathbf{M}}d\xi dxd\tau.
\end{aligned}
\tag{3.75}
$$

Substituting (3.64), (3.65), (3.75) into (3.63) yields (3.62) and the proof of Lemma 3.9 is completed.

By suitably linearly combining (3.50), (3.61), and (3.62), we have by choosing λ and δ_0 sufficiently small and t_0 sufficiently large that

Corollary 3.1 *Under the a priori assumption (3.3), we have*

$$
\begin{aligned}
& \sum_{|\alpha|=1}\int_0^\infty\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{M}|^2+|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx\Big|_0^t+\sum_{|\beta|=2}\int_0^\infty\int_{\mathbf{R}^3}\frac{|\partial^\beta f|^2|}{\mathbf{M}}d\xi dx\Big|_0^t \\
& +\int_0^t\int_0^\infty\left[\sum_{|\alpha|=1}|\partial^\alpha(u_x,\theta_x)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)\sum_{|\alpha|=1}^{2}|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi\right]dxd\tau \\
\le\, & O(1)t_0^{-\frac{1}{4}}+O(1)\left(\delta_0^{\frac{1}{2}}+t_0^{-\frac{1}{2}}\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)\sum_{|\alpha|=1}^{2}|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau \\
& +O(1)\left(\lambda+\delta_0^{\frac{1}{2}}+t_0^{-\frac{1}{2}}\right)\int_0^t\int_0^\infty\left(\sum_{|\alpha|=1}|\partial^\alpha\rho_x|^2+|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)\tilde{\mathbf{G}}^2}{\mathbf{M}}\right)dxd\tau.
\end{aligned}
\tag{3.76}
$$

To recover the estimate on $|\partial^\alpha\rho_x|^2$ with $|\alpha|=1$ in (3.76), similar to that of Lemma 3.6, we need to use the system of conservation law (1.11) again, cf [18] where the same technique was used. For results in this direction, we have

Lemma 3.10 *Under the a priori assumption (3.3), we have*

$$\begin{aligned}\int_0^t\int_0^\infty \rho_{xx}^2 dxd\tau \leq O(1)t_0^{-\frac14}+O(1)\int_0^t\int_0^\infty\left(|(u_{xx},u_{xt},\theta_{xx})|^2+\int_{\mathbf{R}^3}\frac{\mathbf{G}_{xx}^2}{\mathbf{M}}d\xi\right)dxd\tau\\ +O(1)\left(\delta_0+t_0^{-1}\right)\int_0^t\int_0^\infty\left(|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)|^2+\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi\right)dxd\tau\end{aligned}\tag{3.77}$$

and

$$\begin{aligned}\int_0^t\int_0^\infty \rho_{xt}^2 dxd\tau \leq O(1)\int_0^t\int_0^\infty(|\rho_{xx}|^2+|u_{xx}|^2)dxd\tau+O(1)t_0^{-\frac14}\\ +O(1)\left(\delta_0+t_0^{-1}\right)\int_0^t\int_0^\infty|(\tilde\rho_x,\tilde u_x)|^2dxd\tau.\end{aligned}\tag{3.78}$$

Proof. Since

$$\begin{cases}\rho_t+(\rho u_1)_x=0,\\ u_{1t}+u_1u_{1x}+\frac23\theta_x+\frac{2\theta}{3\rho}\rho_x+\int_{\mathbf{R}^3}\frac{\xi_1^2\mathbf{G}_x}{\rho}d\xi=0,\end{cases}\tag{3.79}$$

differentiating $(3.79)_2$ with respect to x and multiplying the resulting identity by ρ_{xx}, we have by integrating the final result with respect to t and x over $[0,t]\times\mathbf{R}_+$ that

$$\begin{aligned}\int_0^t\int_0^\infty\frac{2\theta}{3\rho}\rho_{xx}^2dxd\tau &= -\int_0^t\int_0^\infty\Big[\rho_{xx}u_{1xt}+\rho_{xx}(u_1u_{1x})_x\Big]dxd\tau\\ &\quad-\int_0^t\int_0^\infty\rho_{xx}\left[\frac23\theta_{xx}+\rho_x\left(\frac{2\theta}{3\rho}\right)_x\right]dxd\tau\\ &\quad-\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\rho_{xx}\left[\frac{\xi_1^2\mathbf{G}_{xx}}{\rho}-\frac{|\xi_1|^2\rho_x\mathbf{G}_x}{\rho^2}\right]d\xi dxd\tau\\ &=\sum_{j=26}^{28}I_j.\end{aligned}\tag{3.80}$$

From the Cauchy-Schwarz's inequality, we have

$$\begin{aligned}\Big|I_{26}\Big| &\leq \lambda\int_0^t\int_0^\infty\rho_{xx}^2dxd\tau+O(1)\int_0^t\int_0^\infty|(u_{xt},u_{xx})|^2dxd\tau+O(1)\int_0^t\int_0^\infty|u_{1x}|^4dxd\tau\\ &\leq O(1)t_0^{-1/4}+\lambda\int_0^t\int_0^\infty\rho_{xx}^2dxd\tau+O(1)\int_0^t\int_0^\infty|(u_{xt},u_{xx})|^2dxd\tau\\ &\quad+O(1)(\delta_0+t_0^{-1})\int_0^t\int_0^\infty|\tilde u_x|^2dxd\tau,\\ |I_{27}| &\leq \lambda\int_0^t\int_0^\infty\rho_{xx}^2dxd\tau+O(1)\int_0^t\int_0^\infty\Big[\theta_{xx}^2+|(\rho_x,u_x,\theta_x)|^4\Big]dxd\tau\\ &\leq O(1)t_0^{-1/4}+\lambda\int_0^t\int_0^\infty\rho_{xx}^2dxd\tau\\ &\quad+O(1)\left(\delta_0+t_0^{-1}\right)\int_0^t\int_0^\infty|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)|^2dxd\tau+O(1)\int_0^t\int_0^\infty|\theta_{xx}|^2dxd\tau,\end{aligned}$$

and

$$\Big|I_{28}\Big|\leq\lambda\int_0^t\int_0^\infty|\rho_{xx}|^2dxd\tau+\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_{xx}^2}{\mathbf{M}}d\xi dxd\tau+O(1)\left(\delta_0+t_0^{-1}\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi dxd\tau.$$

Substituting the above inequalities into (3.80), we can deduce (3.77) immediately.

As for ρ_{xt}, since

$$\rho_{xt}=-u\rho_{xx}-\rho u_{xx}-2\rho_xu_x,$$

(3.80) follows immediately from (3.77). This completes the proof of Lemma 3.10.

Based on (3.23), (3.38), (3.39), (3.76), (3.77), and (3.78), we have, cf. [25], that

Corollary 3.2 *Under the a priori assumption (3.3), we have*

$$\begin{aligned}&\int_0^\infty \eta(t)dx+\int_0^t\int_0^\infty\Big(|\partial_x(\tilde\rho,\tilde u,\tilde\theta)|^2+\overline{u}_{1x}|(\tilde\rho,\tilde u_1,\tilde\theta)|^2+\sum_{|\alpha|=1}|\partial^\alpha(\rho_x,u_x,\theta_x)|^2\Big)dxd\tau\\&+\int_0^\infty\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2+\sum\limits_{|\alpha|=1}(|\partial^\alpha\mathbf{M}|^2+|\partial^\alpha\mathbf{G}|^2)+\sum\limits_{|\alpha|=2}|\partial^\alpha f|^2}{\mathbf{M}}d\xi dx+\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\frac{\nu(\xi)\left[\tilde{\mathbf{G}}^2+\sum\limits_{|\alpha|=1}^{2}|\partial^\alpha\mathbf{G}|^2\right]}{\mathbf{M}}d\xi dxd\tau\\&\le O(1)\left(t_0^{-\frac14}+N(0)^2\right)+O(1)\Big(\delta_0+t_0^{-1}\Big)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\left(\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}+\sum_{|\alpha|=1}^{2}\frac{|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_-}\right)d\xi dxd\tau.\end{aligned}\tag{3.81}$$

A direct consequence of (3.81) is

$$\begin{aligned}&\int_0^t\int_0^\infty\left(|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)|^2+\sum_{|\alpha|=1}|\partial^\alpha(\rho_x,u_x,\theta_x)|^2\right)dxd\tau\\&\le O(1)\left(t_0^{-\frac14}+N(0)^2\right)+O(1)\left(\delta_0+t_0^{-1}\right)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\left(\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}+\sum_{|\alpha|=1}^{2}\frac{|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_-}\right)d\xi dxd\tau.\end{aligned}\tag{3.82}$$

Now we consider the energy estimates with respect to the global Maxwellian $\mathbf{M}_-$ and complete the proof of Theorem 3.1. The main difference here is that the fluid part $\mathbf{P}_0(\partial^\alpha\mathbf{M})$ and the non-fluid part $\mathbf{G}$ are no longer orthogonal with respect to $\mathbf{M}_-$, i.e.

$$\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\alpha\mathbf{M})\mathbf{G}}{\mathbf{M}_-}d\xi\neq 0.$$

As a result, there is an extra error term in the form of

$$O(1)\int_0^t\int_0^\infty\left(|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)|^2+\sum_{|\alpha|=1}|\partial^\alpha(\rho_x,u_x,\theta_x)|^2\right)dxd\tau.$$

However, due to (3.82), this term can be suitably controlled. Hence from $(3.11)_5$, we can deduce that (3.1) holds provided that the a priori assumption (3.3) is satisfied.

Based on (3.1), we can close the a priori estimate (3.3) by choosing $\delta_0>0$ sufficiently small and $t_0>0$ sufficiently large such that

$$\begin{cases}N(0)<\varepsilon_0\\ O(1)\Big(t_0^{-\frac14}+\varepsilon_0^2\Big)<\delta_0^2\end{cases}$$

and this completes the proof of Theorem 3.1.

4 The Proof of Theorem 1.1

4.1 Local Existence in $H_x^2\left(L^2_{\xi,\mathcal{M}}\right)$

In this subsection, we show how to construct local solution in the energy space $H_x^2\left(L^2_{\xi,\mathbf{M}_-}\right)$ to the initial boundary value problem (1.1)-(1.3). First, set

$$\mathcal{K}_i(\xi,\xi_*)=\sqrt{\frac{\mathbf{M}(\xi)}{\mathbf{M}_-(\xi_*)}}k_i(\xi,\xi_*)\sqrt{\frac{\mathbf{M}_-(\xi_*)}{\mathbf{M}(\xi)}},\quad i=1,2.\tag{4.1}$$

By using the explicit expression of $k_i(\xi,\xi_*)(i=1,2)$, we have the following lemma. Thus, we omit the proof for brevity.

Lemma 4.1 *If $\theta_- > \frac{\theta}{2} > 0$, then for $i = 1, 2$*

$$\begin{cases} \sup\limits_{\xi\in\mathbf{R}^3}\left\{\int_{\mathbf{R}^3}|\mathcal{K}_i(\xi,\xi_*)|d\xi_*\right\} \leq O(1), \\ \sup\limits_{\xi_*\in\mathbf{R}^3}\left\{\int_{\mathbf{R}^3}|\mathcal{K}_i(\xi,\xi_*)|d\xi\right\} \leq O(1). \end{cases} \tag{4.2}$$

To construct the local solution in $H^2_x\left(L^2_{\xi,\mathbf{M}_-}\right)$ to the initial boundary value problem (1.1)-(1.3), for each (t,x,ξ) with $t>0, x\geq 0$, as in [2], we first construct the backward characteristic line $(X(s;t,x,\xi), E(s;t,x,\xi))$ passing through (t,x,ξ), i.e.

$$\begin{cases} \frac{dX(s;t,x,\xi)}{ds} = E_1(s;t,x,\xi), \\ \frac{dE(s;t,x,\xi)}{ds} = 0, \end{cases} \tag{4.3}$$

$$(X(s;t,x,\xi), E(s;t,x,\xi))|_{s=t} = (x,\xi). \tag{4.4}$$

As in [2], if the characteristic line $(X(s;t,x,\xi), E(s;t,x,\xi))$ intersects with the boundary $x = 0$ at $s = s_0 \in (0,t)$, then we construct the backward characteristic line $(X(s;t,x,\xi), E(s;t,x,\xi))$ for $s < s_0$ by solving (4.3) with

$$(X(s;t,x,\xi), E(s;t,x,\xi))|_{s=s_0} = (0, RE(s_0;t,x,\xi)). \tag{4.5}$$

It is easy to show that

$$X(s;t,x,\xi) = \begin{cases} x+(s-t)\xi_1, & \text{if } \ x-\xi_1 t\geq 0, 0\leq s\leq t, x>0, t>0, \\ x+(s-t)\xi_1, & \text{if } \ x-\xi_1 t< 0, t-\frac{x}{\xi_1}\leq s\leq t, x>0, t>0, \\ -x-(s-t)\xi_1, & \text{if } \ x-\xi_1 t< 0, 0\leq s\leq t-\frac{x}{\xi_1}, x>0, t>0, \end{cases} \tag{4.6}$$

and

$$E(s;t,x,\xi) = \begin{cases} \xi, & x-\xi_1 t\geq 0, 0\leq s\leq t, x>0, t>0, \\ \xi, & x-\xi_1 t< 0, 0<t-\frac{x}{\xi_1}\leq s\leq t, x>0, t>0, \\ R\xi, & x-\xi_1 t< 0, 0\leq s\leq s_0=t-\frac{x}{\xi_1}, x>0, t>0. \end{cases} \tag{4.7}$$

Notice that $(X(s;t,x,\xi), E(s;t,x,\xi))$ has the following properties:

(i) $(X,E)(s;t,x,\xi)$ is piecewise Lipschitz continuous, $X(s;t,x,\xi)\geq 0, |E(s;t,x,\xi)| = |\xi|$;

(ii) The Jacobian determinant of $\frac{\partial(X,E)}{\partial(x,\xi)}$ equals to 1.

Now for any function $F(t,x,\xi)$, define the micro-macro decomposition as before and denoted by:

$$F(t,x,\xi) = \mathbf{M}_F(t,x,\xi) + \mathbf{G}_F(t,x,\xi) = \mathbf{M}_{[\rho^F(t,x),u^F(t,x),\theta^F(t,x)]} + \mathbf{G}_F(t,x,\xi),$$

where $\left(\rho^F(t,x), u^F(t,x), \theta^F(t,x)\right)$ and $\mathbf{G}_F(t,x,\xi)$ are the corresponding fluid and non-fluid components of F.

Define a set of functions:

$$\mathcal{X} = \left\{F(t,x,\xi)\in H^2_x\left(L^2_{\xi,\mathbf{M}_-}\right) : F(t,0,R\xi) = F(t,0,\xi), \quad |||F|||\leq 2\delta_1, \theta_- < \theta_F < 2\theta_-\right\},$$

where $\delta_1 > 0$ is a small constant and the norm $|||\cdot|||$ is defined as:

$$\begin{aligned}|||F|||^2 = &\int_0^\infty \Big|(\rho^F-\bar\rho, u^F-\bar u, \theta^F-\bar\theta)\Big|^2 dx + \int_0^t\int_0^\infty \Big(|\partial_x(\rho^F-\bar\rho, u^F-\bar u, \theta^F-\bar\theta)|^2\\
&+\bar u_{1x}\Big|(\rho^F-\bar\rho, u^F-\bar u, \theta^F-\bar\theta)\Big|^2 + \sum_{|\alpha|=1}\Big|\partial^\alpha\left(\rho^F_x,\theta^F_x,u^F_x\right)\Big|^2\Big)dxd\tau\\
&+\int_0^\infty\int_{\mathbf{R}^3}\tfrac{1}{\mathbf{M}_-}\Big(|\mathbf{G}_F-\overline{\mathbf{G}}|^2+\sum_{|\alpha=1}(|\partial^\alpha\mathbf{M}_F|^2+|\partial^\alpha\mathbf{G}_F|^2)+\sum_{|\alpha|=2}|\partial^\alpha F|^2\Big)d\xi dxd\tau\\
&+\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{1}{\mathbf{M}_-}\Big(|G_F-\overline{\mathbf{G}}|^2+\sum_{|\alpha|=1}^{2}|\partial^\alpha\mathbf{G}|^2\Big)d\xi dxd\tau.\end{aligned}\tag{4.8}$$

Here $\left(\overline{\rho},\overline{u},\overline{\theta}\right)$ and $\overline{\mathbf{G}}$ are the fluid and non-fluid components of $\mathbf{M}_-$.

For $F\in\mathcal{X}$, consider the linear equation

$$f_t+\xi_1 f_x = L_{\mathbf{M}_F}\mathbf{G}+Q(\mathbf{G}_F,\mathbf{G}_F),\tag{4.9}$$

$$\begin{cases} f(t,x,\xi)|_{t=0}=f_0(x,\xi),\\ f(t,0,R\xi)=f(t,0,\xi),\end{cases}\tag{4.10}$$

with $\mathbf{M}_{f_0}=\mathbf{M}_{[\rho_0,u_0,\theta_0]}$ satisfying $\theta_-<\inf\limits_x\theta_0(x)\le\sup\limits_x\theta_0(x)<2\theta_-$.

Let $f=\mathbf{M}+\mathbf{G}$ and $\tilde f=f-\overline{\mathbf{M}}$. Since $\mathbf{G}=f-\mathbf{M}=\tilde f+\left(\overline{\mathbf{M}}-\mathbf{M}\right)$, we have

$$\begin{aligned}\tilde f_t+\xi_1\tilde f_x+\nu(\xi)\tilde f = &-\Big[(\overline{\mathbf{M}}_t+\xi_1\overline{\mathbf{M}}_x)+\nu(\xi)(\overline{\mathbf{M}}-\mathbf{M})\Big]\\
&+\sqrt{\mathbf{M}_F}K_2\left(\tfrac{\mathbf{G}}{\sqrt{\mathbf{M}_F}}\right)-\sqrt{\mathbf{M}_F}K_1\left(\tfrac{\mathbf{G}}{\sqrt{\mathbf{M}_F}}\right)+Q(\mathbf{G}_F,\mathbf{G}_F).\end{aligned}\tag{4.11}$$

Noticing $\nu(\xi)=\nu(|\xi|)$ and $|E(s;t,x,\xi)|=|\xi|$, $\tilde f$ has the following expression:

$$\begin{aligned}\tilde f(t,x,\xi) = &\exp\Big(-\nu(\xi)t\Big)\tilde f_0\Big(X(0,t,x,\xi),E(0;t,x,\xi)\Big)\\
&-\int_0^t\exp\Big(-\nu(\xi)(t-\eta)\Big)\Big[(\overline{\mathbf{M}}_t+\xi_1\overline{\mathbf{M}}_x)+\nu(\xi)(\overline{\mathbf{M}}-\mathbf{M})\Big]\Big(\eta,X(\eta;t,x,\xi),E(\eta;t,x,\xi)\Big)d\eta\\
&+\int_0^t\exp\Big(-\nu(\xi)(t-\eta)\Big)\Big[\sqrt{\mathbf{M}_F}K_2\left(\tfrac{\mathbf{G}}{\sqrt{\mathbf{M}_F}}\right)-\sqrt{\mathbf{M}_F}K_1\left(\tfrac{\mathbf{G}}{\sqrt{\mathbf{M}_F}}\right)\Big]\Big(\eta,X(\eta;t,x,\xi),E(\eta;t,x,\xi)\Big)d\eta\\
&+\int_0^t\exp\Big(-\nu(\xi)(t-\eta)\Big)\Big[Q(\mathbf{G}_F,\mathbf{G}_F)\Big]\Big(\eta,X(\eta;t,x,\xi),E(\eta;t,x,\xi)\Big)d\eta=\sum_{j=29}^{32}I_j.\end{aligned}\tag{4.12}$$

Since $\mathbf{M}_-(\xi)=\mathbf{M}_-(E(s;t,x,\xi))$, we can easily deduce from (4.2) that if $\frac{\theta}{2}<\theta_-<\theta,\theta_->\frac{\overline{\theta}}{2},\frac{\theta_F}{2}<\theta_-<\theta_F$

$$\int_0^\infty\int_{\mathbf{R}^3}\frac{|I_{29}|^2}{\mathbf{M}_-(\xi)}d\xi dx\le O(1)\int_0^\infty\int_{\mathbf{R}^3}\frac{|\tilde f_0|^2}{\mathbf{M}_-}d\xi dx,\tag{4.13}$$

$$\int_0^\infty\int_{\mathbf{R}^3}\frac{|I_{30}|^2}{\mathbf{M}_-(\xi)}d\xi dx\le O(1)\int_0^t\int_0^\infty\left[|(\overline{\rho}_x,\overline{u}_x,\overline{\theta}_x)|^2+\int_{\mathbf{R}^3}\frac{|\overline{\mathbf{M}}-\mathbf{M}|^2}{\mathbf{M}_-}d\xi\right]dxd\tau,\tag{4.14}$$

$$\begin{aligned}&\int_0^\infty\int_{\mathbf{R}^3}\tfrac{|I_{31}|^2}{\mathbf{M}_-(\xi)}d\xi dx\\
&\le\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{M}_F(\xi)}{\mathbf{M}_-(\xi)}\Big|\int_0^t\left(\int_{\mathbf{R}^3}\Big(k_2(\xi;\xi_*)-k_1(\xi;\xi_*)\Big)\tfrac{\mathbf{G}(\xi_*)}{\sqrt{\mathbf{M}_F(\xi_*)}}d\xi_*\right)(\tau,X(\tau),E(\tau))d\tau\Big|^2d\xi dx\\
&\le O(1)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\left(\int_{\mathbf{R}^3}\sum_{i=1}^2|G_i(\xi,\xi_*)|d\xi_*\right)\left(\int_{\mathbf{R}^3}\sum_{i=1}^2|G_i(\xi,\xi_*)|\tfrac{\mathbf{G}^2(\xi_*)}{\mathbf{M}_-(\xi_*)}d\xi_*\right)d\xi dxd\tau\\
&\le O(1)\int_0^t\left[\int_0^\infty\int_{\mathbf{R}^3}\tfrac{\mathbf{G}^2(\xi)}{\mathbf{M}_-}d\xi dx\right]d\tau.\end{aligned}\tag{4.15}$$

For I_{32}, since

$$
\begin{aligned}
\Big|I_{32}\Big| &\le \Big(\int_0^t \exp\Big(-2\nu(\xi)(t-\eta)\Big)\nu(\xi)d\eta\Big)^{\frac{1}{2}}\Big(\int_0^t \frac{1}{\nu(\xi)}[Q(\mathbf{G}_F,\mathbf{G}_F)]^2(\eta,X(\eta),E(\eta))d\eta\Big)^{\frac{1}{2}}\\
&\le O(1)\Big(\int_0^t \frac{1}{\nu(\xi)}[Q(\mathbf{G}_F,\mathbf{G}_F)^2(\eta,X(\eta),E(\eta))d\eta\Big)^{\frac{1}{2}},
\end{aligned}
$$

we have

$$
\begin{aligned}
\int_0^\infty\int_{\mathbf{R}^3}\tfrac{|I_{32}|^2}{\mathbf{M}_-(\xi)}d\xi dx &\le O(1)\int_0^t\int_0^\infty\int_{\mathbf{R}^3}\tfrac{1}{\nu(\xi)}\tfrac{Q(\mathbf{G}_F,\mathbf{G}_F)^2}{\mathbf{M}_-(\xi)}d\xi dx d\tau\\
&\le O(1)\int_0^t\int_0^\infty\Big[\int_{\mathbf{R}^3}\tfrac{\nu(\xi)\mathbf{G}_F^2}{\mathbf{M}_-}d\xi\Big]\Big[\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_F^2}{\mathbf{M}_-}d\xi\Big]dx d\tau\\
&\le O(1)\delta_0^4.
\end{aligned}
\tag{4.16}
$$

Combining (4.13)-(4.16), we have the $L^2_x\Big(L^2_{\xi,\mathbf{M}_-}\Big)$-norm estimate on the solution $\tilde{f}(t,x,\xi)$ to the linear problem (4.9) and (4.10) when the initial data $\tilde{f}_0(x,\xi)\in L^2_x\Big(L^2_{\xi,\mathbf{M}_-}\Big)$. A similar argument holds for the $H^2_x\Big(L^2_{\xi,\mathbf{M}_-}\Big)$-norm estimates.

Define a solutions operator $\mathcal{T}$ for the problem (4.9)-(4.10) by:

$$
f=\mathcal{T}(F). \tag{4.17}
$$

The above estimates imply that if $\Big\|f_0(x,\xi)-\mathbf{M}_{[\overline{\rho}(0,x),\overline{u}(0,x),\overline{\theta}(0,x)]}\Big\|_{H^2_x\Big(L^2_{x,\mathbf{M}_-}\Big)}\le\delta_1$ for some small positive constant δ_1, there exists a $t_1>0$ such that the operator $\mathcal{T}$ defined by (4.17) maps $\mathcal{X}$ to $\mathcal{X}$. Similar argument shows that it is contractive. Therefore, we have the following local existence result.

Theorem 4.1 *For each multi-index $\alpha=(\alpha_1,\alpha_2)$ with $1\le|\alpha|\le 3$, suppose that $|\xi|\partial^\alpha\overline{\mathbf{M}}\in L^2_x\Big(L^2_{\xi,\mathbf{M}_-}\Big)$. If the initial data $f_0(x,\xi)$ satisfy $f_0(x,\xi)-\mathbf{M}_{[\overline{\rho}(0,x),\overline{u}(0,x),\overline{\theta}(0,x)]}\in H^2_x\Big(L^2_{\xi,\mathbf{M}_-}\Big)$ with its $H^2_x\Big(L^2_{\xi,\mathbf{M}_-}\Big)$−norm sufficiently small and $\theta_-<\inf_x\theta_0(x)\le\sup_x\theta_0(x)<2\theta_-$, then the initial boundary value problem (1.1)-(1.3) admits a unique solution $f(t,x)=\mathbf{M}_{[\rho(t,x),u(t,x),\theta(t,x)]}(\xi)+\mathbf{G}(t,x,\xi)$ on $[0,t_1]\times\mathbf{R}_+$ satisfying $\frac{\theta(t,x)}{2}<\theta_-<\theta(t,x)$ for all $(t,x)\in[0,t_1]\times\mathbf{R}_+$. Here t_1 depending only on $\Big\|f_0(x,\xi)-\mathbf{M}_{[\overline{\rho}(0,x),\overline{u}(0,x),\overline{\theta}(0,x)]}\Big\|_{H^2_x\Big(L^2_{\xi,\mathbf{M}_-}\Big)}$*

4.2 The Proof of Theorem 1.1

We now finish the prrof of the main result. The global existence result follows immediately from Theorem 3.1 and Theorem 4.1. To complete the proof of Theorem 1.1, we only need to give the time asymptotic estimate (1.28). For this purpose, notice from (3.1) that

$$
\begin{cases}
\int_0^\infty\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}_-}d\xi dx d\tau\le O(1),\\
\int_0^\infty\Big|\frac{d}{dt}\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}_-}d\xi dx\Big|d\tau\le\int_0^\infty\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2+\mathbf{G}_{xt}}{\mathbf{M}_-}d\xi dx d\tau\le O(1),\\
\int_0^\infty\int_0^\infty\int_{\mathbf{R}^3}\frac{|\partial_x(\mathbf{M}-\mathbf{M}_{[\overline{\rho},\overline{u},\overline{\theta}]})|^2}{\mathbf{M}_-}d\xi dx d\tau\le O(1),\\
\int_0^\infty\Big|\frac{d}{dt}\int_0^\infty\int_{\mathbf{R}^3}\frac{|\partial_x(\mathbf{M}-\mathbf{M}_{[\overline{\rho},\overline{u},\overline{\theta}]})|^2}{\mathbf{M}_-}d\xi dx\Big|d\tau\le\int_0^\infty\int_0^\infty\int_{\mathbf{R}^3}\frac{|\partial_x(\mathbf{M}-\mathbf{M}_{[\overline{\rho},\overline{u},\overline{\theta}]})|^2+|\partial_{xt}(\mathbf{M}-\mathbf{M}_{[\overline{\rho},\overline{u},\overline{\theta}]})|^2}{\mathbf{M}_-}d\xi dx d\tau\\
\qquad\le O(1).
\end{cases}
$$

Consequently

$$\lim_{t\to\infty}\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2+|\partial_x(\mathbf{M}-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]})|^2}{\mathbf{M}_-}d\xi dx=0. \tag{4.18}$$

Since

$$\begin{aligned}\int_{\mathbf{R}^3}\frac{\mathbf{G}^2+|\mathbf{M}-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]}|^2}{\mathbf{M}_-}d\xi &\leq O(1)\int_0^\infty\int_{\mathbf{R}^3}\frac{|\mathbf{G}\mathbf{G}_x|+|\mathbf{M}-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]}||\partial_x(\mathbf{M}-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]})|}{\mathbf{M}_-}d\xi dx\\ &\leq O(1)\left(\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}^2}{\mathbf{M}_-}d\xi dx\right)^{\frac12}\left(\int_0^\infty\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}_-}d\xi dx\right)^{\frac12}\\ &\quad+O(1)\left(\int_0^\infty\int_{\mathbf{R}^3}\frac{|\mathbf{M}-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]}|^2}{\mathbf{M}_-}d\xi dx\int_0^\infty\int_{\mathbf{R}^3}\frac{|\partial_x(\mathbf{M}-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]})|^2}{\mathbf{M}_-}d\xi dx\right)^{\frac12},\end{aligned}$$

we have from (3.1) and (4.18) that

$$\lim_{t\to\infty}\sup_{x\in\mathbf{R}_+}\int_{\mathbf{R}^3}\left(\frac{\mathbf{G}^2+|\mathbf{M}-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]}|^2}{\mathbf{M}_-}\right)(t,x,\xi)d\xi=0.$$

Thus

$$\begin{aligned}\lim_{t\to\infty}\sup_{x\in\mathbf{R}_+}\int_{\mathbf{R}^3}\left(\frac{|f-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]}|^2}{\mathbf{M}_-}\right)(t,x,\xi)d\xi &\leq \lim_{t\to\infty}\sup_{x\in\mathbf{R}_+}\int_{\mathbf{R}^3}\left(\frac{\mathbf{G}^2+|\mathbf{M}-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]}|^2}{\mathbf{M}_-}\right)(t,x,\xi)d\xi\\ &=0.\end{aligned} \tag{4.19}$$

Moreover,

$$\lim_{t\to\infty}\sup_{x\in\mathbf{R}_+}\int_{\mathbf{R}^3}\left(\frac{\left|\mathbf{M}_{[\rho^R,u^R,\theta^R]}-\mathbf{M}_{[\bar\rho,\bar u,\bar\theta]}\right|^2}{\mathbf{M}_-}\right)(t,x,\xi)d\xi=0, \tag{4.20}$$

we have from (4.19) and (4.20) that

$$\lim_{t\to\infty}\sup_{x\in\mathbf{R}_+}\int_{\mathbf{R}^3}\left(\frac{\left|f-\mathbf{M}_{[\rho^R,u^R,\theta^R]}\right|^2}{\mathbf{M}_-}\right)(t,x,\xi)d\xi=0, \tag{4.21}$$

which is (1.28). And this completes the proof of Theorem 1.1.

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