Remarks on the formation and decay of multidimensional shock waves

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Abstract

In this paper, we present formula describing formation and decay of shock wave type solutions in some special cases.

In [1] and [2], in the quadratic and general cases of convex nonlinearity, we consider the process of formation of shock waves for scalar conservation laws in the one-dimensional case. Recall that, in the construction suggested in these papers, the key role is played by the function $u_1(x)$ determined by the implicit equation

$$f'(u_1(x)) = -Kx + b, \tag{1}$$

where $K > 0$ and b are constants, and $f(u)$ is the nonlinear (convex) density of the conservation law.

In the present paper, we generalize this construction to the multidimensional case.

The main point is to generalize Eq. (1). Recall that the function $u_1(x)$ in (1) describes both the shock wave formation and the decay of a nonstable step function (a rarefaction wave type solution).

The problems of formation and decay of step functions are closely related to each other: the change $t \to -t$ allows one to use solutions describing the step function formation to construct solutions describing the step function decay, and conversely.

This procedure is described in detail (in the framework of the technique used there) in [1] in the scalar quadratic case.

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We consider the equation

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(u) = 0, \tag{2}$$

where $f_i(u)$ are smooth functions.

To Eq. (2) there corresponds the system of differential equations (equations of characteristics)

$$\begin{aligned} &\dot{x}_i = f_i'(u), \qquad && x\big|_{t=0} = x_0, \\ &\dot{u} = 0, && u\big|_{t=0} = u_0(x_0). \end{aligned} \tag{3}$$

We have the following obvious assertion.

Lemma 1

$$J \stackrel{\text{def}}{=} \det \left| \frac{\partial x}{\partial x_0} \right| = t \sum_{i=1}^{n} \frac{\partial^2 f_i(u_0)}{\partial u^2} \frac{\partial u_0}{\partial x_{0i}}. \tag{4}$$

The proof readily follows from the relation

$$\frac{d^2 J}{dt^2} = 0.$$

A generalization of (1) is based on the fact that the derivative of the left-hand side of (1) with respect to x is exactly the expression under the sign of sum in (4) in the one-dimensional case.

Namely, suppose that two smooth surfaces Γ_1 and Γ_2, $\Gamma_1 \cap \Gamma_2 = \varnothing$, are given in a connected simply connected domain Ω.

We assume that Γ_i are determined by the equations $x_j^i = \chi_j^i(s)$, $s \in \mathcal{D} \in R^{n-1}$, $i = 1, 2$, $j = 1, \ldots, n$.

We also assume that in Ω there exists a solution $u_1(x)$ of the problem

$$\begin{gathered} \sum_{i=1}^{n} \frac{\partial^2 f_i(u_0)}{\partial u^2}(u_1) \frac{\partial u_1}{\partial x_i} + K = 0, \\ u_1\big|_{\Gamma_1} = U = const, \qquad u_1\big|_{\Gamma_2} = u_0^0 = const, \end{gathered} \tag{5}$$

where $K = K(s) > 0$ is an unknown function which we seek together with $u_1(x)$. It is clear that problem (5) is the required generalization of (1).

Remark 1 The condition $K = K(s)$ means that the function $K(s)$ is constant on the characteristics corresponding to Eq. (5).

Remark 2 The solvbility of (5) means that the vector field $f''(u_0)$ is not singular, i.e.,

$$|f''| \neq 0,$$

which is an analog of the convexity condition.

In the multidimensional case, the function $u_1(x_1, \dots, x_n)$, i.e., the solution of (5), will play the same role as the solution of Eq. (1) in the one-dimensional case.

Problem (5) is equivalent to the following one:

$$\begin{aligned} &\frac{dX_i}{d\tau} = f_i''(u_1), && X_i\big|_{\tau=0} = \chi_i^1(s), \\ &\frac{du_1}{d\tau} = -K, && u_1\big|_{\tau=0} = U, \qquad i = 1, \dots, n. \end{aligned} \tag{6}$$

We have

$$u_i = U - K(s)\tau.$$

Let $\tau_0(s)$ be such that $X(\tau_0(s), s) \in \Gamma_2$, then

$$K(s) = \frac{U - u_0^0}{\tau_0(s)}.$$

For given U and u_0^0, the condition that $K(s)$ is positive implies restrictions on the direction of motion along the trajectories determined by (6), and the fact that problem (5) has a solution means that Γ_1 and Γ_2 are sections of the bundle determined by the trajectories of (6).

To be definite, we assume that $U > u_0^0$. Then for $K > 0$ the motion along the trajectories of (6) must occur from Γ_1 to Γ_2 with increasing τ. Otherwise, problem (5) does not have solutions.

Next, by Ω^- we denote the domain lying “before” Γ_1, i.e., the domain entered by the trajectories of system (6) for $\tau < 0$. By Ω^+ we denote the domain lying “after” Γ_2, i.e., the domain entered by the trajectories of system (6) for $\tau > \tau_0(s)$.

By $H^\pm$ we denote the characteristic functions of the domains $\Omega^\pm$, and by H^0 we denote the characteristic function of the domain $\Omega \setminus \Omega^+ \setminus \Omega^-$.

We consider the equation

$$\frac{\partial u}{\partial t} + \sum \frac{\partial f_i(u)}{\partial x_i} = 0 \tag{7}$$

and set

$$u\Big|_{t=0} = UH^- + u_0^0 H^+ + u_1(x)H^0. \tag{8}$$

Next, we must define the concrete geometry of the problem. For example, we can assume that Γ_1 and Γ_2 are closed and Γ_2 is located in the interior of Γ_1, or conversely. However, we shall not do this, but simply assume that we are interested in the solution of problem (7), (8) in the domain where it can be obtained from the initial condition by using the characteristics.

Clearly, it follows from Lemma 1 and the choice of the function $u_1(x)$ in (8) that the wave turns over on the trajectories of the characteristic system corresponding to (7).

More precisely, the trajectories of the characteristic system corresponding to (7),

$$\frac{dx_i}{dt} = f_i'(u_1), \qquad x_i\Big|_{t=0} = x_0, \qquad i = 1, \dots, n, \tag{9}$$

such that the point $(x_{10}, \dots, x_{n0}) = X(s, \tau)$ belongs to the trajectory of system (6) for some fixed s and $0 \le \tau \le \tau_0(s)$ intersect at $t = t_0(s) = 1/K(s)$ at the point $x = x^*(s)$, $x^*(s) = x(t_0(s), X(s, \tau))$.

Thus, for $t > \max_s t_0(s)$, the evolution of the initial condition (8) gives a shock type solution of the form

$$u = U + H(S(x,t))(u_0^0 - U), \tag{10}$$

where $S \in C^\infty$, $H(s)$ is the Heaviside function, and the set $S(x,t) = 0$ is the shock wave front.

Moreover, at the point $\bar{x}$ at which the jump occurs, we have the inequality

$$u_+ - u_- < 0, \tag{11}$$

where u_+ is the limit value of the solution calculated along the trajectory of system (6) as $x \to \bar{x}$; here x corresponds to the value $\tau > \bar{\tau}$, $\bar{\tau}$ corresponds to $\bar{x}$, and u_- is determined similarly.

Inequality (11) can be treated as the stability condition for the jump of the solution to Eq. (7) in the multidimensional case.

Of course, we here must take into account the above assumption on the direction of motion along the trajectories of system (7).

It is easy to see that the limit u_+ can also be calculated along the vector $f''_u(u_0^0)$, and the limit u_- along the vector $f''_u(U)$.

We agree to denote the limit of $g(x,t)$ as $x \to \bar{x}$ for fixed t along the vector X by

$$(X) \overrightarrow{\lim_{x\to\bar{x}}}\, g(x,t),$$

and the limit of $g(x,t)$ as $x \to \bar{x}$ along the vector X but in the opposite direction by

$$(X) \overleftarrow{\lim_{x\to\bar{x}}}\, g(x,t).$$

Then the stability condition for the jump (11) can be written as

$$(f''_{uu}(u_0^0)) \overleftarrow{\lim_{x\to\bar{x}}}\, u(x,t) - (f''_{uu}(U)) \overrightarrow{\lim_{x\to\bar{x}}}\, u(x,t) < 0. \tag{12}$$

Definition 1 A piecewise constant solution of Eq. (7) of the form (10) is said to be *absolutely nonstable* if the following inequality holds at all points $\bar{z}x \in \{s(\bar{x},t) = 0\}$ for a fixed t:

$$(f''_{uu}(u_0^0)) \lim_{x\to\bar{x}} u(x,t) - (f''_{uu}(U)) \lim_{x\to\bar{x}} u(x,t) > 0. \tag{13}$$

It follows from the above that an absolutely nonstable jump must turn into a solution of the form (8). This construction is completely similar to the one-dimensional case. More precisely, in the one-dimensional case, this is described for the case of quadratic nonlinearity in [1]. The same also holds for the case of general convex nonlinearity in the one-dimensional case [2]. In the multidimensional case, in fact, the above assumptions reduce the problem to the one-dimensional problem along the trajectories of system (6).

We note that system (6) can be easily integrated:

$$\begin{aligned} u_1 &= U - K(s)\tau, \qquad\qquad (14)\\ X_i(\tau,s) &= \chi_i^1(s) + \frac{1}{K}(f_i'(U) - f_i'(U - K\tau)), \qquad i = 1,\dots,n. \end{aligned}$$

Now let $X_0^1 = \chi^1(s_0)$ be an arbitrary point on Γ_1. By $X_0^2 = X(\tau_0(s_0), s_0)$ we denote the point of intersection of the trajectory of system (6) with Γ_2. Let $x(X_0^i, t)$, $i = 1, 2$, be solutions of system (9) such that

$$x\big|_{t=0} = X_0^i.$$

Then, by (14), we have

$$x(X_0^1, t) - x(X_0^2, t) = X_0^1 - X_0^2 + t(f'(U) - f'(u_0^0)) \qquad (15)$$
$$= \frac{1}{K}[f'(U) - f'(u_0^0)](Kt - 1).$$

In the construction of multidimensional nonlinear waves, an important role is played by the level surface Γ^t of the solution [1]. These surfaces are determine by the relations $\Gamma^t = \{t = \psi(x)\}$, where $\psi(x)$ is the desired unknown function whose zero-level surface is assumed to be given (in our problem, these are the surfaces Γ_1 and Γ_2).

Clearly, if, for example, $\Gamma^0 = \Gamma_1$, then Γ_1^t is the set of the endpoints of the trajectories of system (9) starting on Γ_1 at time t. In this case, the function $\psi_1(x)$ is the time required for the trajectory starting at a point $X_{10} \in \Gamma_1$ to come to the point X. Similarly, we determine $\psi_2(x)$ and Γ_2^t.

Let us consider the expression

$$[\psi_1(x) - \psi_2(x)]\Big|_{\Gamma_1^t} \equiv t - \psi_2(x)\Big|_{\Gamma_t^1},$$

where the restriction means that x is a point on Γ_t^1. We have

$$x = X_{10} + \psi_1 f'(U) = X_{20} + \psi_2 f'(u_0^0),$$

where $X_{10} \in \Gamma_1$ and $X_{20} \in \Gamma_2$ are some initial point of the trajectories (9).

We have

$$X_{10} - X_{20} = \psi_2 f'(u_0^0) - \psi_1 d'(U).$$

Next, we have $X_{10} = \chi^1(s)$ for some s. We denote $X_{20} = X_1(\tau_0(s), s)$. By (15), we have

$$(\psi_2 - \psi_1)\Big|_{\Gamma_t^1} f'(U) = X_{02} - X_{20} + \frac{1}{K}[f'(U) - f'(u_0^0)](Kt - 1).$$

Now let $t = 1/K$ and $x \in \Gamma_1^t \cap \Gamma_2^t$. Then $X_{20} = X_{02}$. In general, we have

$$(\psi_2 - \psi_1)\Big|_{\Gamma_t} = \frac{(Kt-1)K^{-1}\langle f'(U), f'(U) - f'(u_0^0)\rangle}{\|f'(U)\|^2},$$

where $\langle , \rangle$ is the inner product in $\mathbb{R}^n$. Similarly, we define the quantity

$$(\psi_2 - \psi_1)\Big|_{\Gamma_{1t}}.$$

Everything said above is an analog of the Introduction in [2]. In the present text, we restrict ourselves to this and only note that we have prepared everything necessary to construct the multidimensional analog of the weak asymptotic solution given in [2], which describes the formation of a shock wave. We shall present this in detail in the next paper.

In conclusion, we formulate the solution of the problem concerning the decay of a nonstable step-function.

Suppose that the initial data for Eq. 2 have the form (10) for $t = 0$, and the vector fields $f'(U)$, $f'(u_0^0)$ and $f''(U)$, $f''(u_0^0)$ are transversal to the surface $\Gamma_0 = \{S(x, 0) = 0\}$. Next, we assume that inequality (13) holds at the points of Γ_0. Then there exists a $\bar{t} > 0$ such that for $t \in [0, \bar{t}]$, the solution of Eq. (2) with the initial condition

$$u\big|_{t=0} = U + H(S(x, 0))(u_0^0 - U)$$

has the form (8).

References

[1] V. Danilov, *Generalized solutions describing singularity interaction*, IJMMS, 29:8 (2002), 481–494.

[2] V. Danilov and D. Mitrovich, *Weak asymptotics of shock wave formation process*, http://www.math.ntnu.no/conservation/2003/078.html