Cauchy Problem for the Vlasov-Poisson-Boltzmann System

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Contents

Abstract

The dynamics of the dilute electrons can be modeled by the fundamental Vlasov-Poisson-Boltzmann system when the electrons interact with themselves through collisions in the self-consistent electric field. In this paper, it is shown that any smooth initial perturbation of a given global Maxwellian leads to a unique global-in-time classical solution when either the mean free path is small or the background charge density is large. And the solution converges to the global Maxwellian when time tends to infinity. To our knowledge, this is the first global existence result on classical solutions not around vacuum to the Cauchy problem for the Vlasov-Poisson-Boltzmann system. The analysis combines the analytic techniques used in the study of conservation laws with the decomposition for the Boltzmann equation introduced [16] through entropy construction revealing new entropy estimates in this physical setting.

1 Introduction

The dilute charged particles (e.g., electrons) in the absence of the magnetic field can be described by the foundamental Vlasov-Poisson-Boltzmann system:

$$\begin{cases} f_t + \xi \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_\xi f = \frac{1}{\kappa} Q(f,f), \\ \Delta_x \Phi = \rho - \rho_0 = \int_{\mathbf{R}^3} f d\xi - \rho_0, \quad |\Phi| \to 0, \text{ as } |x| \to +\infty, \end{cases} \tag{1.1}$$

with initial data given by

$$f(0,x,\xi) = \overline{f}_0(x,\xi),$$

where $f(t,x,\xi)$ is the distribution function for the particles at time $t \geq 0$ located at $x = (x_1,x_2,x_3) \in \mathbf{R}^3$ with velocity $\xi = (\xi_1,\xi_2,\xi_3) \in \mathbf{R}^3$, and the constant $\kappa > 0$ is the Knudsen number proportional to the mean free path. The self-consistent electric potential $\Phi(t,x)$ is coupled with the distribution function $f(t,x,\xi)$ through the Poisson equation. The constant background charge is denoted by $\rho_0 > 0$. The short-range interaction between particles is given by the standard Boltzmann collision operator $Q(f,g)$ for the hard-sphere model

$$Q(f,g)(\xi) \equiv \frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{S}^2_+} \Big(f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi') - f(\xi)g(\xi_*) - f(\xi_*)g(\xi) \Big) |(\xi - \xi_*) \cdot \Omega| \ d\xi_* d\Omega.$$

Here $\mathbf{S}^2_+ = \{\Omega \in \mathbf{S}^2 : \ (\xi - \xi_*) \cdot \Omega \geq 0\}$, and

$$\begin{cases} \xi' = \xi - [(\xi - \xi_*) \cdot \Omega] \, \Omega, \\ \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega] \, \Omega, \end{cases}$$

which represents the relation between velocities ξ', ξ'_* after and the velocities ξ, ξ_* before the collision coming from conservation of momentum and energy.

Some works have been done on the Vlasov-Poisson-Boltzmann system. Global-in-time renormalized solutions with arbitrary amplitude were constructed in [5] and the result has been generalized to the case with boundary in [19]. The long-time behavior of the weak solutions with extra regularity assumptions was studied in [4]. As to the problem on classical solutions, there are some progress only recently. For any smooth periodic initial perturbation of a global Maxwellian that preserves the mass, momentum, and total energy, the first global existence result on the smooth periodic solution was obtained in [11]. As in the whole space, to our knowledge, the only result so far is [13] where the global smooth small-amplitude solutions near vacuum were constructed for a class of "soft" collision kernels. Therefore, our global existence result is new on the Vlasov-Poisson-Boltzmann system near a given global Maxwellian in the whole space.

To state the main result, we first reformulate the problem by the scaling

$$\begin{cases} f(t,x,\xi) \to \frac{\overline{\rho}}{\rho_0} f\left(\frac{\overline{\rho}}{\kappa\rho_0} t, \frac{\overline{\rho}}{\kappa\rho_0} x, \xi \right), \\ \Phi(t,x) \to \Phi\left(\frac{\overline{\rho}}{\kappa\rho_0} t, \frac{\overline{\rho}}{\kappa\rho_0} x \right), \end{cases}$$

where $\overline{\rho}$ is any fixed constant. It is easy to see that the solution after scaling satisfies

$$\begin{cases} f_t + \xi \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_\xi f = Q(f, f), \\ \lambda \Delta_x \Phi = \rho - \overline{\rho} = \int_{\mathbf{R}^3} f d\xi - \overline{\rho}, \quad |\Phi| \to 0, \text{ as } |x| \to +\infty, \\ f(0, x, \xi) = f_0(x, \xi) = \frac{\overline{\rho}}{\rho_0} \overline{f}_0 \left(\frac{\overline{\rho}}{\kappa \rho_0} x, \xi \right), \end{cases} \tag{1.2}$$

with $\lambda = \frac{\rho_0}{\kappa^2 \overline{\rho}}$. In this paper, we will assume $\lambda > 0$ suitably large which means either the Knudsen number $\kappa > 0$, i.e., the mean free path is suffiently small, or the constant background charge $\rho_0 > 0$ is sufficiently large.

A micro-macro decomposition of the Boltzmann equation and its solution was introduced in [16, 18], and the Boltzmann equation can be rewritten into a fluid-type system coupled with an equation for the non-fluid component in [16]. As an illustration of this method, the global existence of classical solutions around a global Maxwellian was proved by using a simple energy method through the construction of entropy-entropy flux pairs in [16]. This method is useful here for the study of the Boltzmann equation with self-induced electric field. It shows that the entropy-entropy flux pair similar to the one for fluid dynamics plays an important role in the lower order energy estimate. And the dissipation coming from the electric field governed by the Poisson equation is crucial to close the a priori estimate which in turn implies that the uniform space-time integrability of the square of the difference between perturbed and unperturbed density function. Notice that the later integral diverges for the Boltzmann equation or even the Navier-Stokes equations without force.

Precisely, we decompose the solution of the Vlasov-Poisson-Boltzmann system $f(t, x, \xi)$ into the macroscopic (fluid) component, i.e., the local Maxwellian $\mathbf{M} = \mathbf{M}(t, x, \xi) = \mathbf{M}_{[\rho,u,\theta]}(\xi)$, and the microscopic (non-fluid) component, i.e., $\mathbf{G} = \mathbf{G}(t, x, \xi)$ as follows:

$$f(t, x, \xi) = \mathbf{M}(t, x, \xi) + \mathbf{G}(t, x, \xi).$$

The local Maxwellian $\mathbf{M}$ is defined by the five conserved quantities, that is, the density $\rho(t, x)$, momentum $m(t, x) = \rho(t, x)u(t, x)$, and energy $\mathbf{E}(t, x) + \frac{1}{2}|u(t, x)|^2$ given by:

$$\begin{cases} \rho(t, x) \equiv \int_{\mathbf{R}^3} f(t, x, \xi) d\xi, \\ m^i(t, x) \equiv \int_{\mathbf{R}^3} \psi_i(\xi) f(t, x, \xi) d\xi \text{ for } i = 1, 2, 3, \\ \left[\rho \left(\mathbf{E} + \frac{1}{2}|u|^2 \right) \right](t, x) \equiv \int_{\mathbf{R}^3} \psi_4(\xi) f(t, x, \xi) d\xi, \end{cases} \tag{1.3}$$

$$\mathbf{M} \equiv \mathbf{M}_{[\rho,u,\theta]}(t, x, \xi) \equiv \frac{\rho(t, x)}{\sqrt{(2\pi R\theta(t, x))^3}} \exp \left(-\frac{|\xi - u(t, x)|^2}{2R\theta(t, x)} \right). \tag{1.4}$$

Here $\theta(t, x)$ is the temperature which is related to the internal energy $\mathbf{E}$ by $\mathbf{E} = \frac{3}{2}R\theta$ with R being the gas constant, and $u(t, x)$ is the fluid velocity. And $\psi_\alpha(\xi)$, $\alpha = 0, 1, \cdots, 4$, are the five collision invariants, cf. [1, 2]:

$$\begin{cases} \psi_0(\xi) \equiv 1, \\ \psi_i(\xi) \equiv \xi_i \quad \text{for} \quad i = 1, 2, 3 \text{ or } \psi = \xi, \\ \psi_4(\xi) \equiv \frac{1}{2}|\xi|^2, \end{cases} \tag{1.5}$$

satisfying

$$\int_{\mathbf{R}^3} \psi_\alpha(\xi) Q(h,g) d\xi = 0, \text{ for } \alpha = 0,1,2,3,4.$$

In the following, we define an inner product in $\xi \in \mathbf{R}^3$ w.r.t. a given Maxwellian $\tilde{\mathbf{M}}$ as:

$$\langle h, g\rangle_{\tilde{\mathbf{M}}} \equiv \int_{\mathbf{R}^3} \frac{1}{\tilde{\mathbf{M}}} h(\xi) g(\xi) d\xi,$$

for functions $h,\ g$ of ξ so that the above integral is well-defined. With respect to the inner product $\langle h, g\rangle_{\mathbf{M}}$, the following functions spanning the space of macroscopic, i.e. fluid components of the solution, are pairwise orthogonal:

$$\begin{cases} \chi_0(\xi;\rho,u,\theta) \equiv \frac{1}{\sqrt{\rho}}\mathbf{M}, \\ \chi_i(\xi;\rho,u,\theta) \equiv \frac{\xi_i - u_i}{\sqrt{R\theta\rho}}\mathbf{M} \text{ for } i = 1,2,3, \\ \chi_4(\xi;\rho,u,\theta) \equiv \frac{1}{\sqrt{6\rho}}\left(\frac{|\xi-u|^2}{R\theta} - 3\right)\mathbf{M}, \\ \langle \chi_i, \chi_j\rangle_{\mathbf{M}} = \delta_{ij}, \quad \text{for } i,j = 0,1,2,3,4. \end{cases} \tag{1.6}$$

By using these five functions, the macroscopic projection $\mathbf{P}_0$ and microscopic projection $\mathbf{P}_1$ are:

$$\begin{cases} \mathbf{P}_0 h \equiv \sum_{j=0}^{4} \langle h, \chi_j\rangle_{\mathbf{M}} \chi_j, \\ \mathbf{P}_1 h \equiv h - \mathbf{P}_0 h. \end{cases} \tag{1.7}$$

Notice that the operators $\mathbf{P}_0$ (and therefore $\mathbf{P}_1$) are orthogonal self-adjoint projections w.r.t. the inner product $\langle \cdot, \cdot\rangle_{\mathbf{M}}$.

Notice that a function $h(\xi)$ is called microscopic or non-fluid if it has no fluid components, that is,

$$\int_{\mathbf{R}^3} h(\xi)\psi_\alpha(\xi) d\xi = 0, \text{ for } \alpha = 0,1,2,3,4. \tag{1.8}$$

It is clear that such a function is in the range of the microscopic projection $\mathbf{P}_1$. Under this decomposition, the solution $f(t,x,\xi)$ of the Vlasov-Poisson-Boltzmann system satisfies,

$$\mathbf{P}_0 f = \mathbf{M}, \qquad \mathbf{P}_1 f = \mathbf{G}.$$

Then by using $f(t,x,\xi) = \mathbf{M}(t,x,\xi) + \mathbf{G}(t,x,\xi)$, the Vlasov-Poisson-Boltzmann system $(1.2)_1$ becomes:

$$(\mathbf{M}+\mathbf{G})_t + \xi\cdot\nabla_x(\mathbf{M}+\mathbf{G}) + \nabla_x\Phi\cdot\nabla_\xi(\mathbf{M}+\mathbf{G}) = \Big(2Q(\mathbf{G},\mathbf{M}) + Q(\mathbf{G},\mathbf{G})\Big). \tag{1.9}$$

By applying $\mathbf{P}_0$ to (1.9), we have

$$\mathbf{M}_t + \mathbf{P}_0\Big(\xi\cdot\nabla_x\mathbf{M}\Big) + \mathbf{P}_0\Big(\xi\cdot\nabla_x\mathbf{G}\Big) + \nabla_x\Phi\cdot\nabla_\xi\mathbf{M} = 0.$$

As usual, the system of five conservation laws can be obtained by taking the inner product of the Vlasov-Poisson-Boltzmann system $(1.2)_1$ with the collision invariants $\psi_\alpha(\xi)$:

$$
\begin{cases}
\rho_t + \mathrm{div}_x\ m = 0, \\
m_{it} + \sum_{j=1}^{3} (u_i m_j)_{x_j} + p_{x_i} - \rho\Phi_{x_i} = -\int_{\mathbf{R}^3} \psi_i\,(\xi\cdot\nabla_x \mathbf{G})\,d\xi, \quad i=1,2,3, \\
\Big[\rho(\frac{1}{2}|u|^2+\mathbf{E})\Big]_t + \sum_{j=1}^{3}\bigg(u_j\Big(\rho\left(\frac{1}{2}|u|^2+\mathbf{E}\right)+p\Big)\bigg)_{x_j} - m\cdot\nabla_x\Phi = -\int_{\mathbf{R}^3}\psi_4\,(\xi\cdot\nabla_x\mathbf{G})\,d\xi.
\end{cases}
\tag{1.10}
$$

Here p is the pressure for the monatomic gases:

$$p = \frac{2}{3}\rho\mathbf{E} = R\rho\theta.$$

Moreover, the microscopic equation for $\mathbf{G}$ is obtained by applying the microscopic projection $\mathbf{P}_1$ to (1.9):

$$\mathbf{G}_t + \mathbf{P}_1\Big(\xi\cdot\nabla_x\mathbf{G} + \xi\cdot\nabla_x\mathbf{M}\Big) + \nabla_x\Phi\cdot\nabla_\xi\mathbf{G} = L_{\mathbf{M}}\mathbf{G} + Q(\mathbf{G},\mathbf{G}), \tag{1.11}$$

i.e.,

$$
\begin{aligned}
\mathbf{G} &= L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\Big) + L_{\mathbf{M}}^{-1}\Big(\mathbf{G}_t + \mathbf{P}_1(\xi\cdot\nabla_x\mathbf{G}) + \nabla_x\Phi\cdot\nabla_\xi\mathbf{G} - Q(\mathbf{G},\mathbf{G})\Big) \\
&:= L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\Big) + \Theta,
\end{aligned}
\tag{1.12}
$$

where

$$L_{\mathbf{M}}g = L_{[\rho,u,\theta]}g \equiv Q\Big(\mathbf{M}+g,\mathbf{M}+g\Big) - Q(g,g).$$

By plugging (1.12) into (1.10), we now have another representation of the Vlasov-Poisson-Boltzmann system which contains a fluid-type system

$$
\begin{cases}
\rho_t + \mathrm{div}_x m = 0, \\
m_{it} + \sum_{j=1}^{3}(u_i m_j)_{x_j} + p_{x_i} - \rho\Phi_{x_i} = -\int_{\mathbf{R}^3}\psi_i\left(\xi\cdot\nabla_x L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\Big)\right)d\xi \\
\qquad\qquad -\int_{\mathbf{R}^3}\psi_i\,(\xi\cdot\nabla_x\Theta)\,d\xi, \quad i=1,2,3, \\
\Big[\rho(\frac{1}{2}|u|^2+\mathbf{E})\Big]_t + \sum_{j=1}^{3}\bigg(u_j\Big(\rho\left(\frac{1}{2}|u|^2+\mathbf{E}\right)+p\Big)\bigg)_{x_j} - m\cdot\nabla_x\Phi \\
\qquad = -\int_{\mathbf{R}^3}\psi_4\left(\xi\cdot\nabla_x L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\Big)\right)d\xi - \int_{\mathbf{R}^3}\psi_4\,(\xi\cdot\nabla_x\Theta)\,d\xi,
\end{cases}
\tag{1.13}
$$

the equation (1.11) for the non-fluid component $\mathbf{G}$ and the Poisson equation $(1.2)_2$ for the electric potential. Notice that if one drops all the terms containing Θ, then it becomes the system of the Navier-Stokes-Poisson equations. Later in this paper, we will work on this reformulated system by applying the analytic techniques used in the study of conservation laws together with those dissipative estimates on the Boltzmann equation.

For preparation, we now recall some properties of the linearized collision operator $L_{\mathbf{M}}$. By definition, $L_{\mathbf{M}}$ is self-adjoint w.r.t. the inner product $\langle h,g\rangle_{\mathbf{M}}$, i.e.,

$$\langle h, L_{\mathbf{M}}g\rangle_{\mathbf{M}} = \langle L_{\mathbf{M}}h, g\rangle_{\mathbf{M}},$$

and the null space N of $L_{\mathbf{M}}$ is spanned by the macroscopic variables:

$$\chi_j, \qquad j=0,\cdots,4.$$

For the hard sphere model, $L_{\mathbf{M}}$ takes the form, cf. [10]

$$(L_{\mathbf{M}}h)(\xi)=-\nu(\xi;\rho,u,\theta)h(\xi)+\sqrt{\mathbf{M}(\xi)}K_{\mathbf{M}}\left(\left(\frac{h}{\sqrt{\mathbf{M}}}\right)(\xi)\right). \tag{1.14}$$

Here $K_{\mathbf{M}}(\cdot)=-K_{1\mathbf{M}}(\cdot)+K_{2\mathbf{M}}(\cdot)$ is a symmetric compact L^2-operator. And the collision frequency $\nu(\xi;\rho,u,\theta)$ and $K_{i\mathbf{M}}(\cdot)$ have the following expressions

$$\begin{cases}\nu(\xi;\rho,u,\theta)=\frac{2\rho}{\sqrt{2\pi R\theta}}\left\{\left(\frac{R\theta}{|\xi-u|}+|\xi-u|\right)\int_0^{|\xi-u|}\exp\left(-\frac{y^2}{2R\theta}\right)dy+R\theta\exp\left(-\frac{|\xi-u|^2}{2R\theta}\right)\right\},\\ k_{1\mathbf{M}}(\xi,\xi_*)=\frac{\pi\rho}{\sqrt{(2\pi R\theta)^3}}|\xi-\xi_*|\exp\left(-\frac{|\xi-u|^2}{4R\theta}-\frac{|\xi_*-u|^2}{4R\theta}\right),\\ k_{2\mathbf{M}}(\xi,\xi_*)=\frac{2\rho}{\sqrt{2\pi R\theta}}|\xi-\xi_*|^{-1}\exp\left(-\frac{|\xi-\xi_*|^2}{8R\theta}-\frac{(|\xi|^2-|\xi_*|^2)^2}{8R\theta|\xi-\xi_*|^2}\right),\end{cases}$$

where $k_{i\mathbf{M}}(\xi,\xi_*)(i=1,2)$ is the kernel of the operator $K_{i\mathbf{M}}(i=1,2)$ respectively, and $\nu(\xi;\rho,u,\theta)\sim(1+|\xi|)$ as $\xi\to\infty$. Furthermore, there exists $\sigma_0(\rho,u,\theta)>0$ such that for any function $h(\xi)\in N^{\perp}$

$$\langle h,L_{\mathbf{M}}h\rangle_{\mathbf{M}}\leq-\sigma_0(\rho,u,\theta)\langle h,h\rangle_{\mathbf{M}},$$

which implies cf. [10]

$$\langle h,L_{\mathbf{M}}h\rangle_{\mathbf{M}}\leq-\sigma(\rho,u,\theta)\langle\nu(\xi)h,h\rangle_{\mathbf{M}}, \tag{1.15}$$

with some constant $\sigma(\rho,u,\theta)>0$.

For later use, notice also that the projections $\mathbf{P}_0$ and $\mathbf{P}_1$ have the following basic properties:

$$\begin{cases}\mathbf{P}_0(\psi_j\mathbf{M})=\psi_j\mathbf{M},\ \mathbf{P}_1(\psi_j\mathbf{M})=0,\ j=0,1,2,3,4,\\ L_{\mathbf{M}}\mathbf{P}_1=\mathbf{P}_1L_{\mathbf{M}}=L_{\mathbf{M}},\ \mathbf{P}_1(Q(h,h))=Q(h,h),\\ L_{\mathbf{M}}\mathbf{P}_0=\mathbf{P}_0L_{\mathbf{M}}=0,\ \mathbf{P}_0(Q(h,h))=0,\\ \langle\psi_j\mathbf{M},h\rangle_{\mathbf{M}}=\langle\psi_j\mathbf{M},\mathbf{P}_0h\rangle_{\mathbf{M}},\ j=0,1,2,3,4,\\ \langle h,L_{\mathbf{M}}g\rangle_{\mathbf{M}}=\langle\mathbf{P}_1h,L_{\mathbf{M}}(\mathbf{P}_1g)\rangle_{\mathbf{M}},\\ \left\langle h,L_{\mathbf{M}}^{-1}(\mathbf{P}_1g)\right\rangle_{\mathbf{M}}=\left\langle L_{\mathbf{M}}^{-1}(\mathbf{P}_1h),\mathbf{P}_1g\right\rangle_{\mathbf{M}}=\left\langle\mathbf{P}_1h,L_{\mathbf{M}}^{-1}(\mathbf{P}_1g)\right\rangle_{\mathbf{M}}.\end{cases}$$

For a fixed temperature $\overline{\theta}>0$, we will study the existence of the classical solutions for (1.2) near the global Maxwellian

$$\overline{\mathbf{M}}=\mathbf{M}_{[\overline{\rho},0,\overline{\theta}]}=\frac{\overline{\rho}}{\sqrt{\left(2\pi R\overline{\theta}\right)^3}}\exp\left(-\frac{|\xi|^2}{2R\overline{\theta}}\right).$$

As in [17], two sets of energy estimates are needed, i.e., the energy estimates w.r.t. the local Maxwellian $\mathbf{M}_{[\rho,u,\theta]}(t,x,\xi)$ and a suitably chosen global Maxwellian $\mathbf{M}_-=\mathbf{M}_{[\rho_-,0,\theta_-]}(\xi)$. For this, a variation of the microscopic $H-$theroem is needed to relate the dissipation estimates with different weights as in Lemma 2.2 of [17]. That is, there exists a positive constant $\eta_0=$

$\eta_0(\rho,u,\theta;\tilde\rho,\tilde u.\tilde\theta)>0$, which is not necessary to be small, such that if $\frac{\theta}{2}<\tilde\theta<\theta$ and $|\rho-\tilde\rho|+|u-\tilde u|+|\theta-\tilde\theta|<\eta_0$, the following microscopic $H-$therem

$$-\int_{\mathbf{R}^3}\frac{\mathbf{G}L_{\mathbf{M}}\mathbf{G}}{\tilde{\mathbf{M}}}d\xi\geq\overline{\sigma}\int_{\mathbf{R}^3}\frac{\nu(\xi)\mathbf{G}^2}{\tilde{\mathbf{M}}}d\xi, \tag{1.16}$$

holds for some positive constant $\overline{\sigma}=\overline{\sigma}(\rho,u,\theta;\tilde\rho,\tilde u,\tilde\theta)>0$ with $\tilde{\mathbf{M}}=\mathbf{M}_{[\tilde\rho,\tilde u,\tilde\theta]}$.

Throughout this paper, we choose positive constants ρ_- and θ_- such that

$$\begin{cases}\rho_-=\overline{\rho},\\ \frac{\overline{\theta}}{2}<\theta_-<\overline{\theta},\\ \left|\theta_--\overline{\theta}\right|<\eta_0.\end{cases} \tag{1.17}$$

It is easy to see that if $\mathbf{M}(t,x,\xi)$ is a small perturbation of $\overline{\mathbf{M}}(\xi)$, (1.16) holds for such chosen ρ_- and θ_- when $\tilde{\mathbf{M}}\equiv\mathbf{M}_-=M_{[\rho_-,0,\theta_-]}$.

The following are the spaces for the solution considered in this paper.

$$\begin{cases}\mathbf{H}^N_{t,x,\xi}\left(\mathbf{R}^3\times\mathbf{R}^3\right)=\left\{f(t,x,\xi)\;\middle|\;\begin{array}{c}\frac{\partial_t^{\gamma_0}\partial_x^\alpha\partial_\xi^\beta\left(f(t,x,\xi)-\overline{\mathbf{M}}(\xi)\right)}{\sqrt{\mathbf{M}_-(\xi)}}\in L^2_{x,\xi}\left(\mathbf{R}^3\times\mathbf{R}^3\right),\\ |\gamma_0|+|\alpha|+|\beta|\leq N\end{array}\right\},\\ \overline{\mathbf{H}}^N=\left\{f(t,x,\xi)\;\middle|\;\begin{array}{c}\frac{\partial_t^{\gamma_0}\partial_x^\alpha\partial_\xi^\beta\left(f(t,x,\xi)-\overline{\mathbf{M}}(\xi)\right)}{\sqrt{\mathbf{M}_-(\xi)}}\in L^2_{x,\xi}\left(\mathbf{R}^3\times\mathbf{R}^3\right),\\ |\gamma_0|\leq1,\quad|\gamma_0|+|\alpha|+|\beta|\leq N\end{array}\right\}.\end{cases}$$

By the above notations, the main result in this paper can be stated as follows

Theorem 1.1 (Main result) *Assume that $f_0(x,\xi)\geq0$ and $N\geq4$. There exist a sufficiently small constant $\varepsilon>0$ and a sufficiently large constant λ_0 such that if*

$$\begin{cases}\lambda>\lambda_0,\qquad\lambda_0\varepsilon<1,\\ \mathcal{E}(f_0)=\left\|\nabla_x\Delta_x^{-1}\left(\rho_0(x)-\overline{\rho}\right)\right\|_{L^2_x(\mathbf{R}^3)}+\sum\limits_{|\alpha|+|\beta|\leq N}\left\|\frac{\partial_x^\alpha\partial_\xi^\beta(f_0(x,\xi)-\overline{\mathbf{M}}(\xi))}{\sqrt{\mathbf{M}_-(\xi)}}\right\|_{L^2_{x,\xi}(\mathbf{R}^3\times\mathbf{R}^3)}\leq\varepsilon,\end{cases} \tag{1.18}$$

then there exists a unique global classical solution $f(t,x,\xi)$ to the Vlasov-Poisson-Boltzmann system (1.2) which satisfies $f(t,x,\xi)\geq0$ and is uniformly bounded in $\overline{\mathbf{H}}^N$. Furthermore,

$$\lim_{t\to\infty}\sup_{x\in\mathbf{R}^3}\sum_{|\alpha|\leq N-4}\int_{\mathbf{R}^3}\frac{\left|\partial_x^\alpha\left(f(t,x,\xi)-\overline{\mathbf{M}}(\xi)\right)\right|^2}{\mathbf{M}_-(\xi)}d\xi=0. \tag{1.19}$$

Remark 1.1 *Notice that in the space $\overline{\mathbf{H}}^N$, the time derivative on $f(t,x,\xi)$ in Theorem 1.1 is at most once. In general, the solutions may not be uniformly bounded in the space $\mathbf{H}^N_{t,x,\xi}\left(\mathbf{R}^3\times\mathbf{R}^3\right)$. However, for any fixed $T>0$, there exists a positive constant $C(T)>0$ such that*

$$\sup_{0\leq t\leq T}\sum_{\gamma_0+|\alpha|+|\beta|\leq N}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\partial_t^{\gamma_0}\partial_x^\alpha\partial_\xi^\beta\left(f(t,x,\xi)-\overline{\mathbf{M}}(\xi)\right)\right|^2}{\mathbf{M}_-(\xi)}d\xi dx\leq C(T). \tag{1.20}$$

Here, we briefly outline the relation and difference between our work and the previous ones done by others in this direction. For the Vlasov-Poisson-Boltzmann system, the behavior of the solutions to the linearized system around a global Maxwellian $\overline{\mathbf{M}}$ was studied in [7, 8]. The global existence in the whole space and for initial data as a small perturbation of vacuum was given in [13] for a class of "soft" potentials. The analysis there relies on the time decay estimate on the electric potential $\Phi(t,x)$ which comes from the $L^1_x\left(\mathbf{R}^3\right)$ −estimate on $\rho(t,x)$ and the assumptions on the collision potentials. In fact, even for the case when the initial data is a small perturbation of the vacuum, the argument in [13] may not hold for the hard sphere potential. As pointed out in [11], this argument can not be used to consider the existence of global classical solution to the Vlasov-Poisson-Boltzmann system near a Maxwellian $\overline{\mathbf{M}}$ because it is very difficult, if not impossible, to obtain the desired $L^1_x\left(\mathbf{R}^3\right)$ −estimate on $\rho(t,x)-\overline{\rho}$.

To construct global-in-time classical solutions around a global Maxwellian, a nonlinear energy method based on the decomposition w.r.t. the global Maxwellian was used in [11] for periodic data. For the perturbation that preserves the mass, momentum, and total energy, the existence of global smooth periodic solutions to the Vlasov-Poisson-Boltzmann system was obtained there. Their analysis is based on a new estimate in the form of

$$\begin{aligned} -\sum_{|\alpha|\leq N}\int_0^t\int_{\mathbf{T}^3}\int_{\mathbf{R}^3}&\frac{\partial^\alpha_x\left(f(\tau,x,\xi)-\overline{\mathbf{M}}(\xi)\right)L_{\overline{\mathbf{M}}}\left[\partial^\alpha_x\left(f(\tau,x,\xi)-\overline{\mathbf{M}}(\xi)\right)\right]}{\overline{\mathbf{M}}(\xi)}d\xi dxd\tau\\ &\geq C\sum_{|\alpha|\leq N}\int_0^t\int_{\mathbf{T}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left|\partial^\alpha_x\left(f(\tau,x,\xi)-\overline{\mathbf{M}}(\xi)\right)\right|^2}{\overline{\mathbf{M}}(\xi)}d\xi dxd\tau. \end{aligned}\tag{1.21}$$

Here $\mathbf{T}^3=[-\pi,\pi]^3$. In fact, for periodic solution, the zero-th order estimate of the solution in (1.21) comes directly from the Poisson equation and the Poincare inequality which give

$$\|\nabla_x\Phi(t,x)\|_{L^2_x(\mathbf{T}^3)}\leq O(1)\|\rho(t,x)-\overline{\rho}\|_{L^2_x(\mathbf{T}^3)}\leq O(1)\|\nabla_x\rho(t,x)\|_{L^2_x(\mathbf{T}^3)}.$$

This together with the conservation laws imply that both $|\nabla_x\Phi(t,x)|^2$ and $(\rho(x,t)-\overline{\rho})^2$ are space-time uniformly integrable. Notice that for the problem on the whole space, the above argument based on Poincare inequality does not hold.

In this paper, our method is based on the decomposition w.r.t. the local Maxwellian as in the case for Boltzmann equation without force studied in [16]. To capture the dissipations on both the fluid and non-fluid components in the solution, we use the above reformulated fluid-type system (1.13) so that we can use techniques from the study in conservation laws. In this way, the behavior of the local Maxwellian is much clear and dissipative effects from the viscosity, heat conductivity and the one from the linearized collision operator on the non-fluid component are better analyzed. Furthermore, this method would be helpful to study the problem for the fluid dynamic limit, i.e., the behavior of the solutions when the Knudsen number tends to zero.

There are two more points noted in the following argument. The first is that the estimates is closed in the space $\overline{\mathbf{H}}^N$, not the usual space $\mathbf{H}^N_{t,x,\xi}\left(\mathbf{R}^3\times\mathbf{R}^3\right)$ as for the Boltzmann equation without force. This is because that there is a lack of the $L^2_{t,x}\left(\mathbf{R}^+\times\mathbf{R}^3\right)$ −estimate on $\nabla_x\Phi(t,x)$ in the whole space. The estimate in the space $\overline{\mathbf{H}}^N$ is based on a new space and time integrability estimate on $|\rho(t,x)-\overline{\rho}|^2$ coming from the dissipative effect of the Poisson equation $(1.2)_2$.

The other is that the energy estimates is worked out both w.r.t. the local Maxwellian $\mathbf{M}(t,x,\xi)$ and the global Maxwellian $\mathbf{M}_-(\xi)$ as in [17] for the study of the nonlinear stability of rarefaction waves. The energy estimate w.r.t. a global Maxwellian is needed to absorb the polynomials in ξ coming from the derivatives of the local Maxwellian because the collision frequency $\nu(\xi)$ in (1.14) is only of the order of $(1+|\xi|)$ for hard sphere model. However, to obtain the higher order energy estimates on the fluid component $\mathbf{M}$, there is no need to use the energy estimate w.r.t. $\mathbf{M}_-$ because all polynomials of ξ (if any) can be absorbed by $\mathbf{M}$ itself.

This rest of the paper is arranged as follows. The microscopic and macroscopic versions of the $H-$theorems will be stated in Section 2. The main energy estimates are analyzed for the case when $N = 4$ in Section 3. The case when $N > 4$ can be discussed similarly. The proof of Theorem 1.1 will be given in Section 4, and the proofs of some technical Lemmas in Section 3 are given in Section 5 for the brevity of presentation.

Notation

Throughout the paper, $O(1)$ and C denote generic positive constants independent of λ and ϵ, $C(\cdot,\cdot)$ denotes a positive constant depending on the quantities in the patenthesis, and μ is a sufficiently small positive constant. Note that all constants may vary from line to line.

For $\gamma = (\gamma_0, \alpha, \beta)$, we use ∂^γ to denote the differential operator $\partial_t^{\gamma_0}\partial_x^\alpha\partial_\xi^\beta$. Here γ_0 is a non-negative integer and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ are multi-indices with length $|\alpha|$ and $|\beta|$, respectively. C_b^a means $\begin{pmatrix} a \\ b \end{pmatrix}$.

In the following energy estimates, we will use the following sets of indice in different cases.

$$
\begin{cases}
\Lambda_1 = \Big\{\gamma = (\gamma_0, \alpha, 0): \quad \gamma_0 \le 1,\ \gamma_0 + |\alpha| \le 4\Big\}, \\
\Lambda_2^j = \Big\{\gamma = (\gamma_0, \alpha, \beta): \quad \gamma_0 \le j,\ \gamma_0 + |\alpha| + |\beta| \le 4\Big\}, \quad j = 1, 2, 3, 4, \\
\Lambda_3 = \Big\{\gamma = (\gamma_0, \alpha, \beta): \quad \gamma \in \Lambda_2^1,\ |\beta| \ge 1\Big\}, \\
\Lambda_4 = \Big\{\gamma = (\gamma_0, \alpha, 0): \quad \gamma_0 \le 1,\ \gamma_0 + |\alpha| \le 3\Big\}, \\
\Lambda_5 = \Big\{\gamma = (0, \alpha, \beta): \quad |\alpha| \ge 1,\ |\beta| \ge 1,\ |\alpha| + |\beta| \le 4\Big\}, \\
\Lambda_6 = \Big\{\gamma = (1, \alpha, \beta): \quad |\beta| \ge 1,\ |\alpha| + |\beta| \le 3\Big\}, \\
\Lambda_7 = \Big\{\gamma = (\gamma_0, \alpha, 0): \quad \gamma_0 \le 1,\ 2 \le \gamma_0 + |\alpha| \le 4\Big\}.
\end{cases}
$$

2 $H-$theorem

The celebrated $H-$theorem of the Boltzmann equation is based on the special property of the bilinear structure of $Q(f, f)$ satisfying

$$\int_{\mathbf{R}^3} Q(f, f) \ln f d\xi \le 0,$$

and the equality holds only when the solution $f(t, x, \xi)$ is a Maxwellian.

Corresponding to the macroscopic and microscopic components, the H-theorem can be viewed in these two aspects. The first kind of dissipation comes from the linearized collision operator $L_{\mathbf{M}}$ acting on the microscopic components stated in (1.15) and (1.16). The second kind of dissipation comes from the nonlinear collision operator in the expression of the viscosity and heat conductivity in the macroscopic level.

In the following, we will first state some inequalities on the nonlinear and linearized collision operators $Q(f, f)$ and $L_{\mathbf{M}}$. The first lemma is from [9].

Lemma 2.1 *There exists a positive constant $C>0$ such that*

$$\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}Q(f,g)^2}{\mathbf{M}}d\xi\leq C\left\{\int_{\mathbf{R}^3}\frac{\nu(\xi)f^2}{\mathbf{M}}d\xi\cdot\int_{\mathbf{R}^3}\frac{g^2}{\mathbf{M}}d\xi+\int_{\mathbf{R}^3}\frac{f^2}{\mathbf{M}}d\xi\cdot\int_{\mathbf{R}^3}\frac{\nu(\xi)g^2}{\mathbf{M}}d\xi\right\}, \tag{2.1}$$

where $\mathbf{M}$ is any Maxwellian such that the above integrals are well defined.

Based on Lemma 2.1, the following result was proved in [17].

Lemma 2.2 *If $\frac{\theta}{2}<\tilde\theta<\theta$, then there exist two positive constants $\overline{\sigma}=\overline{\sigma}(\rho,u,\theta;\tilde\rho,\tilde u,\tilde\theta)$ and $\eta_0=\eta_0(\rho,u,\theta;\tilde\rho,\tilde u,\tilde\theta)$ such that if $|\rho-\tilde\rho|+|u-\tilde u|+|\theta-\tilde\theta|<\eta_0$, we have for $h(\xi)\in N^\perp$,*

$$-\int_{\mathbf{R}^3}\frac{hL_{\mathbf{M}}h}{\tilde{\mathbf{M}}}d\xi\geq\overline{\sigma}\int_{\mathbf{R}^3}\frac{\nu(\xi)h^2}{\tilde{\mathbf{M}}}d\xi.$$

Here $\mathbf{M}\equiv\mathbf{M}_{[\rho,u,\theta]}(t,x,\xi)$ and $\tilde{\mathbf{M}}(t,x,\xi)=\tilde{\mathbf{M}}_{[\tilde\rho,\tilde u,\tilde\theta]}(t,x,\xi)$.

As a direct consequence of Lemma 2.2 and the Cauchy inequality, we have the following corollary (cf. [17]).

Corollary 2.1 *Under the assumptions in Lemma 2.2, we have for $h(\xi)\in N^\perp$,*

$$\begin{cases}\displaystyle\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left|L_{\mathbf{M}}^{-1}h\right|^2d\xi\leq\sigma^{-2}\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}h^2(\xi)}{\mathbf{M}}d\xi,\\ \displaystyle\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}_-}\left|L_{\mathbf{M}}^{-1}h\right|^2d\xi\leq\overline{\sigma}^{-2}\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}h^2(\xi)}{\mathbf{M}_-}d\xi.\end{cases} \tag{2.2}$$

To construct the enetropy-entropy flux pairs to the Vlasov-Poisson-Boltzmann system, we first derive the macroscopic version of the $H-$theorem as the one in [16] for the Boltzmann equation without force. Set

$$-\frac{3}{2}\rho S\equiv\int_{\mathbf{R}^3}\mathbf{M}\ln\mathbf{M}d\xi. \tag{2.3}$$

Direct calculation yields

$$-\frac{3}{2}(\rho S)_t-\frac{3}{2}\mathrm{div}_x(\rho uS)+\nabla_x\left(\int_{\mathbf{R}^3}(\xi\ln\mathbf{M})\mathbf{G}d\xi\right)=\int_{\mathbf{R}^3}\frac{\mathbf{G}\mathbf{P}_1\Big(\xi\cdot\nabla_x\mathbf{M}\Big)}{\mathbf{M}}d\xi, \tag{2.4}$$

and

$$\begin{cases}S=-\frac{2}{3}\ln\rho+\ln(2\pi R\theta)+1,\\ p=\frac{2}{3}\rho\theta=k\rho^{\frac{5}{3}}\exp(S),\\ \mathbf{E}=\theta,\quad R=\frac{2}{3}.\end{cases} \tag{2.5}$$

Remark 2.1 *Note that when the macroscopic entropy S is defined as in (2.3), the gas constant R is normalized to be $\frac{2}{3}$ and in such a case $\mathbf{E}=\theta$.*

An convex entropy-entropy flux pair (η, q) around the global Maxwellian $\overline{\mathbf{M}} = \mathbf{M}_{[\overline{\rho},0,\overline{\theta}]}$ can be given as follows, [16]. Denote the conservation laws (1.10) by

$$\mathbf{m}_t + \mathrm{div}_x \mathbf{n} = -\begin{pmatrix} 0 \\ \int_{\mathbf{R}^3} \psi_1(\xi\cdot\nabla_x \mathbf{G})d\xi \\ \int_{\mathbf{R}^3} \psi_2(\xi\cdot\nabla_x \mathbf{G})d\xi \\ \int_{\mathbf{R}^3} \psi_3(\xi\cdot\nabla_x \mathbf{G})d\xi \\ \int_{\mathbf{R}^3} \psi_4(\xi\cdot\nabla_x \mathbf{G})d\xi \end{pmatrix} + \begin{pmatrix} 0 \\ \rho\Phi_{x_1} \\ \rho\Phi_{x_2} \\ \rho\Phi_{x_3} \\ m\cdot\nabla_x\Phi \end{pmatrix}.$$

Here

$$\begin{cases} \mathbf{m} = (m_0, m_1, m_2, m_3, m_4)^t = \left(\rho, \rho u_1, \rho u_2, \rho u_3, \rho\left(\frac{1}{2}|u|^2 + \theta\right)\right)^t, \\ \mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3), \\ \mathbf{n}_j = (n_0^j, n_1^j, n_2^j, n_3^j, n_4^j)^t \\ \quad = \left(\rho u_j, u_1 m_j + \frac{2}{3}\rho\theta, u_2 m_j + \frac{2}{3}\rho\theta, u_3 m_j + \frac{2}{3}\rho\theta, \rho u_j\left(\frac{1}{2}|u|^2 + \frac{5}{3}\theta\right)\right)^t, j = 1,2,3. \end{cases}$$

Then the entropy-entropy flux pair (η, q) can be defined by

$$\begin{cases} \eta = \overline{\theta}\left\{-\frac{3}{2}\rho S + \frac{3}{2}\overline{\rho}\overline{S} + \frac{3}{2}\nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\overline{\mathbf{m}}}(\mathbf{m}-\overline{\mathbf{m}})\right\}, \\ q_j = \overline{\theta}\left\{-\frac{3}{2}\rho u_j S + \frac{3}{2}\nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\overline{\mathbf{m}}}(\mathbf{n}_j - \overline{\mathbf{n}}_j)\right\}, \ j = 1,2,3. \end{cases} \tag{2.6}$$

Since

$$\begin{cases} (\rho S)_{m_0} = S + \frac{|u|^2}{2\theta} - \frac{5}{3}, \\ (\rho S)_{m_i} = -\frac{u_i}{\theta}, \ i = 1,2,3, \\ (\rho S)_{m_4} = \frac{1}{\theta}, \end{cases}$$

we have

$$\begin{cases} \eta = \frac{3}{2}\left\{\rho\theta - \overline{\theta}\rho S + \rho\left[\left(\overline{S} - \frac{5}{3}\right)\overline{\theta} + \frac{|u|^2}{2}\right] + \frac{2}{3}\overline{\rho}\overline{\theta}\right\}, \\ q_j = u_j\eta + u_j\left(\rho\theta - \overline{\rho}\overline{\theta}\right), j = 1,2,3. \end{cases} \tag{2.7}$$

Notice that for $\mathbf{m}$ in any closed bounded region $\mathcal{D} \subset \Sigma = \{\mathbf{m} : \rho > 0, \theta > 0\}$, there exists a positive constant C depending on $\mathcal{D}$ such that the entropy-entropy flux thus constructed satisfies (cf. [16, 17])

$$C^{-1}|\mathbf{m} - \overline{\mathbf{m}}|^2 \le \eta \le C|\mathbf{m} - \overline{\mathbf{m}}|^2. \tag{2.8}$$

And (η, q_1, q_2, q_3) solves the following partial differential equation

$$\begin{aligned} \eta_t + \mathrm{div}_x q = & -\nabla_x\left(\int_{\mathbf{R}^3}\left(\xi\mathbf{G}\ln\mathbf{M} + \frac{3}{2}\psi_4\xi\mathbf{G}\right)d\xi\right) + \frac{3}{2}m\cdot\nabla_x\Phi \\ & + \int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\mathbf{G}}{\mathbf{M}}d\xi. \end{aligned} \tag{2.9}$$

Integrating (2.9) w.r.t. x over $\mathbf{R}^3$ gives

$$\frac{d}{dt}\int_{\mathbf{R}^3}\eta(t)dx=\frac{3}{2}\int_{\mathbf{R}^3}m\cdot\nabla_x\Phi dx+\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1\left(\xi\cdot\nabla_x\mathbf{M}\right)\mathbf{G}}{\mathbf{M}}d\xi. \tag{2.10}$$

Since

$$\begin{aligned}\int_{\mathbf{R}^3}m\cdot\nabla_x\Phi dx&=-\int_{\mathbf{R}^3}\mathrm{div}_x m\Phi dx=\int_{\mathbf{R}^3}(\rho-\overline{\rho})_t\Phi dx\\&=\lambda\int_{\mathbf{R}^3}\Phi\Delta\Phi_t dx=-\tfrac{\lambda}{2}\tfrac{d}{dt}\int_{\mathbf{R}^3}|\nabla_x\Phi|^2dx,\end{aligned} \tag{2.11}$$

we obtain the entropy estimate

$$\frac{d}{dt}\left\{\int_{\mathbf{R}^3}\left(\eta+\frac{3\lambda}{4}|\nabla_x\Phi|^2\right)dx\right\}=\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1\Big(\xi\cdot\nabla_x\mathbf{M}\Big)\mathbf{G}}{\mathbf{M}}d\xi, \tag{2.12}$$

which is crucial in the later energy estimates on the fluid components of the solutions.

3 Energy estimates

In this section, we will give the entropy estimates for the proof of global existence theorem. For this, we first assume the the following a priori estimate,

$$\begin{aligned}N(t)^2=&\sup_{0\le\tau\le t}\Bigg\{\sum_{\gamma\in\Lambda_1}\int_{\mathbf{R}^3}\Big(|\partial^\gamma(\rho-\overline{\rho},u,\theta-\overline{\theta})(\tau,x)|^2+\lambda|\nabla_x\partial^\gamma\Phi(\tau,x)|^2\Big)dx\\&+\sum_{\gamma\in\Lambda_2^1}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{|\partial^\gamma\mathbf{G}(\tau,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dx\Bigg\}+\sum_{\gamma\in\Lambda_2^1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{\nu(\xi)|\partial^\gamma\mathbf{G}(\tau,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi dxd\tau\\&\le\delta_0^2.\end{aligned} \tag{3.1}$$

Here $\delta_0>0$ is a sufficiently small constant such that $\lambda\delta_0<1$.

First, from the Poisson equation $(1.2)_2$ and the conservation laws (1.10), we know that

$$N(0)\le O(1)\mathcal{E}(f_0), \tag{3.2}$$

and

$$\begin{cases}\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\Phi_t(t,x)|^2\,dx\le\frac{4}{\lambda^2}\int_{\mathbf{R}^3}|\partial_x^\alpha m(t,x)|^2\,dx,\quad|\alpha|\le 4,\\ \int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\Phi(t,x)|^2\,dx\le\frac{4}{\lambda^2}\sum\limits_{|\alpha'|=|\alpha|-1}\int_{\mathbf{R}^3}\left|\partial_x^{\alpha'}(\rho(t,x)-\overline{\rho})\right|^2dx,\quad 1\le|\alpha|\le 5.\end{cases} \tag{3.3}$$

By Sobolev's inequality, (3.1) and (3.3) imply

$$\begin{cases}\left|(\rho(t,x)-\overline{\rho},u(t,x),\theta(t,x)-\overline{\theta})\right|+\sum\limits_{|\alpha|\le1}\Big(|\partial_x^\alpha\partial_t(\rho,u,\theta)|+|\nabla_x\partial_x^\alpha(\rho,u,\theta)|\Big)(t,x)\le O(1)\delta_0,\\ \sum\limits_{|\alpha|\le3}\left|\nabla_x\partial_x^\alpha\Phi(t,x)\right|+\sum\limits_{|\alpha|\le2}\left|\nabla_x\partial_x^\alpha\Phi_t(t,x)\right|\le O(1)\frac{\delta_0}{\lambda}\le O(1)\delta_0,\\ \int_{\mathbf{R}^3}\left\{\frac{1}{\mathbf{M}_-}\left(|\mathbf{G}|^2+\sum\limits_{|\alpha|\le1}(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2)\right)\right\}(t,x,\xi)d\xi\le O(1)\delta_0^2.\end{cases} \tag{3.4}$$

The weighted integrals of the collision operators $Q\left(\partial^\gamma\mathbf{G},\partial^{\gamma'}\mathbf{G}\right)$ and $Q\left(\partial^\gamma\mathbf{M},\partial^{\gamma'}\mathbf{G}\right)$ w.r.t $\mathbf{M}$ and $\mathbf{M}_-$ are given in the following lemma.

Lemma 3.1 *Under the assumption (3.1), for each $\gamma\in\Lambda_1,\gamma'\in\Lambda_1,|\gamma|+|\gamma'|\le 4$, we have the following estimates w.r.t. the weight $\mathbf{M}$,*

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}\left|Q\left(\partial^\gamma\mathbf{G},\partial^{\gamma'}\mathbf{G}\right)\right|^2}{\mathbf{M}}d\xi dxd\tau\\ &\le O(1)\delta_0^2\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left(|\partial^\gamma\mathbf{G}|^2+|\partial^{\gamma'}\mathbf{G}|^2\right)}{\mathbf{M}}d\xi dxd\tau,\end{aligned}\tag{3.5}$$

and for $|\gamma|>0$,

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}\left|Q\left(\partial^\gamma\mathbf{M},\partial^{\gamma'}\mathbf{G}\right)\right|^2}{\mathbf{M}}d\xi dxd\tau\\ &\le O(1)\delta_0^2\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau+O(1)\delta_0^2\sum_{|\alpha|\le|\gamma|-1}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2\,dxd\tau.\end{aligned}\tag{3.6}$$

The corresponding estimates w.r.t. the weight $\mathbf{M}_-$ are different when $|\gamma|\ge 3$ as follows.

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}\left|Q\left(\partial^\gamma\mathbf{G},\partial^{\gamma'}\mathbf{G}\right)\right|^2}{\mathbf{M}_-}d\xi dxd\tau\\ &\le\begin{cases}O(1)\delta_0^2\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left(|\partial^\gamma\mathbf{G}|^2+|\partial^{\gamma'}\mathbf{G}|^2\right)}{\mathbf{M}_-}d\xi dxd\tau, & \text{if } \max\{|\gamma|,|\gamma'|\}\le 2,\\ O(1)\delta_0^2\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_x^\alpha\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau, & \text{if } \max\{|\gamma|,|\gamma'|\}=|\gamma|\ge 3,\end{cases}\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}\left|Q\left(\partial^\gamma\mathbf{M},\partial^{\gamma'}\mathbf{G}\right)\right|^2}{\mathbf{M}_-}d\xi dxd\tau\\ &\le\begin{cases}O(1)\delta_0^2\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau, & \text{if } 0<|\gamma|\le 2,\\ O(1)\delta_0^2\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_x^\alpha\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau, & \\ \quad+O(1)\delta_0^2\sum_{|\alpha|\le|\gamma|-1}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2\,dxd\tau, & \text{if } |\gamma|\ge 3.\end{cases}\end{aligned}\tag{3.8}$$

Proof: We only prove (3.5) and (3.8), and the proofs for the others are similar. First, for (3.5), (3.4) together with the fact that $\frac{\theta}{2}<\theta_-<\theta$ imply

$$\sum_{\gamma\in\Lambda_2^1,|\gamma|\le 2}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\le\sum_{\gamma\in\Lambda_2^1,|\gamma|\le 2}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi\le O(1)\delta_0^2.$$

With this and Lemma 2.1, we have from $|\gamma|+|\gamma'|\le 4$ that

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}\left|Q\left(\partial^\gamma\mathbf{G},\partial^{\gamma'}\mathbf{G}\right)\right|^2}{\mathbf{M}}d\xi dxd\tau\le O(1)\int_0^t\int_{\mathbf{R}^3}\left(\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}}d\xi\right.\\ &\left.+\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\int_{\mathbf{R}^3}\frac{|\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dxd\tau\le O(1)\delta_0^2\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left(|\partial^\gamma\mathbf{G}|^2+|\partial^{\gamma'}\mathbf{G}|^2\right)}{\mathbf{M}}d\xi dxd\tau,\end{aligned}$$

which yields (3.5).

Now for (3.8), we have from Lemma 2.1 again that

$$\begin{aligned}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}\left|Q\left(\partial^\gamma\mathbf{M},\partial^{\gamma'}\mathbf{G}\right)\right|^2}{\mathbf{M}_-}d\xi dxd\tau &\le O(1)\int_0^t\int_{\mathbf{R}^3}\left(\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{M}|^2}{\mathbf{M}_-}d\xi\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right.\\ &\quad\left.+\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{M}|^2}{\mathbf{M}_-}d\xi\int_{\mathbf{R}^3}\frac{|\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)dxd\tau\\ &= I_1^{\gamma,\gamma'}+I_2^{\gamma,\gamma'}.\end{aligned}\tag{3.9}$$

If $1\le|\gamma|\le 2$, we have from $\gamma,\gamma'\in\Lambda_1$ and (3.4) that

$$I_1^{\gamma,\gamma'}+I_2^{\gamma,\gamma'}\le O(1)\delta_0^2\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau.$$

On the other hand, if $|\gamma|\ge 3$, we must have $|\gamma'|\le 1$ because $|\gamma|+|\gamma'|\le 4$. Thus from (3.4), the fact that $\frac{\theta}{2}<\theta_-<\theta$, $I_1^{\gamma,\gamma'}$ satisfies

$$\begin{aligned}I_2^{\gamma,\gamma'}&\le O(1)\delta_0^2\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{M}|^2}{\mathbf{M}_-}d\xi dxd\tau\\ &\le O(1)\delta_0^2\sum_{|\alpha|\le|\gamma|-1}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2dxd\tau.\end{aligned}$$

Here we have used the fact that $|\gamma|\ge 3$, the conservation laws (1.10) and (3.3).

For $I_1^{\gamma,\gamma'}$, by using the indentity

$$f^2(x_1,x_2,x_3)=2\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{x_3}(ff_{x_1})_{x_2x_3}\,dx_1dx_2dx_3\le\sum_{|\alpha|\le 3}\int_{\mathbf{R}^3}|\partial_x^\alpha f|^2dx,$$

we have from the a priori assumption (3.1) that

$$\int_0^t\left(\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)d\tau\le O(1)\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^{\gamma'}\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau.$$

Thus,

$$\begin{aligned}I_1^{\gamma,\gamma'}&\le O(1)\sup_{0\le\tau\le t,x\in\mathbf{R}^3}\left\{\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{M}|^2}{\mathbf{M}_-}d\xi dx\right\}\int_0^t\left(\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)d\tau\\ &\le O(1)\delta_0^2\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_x^\alpha\partial^{\gamma'}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau.\end{aligned}$$

The above estimates on $I_1^{\gamma,\gamma'}$ and $I_2^{\gamma,\gamma'}$ give (3.8) from (3.9). This completes the proof of Lemma 3.1.

Remark 3.1 *As a direct consequence of Lemma 3.1, we have the following estimates based on the bilinear forms of the $Q(f,g)$ and $\mathbf{L}_{\mathbf{M}}(h)$.*

$$\begin{aligned}&\sum_{\gamma\in\Lambda_1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}|\partial^\gamma Q(\mathbf{G},\mathbf{G})|^2}{\tilde{\mathbf{M}}}d\xi dxd\tau\\ &\le O(1)\delta_0^2\sum_{\gamma\in\Lambda_1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\tilde{\mathbf{M}}}d\xi dxd\tau,\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}
&\sum_{\gamma\in\Lambda_1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}|\partial^\gamma(L_{\mathbf{M}}\mathbf{G})-L_{\mathbf{M}}(\partial^\gamma\mathbf{G})|^2}{\tilde{\mathbf{M}}}d\xi dxd\tau\\
&\le O(1)\delta_0^2\sum_{\gamma\in\Lambda_1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\tilde{\mathbf{M}}}d\xi dxd\tau\\
&\quad+O(1)\delta_0^2\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2dxd\tau.
\end{aligned}\tag{3.11}$$

Here $\tilde{\mathbf{M}}$ *can be taken as* $\mathbf{M}$ *or* $\mathbf{M}_-$.

With the above estimates, we now give the energy estimates on the solution in the following three subsections. The first one is on the estimates on the entropy $\eta(\rho,u,\theta)$ and the non-fluid component $\mathbf{G}$. And the other two are on the derivatives w.r.t. the weight of the local Maxwellian $\mathbf{M}$, and the derivatives w.r.t. the weight of the global Maxwellian $\mathbf{M}_-$, respectively.

3.1 Lower order estimates

In this subsection, we will give the energy estimates on the entropy $\eta(\rho,u,\theta)$ and the non-fluid component $\mathbf{G}(t,x,\xi)$.

First, integrating (2.12) w.r.t. x over $\mathbf{R}^3$ yields

$$\int_{\mathbf{R}^3}\left(\eta+\frac{3\lambda}{4}|\nabla_x\Phi|^2\right)dx\Big|_0^t=\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\mathbf{G}}{\mathbf{M}}d\xi dxd\tau.\tag{3.12}$$

From (1.12) and the fact that there exists a positive constant $C>0$ such that

$$\begin{aligned}
-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\Big)}{\mathbf{M}}d\xi dxd\tau &\ge C\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})|^2}{\mathbf{M}}d\xi dxd\tau\\
&\ge C\int_0^t\int_{\mathbf{R}^3}|\nabla_x(u,\theta)|^2dxd\tau,
\end{aligned}$$

we have from Lemma 2.1 and Corollary 2.1 that

$$\begin{aligned}
&\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\mathbf{G}}{\mathbf{M}}d\xi dxd\tau\\
&=\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\Big)}{\mathbf{M}}d\xi dxd\tau\\
&\quad+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})L_{\mathbf{M}}^{-1}(\Theta)}{\mathbf{M}}d\xi dxd\tau\\
&\le -C\int_0^t\int_{\mathbf{R}^3}|\nabla_x(u,\theta)|^2dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\mathbf{G}_t|^2+|\nabla_x\mathbf{G}|^2+\delta_0^2\Big(|\nabla_\xi\mathbf{G}|^2+|\mathbf{G}|^2\Big)\right)d\xi dxd\tau.
\end{aligned}\tag{3.13}$$

Substituting (3.13) into (3.12) yields

$$\begin{aligned}
&\int_{\mathbf{R}^3}\left(\eta+\lambda|\nabla_x\Phi|^2\right)(t)dx+\int_0^t\int_{\mathbf{R}^3}|\nabla_x(u,\theta)|^2dxd\tau\\
&\le O(1)N(0)^2+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\mathbf{G}_t|^2+|\nabla_x\mathbf{G}|^2+\delta_0^2\Big(|\nabla_\xi\mathbf{G}|^2+|\mathbf{G}|^2\Big)\right)d\xi dxd\tau.
\end{aligned}\tag{3.14}$$

For the non-fluid component $\mathbf{G}$, multiplying (1.11) by $\frac{\mathbf{G}}{\mathbf{M}}$ and integrating the result w.r.t. t, x, and ξ over $[0,t]\times\mathbf{R}^3\times\mathbf{R}^3$, we have from (1.15), Lemma 3.1, and Cauchy-Schwarz's inequality that

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
\leq\ & O(1)N(0)^2+O(1)\delta_0^2\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&+O(1)\int_0^t\int_{\mathbf{R}^3}\left(|\nabla_x(u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\mathbf{G}|^2}{\mathbf{M}}\right)dxd\tau.
\end{aligned}
\tag{3.15}
$$

Similarly, if we replace the weight $\mathbf{M}$ by the global Maxwellian $\mathbf{M}_-$, we have

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
\leq\ & O(1)N(0)^2+O(1)\int_0^t\int_{\mathbf{R}^3}\left(|\nabla_x(u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\mathbf{G}|^2}{\mathbf{M}_-}\right)dxd\tau.
\end{aligned}
\tag{3.16}
$$

(3.14)-(3.16) give the complete lower order energy estimates.

3.2 Higher order energy estimates w.r.t. M

In this subsection, we will consider higher order energy estimates, i.e., $\partial^\gamma\mathbf{M}$, $\partial^\gamma\mathbf{G}$, and $\partial^\gamma f$ for $|\gamma|\geq 1$ w.r.t. the local Maxwellian $\mathbf{M}$. Since the proofs are tedious and technical, we put their proofs in the appendix for the brevity of the presentation.

First, for $\partial_x^\alpha\mathbf{M}$ with $1\leq|\alpha|\leq 3$, we have the following lemma.

Lemma 3.2 *Under the assumptions in Lemma 3.1, we have for $j=1,2,3$ that*

$$
\begin{aligned}
&\sum_{|\alpha|=j}\int_{\mathbf{R}^3}\left(\lambda\left|\nabla_x\partial_x^\alpha\Phi\right|^2+\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha\mathbf{M}|^2}{\mathbf{M}}d\xi\right)dx+\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha(u,\theta)\right|^2dxd\tau\\
\leq\ & O(1)N(0)^2+O(1)\delta_0\sum_{|\alpha|\leq j}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha(\rho,u,\theta)\right|^2dxd\tau\\
&+O(1)\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left(|\partial_x^\alpha\mathbf{G}_t|^2+|\nabla_x\partial_x^\alpha\mathbf{G}|^2\right)}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{|\alpha|\leq j}|\nabla_\xi\partial_x^\alpha\mathbf{G}|^2+\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha\mathbf{G}|^2\right)d\xi dxd\tau\\
&+O(1)\delta_{1,j}\sum_{|\alpha|=1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha\Phi\right|^2dxd\tau,
\end{aligned}
\tag{3.17}
$$

where we have used the assumption that $\delta_0\lambda<1$. Here and in the following $\delta_{i,j}$ is the Kronecker symbol, i.e.,

$$
\delta_{i,j}=\begin{cases}0, & i\neq j,\\ 1, & i=j.\end{cases}
$$

Secondly, for $\partial^\gamma\mathbf{G}$ with $\gamma\in\Lambda_1, |\gamma|\leq 3$, we have the following lemma.

Lemma 3.3 *Under the assumptions in Lemma 3.1, we have for $j=0,1,2$ that*

$$
\begin{aligned}
&\sum_{|\alpha|=j}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leq O(1)N(0)^2+O(1)\delta_0\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|=j+1}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\delta_0\sum_{|\alpha|\leq j}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2\right)dxd\tau\\
&\quad+O(1)\delta_0\sum_{|\alpha|\leq j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\partial_x^\alpha\mathbf{G}|^2+|\nabla_\xi\partial_x^\alpha\mathbf{G}|^2\right)d\xi dxd\tau,
\end{aligned}
\tag{3.18}
$$

and

$$
\begin{aligned}
&\sum_{|\alpha|=j}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leq O(1)N(0)^2+O(1)\delta_0\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|=j+1}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\delta_0\sum_{|\alpha|\leq j}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2\right)dxd\tau\\
&\quad+O(1)\sum_{|\alpha|=j+1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_t\partial_x^\alpha\mathbf{G}|^2\right)d\xi dxd\tau\\
&\quad+O(1)\delta_0\sum_{|\alpha|\leq j}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\partial_t\partial_x^\alpha\mathbf{G}|^2+|\nabla_x\partial_x^\alpha\mathbf{G}|^2\right)d\xi dxd\tau\\
&\quad+O(1)\delta_{1,j+1}\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\Phi|^2dxd\tau\\
&\quad+O(1)\left(1-\delta_{1,j+1}\right)\lambda^{-2}\sum_{|\alpha|=j}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\rho|^2dxd\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{|\alpha|\leq j}|\nabla_\xi\partial_x^\alpha\mathbf{G}|^2+\sum_{|\alpha|<j}|\nabla_\xi\partial_t\partial_x^\alpha\mathbf{G}|^2\right)d\xi dxd\tau.
\end{aligned}
\tag{3.19}
$$

A suitable linear combination of (3.17), (3.18), and (3.19) yields the following estimates.

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\left(\lambda\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2+\int_{\mathbf{R}^3}\frac{|\nabla_x\mathbf{M}|^2+|\nabla_x\mathbf{G}|^2+|\mathbf{G}_t|^2}{\mathbf{M}}d\xi\right)dx\\
&+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|=1}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\nabla_x\mathbf{G}|^2+|\mathbf{G}_t|^2\right)d\xi\right)dxd\tau\\
&\leq O(1)N(0)^2+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\mathbf{G}|^2+|\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\sum_{|\alpha|=1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\partial_t\partial_x^\alpha\mathbf{G}|^2+|\nabla_x\partial_x^\alpha\mathbf{G}|^2\right)d\xi dxd\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 1}|\nabla_x\partial_x^\alpha\rho|^2+|\nabla_x(u,\theta)|^2\right)dxd\tau+O(1)\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\Phi|^2dxd\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{|\beta|\leq 1}|\nabla_x\partial_\xi^\beta\mathbf{G}|^2+\sum_{|\alpha|\leq 4}|\partial_x^\alpha\mathbf{G}|^2+|\nabla_\xi\mathbf{G}|^2+|\nabla_\xi\mathbf{G}_t|^2\right)d\xi dxd\tau,
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\left(\lambda\sum_{|\alpha|=3}|\partial_x^\alpha\Phi|^2+\sum_{|\alpha|=1}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{M}|^2+|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2}{\mathbf{M}}d\xi\right)dx\\
&+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|=2}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\sum_{|\alpha|=1}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\right)d\xi\right)dxd\tau\\
\leq{}& O(1)N(0)^2+O(1)\delta_0\sum_{|\alpha|=1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&+O(1)\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\partial_t\partial_x^\alpha\mathbf{G}|^2+|\nabla_x\partial_x^\alpha\mathbf{G}|^2\right)d\xi dxd\tau\\
&+O(1)\left(\delta_0+\lambda^{-2}\right)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq2}|\nabla_x\partial_x^\alpha\rho|^2+\sum_{|\alpha|\leq1}|\nabla_x\partial_x^\alpha(u,\theta)|^2\right)dxd\tau\\
&+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{|\alpha|\leq2}|\nabla_\xi\partial_x^\alpha\mathbf{G}|^2+\sum_{|\alpha|\leq4}|\partial_x^\alpha\mathbf{G}|^2+\sum_{|\beta|\leq1}|\partial_\xi^\beta\mathbf{G}_t|^2\right)d\xi dxd\tau,
\end{aligned}
\tag{3.21}
$$

and

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\left(\lambda\sum_{|\alpha|=4}|\partial_x^\alpha\Phi|^2+\sum_{|\alpha|=2}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{M}|^2+|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2}{\mathbf{M}}d\xi\right)dx\\
&+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|=3}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\sum_{|\alpha|=2}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\right)d\xi\right)dxd\tau\\
\leq{}& O(1)N(0)^2+O(1)\delta_0\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&+O(1)\sum_{|\alpha|=3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\partial_t\partial_x^\alpha\mathbf{G}|^2+|\nabla_x\partial_x^\alpha\mathbf{G}|^2\right)d\xi dxd\tau\\
&+O(1)\lambda^{-2}\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\rho|^2dxd\tau+O(1)\delta_0\sum_{|\alpha|\leq3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2dxd\tau\\
&+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{|\alpha|\leq3}\left(|\nabla_\xi\partial_x^\alpha\mathbf{G}|^2+|\nabla_x\partial_x^\alpha\mathbf{G}|^2\right)\right.\\
&\qquad\left.+\sum_{|\alpha|\leq1}\left(|\partial_x^\alpha\mathbf{G}_t|^2+|\nabla_\xi\partial_x^\alpha\mathbf{G}_t|^2\right)\right)d\xi dxd\tau.
\end{aligned}
\tag{3.22}
$$

Combining (3.14), (3.15), (3.20), (3.21), and (3.22) gives the following estimate.

Corollary 3.1 *Under the assumptions listed in Lemma 3.1, we have*

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\left(\eta+\lambda\sum_{1\leq|\alpha|\leq4}|\partial_x^\alpha\Phi|^2+\int_{\mathbf{R}^3}\frac{1}{\mathbf{M}}\left(\sum_{|\alpha|\leq2}|\nabla_x\partial_x^\alpha\mathbf{M}|^2+\sum_{\gamma\in\Lambda_4}|\partial^\gamma\mathbf{G}|^2\right)d\xi\right)dx\\
&+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq3}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\sum_{\gamma\in\Lambda_4}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dxd\tau\\
\leq{}& O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&+O(1)\sum_{|\alpha|=3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\right)d\xi dxd\tau\\
&+O(1)\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\Phi|^2dxd\tau+O(1)\left(\delta_0+\lambda^{-2}\right)\sum_{|\alpha|\leq3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau\\
&+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{|\alpha|\leq3}|\nabla_\xi\partial_x^\alpha\mathbf{G}|^2+\sum_{|\alpha|\leq1}|\nabla_\xi\partial_x^\alpha\mathbf{G}_t|^2+\sum_{|\beta|\leq1}|\partial_\xi^\beta\mathbf{G}_t|^2\right)d\xi dxd\tau.
\end{aligned}
\tag{3.23}
$$

Now we turn to the estimates including the derivatives w.r.t. the velocity ξ on the solutions $f(t,x,\xi)$ to the Vlasov-Poisson-Boltzmann system (1.2). It is clear that we only need to obtain the corresponding estimates on the non-fluid component. The main idea is to reduce the derivatives w.r.t. ξ to the estimates on the derivatives of the non-fluid part w.r.t. the space and time variables. The following lemma is about the estimates on $\partial^\gamma \mathbf{G}$ with $\gamma \in \Lambda_3$.

Lemma 3.4 *Under the assumptions in Lemma 3.1, we have*

$$\begin{aligned}
&\sum_{\gamma\in\Lambda_3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\sum_{\gamma\in\Lambda_3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leq O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\Phi|^2dxd\tau\\
&\quad+O(1)\sum_{|\alpha|\leq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\mathbf{G}|^2+\sum_{|\alpha|\leq 3}\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\right)\right)d\xi dxd\tau.
\end{aligned}\tag{3.24}$$

Note that the proof of the above lemma is based on the following three estimates on the derivatives of $\mathbf{G}$ w.r.t. ξ and their proofs can be found in the Appendix.

$$\begin{aligned}
&\sum_{|\beta|\leq 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_\xi^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\sum_{|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_\xi^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leq O(1)N(0)^2+O(1)\delta_0\sum_{|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_\xi^\beta\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}^3}|\nabla_x(u,\theta)|^2dxd\tau\\
&\quad+O(1)\sum_{|\alpha|\leq 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau,
\end{aligned}\tag{3.25}$$

$$\begin{aligned}
&\sum_{\gamma\in\Lambda_5}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\sum_{\gamma\in\Lambda_5}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leq O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_5}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{1\leq|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\delta_0\left(|\nabla_x(u,\theta)|^2+\sum_{|\alpha|\leq 2}|\nabla_x\partial_x^\alpha\rho|^2\right)\right)dxd\tau\\
&\quad+O(1)\sum_{1\leq|\alpha|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\delta_0\sum_{1\leq|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_\xi^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau,
\end{aligned}\tag{3.26}$$

and

$$
\begin{aligned}
&\sum_{\gamma\in\Lambda_6}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}}d\xi dx+\sum_{\gamma\in\Lambda_6}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leq O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_6}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau+O(1)\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\Phi|^2dxd\tau\\
&\quad+O(1)\sum_{|\alpha|\leq 3}\int_0^t\int_{\mathbf{R}^3}\left(|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\right)}{\mathbf{M}}d\xi\right)dxd\tau\\
&\quad+O(1)\delta_0\sum_{|\beta|\geq 1,|\alpha|+|\beta|\leq 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_x^\alpha\partial_\xi^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau.
\end{aligned}
\tag{3.27}
$$

The following is a direct consequence of (3.23) and (3.24).

Corollary 3.2 *Under the assumptions listed in Lemma 3.1, we can deduce that*

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\left(\eta+\lambda\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha\Phi|^2+\int_{\mathbf{R}^3}\frac{1}{\mathbf{M}}\left(\sum_{|\alpha|\leq 2}|\nabla_x\partial_x^\alpha\mathbf{M}|^2+\sum_{\gamma\in\Lambda_3\cup\Lambda_4}|\partial^\gamma\mathbf{G}|^2\right)d\xi\right)dx\\
&\quad+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\sum_{\gamma\in\Lambda_3\cup\Lambda_4}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dxd\tau\\
&\leq O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_3\cup\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\sum_{|\alpha|=3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\right)d\xi dxd\tau\\
&\quad+O(1)\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\Phi|^2dxd\tau\\
&\quad+O(1)\left(\delta_0+\lambda^{-2}\right)\sum_{|\alpha|\leq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau.
\end{aligned}
\tag{3.28}
$$

To obtain the 4-th order derivatives w.r.t. space variables on $\mathbf{G}$, we need to work on the original Vlasov-Poisson-Boltzmann equation to avoid the appearance of the 5-th order derivatives. This can be summarized in the following lemma.

Lemma 3.5 *Under the assumptions in Lemma 3.1, we have*

$$
\begin{aligned}
&\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma f|^2}{\mathbf{M}}d\xi dx+\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leq O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\left(\delta_0+\lambda^{-1}\right)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\mathbf{G}|^2+|\nabla_x\mathbf{G}|^2+|\mathbf{G}_t|^2\right)d\xi dxd\tau\\
&\quad+O(1)\delta_0\sum_{\gamma\in\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_\xi\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau.
\end{aligned}
\tag{3.29}
$$

Remark 3.2 *To obtain (3.29), we use the property that the fluid part and the non-fluid part are orthogonal w.r.t. the local Maxwellian* $\mathbf{M}$*, especially the folowing identity*

$$\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma\mathbf{M})\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi=\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma\mathbf{M})\partial^\gamma(Q(\mathbf{G},\mathbf{G}))}{\mathbf{M}}d\xi=0. \tag{3.30}$$

By (3.28) and (3.29), we can now complete the energy estimates w.r.t. $\mathbf{M}$. First, from

$$\mathbf{M}_t+\mathbf{P}_0\Big(\xi\cdot\nabla_x\mathbf{M}\Big)+\mathbf{P}_0\Big(\xi\cdot\nabla_x\mathbf{G}\Big)+\nabla_x\Phi\cdot\nabla_\xi\mathbf{M}=0,$$

we have

$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{M}_t|^2}{\mathbf{M}}d\xi dx\leq O(1)\int_{\mathbf{R}^3}\left(|\nabla_x\Phi|^2+\int_{\mathbf{R}^3}\frac{|\nabla_x\mathbf{M}|^2+|\nabla_x\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dx. \tag{3.31}$$

Secondly, we have from the conservation laws (1.10) that

$$\begin{aligned}\sum_{|\alpha|\leq3}\int_{\mathbf{R}^3}|\partial_t\partial_x^\alpha(\rho,u,\theta)|^2dx\leq{}&O(1)\sum_{|\alpha|\leq3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx\\&+O(1)\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq1}|\nabla_x\partial_x^\alpha\Phi|^2+\sum_{|\alpha|\leq3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2\right)dx,\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}&\sum_{|\alpha|\leq2}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_t\partial_x^\alpha(\rho,u,\theta)|^2dxd\tau\\&\leq O(1)\sum_{|\alpha|\leq3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{\nu(\xi)|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\&\quad+O(1)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2+\sum_{|\alpha|\leq3}|\nabla_x\partial_x^\alpha(u,\theta)|^2\right)dxd\tau\\&\quad+O(1)\delta_0\sum_{|\alpha|\leq3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau.\end{aligned} \tag{3.33}$$

On the other hand, since

$$\int_{\mathbf{R}^3}\frac{|\partial^\gamma f|^2}{\mathbf{M}}d\xi=\int_{\mathbf{R}^3}\frac{|\mathbf{P}_0(\partial^\gamma\mathbf{M})|^2+|\mathbf{P}_1(\partial^\gamma\mathbf{M})+\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\geq\int_{\mathbf{R}^3}\frac{|\mathbf{P}_0(\partial^\gamma\mathbf{M})|^2}{\mathbf{M}}d\xi, \tag{3.34}$$

we have by induction and the estimates (3.31) and (3.32) that

$$\begin{aligned}&\sum_{|\alpha|\leq3}\int_{\mathbf{R}^3}\left(|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+|\partial_t\partial_x^\alpha(\rho,u,\theta)|^2\right)dx\\&\leq O(1)\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq1}|\nabla_x\partial_x^\alpha\Phi|^2+\sum_{|\alpha|\leq3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{\gamma\in\Lambda_2^1}\int_{\mathbf{R}^3}\tfrac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dx\\&\leq O(1)\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{|\mathbf{P}_0(\partial^\gamma\mathbf{M})|^2}{\mathbf{M}}d\xi dx+O(1)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{|\nabla_x\mathbf{M})|^2}{\mathbf{M}}d\xi dx\\&\quad+O(1)\sum_{\gamma\in\Lambda_2^1}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx+O(1)\sum_{|\alpha|\leq1}\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\Phi|^2dx.\end{aligned} \tag{3.35}$$

Multiplying (3.28) and (3.29) by λ_1 and λ_1^2 respectively, adding the results, and by choosing $\lambda_1 > 0$ sufficiently large, we have from (3.31), (3.33), (3.34), and (3.35) that

$$\begin{aligned}
&\int_{\mathbf{R}^3}\left(\eta+\lambda\sum_{|\alpha|\le 3}|\nabla_x\partial_x^\alpha\Phi|^2+\sum_{|\alpha|\le 3}\left(|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+|\partial_t\partial_x^\alpha(\rho,u,\theta)|^2\right)\right)dx\\
&+\sum_{\gamma\in\Lambda_2^1}\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\right)\\
&+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\le 3}|\nabla_x\partial_x^\alpha(u,\theta)|^2+\sum_{|\alpha|\le 2}|\nabla_x\partial_t\partial_x^\alpha(u,\theta)|^2\right)dxd\tau\\
\le\ & O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_2^1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\tfrac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&+O(1)\sum_{|\alpha|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^\alpha\Phi|^2dxd\tau+O(1)(\delta_0+\lambda^{-1})\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau.
\end{aligned}\tag{3.36}$$

To recover the estimates on $\nabla_x\partial_x^\alpha\rho$ in (3.36), we need to use the conservation laws (1.10) as in [15]. For this, since

$$\frac{2\theta}{3\rho}\nabla_x\rho-\nabla_x\Phi=-u_t-u\cdot\nabla_x u-\frac{2}{3}\nabla_x\theta-\int_{\mathbf{R}^3}\frac{\psi(\xi\cdot\nabla_x\mathbf{G})}{\rho}d\xi,\tag{3.37}$$

for $|\alpha|\le 3$, we have

$$\begin{aligned}
&\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}\tfrac{2\theta}{3\rho}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau\\
=\ &\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\rho\cdot\nabla_x\partial_x^\alpha\Phi dxd\tau\\
&-\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\rho\cdot\partial_x^\alpha u_t dxd\tau\\
&-\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\rho\cdot\partial_x^\alpha(u\cdot\nabla_x u)dxd\tau\\
&-\tfrac{2}{3}\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\rho\cdot\nabla_x\partial_x^\alpha\theta dxd\tau\\
&-\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\rho\psi\xi\cdot\partial_x^\alpha\left(\tfrac{\nabla_x\mathbf{G}}{\rho}\right)dxd\tau\\
&-\sum_{|\alpha|\le 3}\left(1-\delta_{1,|\alpha|+1}\right)\sum_{0<\alpha'\le\alpha}C_\alpha^{\alpha'}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha\rho\cdot\nabla_x\partial_x^{\alpha-\alpha'}\rho\partial_x^{\alpha'}\left(\tfrac{2\theta}{3\rho}\right)dxd\tau\\
=\ &\sum_{j=1}^{6}I_j,
\end{aligned}\tag{3.38}$$

where I_js are the corresponding terms in the above equation.

Now we estimate $I_j(1\le j\le 6)$ term by term. First, for I_1, we have from the Poisson equation $(1.2)_2$ that

$$\begin{aligned}
I_1&=\lambda\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}\nabla_x\partial_x^\alpha(\Delta_x\Phi)\cdot\nabla_x\partial_x^\alpha\Phi dxd\tau\\
&=-\tfrac{\lambda}{2}\sum_{1\le|\alpha|\le 4}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\Phi|^2dxd\tau.
\end{aligned}\tag{3.39}$$

For I_2, if $|\alpha|=0$, we have

$$\begin{aligned}
I_2 &= -\int_0^t\int_{\mathbf{R}^3}\nabla_x\rho\cdot u_t dxd\tau \\
&= -\int_{\mathbf{R}^3}u\cdot\nabla_x\rho dx\Big|_0^t+\int_0^t\int_{\mathbf{R}^3}u\cdot\nabla_x\rho_t dxd\tau \\
&= -\int_{\mathbf{R}^3}u\cdot\nabla_x\rho dx\Big|_0^t+\int_0^t\int_{\mathbf{R}^3}\mathrm{div}_x u\,\mathrm{div}_x m dxd\tau \\
&\le O(1)N(0)^2+O(1)\int_{\mathbf{R}^3}|(u,\nabla_x\rho)|^2dx \\
&\quad +O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}|\nabla_x\rho|^2dxd\tau+O(1)\int_0^t\int_{\mathbf{R}^3}|\nabla_x u|^2dxd\tau.
\end{aligned}$$

And for $|\alpha|\ge 1$, we have

$$I_2\le\mu\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau+O(1)\sum_{|\alpha|\le 2}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_t\partial_x^\alpha u|^2dxd\tau.$$

Hence,

$$\begin{aligned}
I_2 &\le O(1)N(0)^2+O(1)\int_{\mathbf{R}^3}|(u,\nabla_x\rho)|^2dx \\
&\quad +(\mu+O(1)\delta_0)\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau \\
&\quad +O(1)\sum_{|\alpha|\le 2}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_t\partial_x^\alpha u|^2dxd\tau.
\end{aligned}\tag{3.40}$$

Similarly, we can prove

$$I_3\le O(1)\delta_0\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(\rho,u)|^2dxd\tau,\tag{3.41}$$

$$I_4\le\mu\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau+O(1)\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\theta|^2dxd\tau,\tag{3.42}$$

$$I_5\le\mu\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau+O(1)\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau,\tag{3.43}$$

and

$$I_6\le O(1)\delta_0\sum_{|\alpha|\le 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(\rho,\theta)|^2dxd\tau.\tag{3.44}$$

Substituting (3.39)-(3.44) into (3.38), we have by choosing $\mu > 0$ sufficiently small that

$$\begin{aligned}
&\sum_{|\alpha|\leq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\rho|^2dxd\tau+\lambda\sum_{1\leq|\alpha|\leq 4}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\Phi|^2dxd\tau\\
&\leq O(1)N(0)^2+O(1)\int_{\mathbf{R}^3}|(u,\nabla_x\rho)|^2dx\\
&\quad+O(1)\sum_{|\alpha|\leq 2}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_t\partial_x^\alpha u|^2dxd\tau\\
&\quad+O(1)\sum_{|\alpha|\leq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(u,\theta)|^2dxd\tau\\
&\quad+O(1)\sum_{|\alpha|\leq 3}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau.
\end{aligned}\tag{3.45}$$

If we choose $\lambda > 0$ sufficiently large, a suitably linear combination of (3.36) and (3.45) gives the following corollary.

Corollary 3.3 *There exists a sufficiently large positive constant $\lambda_0 > 1$ such that for $\lambda > \lambda_0$, we have*

$$\begin{aligned}
&\int_{\mathbf{R}^3}\left(\eta+\lambda\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha\Phi|^2+\sum_{|\alpha|\leq 3}\left(|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+|\partial_t\partial_x^\alpha(\rho,u,\theta)|^2\right)\right)dx\\
&+\sum_{\gamma\in\Lambda_2^1}\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\right)\\
&+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|\leq 2}|\nabla_x\partial_t\partial_x^\alpha(\rho,u,\theta)|^2+\lambda\sum_{1\leq|\alpha|\leq 4}|\nabla_x\partial_x^\alpha\Phi|^2\right)dxd\tau\\
&\leq O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_2^1}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau.
\end{aligned}\tag{3.46}$$

3.3 Higher order energy estimates w.r.t. $\mathbf{M}_-$

In this subsection, we will consider certain higher order energy estimates w.r.t. the global Maxwellian $\mathbf{M}_- = \mathbf{M}_{[\rho_-,0,\theta_-]}$ in order to close the estimate (3.46). Compared to those w.r.t. the local Maxwellian $\mathbf{M}$, the only difference is that the fluid part and the non-fluid part are no longer orthogonal w.r.t. the global Maxwellian $\mathbf{M}_-$, for example,

$$\left\{\begin{aligned}
&\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma\mathbf{M})\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}_-}d\xi\neq 0,\\
&\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma\mathbf{M})\partial^\gamma(Q(\mathbf{G},\mathbf{G}))}{\mathbf{M}_-}d\xi\neq 0.
\end{aligned}\right.\tag{3.47}$$

As a result, there is an extra error term in the form of

$$\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|\leq 2}|\nabla_x\partial_t\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{1\leq|\alpha|\leq 4}|\nabla_x\partial_x^\alpha\Phi|^2\right)dxd\tau.\tag{3.48}$$

Noticing this difference, we have by repeating the procedure for (3.46) to obtain

$$\begin{aligned}
&\int_{\mathbf{R}^3}\left(\eta+\lambda\sum_{|\alpha|\le 3}|\nabla_x\partial_x^\alpha\Phi|^2+\sum_{|\alpha|\le 3}\left(|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+|\partial_t\partial_x^\alpha(\rho,u,\theta)|^2\right)\right)dx\\
&\quad+\sum_{\gamma\in\Lambda_2^1}\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\right)\\
&\le O(1)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\le 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|\le 2}|\nabla_x\partial_t\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{1\le|\alpha|\le 4}|\nabla_x\partial_x^\alpha\Phi|^2\right)dxd\tau\\
&\quad+O(1)N(0)^2.
\end{aligned}\tag{3.49}$$

Combining (3.49) with (3.46), we finally obtain

$$\begin{aligned}
&\int_{\mathbf{R}^3}\left(\eta+\lambda\sum_{|\alpha|\le 3}|\nabla_x\partial_x^\alpha\Phi|^2+\sum_{|\alpha|\le 3}\left(|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+|\partial_t\partial_x^\alpha(\rho,u,\theta)|^2\right)\right)dx\\
&\quad+\sum_{\gamma\in\Lambda_2^1}\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\right)\\
&\quad+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\le 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|\le 2}|\nabla_x\partial_t\partial_x^\alpha(\rho,u,\theta)|^2+\lambda\sum_{1\le|\alpha|\le 4}|\nabla_x\partial_x^\alpha\Phi|^2\right)dxd\tau\\
&\le O(1)N(0)^2\le O(1)\mathcal{E}(f_0)^2,
\end{aligned}\tag{3.50}$$

which closes the a priori estimate (3.1) provided that $\varepsilon>0$ is sufficiently small satisfying

$$\begin{cases}\mathcal{E}(f_0)<\varepsilon,\\ O(1)\varepsilon^2<\delta_0^2.\end{cases}\tag{3.51}$$

Remark 3.3 *(3.50) and the Poisson equation $(1.2)_2$ imply that,*

$$\int_0^\infty\int_{\mathbf{R}^3}|\rho(t,x)-\overline{\rho}|^2\,dxd\tau\le O(1)\varepsilon^2,$$

which is not true for the Boltzmann equation without force.

4 The proof of Theorem 1.1

We are now ready to prove the main result Theorem 1.1. The idea is to use the continuity argument to extend the local solution to all time by the closed a priori estimate. To do so, we first need to have the local existence of solutions to the Vlasov-Poisson-Boltzmann system (1.2) in the space

$$\mathbf{H}^4_{x,\xi}([0,T])=\left\{f(t,x,\xi):\ \left|\ \begin{array}{c}\frac{\partial_x^\alpha\partial_\xi^\beta(f(t,x,\xi)-\overline{\mathbf{M}}(\xi))}{\sqrt{\mathbf{M}_-(\xi)}}\in C\left([0,T),L^2_{x,\xi}\left(\mathbf{R}^3\times\mathbf{R}^3\right)\right)\\ |\alpha|+|\beta|\le 4\end{array}\right.\right\}.\tag{4.1}$$

Here $T>0$ is some positive constant.

For periodic initial data, the corresponding local existence result was given in [11]. By a straightforward modification of the argument there, we have the following local existence result for the Vlasov-Poisson-Boltzmann system (1.2) on the whole space. Thus, we omit the proof for brevity.

Lemma 4.1 (Local existence) *For any sufficiently small constant $M>0$, there exists a positive constant $T^*(M)>0$ such that if*

$$\mathcal{E}(f_0)=\left\|\nabla_x\Delta_x^{-1}\Big(\rho_0(x)-\overline{\rho}\Big)\right\|_{L^2_x(\mathbf{R}^3)}+\sum_{|\alpha|+|\beta|\le 4}\left\|\frac{\partial_x^\alpha\partial_\xi^\beta(f_0(x,\xi)-\overline{\mathbf{M}}(\xi))}{\sqrt{\mathbf{M}_-(\xi)}}\right\|_{L^2_{x,\xi}(\mathbf{R}^3\times\mathbf{R}^3)}\le\frac{M}{2},$$

then there is a unique classical solution $f(t,x,\xi)\in\mathbf{H}^4_{x,\xi}([0,T^*(M)))$ to the Vlasov-Poisson-Boltzmann system (1.2) on $[0,T^*(M))\times\mathbf{R}^3\times\mathbf{R}^3$ such that $f(t,x,\xi)\ge 0$ and

$$\sup_{0\le t\le T^*(M)}\sum_{|\alpha|+|\beta|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha\partial_\xi^\beta(f(t,x,\xi)-\overline{M}(\xi))|^2}{\mathbf{M}_-(\xi)}d\xi dx\le M.$$

By using this local existence result and the energy estimates obtained in Section 3, we can conclude that the Vlasov-Poisson-Boltzmann system (1.2) has a unique global classical solution $f(t,x,\xi)\in\overline{\mathbf{H}}^4$ satisfying $f(t,x,\xi)\ge 0$.

To complete the proof of Theorem 1.1, we show that (1.19) holds. In fact, we have from (3.50) that

$$\begin{cases}\displaystyle\sum_{|\alpha|\le 3}\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\le O(1),\\ \displaystyle\sum_{|\alpha|\le 3}\int_0^\infty\left|\frac{d}{dt}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx\right|d\tau\le O(1)\sum_{|\alpha|\le 3}\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2}{\mathbf{M}_-}d\xi dxd\tau\\ \qquad\le O(1),\\ \displaystyle\sum_{|\alpha|\le 2}\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{M}|^2}{\mathbf{M}_-}d\xi dxd\tau\le O(1),\\ \displaystyle\sum_{|\alpha|\le 2}\int_0^\infty\left|\frac{d}{dt}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{M}|^2}{\mathbf{M}_-}d\xi dx\right|d\tau\le O(1)\sum_{|\alpha|\le 2}\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha\mathbf{M}|^2+|\nabla_x\partial_x^\alpha\mathbf{M}_t|^2}{\mathbf{M}_-}d\xi dxd\tau\\ \qquad\le O(1).\end{cases}\tag{4.2}$$

Consequently

$$\lim_{t\to\infty}\sum_{|\alpha|\le 2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^\alpha(\mathbf{M}-\overline{\mathbf{M}})|^2+|\nabla_x\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx=0.\tag{4.3}$$

Since

$$\begin{aligned}\int_{\mathbf{R}^3}\frac{|\mathbf{M}-\overline{\mathbf{M}}|^2+|\mathbf{G}|^2}{\mathbf{M}_-}d\xi\le{}& O(1)\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{M}-\overline{\mathbf{M}}|^2+|\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)^{\frac12}\left(\sum_{|\alpha|=3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha\mathbf{M}|^2+|\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)^{\frac12}\\ &+O(1)\left(\sum_{|\alpha|=1}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha\mathbf{M}|^2+|\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)^{\frac12}\left(\sum_{|\alpha|=2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha\mathbf{M}|^2+|\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)^{\frac12},\end{aligned}$$

we have from (4.3) that

$$\lim_{t\to\infty}\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}\left(\frac{|\mathbf{M}-\overline{\mathbf{M}}|^2+|\mathbf{G}|^2}{\mathbf{M}_-}\right)(t,x,\xi)d\xi=0.\tag{4.4}$$

Thus

$$\lim_{t\to\infty}\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|f(t,x,\xi)-\overline{\mathbf{M}}(\xi)|^2}{\mathbf{M}_-(\xi)}d\xi \le O(1)\lim_{t\to\infty}\sup_{x\in\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{M}(t,x,\xi)-\overline{\mathbf{M}}(\xi)|^2+|\mathbf{G}(t,x,\xi)|^2}{\mathbf{M}_-(\xi)}d\xi = 0,$$

which is (1.19). And this completes the proof of Theorem 1.1.

Finally, we estimate the $\mathbf{H}^4_{t,x,\xi}(\mathbf{R}^3\times\mathbf{R}^3)$ norm of the solution thus obtained. That is, we want to prove the estimate (1.20) in Remark 1.1.

By (3.3) and (3.50), we have from the conservation laws (1.10) that

$$\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\le 3}|\partial_t\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|\le 2}|\partial_t^2\partial_x^\alpha(\rho,u,\theta)|^2\right)dx\le O(1). \tag{4.5}$$

Thus for any fixed $T>0$, the global solution $f(t,x,\xi)$ to the Vlasov-Poisson-Boltzmann system (1.2) satisfies

$$\int_0^T\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\le 3}|\partial_t\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|\le 2}|\partial_t^2\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{\gamma\in\Lambda_2^1}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)dxd\tau \le C(T). \tag{4.6}$$

Based on (4.6), similar to the proof for (3.28), we have

$$\sum_{\gamma\in\Lambda_2^2}\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma(f-\overline{\mathbf{M}})|^2}{\mathbf{M}_-}d\xi dx+\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma(f-\overline{\mathbf{M}})|^2}{\mathbf{M}_-}d\xi dxd\tau\right)\le C(T). \tag{4.7}$$

(4.7) together with (3.50) imply

$$\sum_{\gamma\in\Lambda_2^2}\int_{\mathbf{R}^3}\left(|\partial^\gamma(\rho-\overline{\rho},u,\theta-\overline{\theta})|^2+\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)dx+\sum_{\gamma\in\Lambda_2^2}\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau \le C(T). \tag{4.8}$$

This with the conservation laws (1.10) yields

$$\begin{aligned}&\int_{\mathbf{R}^3}\left(\sum_{\gamma\in\Lambda_2^3}|\partial^\gamma(\rho-\overline{\rho},u,\theta-\overline{\theta})|^2+\sum_{\gamma\in\Lambda_2^2}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)dx\\&\quad+\sum_{\gamma\in\Lambda_2^2}\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\&\le C(T).\end{aligned} \tag{4.9}$$

Therefore,

$$\begin{aligned}&\int_0^T\int_{\mathbf{R}^3}\left(\sum_{\gamma\in\Lambda_2^3}|\partial^\gamma(\rho-\overline{\rho},u,\theta-\overline{\theta})|^2+\sum_{\gamma\in\Lambda_2^2}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi\right)dx\\&\quad+\sum_{\gamma\in\Lambda_2^2}\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\&\le C(T).\end{aligned} \tag{4.10}$$

With (4.10), the similar argument for (3.28) leads to

$$\sum_{\gamma\in\Lambda_2^3}\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma(f-\overline{\mathbf{M}})|^2}{\mathbf{M}_-}d\xi dx+\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma(f-\overline{\mathbf{M}})|^2}{\mathbf{M}_-}d\xi dxd\tau\right)\le C(T). \tag{4.11}$$

Moreover, the same argument gives

$$\sum_{\gamma\in\Lambda_2^4}\left(\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma(f-\overline{\mathbf{M}})|^2}{\mathbf{M}_-}d\xi dx+\int_0^T\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma(f-\overline{\mathbf{M}})|^2}{\mathbf{M}_-}d\xi dxd\tau\right)\le C(T), \tag{4.12}$$

which is exactly (1.20).

5 Appendix

In the last section, we will give the proofs of the Lemma 3.2-3.5. Since the proof of Lemma 3.4 is essentially the same as the one for Lemma 3.3, we will only prove Lemma 3.2, 3.3, and 3.5 in the following subsections respectively.

5.1 The proof of Lemma 3.2

Since the local Maxwellian $\mathbf{M}$ solves

$$\mathbf{M}_t+\mathbf{P}_0\Big(\xi\cdot\nabla_x\mathbf{M}\Big)+\nabla_x\Phi\cdot\nabla_\xi\mathbf{M}=-\mathbf{P}_0\left(\xi\cdot\nabla_x\left(L_{\mathbf{M}}^{-1}\left(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\right)\right)\right)-\mathbf{P}_0\left(\xi\cdot\nabla_x\Theta\right),$$

we have by applying $\partial_x^\alpha(1\le|\alpha|\le 3)$ to the above equation and integrating its product with $\frac{\partial^\alpha\mathbf{M}}{\mathbf{M}}$ over $[0,t]\times\mathbf{R}^3\times\mathbf{R}^3$ that

$$\begin{aligned}\frac12\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha\mathbf{M}|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}=&-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\Big(\frac{|\partial_x^\alpha\mathbf{M}|^2}{\mathbf{M}^2}\mathbf{M}_t+\frac{\partial_x^\alpha\mathbf{M}}{\mathbf{M}}\partial_x^\alpha[\mathbf{P}_0(\xi\cdot\nabla_x\mathbf{M})]\Big)d\xi dxd\tau\\&-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\mathbf{M}\partial_x^\alpha\big\{\mathbf{P}_0\big\{\xi\cdot\nabla_x\big[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))\big]\big\}\big\}}{\mathbf{M}}d\xi dxd\tau\\&-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\mathbf{M}\partial_x^\alpha\big(\nabla_x\Phi\cdot\nabla_\xi\mathbf{M}\big)}{\mathbf{M}}d\xi dxd\tau\\&-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_x^\alpha\mathbf{M}\partial_x^\alpha\{\mathbf{P}_0(\xi\cdot\nabla_x\Theta)\}}{\mathbf{M}}d\xi dxd\tau\\:=&\sum_{j=7}^{10}I_j,\end{aligned}\tag{5.1}$$

where I_7-I_{10} are the corresponding terms in the equation.

In the following, we estimate $I_j(j=7,8,9,10)$ term by term. First, the a priori estimate (3.1) and properties of the operators $\mathbf{P}_0$ and $\mathbf{P}_1$ give

$$|I_7|\le O(1)\delta_0\sum_{|\alpha'|\le|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau. \tag{5.2}$$

Notice that

$$\begin{aligned}I_8=&-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial_x^\alpha\mathbf{M})\partial_x^\alpha\big\{\mathbf{P}_0\big\{\xi\cdot\nabla_x\big[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))\big]\big\}\big\}}{\mathbf{M}}d\xi dxd\tau\\&-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\partial_x^\alpha\mathbf{M})\partial_x^\alpha\big\{\mathbf{P}_0\big\{\xi\cdot\nabla_x\big[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))\big]\big\}\big\}}{\mathbf{M}}d\xi dxd\tau\\:=&\,I_8^1+I_8^2,\end{aligned}$$

and

$$\partial_x^\alpha\left\{L_{\mathbf{M}}^{-1}h\right\} = L_{\mathbf{M}}^{-1}(\partial_x^\alpha h) - \sum_{j=0}^{|\alpha|-1}\sum_{|\alpha_j|=j} C_{\alpha_j} L_{\mathbf{M}}^{-1}\left(Q\left(\partial_x^{\alpha_j}\left(L_{\mathbf{M}}^{-1}h\right), \partial_x^{\alpha-\alpha_j}\mathbf{M}\right)\right),$$

where C_{α_j} are some positive constants. Then,

$$\begin{aligned}
I_8^1 &= -\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial_x^\alpha\mathbf{M})\partial_x^\alpha\left\{\xi\cdot\nabla_x\left[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))\right]\right\}}{\mathbf{M}}d\xi dx d\tau\\
&= \int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1[\xi\cdot\nabla_x(\mathbf{P}_0(\partial_x^\alpha\mathbf{M}))]\partial_x^\alpha\left\{L_{\mathbf{M}}^{-1}[\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})]\right\}}{\mathbf{M}}d\xi dx d\tau\\
&\quad+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial_x^\alpha\mathbf{M})\partial_x^\alpha\left\{L_{\mathbf{M}}^{-1}[\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})]\right\}}{\mathbf{M^2}}\xi\cdot\nabla_x\mathbf{M}d\xi dx d\tau\\
&\le -d\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha(u,\theta)\right|^2 dx d\tau + O(1)\delta_0\sum_{|\alpha'|\le|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2 dx d\tau,
\end{aligned}$$

and

$$\begin{aligned}
I_8^2 &= \int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_x^{\alpha_1'}(\mathbf{P}_1(\partial_x^\alpha\mathbf{M}))\partial_x^{\alpha-\alpha_1'}\left\{\mathbf{P}_0\left\{\xi\cdot\nabla_x\left[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))\right]\right\}\right\}}{\mathbf{M}}d\xi dx d\tau\\
&\quad-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\partial_x^\alpha\mathbf{M})\partial_x^{\alpha-\alpha_1'}\left\{\mathbf{P}_0\left\{\xi\cdot\nabla_x\left[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))\right]\right\}\right\}}{\mathbf{M}^2}\partial_x^{\alpha_1'}\mathbf{M}d\xi dx d\tau\\
&\le \tfrac{d}{3}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha(u,\theta)\right|^2 dx d\tau + O(1)\delta_0\sum_{|\alpha'|\le|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2 dx d\tau.
\end{aligned}$$

Here d is a positive constant coming from the microscopic $H-$theorem (1.15) and $\alpha_1' = (1,0,0)$, $(01,0)$ or $(0,0,1)$ depending on α.

Hence

$$\begin{aligned}
I_8 &\le -\tfrac{2d}{3}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha(u,\theta)\right|^2 dx d\tau\\
&\quad+O(1)\delta_0\sum_{|\alpha'|\le|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2 dx d\tau.
\end{aligned}\tag{5.3}$$

As to I_{10}, from Lemma 3.1 and the a priori estimate (3.1), there exists a sufficiently small constant $\mu > 0$ such that

$$\begin{aligned}
|I_{10}| &\le \mu\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^\alpha(u,\theta)\right|^2 dx d\tau + O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\left(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\right)}{\mathbf{M}}d\xi dx d\tau\\
&\quad+O(1)\delta_0\sum_{|\alpha'|\le|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial^{\alpha'}(\rho,u,\theta)\right|^2 dx d\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{\alpha'\le\alpha}|\nabla_\xi\partial_x^{\alpha'}\mathbf{G}|^2 + \sum_{|\alpha'|\le 3}|\nabla_x\partial_x^{\alpha'}\mathbf{G}|^2\right)d\xi dx d\tau.
\end{aligned}\tag{5.4}$$

Finally, we estimate I_9. For this, we first notice that

$$\begin{aligned}
I_9 &= -\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial_x^\alpha\mathbf{M})\partial_x^\alpha\left(\nabla_x\Phi\cdot\nabla_\xi\mathbf{M}\right)}{\mathbf{M}}d\xi dx d\tau\\
&\quad-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\partial_x^\alpha\mathbf{M})\partial_x^\alpha\left(\nabla_x\Phi\cdot\nabla_\xi\mathbf{M}\right)}{\mathbf{M}}d\xi dx d\tau\\
&= J_9^1 + J_9^2.
\end{aligned}\tag{5.5}$$

It is straightforward to show that

$$J_9^2 \le O(1)\delta_0 \sum_{|\alpha'|\le|\alpha|-1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(\rho,u,\theta)\right|^2 dxd\tau. \tag{5.6}$$

And J_9^1 can be estimated as follows.

Note that

$$\begin{aligned} J_9^1 &= -\sum_{\alpha'\le\alpha} C_\alpha^{\alpha'} \int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\mathbf{P}_0(\partial_x^\alpha \mathbf{M})}{\mathbf{M}} \left(\nabla_x \partial_x^{\alpha'}\Phi \cdot \nabla_\xi \partial_x^{\alpha-\alpha'}\mathbf{M}\right) d\xi dxd\tau \\ &= \sum_{\alpha'\le\alpha} J_9^{1,\alpha'}. \end{aligned} \tag{5.7}$$

For $\alpha' < \alpha$, we have

$$J_9^{1,\alpha'} \le O(1)\delta_0 \sum_{|\alpha'|\le|\alpha|-1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(\rho,u,\theta)\right|^2 dxd\tau. \tag{5.8}$$

For the case $\alpha' = \alpha$, we have

$$\begin{aligned} J_9^{1,\alpha} &= -\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\mathbf{P}_0(\partial_x^\alpha \mathbf{M})}{\mathbf{M}} \left(\nabla_x \partial_x^\alpha\Phi \cdot \nabla_\xi \mathbf{M}\right) d\xi dxd\tau \\ &\le -\int_0^t \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \left(\nabla_x \partial_x^\alpha\Phi \cdot \nabla_\xi \mathbf{M}\right) \left[\left(\frac{\partial_x^\alpha\rho}{\rho} - \frac{3\partial_x^\alpha\theta}{2\theta}\right) + \frac{3}{2\theta}\left((\xi-u)\cdot\partial_x^\alpha u + \frac{|\xi-u|^2}{2\theta}\partial_x^\alpha\theta\right)\right] d\xi dxd\tau \\ &\quad + O(1)\delta_0 \sum_{|\alpha'|\le|\alpha|-1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(\rho,u,\theta)\right|^2 dxd\tau \\ &= O(1)\delta_0 \sum_{|\alpha'|\le|\alpha|-1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(\rho,u,\theta)\right|^2 dxd\tau + \frac{3}{2}\int_0^t \int_{\mathbf{R}^3} \frac{\rho\partial_x^\alpha u\cdot\nabla_x\partial_x^\alpha\Phi}{\theta} dxd\tau \\ &\le O(1)\delta_0 \sum_{|\alpha'|\le|\alpha|-1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(\rho,u,\theta)\right|^2 dxd\tau - \frac{3}{2}\int_0^t \int_{\mathbf{R}^3} \frac{\mathrm{div}_x(\rho\partial_x^\alpha u)\partial_x^\alpha\Phi}{\theta} dxd\tau. \end{aligned} \tag{5.9}$$

Since

$$\mathrm{div}_x\left(\rho\partial_x^\alpha u\right) = \mathrm{div}_x\Big(\partial_x^\alpha(\rho u)\Big) + \sum_{0<\alpha'<\alpha} C_\alpha^{\alpha'} \mathrm{div}_x\left(\partial_x^{\alpha'}\rho\partial_x^{\alpha-\alpha'}u\right) + \mathrm{div}_x\left(\partial^\alpha u\right),$$

we have from the a priori assumption (3.1) and (3.3) that

$$\left|\sum_{0<\alpha'<\alpha} \int_0^t \int_{\mathbf{R}^3} \frac{\partial_x^\alpha\Phi\mathrm{div}_x\left(\partial_x^{\alpha'}\rho\partial_x^{\alpha-\alpha'}u\right)}{\theta} dxd\tau\right| \le O(1)\delta_0 \sum_{|\alpha'|\le|\alpha|-1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(\rho,u,\theta)\right|^2 dxd\tau,$$

$$\begin{aligned} -\frac{3}{2}\int_0^t \int_{\mathbf{R}^3} \frac{\partial_x^\alpha\Phi\mathrm{div}_x(\partial_x^\alpha\rho u)}{\theta} dxd\tau &= \frac{3}{2}\int_0^t \int_{\mathbf{R}^3} \frac{(\theta\nabla_x\partial_x^\alpha\Phi - \nabla_x\theta\partial_x^\alpha\Phi)\cdot(\partial_x^\alpha\rho u)}{\theta^2} dxd\tau \\ &\le O(1)\delta_0 \sum_{|\alpha'|\le|\alpha|-1} \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x \partial_x^{\alpha'}(\rho,u,\theta)\right|^2 dxd\tau \\ &\quad + O(1)\delta_0 \int_0^t \int_{\mathbf{R}^3} \left|\nabla_x\partial_x^\alpha\Phi\right|^2 dxd\tau, \end{aligned}$$

and

$$
\begin{aligned}
&-\tfrac{3}{2}\int_0^t\int_{\mathbf{R}^3}\tfrac{\partial_x^\alpha\Phi\partial_x^\alpha \mathrm{div}_x(\rho u)}{\theta}dxd\tau\\
=&\tfrac{3}{2}\int_0^t\int_{\mathbf{R}^3}\tfrac{\partial_x^\alpha\Phi\partial_x^\alpha\rho_t}{\theta}dxd\tau\\
=&\tfrac{3}{2}\lambda\int_0^t\int_{\mathbf{R}^3}\tfrac{\partial_x^\alpha\Phi\partial_x^\alpha\Delta_x\Phi_t}{\theta}dxd\tau\\
=&-\tfrac{3}{4}\lambda\int_{\mathbf{R}^3}\tfrac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta}dx\Big|_0^t-\tfrac{3}{4}\lambda\int_0^t\int_{\mathbf{R}^3}\tfrac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta^2}\theta_t dxd\tau\\
&+\tfrac{3}{2}\lambda\int_0^t\int_{\mathbf{R}^3}\tfrac{\partial_x^\alpha\Phi\nabla_x\theta\cdot\nabla_x\partial_x^\alpha\Phi_t}{\theta^2}dxd\tau\\
\le&-\tfrac{3}{4}\lambda\int_{\mathbf{R}^3}\tfrac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta}dx\Big|_0^t+O(1)\delta_0\lambda\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\Phi|^2dxd\tau\\
&+O(1)\delta_0\sum_{|\alpha'|\le|\alpha|-1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau.
\end{aligned}
$$

Thus

$$
\begin{aligned}
-\tfrac{3}{2}\int_0^t\int_{\mathbf{R}^3}\tfrac{\mathrm{div}_x(\rho\partial_x^\alpha u)\partial_x^\alpha\Phi}{\theta}dxd\tau\le&-\tfrac{3}{4}\lambda\int_{\mathbf{R}^3}\tfrac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta}dx\Big|_0^t\\
&+O(1)\delta_0\lambda\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\Phi|^2dxd\tau\\
&+O(1)\delta_0\sum_{|\alpha'|\le|\alpha|-1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau.
\end{aligned}\tag{5.10}
$$

Plugging (5.10) into (5.9) yields

$$
\begin{aligned}
J_9^{1,\alpha}\le&-\tfrac{3}{4}\lambda\int_{\mathbf{R}^3}\tfrac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta}dx\Big|_0^t+O(1)\delta_0\lambda\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\Phi|^2dxd\tau\\
&+O(1)\delta_0\sum_{|\alpha'|\le|\alpha|-1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau.
\end{aligned}\tag{5.11}
$$

Combining (5.7), (5.8), and (5.11), we have

$$
\begin{aligned}
J_9^{1}\le&-\tfrac{3}{4}\lambda\int_{\mathbf{R}^3}\tfrac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta}dx\Big|_0^t+O(1)\delta_0\lambda\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\Phi|^2dxd\tau\\
&+O(1)\delta_0\sum_{|\alpha'|\le|\alpha|-1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau.
\end{aligned}\tag{5.12}
$$

Therefore, (5.5), (5.6), and (5.12) give

$$
\begin{aligned}
I_9\le&-\tfrac{3}{4}\lambda\int_{\mathbf{R}^3}\tfrac{|\nabla_x\partial_x^\alpha\Phi|^2}{\theta}dx\Big|_0^t+O(1)\delta_0\lambda\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha\Phi|^2dxd\tau\\
&+O(1)\delta_0\sum_{|\alpha'|\le|\alpha|-1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau.
\end{aligned}\tag{5.13}
$$

Combining (5.1), (5.2), (5.3), (5.4), and (5.13) gives

$$\begin{aligned}
&\int_{\mathbf{R}^3}\left(\lambda\,|\nabla_x\partial_x^\alpha\Phi|^2+\int_{\mathbf{R}^3}\frac{|\partial_x^\alpha\mathbf{M}|^2}{\mathbf{M}}d\xi\right)dx+\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(u,\theta)|^2\,dxd\tau\\
&\le O(1)N(0)^2+O(1)\delta_0\sum_{|\alpha'|\le|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)\big(|\nabla_x\partial_x^\alpha\mathbf{G}|^2+|\partial_x^\alpha\mathbf{G}_t|^2\big)}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{\alpha'\le\alpha}\left|\nabla_\xi\partial_x^{\alpha'}\mathbf{G}\right|^2+\sum_{|\alpha'|\le3}\left|\nabla_x\partial_x^{\alpha'}\mathbf{G}\right|^2\right)d\xi dxd\tau\\
&\quad+O(1)\delta_{1,|\alpha|}\delta_0\lambda\sum_{|\alpha'|=2}\int_0^t\int_{\mathbf{R}^3}\left|\partial_x^{\alpha'}\Phi\right|^2dxd\tau.
\end{aligned}\tag{5.14}$$

By (5.14) and noticing $\delta_0\lambda<1$, we can obtain (3.17) by summing it over α with $|\alpha|=j$ for $j=1,2,3$ respectively. This completes the proof of Lemma 3.2.

5.2 The proof of Lemma 3.3

As for Lemma 3.3. We only need to prove (3.19) because it is easier to prove (3.18).

Applying $\partial_t\partial_x^\alpha(|\alpha|\le2)$ to (1.11) and integrating its product with $\frac{\partial_t\partial_x^\alpha\mathbf{G}}{\mathbf{M}}$ over $[0,t]\times\mathbf{R}^3\times\mathbf{R}^3$ yield

$$\begin{aligned}
\frac12\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx\Big|_0^t
&=-\frac12\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}^2}\mathbf{M}_t d\xi dxd\tau\\
&\quad-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\cdot\partial_t\partial_x^\alpha(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M}))}{\mathbf{M}}d\xi dxd\tau\\
&\quad-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\cdot\partial_t\partial_x^\alpha(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{G}))}{\mathbf{M}}d\xi dxd\tau\\
&\quad-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\cdot\partial_t\partial_x^\alpha\big(\nabla_x\Phi\cdot\nabla_\xi\mathbf{G}\big)}{\mathbf{M}}d\xi dxd\tau\\
&\quad-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\cdot\partial_t\partial_x^\alpha(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi dxd\tau\\
&\quad-\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\cdot\partial_t\partial_x^\alpha(Q(\mathbf{G},\mathbf{G}))}{\mathbf{M}}d\xi dxd\tau\\
&:=\sum_{j=11}^{16}I_j,
\end{aligned}\tag{5.15}$$

where $I_{11}-I_{16}$ are the corresponding terms in the above equation.

Similar to the proof of Lemma 3.2, we have

$$|I_{11}|\le O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau,\tag{5.16}$$

$$
\begin{aligned}
|I_{13}| \leq\ & \mu\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\partial_x^\alpha\mathbf{G}_t|^2}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)\delta_0\sum_{|\alpha'|\leq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)(|\partial_t\partial_x^{\alpha'}\mathbf{G}|^2+|\nabla_x\partial_x^{\alpha'}\mathbf{G}|^2)}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)\delta_0\sum_{|\alpha'|\leq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau,
\end{aligned}
\tag{5.17}
$$

$$
\begin{aligned}
|I_{15}| =\ & \int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\cdot L_{\mathbf{M}}(\partial_t\partial_x^\alpha\mathbf{G})}{\mathbf{M}}d\xi dxd\tau\\
&+2\sum_{\alpha'<\alpha}C_\alpha^{\alpha'}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\cdot Q\left(\partial_x^{\alpha-\alpha'}\mathbf{M},\partial_x^{\alpha'}\mathbf{G}_t\right)}{\mathbf{M}}d\xi dxd\tau\\
&+2\sum_{\alpha'\leq\alpha}C_\alpha^{\alpha'}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\cdot Q\left(\partial_x^{\alpha-\alpha'}\mathbf{M}_t,\partial_x^{\alpha'}\mathbf{G}\right)}{\mathbf{M}}d\xi dxd\tau\\
\leq\ & -\frac{\sigma}{2}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)\delta_0\sum_{|\alpha'|\leq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau\\
&+O(1)\delta_0\sum_{|\alpha'|<|\alpha|}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)(|\partial_t\partial_x^{\alpha'}\mathbf{G}|^2+|\nabla_x\partial_x^{\alpha'}\mathbf{G}|^2)}{\mathbf{M}}d\xi dxd\tau,
\end{aligned}
\tag{5.18}
$$

and

$$
\begin{aligned}
|I_{16}| \leq\ & \mu\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)\sum_{\alpha'\leq\alpha}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)^{-1}\left|Q\left(\partial_x^{\alpha-\alpha'}\mathbf{G},\partial_x^{\alpha'}\mathbf{G}_t\right)\right|^2}{\mathbf{M}}d\xi dxd\tau\\
\leq\ & \mu\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)\delta_0\sum_{|\alpha'|\leq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)(|\partial_t\partial_x^{\alpha'}\mathbf{G}|^2+|\nabla_x\partial_x^{\alpha'}\mathbf{G}|^2)}{\mathbf{M}}d\xi dxd\tau.
\end{aligned}
\tag{5.19}
$$

To estimate I_{14}, we first notice that

$$
\begin{aligned}
I_{14} =\ & -\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}(\nabla_x\Phi\cdot\nabla_\xi(\partial_t\partial_x^\alpha\mathbf{G}))}{\mathbf{M}}d\xi dxd\tau\\
&-\sum_{\alpha'\leq\alpha}C_\alpha^{\alpha'}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}(\nabla_x\partial_x^{\alpha'}\Phi_t\cdot\nabla_\xi\partial_x^{\alpha-\alpha'}\mathbf{G})}{\mathbf{M}}d\xi dxd\tau\\
&-\sum_{\alpha'<\alpha}C_\alpha^{\alpha'}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}(\nabla_x\partial_x^{\alpha-\alpha'}\Phi\cdot\nabla_\xi\partial_x^{\alpha'}\mathbf{G}_t)}{\mathbf{M}}d\xi dxd\tau\\
:=\ & \sum_{j=1}^3 J_{14}^j,
\end{aligned}
$$

where $J_{14}^1-J_{14}^3$ are the corresponding terms in the above equation.

Since $|\alpha|\le 2$, from (3.1), (3.3), and (3.4), we have

$$\begin{aligned}J_{14}^1&=-\tfrac{1}{2}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_t\partial_x^\alpha\mathbf{G}|^2\nabla_x\Phi\cdot\nabla_\xi\mathbf{M}}{\mathbf{M}}d\xi dxd\tau\\&\le O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau,\end{aligned}$$

$$\begin{aligned}J_{14}^2\le{}&\mu\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\&+O(1)\sum_{\alpha'\le\alpha}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_x\partial_x^{\alpha'}\Phi_t|^2|\nabla_\xi\partial_x^{\alpha-\alpha'}\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\\le{}&\mu\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\&+O(1)\delta_0\sum_{\alpha'\le\alpha}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_\xi\partial_x^{\alpha'}\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau,\end{aligned}$$

$$J_{14}^3\le\mu\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau+O(1)\delta_0\sum_{\alpha'<\alpha}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_\xi\partial^{\alpha'}\mathbf{G}_t|^2}{\mathbf{M}}d\xi dxd\tau.$$

Consequently

$$\begin{aligned}I_{14}\le{}&(\mu+O(1)\delta_0)\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\&+O(1)\delta_0\sum_{\alpha'<\alpha}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_\xi\partial_x^{\alpha'}\mathbf{G}_t|^2}{\mathbf{M}}d\xi dxd\tau\\&+O(1)\delta_0\sum_{\alpha'\le\alpha}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\nabla_\xi\partial_x^{\alpha'}\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau.\end{aligned}\tag{5.20}$$

Finally, we estimate I_{12} which is the most difficult part in proving (3.19).
Since

$$\begin{aligned}\partial_t\partial_x^\alpha\Big(\mathbf{P}_1(\xi\cdot\nabla_x\mathbf{M})\Big)={}&\xi\cdot\nabla_x\partial_x^\alpha\Big(-\nabla_x\Phi\cdot\nabla_\xi\mathbf{M}-\mathbf{P}_0(\xi\cdot\nabla_x\mathbf{M})-\mathbf{P}_0(\xi\cdot\nabla_x\mathbf{G})\Big)\\&-\sum_{j=0}^4\partial_x^\alpha\left(\Big\langle\xi\cdot\nabla_x\mathbf{M},\big(\tfrac{\chi_j}{\mathbf{M}}\big)_t\mathbf{M}\Big\rangle_{\mathbf{M}}\chi_j+\langle\xi\cdot\nabla_x\mathbf{M},\chi_j\rangle_{\mathbf{M}}\chi_{jt}\right),\end{aligned}$$

we have

$$\begin{aligned}I_{12}={}&\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\xi\cdot\nabla_x\partial_x^\alpha(\nabla_x\Phi\cdot\nabla_\xi\mathbf{M})}{\mathbf{M}}d\xi dxd\tau\\&+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\xi\cdot\nabla_x\partial_x^\alpha(\mathbf{P}_0(\xi\cdot\nabla_x\mathbf{M}))}{\mathbf{M}}d\xi dxd\tau\\&+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\xi\cdot\nabla_x\partial_x^\alpha(\mathbf{P}_0(\xi\cdot\nabla_x\mathbf{G}))}{\mathbf{M}}d\xi dxd\tau\\&+\sum_{j=0}^4\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial_t\partial_x^\alpha\mathbf{G}\partial_x^\alpha\Big(\big\langle\xi\cdot\nabla_x\mathbf{M},\big(\frac{\chi_j}{\mathbf{M}}\big)_t\mathbf{M}\big\rangle_{\mathbf{M}}\chi_j+\langle\xi\cdot\nabla_x\mathbf{M},\chi_j\rangle_{\mathbf{M}}\chi_{jt}\Big)}{\mathbf{M}}d\xi dxd\tau\\:={}&\sum_{k=1}^4J_{12}^k,\end{aligned}\tag{5.21}$$

where $J_{12}^1-J_{12}^4$ are the corresponding terms in the above equation.

$J_{12}^k (k = 1, 2, 3, 4)$ can be estimated as follows. First, by Cauchy-Schwarz inequality, (3.1), (3.3), and (3.4), we have from $|\alpha| \leq 2$ that

$$\begin{aligned} J_{12}^1 \leq & \mu \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\ & + O(1) \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\nabla_x \partial_x^\alpha (\nabla_x \Phi \cdot \nabla_\xi \mathbf{M})|^2}{\mathbf{M}} d\xi dx d\tau \\ \leq & \mu \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\ & + O(1) \delta_{1,|\alpha|+1} \sum_{|\alpha'|=2} \int_0^t \int_{\mathbf{R}^3} |\partial_x^{\alpha'} \Phi|^2 dx d\tau \\ & + O(1) \left(1 - \delta_{1,|\alpha|+1}\right) \lambda^{-2} \sum_{|\alpha'|=|\alpha|} \int_0^t \int_{\mathbf{R}^3} |\partial_x^{\alpha'} \rho|^2 dx d\tau \\ & + \delta_0 \sum_{|\alpha'| \leq |\alpha|} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^{\alpha'} (\rho, u, \theta)|^2 dx d\tau, \end{aligned} \tag{5.22}$$

$$\begin{aligned} J_{12}^2 = & \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\partial_t \partial_x^\alpha \mathbf{G} \mathbf{P}_1 (\xi \cdot \nabla_x \partial_x^\alpha (\mathbf{P}_0 (\xi \cdot \nabla_x \mathbf{M})))}{\mathbf{M}} d\xi dx d\tau \\ \leq & \mu \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\ & + O(1) \sum_{|\alpha'|=1+|\alpha|} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^{\alpha'} (u, \theta)|^2 dx d\tau \\ & + O(1) \delta_0 \sum_{|\alpha'| \leq |\alpha|} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^{\alpha'} (\rho, u, \theta)|^2 dx d\tau, \end{aligned} \tag{5.23}$$

and

$$\begin{aligned} J_{12}^3 \leq & \mu \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\partial_t \partial_x^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\ & + O(1) \sum_{|\alpha'|=1+|\alpha|} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\nabla_x \partial_x^{\alpha'} \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\ & + O(1) \delta_0 \sum_{|\alpha'| \leq |\alpha|} \int_0^t \int_{\mathbf{R}^3} |\nabla_x \partial_x^{\alpha'} (\rho, u, \theta)|^2 dx d\tau \\ & + O(1) \delta_0 \sum_{|\alpha'| \leq |\alpha|} \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\nu(\xi)|\nabla_x \partial_x^{\alpha'} \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau. \end{aligned} \tag{5.24}$$

As for J_{12}^4, since

$$\begin{aligned} & \partial_x^\alpha \left(\left\langle \xi \cdot \nabla_x \mathbf{M}, (\tfrac{\chi_j}{\mathbf{M}})_t \mathbf{M} \right\rangle_{\mathbf{M}} \chi_j + \langle \xi \cdot \nabla_x \mathbf{M}, \chi_j \rangle_{\mathbf{M}} \chi_{jt} \right) \\ = & \sum_{\alpha' + \alpha'' \leq \alpha} C_\alpha^{\alpha', \alpha''} \left\langle \xi \cdot \nabla_x \partial_x^{\alpha'} \mathbf{M}, \partial_x^{\alpha''} (\tfrac{\chi_j}{\mathbf{M}})_t \mathbf{M} \right\rangle_{\mathbf{M}} \partial_x^{\alpha - \alpha' - \alpha''} \chi_j \\ & + \sum_{\alpha' + \alpha'' \leq \alpha} C_\alpha^{\alpha', \alpha''} \left\langle \xi \cdot \nabla_x \partial_x^{\alpha'} \mathbf{M}, \partial_x^{\alpha''} (\tfrac{\chi_j}{\mathbf{M}}) \mathbf{M} \right\rangle_{\mathbf{M}} \partial_x^{\alpha - \alpha' - \alpha''} \chi_{jt}, \end{aligned}$$

we have

$$
\begin{aligned}
J_{12}^4 \leq & \mu\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau \\
& +O(1)\sum_{j=0}^4\sum_{\alpha'+\alpha''\leq\alpha}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\left\langle\xi\cdot\nabla_x\partial_x^{\alpha'}\mathbf{M},\partial_x^{\alpha''}\left(\frac{\chi_j}{\mathbf{M}}\right)_t\mathbf{M}\right\rangle_{\mathbf{M}}\partial_x^{\alpha-\alpha'-\alpha''}\chi_j\right|^2}{\mathbf{M}}d\xi dxd\tau \\
& +O(1)\sum_{j=0}^4\sum_{\alpha'+\alpha''\leq\alpha}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\left|\left\langle\xi\cdot\nabla_x\partial_x^{\alpha'}\mathbf{M},\partial_x^{\alpha''}\left(\frac{\chi_j}{\mathbf{M}}\right)\mathbf{M}\right\rangle_{\mathbf{M}}\partial_x^{\alpha-\alpha'-\alpha''}\chi_{jt}\right|^2}{\mathbf{M}}d\xi dxd\tau \\
= & K_1+K_2+K_3.
\end{aligned}
\tag{5.25}
$$

If $|\alpha'|\geq 1$, then $|\alpha''|\leq 1, |\alpha-\alpha'-\alpha''|\leq 1$ because $|\alpha|\leq 2$. And we have from the conservation laws (1.10), (3.1), and (3.4) that

$$
K_2\leq O(1)\delta_0\sum_{|\alpha'|\leq|\alpha|}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)|^2dxd\tau.
$$

If $\alpha'=0$, we need to consider the following three cases:

$$
\begin{cases}
(a). & \alpha''=0,\\
(b). & |\alpha''|=1 \text{ (then } |\alpha-\alpha''|\leq 1),\\
(c). & |\alpha''|=|\alpha|=2.
\end{cases}
$$

For cases (a) and (b), the same argument to the case of $|\alpha'|\geq 1$ gives

$$
K_2\leq O(1)\delta_0\sum_{|\alpha'|\leq|\alpha|}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)|^2dxd\tau.
$$

For the case (c),

$$
\begin{aligned}
|K_2|\leq & O(1)\sum_{|\alpha'|\leq 1}\int_0^t\int_{\mathbf{R}^3}|\nabla_x(\rho,u,\theta)|^2\left|\partial_t\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau \\
& +O(1)\sum_{|\alpha'|=2}\int_0^t\int_{\mathbf{R}^3}|\nabla_x(\rho,u,\theta)|^2\left|\partial_t\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau \\
\leq & O(1)\delta_0\sum_{|\alpha'|\leq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)\right|^2dxd\tau.
\end{aligned}
$$

Here we have used (3.1), (3.3), (3.4), and the conservation laws (1.10).

Thus we have

$$
K_2\leq O(1)\delta_0\sum_{|\alpha'|\leq|\alpha|}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)|^2dxd\tau. \tag{5.26}
$$

Similar estimate holds for K_3.

Consequently

$$
\begin{aligned}
J_{12}^4\leq & \mu\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau \\
& +O(1)\delta_0\sum_{|\alpha'|\leq|\alpha|}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)|^2dxd\tau.
\end{aligned}
\tag{5.27}
$$

By plugging (5.22), (5.23), (5.24), and (5.27) into (5.21), we have

$$
\begin{aligned}
I_{12} \leq{} & \mu\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)\sum_{|\alpha'|=1+|\alpha|}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\partial_x^{\alpha'}\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)\delta_0\sum_{|\alpha'|\leq|\alpha|}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)|^2dxd\tau\\
&+O(1)\delta_0\sum_{|\alpha'|\leq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_x\partial_x^{\alpha'}\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)\sum_{|\alpha'|=|\alpha|+1}\int_0^t\int_{\mathbf{R}^3}\left|\nabla_x\partial_x^{\alpha'}(u,\theta)\right|^2dxd\tau\\
&+O(1)\delta_{1,|\alpha|+1}\sum_{|\alpha'|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^{\alpha'}\Phi|^2dxd\tau\\
&+O(1)\left(1-\delta_{1,|\alpha|+1}\right)\lambda^{-2}\sum_{|\alpha'|=|\alpha|}\int_0^t\int_{\mathbf{R}^3}|\partial_x^{\alpha'}\rho|^2dxd\tau.
\end{aligned}
\tag{5.28}
$$

Combining (5.15)-(5.20) with (5.28), we obtain

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leq O(1)N(0)^2+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial_t\partial_x^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau\\
&+O(1)\sum_{|\alpha'|=1+|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left(|\nabla_x\partial_x^{\alpha'}(u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)(|\nabla_x\partial_x^{\alpha'}\mathbf{G}|^2+|\partial_x^{\alpha'}\mathbf{G}_t|^2)}{\mathbf{M}}d\xi\right)dxd\tau\\
&+O(1)\delta_0\sum_{|\alpha'|\leq|\alpha|}\int_0^t\int_{\mathbf{R}^3}\left(|\nabla_x\partial_x^{\alpha'}(\rho,u,\theta)|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)(|\nabla_x\partial_x^{\alpha'}\mathbf{G}|^2+|\partial_x^{\alpha'}\mathbf{G}_t|^2)}{\mathbf{M}}d\xi\right)dxd\tau\\
&+O(1)\delta_{1,1+|\alpha|}\sum_{|\alpha'|=2}\int_0^t\int_{\mathbf{R}^3}|\partial_x^{\alpha'}\Phi|^2dxd\tau\\
&+O(1)\left(1-\delta_{1,1+|\alpha|}\right)\lambda^{-2}\sum_{|\alpha'|=|\alpha|}\int_0^t\int_{\mathbf{R}^3}|\partial_x^{\alpha'}\rho|^2dxd\tau\\
&+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(\sum_{|\alpha'|\leq|\alpha|}\left|\nabla_\xi\partial_x^{\alpha'}\mathbf{G}\right|^2+\sum_{|\alpha'|<|\alpha|}\left|\nabla_\xi\partial_x^{\alpha'}\mathbf{G}_t\right|^2\right)d\xi dxd\tau.
\end{aligned}
\tag{5.29}
$$

(3.19) follows directly from (5.29). This completes the proof of Lemma 3.3.

5.3 The proof of Lemma 3.5

For Lemma 3.5, by applying $\partial^\gamma(\gamma\in\Lambda_7)$ to $(1.2)_1$, multiplying it by $\frac{\partial^\gamma f}{\mathbf{M}}$, and integrating the final equation w.r.t. t, x, and ξ over $[0,t]\times\mathbf{R}^3\times\mathbf{R}^3$, we have

$$
\begin{aligned}
&\frac{1}{2}\sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma f|^2}{\mathbf{M}}d\xi dx\Big|_0^t=-\frac{1}{2}\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma f|^2}{\mathbf{M}^2}\left(\mathbf{M}_t+\xi\cdot\nabla_x\mathbf{M}\right)d\xi dxd\tau\\
&-\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma f\partial^\gamma(\nabla_x\cdot\nabla_\xi f)}{\mathbf{M}}d\xi dxd\tau+\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma f\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi dxd\tau\\
&+\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma f\partial^\gamma(Q(\mathbf{G},\mathbf{G}))}{\mathbf{M}}d\xi dxd\tau:=\sum_{j=17}^{20}I_j,
\end{aligned}
\tag{5.30}
$$

where $I_{17}-I_{20}$ are the corresponding terms in the above equation.

Now we estimate $I_j (j=17,18,19,20)$ term by term. First from Lemma 3.1, (3.1), (3.3), and (3.4), we have

$$
\begin{aligned}
I_{17} &\leq O(1)\delta_0 \sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma f|^2}{\mathbf{M}_-}d\xi dx d\tau\\
&\leq O(1)\delta_0 \sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}_-}d\xi dx d\tau\\
&\quad +O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau,
\end{aligned}
\tag{5.31}
$$

$$
\begin{aligned}
I_{19} &= \sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{(\mathbf{P}_1(\partial^\gamma\mathbf{M})+\mathbf{P}_1(\partial^\gamma\mathbf{G}))\partial^\gamma(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}_-}d\xi dx d\tau\\
&\leq -\frac{\sigma}{2}\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau\\
&\quad +O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\mathbf{G}|^2+|\nabla_x\mathbf{G}|^2+|\mathbf{G}_t|^2\right)d\xi dx d\tau\\
&\quad +O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau,
\end{aligned}
\tag{5.32}
$$

and

$$
\begin{aligned}
I_{20} &= \sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{(\mathbf{P}_1(\partial^\gamma\mathbf{M})+\mathbf{P}_1(\partial^\gamma\mathbf{G}))\partial^\alpha(Q(\mathbf{G},\mathbf{G})}{\mathbf{M}_-}d\xi dx d\tau\\
&\leq \mu\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dx d\tau\\
&\quad +O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\mathbf{G}|^2+|\nabla_x\mathbf{G}|^2+|\mathbf{G}_t|^2\right)d\xi dx d\tau\\
&\quad +O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau.
\end{aligned}
\tag{5.33}
$$

For I_{18}, note that

$$
\begin{aligned}
I_{18} &= -\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma f\nabla_x\Phi\cdot\nabla_\xi\partial^\gamma f}{\mathbf{M}}d\xi dx d\tau\\
&\quad -\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma f\nabla_x\partial^\gamma\Phi\cdot\nabla_\xi f}{\mathbf{M}}d\xi dx d\tau\\
&\quad -\sum_{\gamma\in\Lambda_7}\sum_{0<\gamma'<\gamma}C_\gamma^{\gamma'}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma f\nabla_x\partial^{\gamma-\gamma'}\Phi\cdot\nabla_\xi\partial^{\gamma'} f}{\mathbf{M}}d\xi dx d\tau\\
&:=\sum_{i=1}^3 J_{18}^i,
\end{aligned}
\tag{5.34}
$$

where $J_{18}^1-J_{18}^3$ are the corresponding terms in the above equation. We have from the conservation laws (1.10), (3.1), (3.3), and (3.4) that

$$
\begin{aligned}
J_{18}^1 &= -\frac{1}{2}\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma f|^2}{\mathbf{M}^2}\nabla_x\Phi\cdot\nabla_\xi\mathbf{M}d\xi dxd\tau\\
&\leq O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma f|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leq O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma \mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau.
\end{aligned}
\tag{5.35}
$$

For J_{18}^2, notice that

$$
\begin{aligned}
J_{18}^2 &= -\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma\mathbf{M})\nabla_x\partial^\gamma\Phi\cdot\nabla_\xi\mathbf{M}}{\mathbf{M}}d\xi dxd\tau\\
&\quad-\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\partial^\gamma\mathbf{M})\nabla_x\partial^\gamma\Phi\cdot\nabla_\xi\mathbf{G}}{\mathbf{M}}d\xi dxd\tau\\
&\quad-\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\mathbf{G}\nabla_x\partial^\gamma\Phi\cdot\nabla_\xi\mathbf{G}}{\mathbf{M}}d\xi dxd\tau\\
&:= K_4+K_5+K_6,
\end{aligned}
\tag{5.36}
$$

we have from (3.1), (3.3), and (3.4) that

$$
\begin{aligned}
K_4 &\leq \frac{1}{\lambda}\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\mathbf{P}_0(\partial^\gamma\mathbf{M})|^2}{\mathbf{M}}d\xi dxd\tau+O(1)\lambda\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\frac{|\nabla_x\partial^\gamma\Phi|^2}{\mathbf{M}}dxd\tau\\
&\leq O(1)\lambda^{-1}\sum_{|\alpha|\leq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2dxd\tau,
\end{aligned}
$$

$$
K_5\leq O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2+\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_\xi\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dxd\tau,
$$

and

$$
\begin{aligned}
K_6 &\leq O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_\xi\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\delta_0\sum_{|\alpha|\leq 3}\int_0^t\int_{\mathbf{R}^3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2dxd\tau.
\end{aligned}
$$

Thus

$$
\begin{aligned}
J_{18}^2 &\leq O(1)\left(\delta_0+\lambda^{-1}\right)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau\\
&\quad+O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau+O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_\xi\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau.
\end{aligned}
\tag{5.37}
$$

Finally, for J_{18}^3, since

$$\begin{aligned} J_{18}^3 = & -\sum_{\gamma\in\Lambda_7}\sum_{0<\gamma'<\gamma} C_\gamma^{\gamma'}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma\mathbf{M})\nabla_x\partial^{\gamma-\gamma'}\Phi\cdot\nabla_\xi\left(\mathbf{P}_0\left(\partial^{\gamma'}\mathbf{M}\right)\right)}{\mathbf{M}}d\xi dxd\tau \\ & -\sum_{\gamma\in\Lambda_7}\sum_{0<\gamma'<\gamma} C_\gamma^{\gamma'}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\gamma\mathbf{M})\nabla_x\partial^{\gamma-\gamma'}\Phi\cdot\nabla_\xi\left(\mathbf{P}_1\left(\partial^{\gamma'}\mathbf{M}\right)+\partial^{\gamma'}\mathbf{G}\right)}{\mathbf{M}}d\xi dxd\tau \\ & -\sum_{\gamma\in\Lambda_7}\sum_{0<\gamma'<\gamma} C_\gamma^{\gamma'}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\partial^\gamma\mathbf{G}\nabla_x\partial^{\gamma-\gamma'}\Phi\cdot\nabla_\xi\left(\partial^{\gamma'}\mathbf{G}\right)}{\mathbf{M}}d\xi dxd\tau \\ := & K_7+K_8+K_9, \end{aligned} \tag{5.38}$$

we also have from the conservation laws (1.10), (3.1), (3.3), and (3.4) that

$$K_7 \leq O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau,$$

$$\begin{aligned} K_8 \leq & O(1)\delta_0\sum_{\gamma\in\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_\xi\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau \\ & +O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau, \end{aligned}$$

and

$$\begin{aligned} K_9 \leq & O(1)\delta_0\sum_{\gamma\in\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_\xi\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau \\ & +O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau. \end{aligned}$$

Consequently

$$\begin{aligned} J_{18}^2 \leq & O(1)\delta_0\sum_{\gamma\in\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_\xi\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau+O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau \\ & +O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau. \end{aligned} \tag{5.39}$$

Combining (5.35), (5.37), and (5.39) yields

$$\begin{aligned} I_{18} \leq & O(1)\delta_0\sum_{\gamma\in\Lambda_4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\nabla_\xi\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau+O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau \\ & +O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau. \end{aligned} \tag{5.40}$$

Substituting (5.31), (5.32), (5.33), and (5.40) into (5.30), we finally obtain

$$\begin{aligned} & \sum_{\gamma\in\Lambda_7}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma f|^2}{\mathbf{M}}d\xi dx+\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau \\ & \leq O(1)N(0)^2+O(1)\delta_0\sum_{\gamma\in\Lambda_7}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^\gamma\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau \\ & \quad +O(1)\left(\delta_0+\lambda^{-1}\right)\int_0^t\int_{\mathbf{R}^3}\left(\sum_{|\alpha|\leq 3}|\nabla_x\partial_x^\alpha(\rho,u,\theta)|^2+\sum_{|\alpha|=2}|\partial_x^\alpha\Phi|^2\right)dxd\tau \\ & \quad +O(1)\delta_0\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)}{\mathbf{M}}\left(|\mathbf{G}|^2+|\nabla_\xi\mathbf{G}|^2+|\mathbf{G}_t|^2+\sum_{\gamma\Lambda_4}|\nabla_\xi\partial^\gamma\mathbf{G}|^2\right)d\xi dxd\tau. \end{aligned} \tag{5.41}$$

This is (3.29) and completes the proof of Lemma 3.5.

Acknowledgment

The research of the Tong Yang was supported by RGC Competitive Earmarked Research Grant CityU 102703. The research of the Huijiang Zhao was supported by a grant from the Chinese Academy of Sciences entitled "Yin Jin Guo Wai Jie Chu Ren Cai Ji Jin", the National Key Program for Basic Research of China under contract 2002CCA03700, and the JSPS Research Fellowship for Foreign Researchers. The research was also supported in part by the National Natural Science Foundation of China under contract 10329101.

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