CONTINUOUS DEPENDENCE ESTIMATES FOR VISCOSITY SOLUTIONS OF INTEGRO-PDES

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Abstract. We present a general framework for deriving continuous dependence estimates for, possibly polynomially growing, viscosity solutions of fully nonlinear degenerate parabolic integro-PDEs. We use this framework to provide explicit estimates for the continuous dependence on the coefficients and the "Lévy measure" in the Bellman/Isaacs integro-PDEs arising in stochastic control/differential games. Moreover, these explicit estimates are used to prove regularity results and rates of convergence for some singular perturbation problems. Finally, we illustrate our results on some integro-PDEs arising when attempting to price European/American options in an incomplete stock market driven by a geometric Lévy process. Many of the results obtained herein are new even in the convex case where stochastic control theory provides an alternative to our pure PDE methods.

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1. Introduction

The theory of viscosity solutions for fully nonlinear degenerate elliptic/parabolic PDEs is now highly developed [4, 5, 18, 25]. In recent years we have witnessed an interest in extending viscosity solution theory to integro-PDEs [1, 2, 3, 6, 11, 12, 13, 36, 42, 44, 46, 47, 50, 51]. Such non-local equations occur in the theory of optimal control of jump-diffusion (Lévy) processes and find many applications in mathematical finance, see, e.g., [1, 2, 3, 11, 12, 13, 27] and the references cited therein. We refer to the books [28, 29] for an investigation of integro-PDEs by completely different methods.

In this paper we are interested in "continuous dependence on the nonlinearities" estimates and various consequences of such estimates for viscosity solutions of fully nonlinear degenerate parabolic integro-PDEs. To be as general as possible, we write these equations in the form

$$\begin{aligned} u_t(t,x) + F(t,x,u(t,x),Du(t,x),D^2u(t,x),u(t,\cdot)) = 0 \quad \text{in} \quad Q_T, \\ u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where $Q_T := (0,T) \times \mathbb{R}^N$ and $F : \overline{Q}_T \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \times C_p^2(\mathbb{R}^N) \to \mathbb{R}$ is a given *functional*. Here $\mathbb{S}^N$ denotes the space of symmetric $N \times N$ real valued matrices, and $C_p^2(\mathbb{R}^N)$ denotes the space of $C^2(\mathbb{R}^N)$ functions with polynomial growth of order $p \geq 0$ at infinity.

These equations are non-local as is indicated by the $u(t,\cdot)$-term in (1.1). A simple example of such an equation is

$$u_t - \int_{\mathbb{R}^M \setminus \{0\}} [u(\cdot,\cdot+z) - u - zDu]\, \pi(dz) = 0 \quad \text{in} \quad Q_T, \tag{1.2}$$

where $\pi(dz)$ is a positive Radon measure on $\mathbb{R}^M \setminus \{0\}$ (the so-called Lévy measure) with a singularity at the origin satisfying

$$\int_{\mathbb{R}^M \setminus \{0\}} \left(z^2 \, \mathbf{1}_{B(0,1)} + |z|^p \, \mathbf{1}_{B(0,1)^c} \right) \pi(dz) < \infty. \tag{1.3}$$

Note that the Lévy measure integrates functions with p-th order polynomial growth at infinity. In view of (1.3) and a Taylor expansion of the integrand, the integro operator in (1.2) is well defined on $C_p^2(\mathbb{R}^N)$. Moreover, it is clear that the integro operator in (1.2) acts as a non-local second order term, and for that reason the "order" of the integro operator is said to be two. If $|z|^2$ in (1.3) is replaced by $|z|$, this changes the order of the integro operator from two to one, since then it acts just like a non-local first order term. Finally, if $|z|^2$ in (1.3) is replaced by 1 (i.e., $\pi(dz)$ is a bounded measure), then the integro operator in (1.2) is said to be bounded or of order zero, and in this case the integro operator acts like a non-local zeroth order term.

An important example of a non-local equation of the form (1.1) is the non-convex Isaacs equations associated with zero-sum, two-player stochastic differential games (see, e.g., [26] for the case *without* jumps)

$$u_t + \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \left\{ -\mathcal{L}^{\alpha,\beta} u - \mathcal{B}^{\alpha,\beta} u + f^{\alpha,\beta} \right\} = 0 \quad \text{in} \quad Q_T, \tag{1.4}$$

where $\mathcal{A}$ and $\mathcal{B}$ are compact metric spaces and for any sufficiently regular ϕ

$$\begin{cases} \mathcal{L}^{\alpha,\beta}\phi(t,x) = \operatorname{tr}\left[a^{\alpha,\beta}(t,x)D^2\phi\right] + b^{\alpha,\beta}(t,x)D\phi - c^{\alpha,\beta}(t,x)\phi, \\ \quad a^{\alpha,\beta}(t,x) = \dfrac{1}{2}\sigma^{\alpha,\beta}(t,x)\sigma^{\alpha,\beta^T}(t,x) \geq 0, \\ \mathcal{B}^{\alpha,\beta}\phi(t,x) = \displaystyle\int_{\mathbb{R}^M\setminus\{0\}} \left[\phi(t,x+j^{\alpha,\beta}(t,x,z)) - \phi - j^{\alpha,\beta}(t,x,z)D\phi\right]\pi(dz). \end{cases} \tag{1.5}$$

Here tr and ^{T} denote the trace and transpose of matrices. The Lévy measure $\pi(dz)$ is a positive Radon measure on $\mathbb{R}^M \setminus \{0\}$, $M \geq 1$, satisfying a condition similar to (1.3), see (A0) and (A4) in Section 4. Also see Section 4 for the (standard) regularity assumptions on the coefficients, σ, b, c, and η. We remark that if $\mathcal{A}$ is a singleton, then equation (1.4) becomes the convex Bellman equation associated with optimal control of Lévy (jump-diffusion) processes over a finite horizon (see, e.g., [42, 44] and the references therein). Henceforth we will refer to (1.4) simply as the "Bellman/Isaacs equation".

The general problem we are confronted with here is to find an upper bound on the difference between a viscosity subsolution u of (1.1) and a viscosity supersolution $\bar{u}$ of (1.1) with F replaced by another nonlinear functional $\bar{F}$ satisfying the same assumptions as F. The sought upper bound for $u-\bar{u}$ should be expressed in terms of "$F-\bar{F}$". Let us give an explicit example of the type of results that can be obtained with our general continuous dependence framework for integro-PDEs (1.1). Let u be a viscosity subsolution of (1.2) and let $\bar{u}$ be a viscosity supersolution of

$$\bar{u}_t - \int_{\mathbb{R}^M\setminus\{0\}} \left[\bar{u}(\cdot,\cdot+z) - \bar{u} - zD\bar{u}\right]\bar{\pi}(dz) = 0 \quad \text{in} \quad Q_T, \tag{1.6}$$

where $\bar{\pi}(dz)$ is another Lévy measure satisfying (1.3). For simplicity, suppose that the viscosity sub- and supersolutions are bounded, the initial values are zero, and that the Lévy measures admit densities (which is the typical case in finance applications, see Section 6), i.e.,

$$\pi(dz) = m(z)\,dz, \qquad \bar{\pi}(dz) = \bar{m}(z)\,dz,$$

for some functions $m(z)$ and $\bar{m}(z)$ that may have second order singularities at the origin. Our continuous dependence result then yields for any $(t,x) \in \overline{Q}_T$

$$(u-\bar{u})(t,x) \leq C\sqrt{T\int_{\mathbb{R}^N\setminus\{0\}} |z|^2\,|(m-\bar{m})(z)|\,dz}. \tag{1.7}$$

In other words, the difference between u and $\bar{u}$ is expressed in terms of a weighted L^1 norm of the difference between the Lévy densities m and $\bar{m}$. Note that it is important that the L^1 norm is weighted with the function $|z|^2$, as the densities may have second order singularities at the origin. The reason for the "square-root" is that the estimate is robust with respect to the smoothness of u and $\bar{u}$. If u and $\bar{u}$ are both viscosity solutions, then, by reversing the roles of u and $\bar{u}$, we obtain an estimate for $|u-\bar{u}|$. Results similar to (1.7) will be stated for the Bellman/Isaacs equation (1.4) (where also the parameters $\sigma^{\alpha,\beta}$, $b^{\alpha,\beta}$, $c^{\alpha,\beta}$, $j^{\alpha,\beta}$ are varied) as well as some integro-PDEs arising in option pricing theory in financial markets driven by Lévy processes. To our knowledge, explicit continuous dependence estimates like (1.7) have not appeared in the literature before. Moreover, compared to our previous work [34, 35], the results obtained herein are new even in the pure PDE

case, since we allow for growth in the solutions and hence our results can be applied to the PDEs (and integro-PDEs) arising in finance applications. We will come back to a finance application of our results in the last section of this paper.

Let us mention that continuous dependence estimates are relevant when it comes to determining the regularity of viscosity solutions and obtaining explicit error estimates for approximate solutions. We will provide examples of both aspects. In particular, we derive error estimates for the vanishing viscosity and vanishing jump viscosity methods for the Bellman/Isaacs equation (1.4) as well as for another singular perturbation problem studied first in [40, 37] in a simpler context. The case of numerical methods is more difficult and some of the first results in that direction for the pure PDE version of the convex Bellman equation can be found in [7, 8, 33, 39]. We anticipate that the continuous dependence estimates herein, together with the ideas in [7, 8, 33], can be used to derive error estimates for the Bellman equation of controlled jump-diffusion processes. We intend to investigate this in a future paper. Although we do not pursue this here, let us also mention that estimates like (1.7) may be relevant to the calibration (inverse) problem for finance models based on Lévy processes, e.g., the problem of determining the Lévy densities using, among other things, empirical data.

Let us now put the present paper in a proper perspective regarding previous literature on continuous dependence estimates for viscosity solutions of pure PDEs. The case of first order time-dependent Hamilton-Jacobi equations is treated in [52]. For second order PDEs, an applications of the comparison principle [18] gives a useful continuous dependence estimate when, for example, $\bar{F}$ is of the form $\bar{F} = F + h$ for some function $h = h(x)$. In general, the estimate provided by the comparison principle is limited in the sense that it cannot, for example, be used to obtain a convergence rate for the vanishing viscosity method. Continuous dependence estimates for degenerate parabolic equations that imply, among other things, a rate of convergence for the vanishing viscosity method have appeared recently in [16] (see also [30]) and [34, 35]. In particular, [34] and [35] contain results that are general enough to include the Bellman equation associated with optimal control of degenerate diffusion processes as well as the Isaacs equation of zero-sum two-player stochastic differential games. Recently a modification of our continuous dependence estimate in [34], accounting for sub-quadratic growing solutions, was used as one key step in the proof in [14] of the x-Hölder regularity of the gradient of solutions to fully nonlinear uniformly parabolic equations.

This paper is organized as follows: Section 2 is devoted to preliminary material related to viscosity solutions and in particular the statement of an "Ishii Lemma" for parabolic integro-PDEs (the elliptic version was proved recently in [36]). In Section 3 we state and prove our general continuous dependence theorem, which is applied to the Bellman/Isaacs equation (with bounded as well as unbounded viscosity solutions) in Section 4. In Section 5 we present several applications to the Bellman/Isaacs equation that include, among other things, regularity results and error estimates for some singular perturbation problems. Finally, in Section 6 we illustrate our results on some integro-PDEs for pricing European/American options in an incomplete geometric Lévy stock market.

Notations. We end this introduction by collecting some notations that will be used throughout this paper. If x, y belong to an ordered set, then we let $x \vee y$ and $x \wedge y$ denote $\max(x, y)$ and $\min(x, y)$ respectively. If x belong to $U \subset \mathbb{R}^n$ and $r > 0$,

then $B(x,r)$ denotes the ball $\{x \in U : |x| < r\}$. We use the notation $\mathbf{1}_U$ for the function that is 1 in U and 0 outside. By a modulus ω, we mean a positive, nondecreasing, continuous, sub-additive function which is zero at the origin. In the space of symmetric matrices $\mathbb{S}^N$ we denote by $\leq$ the usual ordering (i.e., $X \in \mathbb{S}^N$, $0 \leq X$ means that X positive semidefinite) and by $|\cdot|$ the spectral radius norm (i.e., the maximum of the absolute values of the eigenvalues).

Let ν be a signed measure. We denote by ν^+ and ν^- its positive and negative part, so that $\nu = \nu^+ - \nu^-$ (the Jordan decomposition). The absolute value or total variation of ν is $|\nu| = \nu^+ + \nu^-$. If ν_1 and ν_2 are positive measures, we may define the maximum as follows,

$$\nu_1 \vee \nu_2 := \left(\frac{d\nu_1}{d(\nu_1+\nu_2)} \vee \frac{d\nu_2}{d(\nu_1+\nu_2)} \right)(\nu_1+\nu_2),$$

where the derivatives are Radon-Nikodym derivatives. If there are functions f_1, f_2 and a measure ν such that $\nu_i = f_i\nu$ for $i = 1,2$ then $\nu_1 \vee \nu_2 = (f_1 \vee f_2)\nu$.

Let $C^n(\Omega)$ $n = 0,1,2$ denote the spaces of n times continuously differentiable functions on Ω, and let $C^{1,2}((0,T)\times\Omega)$ denote the space of once in time and twice in space Ω continuously differentiable functions. We let $USC(\Omega)$ and $LSC(\Omega)$ denote the spaces of upper and lower semicontinuous functions on Ω, and $SC(\Omega) = USC(\Omega) \cup LSC(\Omega)$. A lower index p denotes the polynomial growth at infinity, so $C^n_p(\Omega)$, $C^{1,2}_p((0,T)\times\Omega)$, $USC_p(\Omega)$, $LSC_p(\Omega)$, $SC_p(\Omega)$ consist of functions f from $C^n(\Omega)$, $C^{1,2}((0,T)\times\Omega)$, $USC(\Omega)$, $LSC(\Omega)$, $SC(\Omega)$, respectively, satisfying the growth condition

$$|f(x)| \leq C(1+|x|)^p \text{ for all } x \in \Omega \text{ (uniformly in } t \text{ if } f \text{ depends on time).}$$

Associated to these spaces are weighted L^∞ norms which we define as follows:

$$|f|_{0,r} = \sup_{x\in\Omega} \frac{|f(x)|}{(1+|x|)^r} \qquad \text{and} \qquad |g|_{0,r} = \sup_{t\in(0,T)} |g(t,\cdot)|_{0,r}$$

for every $r \in \mathbb{R}$ and every locally bounded function f on Ω and g on $(0,T)\times\Omega$. Finally, we let $|\cdot|_0 = |\cdot|_{0,0}$.

2. Viscosity solution theory for integro-PDEs.

In this section we provide some background material for viscosity solutions of integro-PDEs that will be needed in the preceding sections. The class of equations that we cover contains both second order PDEs and up to order two integro operators. This generality has been considered earlier by [6, 44] using directly the "maximum principle for semicontinuous functions" [17]. However, although this approach yields the correct results, it has not been justified in general (see [36]).

In [36], the authors justify a slightly different approach which uses a suitably adapted non-local "maximum principle for semicontinuous functions" or Ishii's Lemma, see Theorem 2.2 below. Here, we will use the abstract formulation given in [36] to derive continuous dependence estimates for (1.1).

For every $t \in [0,T], x, y \in \mathbb{R}^N, r, s \in \mathbb{R}, X, Y \in \mathbb{S}^N$, and $\phi, \phi_k, \psi \in C^{1,2}_p(Q_T)$ we will use the following assumptions on (1.1):

(C1) The function $(t,x,r,q,X) \mapsto F(t,x,r,q,X,\phi(t,\cdot))$ is continuous, and if $(t_k,x_k) \to (t,x), D^n\phi_k \to D^n\phi$ locally uniformly in Q_T for $n = 0,1,2$, and $|\phi_k(t,x)| \leq C(1+|x|)^p$ (C independent of k and (t,x)), then

$$F(t_k, x_k, r, q, X, \phi_k(t_k, \cdot)) \to F(t, x, r, q, X, \phi(t, \cdot)).$$

(C2) If $X \leq Y$ and $(\phi - \psi)(t, \cdot)$ has a global maximum at x, then
$$F(t, x, r, q, X, \phi(t, \cdot)) \geq F(t, x, r, q, Y, \psi(t, \cdot)).$$

(C3) There is a $\gamma \in \mathbb{R}$ (independent of r, s, t, x, q, X, ϕ) such that if $r \leq s$, then
$$\gamma(r - s) \leq F(t, x, r, q, X, \phi(t, \cdot)) - F(t, x, s, q, X, \phi(t, \cdot)).$$

(C4) For every constant $C \in \mathbb{R}$,
$$F(t, x, r, q, X, \phi(t, \cdot) + C) = F(t, x, r, q, X, \phi(t, \cdot)).$$

Remark 2.1. The constants γ in (C3) can be assumed to be non-negative. This can be seen by performing an exponential in time scaling of the solution of (1.1).

Definition 2.1 (Test functions)**.** $v \in USC_p(Q_T)$ ($v \in LSC_p(Q_T)$) is a *viscosity subsolution* (*viscosity supersolution*) of (1.1) if for every $(t, x) \in Q_T$ and $\phi \in C_p^{1,2}(Q_T)$ such that (t, x) is a global maximizer (global minimizer) for $v - \phi$,

$$\phi_t(t, x) + F(t, x, v(t, x), D\phi(t, x), D^2\phi(t, x), \phi(t, \cdot)) \leq 0 \ (\geq 0).$$

We say that v is a *viscosity solution* of (1.1) if v is both a sub- and supersolution of (1.1).

Note that viscosity solutions according to this definition are continuous, and that this concept of solutions is an extension of classical solutions. Furthermore, without changing the (sub/super) solutions, we may in this definition assume strict maxima and that $u = \phi$ at the maximum. See [36] for simple proofs of these statements and more remarks on this abstract formulation.

Next we introduce an alternative definition of viscosity solutions that is needed for proving comparison and uniqueness results. For every $\kappa \in (0, 1)$, assume that we have a function

$$F_\kappa : Q_T \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \times SC_p(Q_T) \times C^{1,2}(Q_T) \to \mathbb{R}$$

satisfying the following list of assumptions for every $t \in [0, T], x, y \in \mathbb{R}^N, r, s \in \mathbb{R}, q \in \mathbb{R}^N, X, Y \in \mathbb{S}^N, u, -v \in USC_p(Q_T), w \in SC_p(Q_T)$, and $\phi, \phi_k, \psi, \psi_k \in C_p^{1,2}(Q_T)$:

(F0) $F_\kappa(t, x, \phi(t, x), D\phi(t, x), D^2\phi(t, x), \phi(t, \cdot), \phi(t, \cdot))$
$$= F(t, x, \phi(t, x), D\phi(t, x), D^2\phi(t, x), \phi(t, \cdot)).$$

(F1) The function F in (F0) satisfy (C1).

(F2) If $X \leq Y$ and both $(u - v)(t, \cdot)$ and $(\phi - \psi)(t, \cdot)$ have global maxima at x, then $F_\kappa(t, x, r, q, X, u(t, \cdot), \phi(t, \cdot)) \geq F_\kappa(t, x, r, q, Y, v(t, \cdot), \psi(t, \cdot))$.

(F3) The function F in (F0) satisfy (C3).

(F4) For all constants $C_1, C_2 \in \mathbb{R}$,
$$F_\kappa(t, x, r, q, X, w(t, \cdot) + C_1, \phi(t, \cdot) + C_2) = F_\kappa(t, x, r, q, X, w(t, \cdot), \phi(t, \cdot)).$$

(F5) If $\psi_k(t, \cdot) \to w(t, \cdot)$ a.e. in $\mathbb{R}^N$ and $|\psi_k(t, x)| \leq C(1 + |x|^p)$, then
$$F_\kappa(t, x, r, q, X, \psi_k(t, \cdot), \phi(t, \cdot)) \to F_\kappa(t, x, r, q, X, u(t, \cdot), \phi(t, \cdot)).$$

Remark 2.2. If (F0) – (F4) hold, then (C1) – (C4) also hold.

Lemma 2.1 (Alternative definition)**.** *Assume there exists F_κ satisfying (F0) – (F2), (F4), and (F5) for every $\kappa \in (0,1)$. Then $v \in USC_p(Q_T)$ ($v \in LSC_p(Q_T)$) is a viscosity subsolution (viscosity supersolution) of* (1.1) *if and only if for every $(t,x) \in Q_T$ and $\phi \in C^{1,2}(Q_T)$ such that (t,x) is a global maximizer (global minimizer) for $v - \phi$, and for every $\kappa \in (0,1)$,*

$$\phi_t(t,x) + F_\kappa(t,x,v(t,x),D\phi(t,x),D^2\phi(t,x),v(t,\cdot),\phi(t,\cdot)) \le 0 \ (\ge 0).$$

The proof is similar to that in Sayah [46], see also [6, 36]. The next theorem replaces the maximum principle for semicontinuos functions (cf. [17, 18]) when working with integro-PDEs.

Theorem 2.2. *Let $u, -v \in USC_p(Q_T)$, $u(t,x), -v(t,x) \le C(1+|x|^2)$, solve in the viscosity sense*

$$u_t + F(t,x,u,Du,D^2u,u(\cdot)) \le 0 \quad \textit{and} \quad v_t + G(t,x,v,Dv,D^2v,v(\cdot)) \ge 0,$$

where F and G satisfies (C1) – (C4). Let $\phi \in C^{1,2}((0,T)\times\mathbb{R}^N\times\mathbb{R}^N)$ and $(\bar t,\bar x,\bar y) \in (0,T)\times\mathbb{R}^N\times\mathbb{R}^N$ be such that

$$u(t,x) - v(t,y) - \phi(t,x,y)$$

has a global maximum at $(\bar t,\bar x,\bar y)$. Furthermore assume that in a neighborhood of $(\bar t,\bar x,\bar y)$ there are continuous functions $g_0 : [0,T]\times\mathbb{R}^{2N} \to \mathbb{R}, g_1, g_2 : \overline{Q}_T \to \mathbb{S}^N$ with $g_0(\bar t,\bar x,\bar y) > 0$, satisfying

$$D^2\phi \le g_0(t,x,y)\begin{pmatrix} I & -I \\ -I & I\end{pmatrix} + \begin{pmatrix} g_1(t,x) & 0 \\ 0 & g_2(t,y)\end{pmatrix}.$$

If in addition for every $\kappa \in (0,1)$ there exist F_κ and G_κ satisfying (F0) – (F5), then for any $\bar\gamma \in (0,\frac12)$ there are $a,b \in \mathbb{R}$ and $X,Y \in \mathbb{S}^N$ satisfying

$$a - b = \phi_t(\bar t,\bar x,\bar y)$$

and

(2.1)
$$-\frac{g_0(\bar t,\bar x,\bar y)}{\bar\gamma}\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y\end{pmatrix} - \begin{pmatrix} g_1(\bar t,\bar x) & 0 \\ 0 & g_2(\bar t,\bar y)\end{pmatrix} \le \frac{g_0(\bar t,\bar x,\bar y)}{1-2\bar\gamma}\begin{pmatrix} I & -I \\ -I & I\end{pmatrix}$$

such that

$$a + F_\kappa(\bar t,\bar x,u(\bar t,\bar x),D_x\phi(\bar t,\bar x,\bar y),X,u(\bar t,\cdot),\phi(\bar t,\cdot,\bar y)) \le 0 \quad \textit{and} \tag{2.2}$$

$$b + G_\kappa(\bar t,\bar y,v(\bar y),-D_y\phi(\bar t,\bar x,\bar y),Y,v(\bar t,\cdot),-\phi(\bar t,\bar x,\cdot)) \ge 0. \tag{2.3}$$

Outline of proof. The theorem is essentially a special case of the corresponding elliptic result Theorem 4.8 in [36]. This follows from the procedure of Section 3 in Crandall and Ishii [17] that we will repeat here for the readers' convenience.

We may assume that the maximum is strict. Then the function

$$u(t,x) - v(s,y) - \phi(t,x,y) - \frac{1}{\delta}(t-s)^2,$$

will have a global maximum at some point $(\tilde t,\tilde s,\tilde x,\tilde y) \in [0,T]^2\times\mathbb{R}^{2N}$. Furthermore, as $\delta \to 0$, along a subsequence $(\tilde t,\tilde s,\tilde x,\tilde y) \to (\bar t,\bar t,\bar x,\bar y)$ and $\frac1\delta(\tilde t-\tilde s)^2 \to 0$. Choosing δ small enough, we have $(\tilde t,\tilde s) \in (0,T)^2$. Letting $\psi(t,s,x,y) := \phi(t,x,y) + \frac1\delta(t-s)^2$, it is not difficult to see that

$$\psi_t - \psi_s = \phi_t \quad \text{and} \quad D^n\psi = D^n\phi \quad (n=1,2).$$

With this in mind, we apply the elliptic result (Lemma 7.8) in [36]. The result is the existence of two matrices $\tilde{X}, \tilde{Y} \in \mathbb{S}^N$ satisfying

$$-\frac{g_0(\tilde{t},\tilde{x},\tilde{y})}{\bar{\gamma}}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} \tilde{X} & 0 \\ 0 & -\tilde{Y} \end{pmatrix} - \begin{pmatrix} g_1(\tilde{t},\tilde{x}) & 0 \\ 0 & g_2(\tilde{t},\tilde{y}) \end{pmatrix} \leq \frac{g_0(\tilde{t},\tilde{x},\tilde{y})}{1-2\bar{\gamma}}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

such that

$$\tilde{a} + F(\tilde{t},\tilde{x},u(\tilde{t},\tilde{x}),D_x\phi(\tilde{t},\tilde{x},\tilde{y}),\tilde{X},\phi(\tilde{t},\cdot,\tilde{y})) \leq 0 \quad \text{and}$$
$$\tilde{b} + G(\tilde{s},\tilde{y},v(\tilde{y}),-D_y\phi(\tilde{t},\tilde{x},\tilde{y}),\tilde{Y},-\phi(\tilde{t},\tilde{x},\cdot)) \geq 0,$$

where $\tilde{a} := \psi_t(\tilde{t},\tilde{s},\tilde{x},\tilde{y})$ and $\tilde{b} := \psi_s(\tilde{t},\tilde{s},\tilde{x},\tilde{y})$. Observe that we use the F/G-formulation, and not the F_κ/G_κ-formulation at this point. Also note that by (C4) the $(t-s)^2$ part in ψ does not appear in the non-local part in the above inequalities because it is a constant w.r.t. x and y.

The inequalities give upper bounds on $\tilde{a}$ and $-\tilde{b}$, and since $\tilde{a}-\tilde{b} = \phi_t(\tilde{t},\tilde{x},\tilde{y})$, the two sequences are bounded in δ. We may therefore extract converging subsequences of $\tilde{a},\tilde{b},\tilde{X},\tilde{Y}$ as $\delta \to 0$. Denoting the limits by a,b,X,Y, we obtain the result by sending $\delta \to 0$ along this subsequence, using (semi) continuity of all involved functions.

The final step is to show that a similar result holds in the F_κ/G_κ formulation. We omit this easy step and refer the interested reader to the proof of Theorem 4.8 in [36], see also Lemma 2.1 above. □

Remark 2.3. The technical condition $u(x), -v(x) \leq C(1+|x|^2)$ is an artifact of the method used to prove Theorem 4.8 in [36]. It does not seem easy to remove. In practice, however, it creates no difficulties.

Remark 2.4. Using the notation of [18], we note that

$$(a, D_x\phi(\bar{t},\bar{x},\bar{y}), X) \in \overline{\mathcal{J}}^{2,+}u(\bar{t},\bar{x}) \qquad \text{and} \qquad (b, -D_y\phi(\bar{t},\bar{x},\bar{y}), Y) \in \overline{\mathcal{J}}^{2,-}v(\bar{t},\bar{y}).$$

But as opposed the pure PDE case, a priori we do not know that the viscosity inequalities hold for elements in $\overline{\mathcal{J}}^{2,+}u(\bar{t},\bar{x})$ and $\overline{\mathcal{J}}^{2,-}v(\bar{t},\bar{y})$ respectively, see [36] for a discussion of this point in the elliptic setting.

3. Continuous Dependence Estimates.

In this section we formulate and prove an abstract continuous dependence estimate for Integro-PDEs. It is a pointwise estimate which may have polynomial growth in the space variable x. As will be explained in the following, this result is an extension of results in [35] (see also [34, 16]) in two directions: (i) We have equations with an integro operator and (ii) we allow for (polynomial) growth in the estimates. In the next sections we will see how this rather complicated and abstract result can be used to obtain new continuous dependence estimates for the Bellman/Isaacs and Black-Scholes type equations.

The following crucial condition can be thought of as a "continuous dependence" version of condition (3.14) in the User's Guide [18]. For every $\kappa \in (0,1), t \in [0,T], x,y \in \mathbb{R}^N, r,s \in \mathbb{R}, q \in \mathbb{R}^N, X,Y \in \mathbb{S}^N, u,-v \in USC_m(Q_T)$, and $\phi \in C^{1,2}(Q_T)$ we assume:

(F6) Let $\alpha,\varepsilon,\lambda > 0$, $p \geq 2$, and define

$$\phi(t,x,y) = e^{\lambda t}\frac{\alpha}{2}|x-y|^2 + e^{\lambda t}\frac{\varepsilon}{p}(|x|^p + |y|^p).$$

There are constants $\eta_1, \dots, \eta_4, p_1, \dots, p_4, p_s, K_1, K_2, K_3 \geq 0$ independent of $\alpha, \varepsilon, \lambda, t$, and a modulus $m_{\alpha,\varepsilon}$ (depending on α, ε) such that whenever $u(t,x) - v(t,y) - \phi(t,x,y)$ has a global maximum at $(\bar{t}, \bar{x}, \bar{y})$,

$$\begin{aligned}
&\bar{F}_\kappa\left(\bar{t}, \bar{y}, r, e^{\lambda\bar{t}}\alpha(\bar{x}-\bar{y}) - e^{\lambda\bar{t}}\varepsilon\bar{y}|\bar{y}|^{p-2}, Y, v(\bar{t},\cdot), -\phi(\bar{t},\bar{x},\cdot)\right)\\
&- F_\kappa\left(\bar{t}, \bar{x}, r, e^{\lambda\bar{t}}\alpha(\bar{x}-\bar{y}) + e^{\lambda\bar{t}}\varepsilon\bar{x}|\bar{x}|^{p-2}, X, u(\bar{t},\cdot), \phi(\bar{t},\cdot,\bar{y})\right)\\
&\quad\leq \sum_{i=1}^{2}(1+|\bar{x}|+|\bar{y}|)^{p_i}\eta_i + \alpha\sum_{i=3}^{4}(1+|\bar{x}|+|\bar{y}|)^{2p_i}\eta_i^2\\
&\qquad + K_1(1+|\bar{x}|+|\bar{y}|)^{p_s}|\bar{x}-\bar{y}| + K_2 e^{\lambda\bar{t}}\alpha|\bar{x}-\bar{y}|^2\\
&\qquad + K_3 e^{\lambda\bar{t}}\varepsilon\left(1+|\bar{x}|^p+|\bar{y}|^p\right) + m_{\alpha,\varepsilon}(\kappa),
\end{aligned}$$

for every $|r| \leq |u|_0 \wedge |\bar{u}|_0$, and X, Y satisfying

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 2e^{\lambda\bar{t}}\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + e^{\lambda\bar{t}}\varepsilon(p-1)\begin{pmatrix} |\bar{x}|^{p-2}I & 0 \\ 0 & |\bar{y}|^{p-2}I \end{pmatrix}. \tag{3.1}$$

The matrix inequality above corresponds to the second inequality in (2.1) when $\bar{\gamma} = 1/4$ and ϕ is as defined above.

Theorem 3.1 (Continuous Dependence Estimate)**.** *Let $p \geq 2$ and $m < p$, let $F, \bar{F}$ and $F_\kappa, \bar{F}_\kappa, \kappa \in (0,1)$ be functions satisfying assumptions (C1) – (C4) and (F0) – (F6) respectively, and let $u, -\bar{u} \in USC_m(Q_T)$ satisfy in the viscosity sense*

$$u_t(t,x) + F(t,x,u(t,x),Du(t,x),D^2u(t,x),u(t,\cdot)) \leq 0 \quad \textit{and}$$
$$\bar{u}_t(t,x) + \bar{F}(t,x,\bar{u}(t,x),D\bar{u}(t,x),D^2\bar{u}(t,x),\bar{u}(t,\cdot)) \geq 0.$$

Furthermore, let $p_0 \geq 0$ (p_0 is used in (3.2)), assume (F6) holds with

$$p > 2\max(p_0, \dots, p_4, p_s),$$

and assume

$$|Du(0,x)|, |D\bar{u}(0,x)| \leq K_4(1+|x|+|y|)^{p_s} \quad a.e.$$

Then there is a constant $C > 0$ (depending only on $K_1, \dots, K_4, p_0, \dots, p_4, p_s, p, T$) such that for every $(t,x) \in \overline{Q}_T$:

$$\begin{aligned}
u(t,x) - \bar{u}(t,x) &\leq C(1+|x|)^{p_0}\left|(u(0,\cdot)-\bar{u}(0,\cdot))^+\right|_{0,p_0}\\
&+ C\sum_{i=1}^{2}T^{1-\frac{p_i}{p}}(1+|x|)^{p_i}\eta_i + C\sum_{i=3}^{4}T^{\frac{1}{2}-\frac{p_i}{p}}(1+|x|)^{p_i+p_s}\eta_i.
\end{aligned} \tag{3.2}$$

Before giving the proof we give some remarks and corollaries.

Remark 3.1. We have not specified the various constants in Theorem 3.1, but it is possible to get bounds on them by tracing them in the proof below. However, getting optimal bounds would be difficult from the present proof because of the complexity, all the approximations used, and arbitrariness of the form that one factor/term can be decreased at the expense of increasing another factor/term.

However, if all the constants p's and K's are independent of T, it follows from the proof that the various constants C can be chosen to be positive, finite, continuous in T, and strictly positive in the limit as $T \to 0$. In addition, it follows that whenever one of the exponents p_0, p_1, p_2 is equal to 0, we may take the corresponding C in (3.2) to be 1.

Let us now consider a special case where u and $\bar{u}$ are bounded and there is no growth in the data, i.e., $m = p_0 = \cdots = p_4 = p_s = 0$.

Corollary 3.2. *Assume that the assumptions of Theorem 3.1 are satisfied with $m = p_0 = \cdots = p_4 = p_s = 0$ and $\eta_2 = \eta_4 = 0$. Then there is a constant $C > 0$ such that*

$$|(u - \bar{u})^+|_0 \leq |(u(0,\cdot) - \bar{u}(0,\cdot))^+|_0 + T\eta_1 + CT^{1/2}\eta_3.$$

This corollary is an extension of Theorem 2.1 in [35] to Integro-PDEs. The coefficient 1 in front of the $T\eta_1$-term is explained in Remark 3.1. Next we consider the case where u and $\bar{u}$ are both continuous. Theorem 3.1 gives an upper bound on

$$u(t,x) - \bar{u}(t,x)$$

valid for all $t \in [0,T)$ and $x \in \mathbb{R}^N$. Furthermore, this bound is independent of t, so by sending $t \to T$ and using continuity the same bound also holds for

$$u(T,x) - \bar{u}(T,x).$$

Renaming T to t we then have the following result:

Corollary 3.3. *(a) Assume that the assumptions of Theorem 3.1 hold and in addition that $u, \bar{u} \in C(\overline{Q}_T)$. Then there is a constant $C > 0$ (independent of t) such that for every $(t,x) \in \overline{Q}_T$,*

$$\begin{aligned} u(t,x) - \bar{u}(t,x) &\leq C(1+|x|)^{p_0} \left|(u(0,\cdot) - \bar{u}(0,\cdot))^+\right|_{0,p_0} \\ &+ C\sum_{i=1}^{2} t^{1-\frac{p_i}{p}}(1+|x|)^{p_i}\eta_i + C\sum_{i=3}^{4} t^{\frac{1}{2}-\frac{p_i}{p}}(1+|x|)^{p_i+p_s}\eta_i. \end{aligned}$$

(b) Assume that the assumptions of Corollary 3.2 hold and in addition that $u, \bar{u} \in C(\overline{Q}_T)$. Then there is a constant $C > 0$ (independent of t) such that

$$u(t,x) - \bar{u}(t,x) \leq |(u(0,\cdot) - \bar{u}(0,\cdot))^+|_0 + t\eta_1 + Ct^{1/2}\eta_3.$$

That fact that the constants C can be chosen independently of t follows from Remark 3.1. Take as new constants the maximum over $[0,T]$ of the t-depending C's given by Theorem 3.1.

Remark 3.2. Notice the time dependence in the estimate in Corollary 3.3 (a). It differs from the time dependency in Corollary 3.3 (b) when $p_i > 0$ for at least one $i \in \{1,2,3,4\}$. This is an effect of the growth in the data (and hence in the solutions).

In the above bounds on $u - \bar{u}$, p behaves like a free parameter. It may vary between its lower bound and any number p for which the non-local part of the equation is well-defined (so no restrictions for pure PDEs!). If we were allowed to send $p \to \infty$, we would obtain the T-exponents (t-exponents) 1 and 1/2. However, our estimates do not allow this, since the way we do the proof, at least some of the constants C will blow up as $p \to \infty$.

Remark 3.3. The complicated condition (F6) is a natural "structure condition" leading to continuous dependence estimates in the viscosity solutions setting. The use of this condition will be clearer in the next section where we derive both known and new continuous dependence results for Bellman/Isaacs equations under assumptions that include the Black-Scholes equation. The new features here consist of estimates on the integro operators and allowing for estimates with growth.

Growth in the estimates arise naturally when studying Black-Scholes type of equations where the underlying stochastic process is an exponential Lévy process. In the following sections, we will present examples where some or all of the exponents $p_0, \cdots, p_4, p_s$ are different from 0.

Finally, we remark that Theorem 3.1 allows for four error terms $\eta_1, \dots, \eta_4$ (with corresponding $p_1, \dots, p_4$). In Corollary 3.2 and in [35], only two terms were used. One could consider any number of such error terms η, both in the above theorem and in applications, but in this paper we confine ourselves to situations where up to four error terms are sufficient.

Now we turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. We may assume that $\gamma \geq 0$, see Remark 2.1. Let us start by defining the following quantities

$$\psi(t,x,y) := u(t,x) - \bar{u}(t,y) - \phi(t,x,y) - \frac{\delta\sigma}{T}t - \frac{\bar{\varepsilon}}{T-t}$$

where $\delta, \bar{\varepsilon} \in (0,1)$ and

$$\sigma_0 := \sup_{x,y\in\mathbb{R}^N} \left\{ u(0,x) - \bar{u}(0,y) - \phi(0,x,y) - \frac{\bar{\varepsilon}}{T} \right\}^+,$$

$$\sigma := \sup_{\substack{t\in[0,T)\\ x,y\in\mathbb{R}^N}} \left\{ u(t,x) - \bar{u}(t,y) - \phi(t,x,y) - \frac{\bar{\varepsilon}}{T-t} \right\} - \sigma_0.$$

By the continuity of ψ, precompactness of sets of the type $\{\psi(t,x,y) > k\}$, and the penalization term $\frac{\bar{\varepsilon}}{T-t}$, there exists $t_0 \in [0,T)$, $x_0, y_0 \in \mathbb{R}^N$ such that

$$\sup_{t\in[0,T),x,y\in\mathbb{R}^N} \psi(t,x,y) = \psi(t_0,x_0,y_0).$$

We want an upper bound on $\sigma + \sigma_0$, and we start by deriving a positive upper bound for σ. We may therefore assume that $\sigma > 0$. This implies that $t_0 > 0$, since on one hand

$$\psi(t_0,x_0,y_0) \geq \sigma + \sigma_0 - \delta\sigma > \sigma_0,$$

while on the other hand $t_0 = 0$ would imply $\psi(t_0,x_0,y_0) \leq \sigma_0$, which is a contradiction.

We can now apply Theorem 2.2 (with $\bar{\gamma} = 1/4$) to conclude that there are numbers $a, b \in \mathbb{R}$ satisfying $a - b = \phi_t(t_0,x_0,y_0) + \frac{\delta\sigma}{T} + \frac{\bar{\varepsilon}}{(T-t)^2}$, and symmetric matrices $X, Y \in \mathbb{S}^N$ satisfying inequality (3.1) such that the following inequality holds

$$\begin{aligned} a - b \leq\ & \bar{F}_\kappa(t_0,y_0,\bar{u}(t_0,y_0),-D_y\phi(t_0,x_0,y_0),Y,\bar{u}(t_0,\cdot),-\phi(t_0,x_0,\cdot)) \\ & -F_\kappa(t_0,x_0,u(t_0,x_0),D_x\phi(t_0,x_0,y_0),X,u(t_0,\cdot),\phi(t_0,\cdot,y_0)). \end{aligned}$$

Since $\sigma > 0$ it follows that $u(t_0,x_0) \geq \bar{u}(t_0,y_0)$, so after using (F3) with $\gamma \geq 0$, (F6), and the above inequality, we have

$$\begin{aligned} & \delta\frac{\sigma}{T} + \frac{\lambda}{2}e^{\lambda t_0}\alpha|x_0 - y_0|^2 + \frac{\lambda}{p}e^{\lambda t_0}\varepsilon\left(|x_0|^p + |y_0|^p\right) \\ & \leq \sum_{i=1}^{2}(1 + |x_0| + |y_0|)^{p_i}\eta_i + \alpha\sum_{i=3}^{4}(1 + |x_0| + |y_0|)^{2p_i}\eta_i^2 \end{aligned}$$

$$
\begin{aligned}
&+ K_1(1+|x_0|+|y_0|)^{p_s}|x_0-y_0| + K_2 e^{\lambda t_0}\alpha|x_0-y_0|^2 \\
&+ K_3 e^{\lambda t_0}\varepsilon\,(1+|x_0|^p+|y_0|^p) + m_{\alpha,\varepsilon}(\kappa).
\end{aligned}
$$

We send $\kappa \to 0$ and choose λ to satisfy

$$\lambda = 2(K_2+1) \vee p(K_3+1)$$

(the number $+1$ is an arbitrarily chosen positive number) and obtain

$$
\begin{aligned}
\delta\frac{\sigma}{T} &\le \sum_{i=1}^{2}(1+|x_0|+|y_0|)^{p_i}\eta_i + \alpha\sum_{i=3}^{4}(1+|x_0|+|y_0|)^{2p_i}\eta_i^2 \\
&+ K_1(1+|x_0|+|y_0|)^{p_s}|x_0-y_0| - e^{\lambda t_0}\alpha|x_0-y_0|^2 - e^{\lambda t_0}\varepsilon\,(1+|x_0|^p+|y_0|^p)\,.
\end{aligned}
$$

Then we send $\delta \to 1$, maximize w.r.t. $|x_0-y_0|$, and use

$$3^{-p+1}\,(1+|x_0|+|y_0|)^p \le 1+|x_0|^p+|y_0|^p$$

to obtain

$$
\begin{aligned}
\frac{\sigma}{T} &\le \sum_{i=1}^{2}(1+|x_0|+|y_0|)^{p_i}\eta_i + \alpha\sum_{i=3}^{4}(1+|x_0|+|y_0|)^{2p_i}\eta_i^2 \\
&\qquad + C\alpha^{-1}(1+|x_0|+|y_0|)^{2p_s} - C\varepsilon\,(1+|x_0|+|y_0|)^p \\
&:= \sum_{i=1}^{2}A_i(r) + \sum_{i=3}^{4}A_i(r) + A_5(r) - C\varepsilon r^p \quad \text{where} \quad r = 1+|x_0|+|y_0|.
\end{aligned}
$$

Now let r_i denote the maximum point of

$$A_i(r) - \frac{1}{5}C\varepsilon r^p,$$

for $i = 1,2,\dots,5$. That is

$$r_i = C\left(\frac{\eta_i}{\varepsilon}\right)^{\frac{1}{p-p_i}},\ i=1,2; \quad r_i = C\left(\frac{\alpha\eta_i^2}{\varepsilon}\right)^{\frac{1}{p-2p_i}},\ i=3,4; \quad r_5 = C(\varepsilon\alpha)^{-\frac{1}{p-2p_s}}.$$

Then we have

$$
\begin{aligned}
\sigma &\le T\sum_{i=1}^{5}\left(A_i(r_i) - \frac{1}{5}Cr_i^p\right) \\
&= CT\sum_{i=1}^{2}\varepsilon^{-\frac{p_i}{p-p_i}}\eta_i^{\frac{p}{p-p_i}} + CT\sum_{i=3}^{4}\varepsilon^{-\frac{2p_i}{p-2p_i}}(\alpha\eta_i^2)^{\frac{p}{p-2p_i}} + CT\varepsilon^{-\frac{2p_s}{p-2p_s}}\alpha^{-\frac{p}{p-2p_s}}.
\end{aligned}
$$

Now we need an estimate of σ_0. Using the regularity of the initial values and a similar optimization procedure as we used above, we obtain

$$\sigma_0 \le C\varepsilon^{-\frac{p_0}{p-p_0}}\left|\frac{(u(0,\cdot)-\bar{u}(0,\cdot))^+}{1+|\cdot|^{p_0}}\right|_0^{\frac{p}{p-p_0}} + C\varepsilon^{-\frac{2p_s}{p-2p_s}}\alpha^{-\frac{p}{p-2p_s}}.$$

By the calculations above we have

$$
\begin{aligned}
\sigma+\sigma_0 &\le C\varepsilon^{-\frac{p_0}{p-p_0}}M_0^{\frac{p}{p-p_0}} + CT\sum_{i=1}^{2}\varepsilon^{-\frac{p_i}{p-p_i}}\eta_i^{\frac{p}{p-p_i}} \\
&\qquad + CT\sum_{i=3}^{4}\varepsilon^{-\frac{2p_i}{p-2p_i}}(\alpha\eta_i^2)^{\frac{p}{p-2p_i}} + C\varepsilon^{-\frac{2p_s}{p-2p_s}}\alpha^{-\frac{p}{p-2p_s}}
\end{aligned}
$$

$$:= B_0 + \sum_{i=1}^{2} B_i + \sum_{i=3}^{4} B_i(\alpha) + B_5(\alpha),$$

where M_0 denotes the weighted norm of the initial conditions. Note that this expression holds for all positive α. We proceed to obtain an upper bound on $\sigma+\sigma_0$ that does not depend on α by choosing a suboptimal α. Let α_3 and α_4 respectively denote the minimum points of

$$B_i(\alpha) + B_5(\alpha) = CT\varepsilon^{-\frac{2p_i}{p-2p_i}}(\alpha\eta_i^2)^{\frac{p}{p-2p_i}} + C\varepsilon^{-\frac{2p_s}{p-2p_s}}\alpha^{-\frac{p}{p-2p_s}},$$

for $i = 3$ and $i = 4$, i.e.

$$\alpha_i = CT^{-\frac{(p-2p_i)(p-2p_s)}{2p(p-p_i-p_s)}}\eta_i^{\frac{p-2p_s}{p-p_i-p_s}}\varepsilon^{\frac{p_i-p_s}{p-p_i-p_s}} \qquad \text{for} \quad i = 3, 4.$$

Then set

$$\bar{\alpha} = \min\{\alpha_3, \alpha_4\}$$

and note that since $\bar{\alpha} \le \alpha_3$ and $\bar{\alpha} \le \alpha_4$, the definitions of $\bar{\alpha}, \alpha_3, \alpha_4$ lead to the following bound

$$\begin{aligned}
\sigma + \sigma_0 &\le B_1 + \sum_{i=1}^{2} B_i + \sum_{i=3}^{4} B_i(\bar{\alpha}) + B_5(\bar{\alpha}) \\
&\le B_1 + \sum_{i=1}^{2} B_i + \sum_{i=3}^{4} B_i(\alpha_i) + B_5(\bar{\alpha}) \\
&= C\varepsilon^{-\frac{p_0}{p-p_0}} M_0^{\frac{p}{p-p_0}} + CT\sum_{i=1}^{2} \varepsilon^{-\frac{p_i}{p-p_i}}\eta_i^{\frac{p}{p-p_i}} \\
&\quad + C\sum_{i=3}^{4} T^{\frac{p-2p_i}{2p-2p_i-2p_s}}\eta_i^{\frac{p}{p-p_i-p_s}}\varepsilon^{-\frac{p_i+p_s}{p-p_i-p_s}} \\
&:= A_0(\varepsilon) + \sum_{i=1}^{2} A_i(\varepsilon) + \sum_{i=3}^{4} A_i(\varepsilon),
\end{aligned} \tag{3.3}$$

which holds for any $\varepsilon > 0$.

To complete the proof, we use the definition of σ to see that

$$u(t,x) - \bar{u}(t,x) - \frac{2\varepsilon}{p}e^{\lambda t}|x|^p - \frac{\bar{\varepsilon}}{T-t} \le \sigma + \sigma_0$$

for any $(t,x) \in Q_T$. We send $\bar{\varepsilon} \to 0$, use $|x|^p \le (1+|x|)^p$, and use the bound (3.3), to see that is

$$\begin{aligned}
u(t,x) - \bar{u}(t,x) &\le \sigma + \sigma_0 + \frac{2\varepsilon}{p}e^{\lambda T}(1+|x|)^p \\
&\le A_0(\varepsilon) + \sum_{i=1}^{2} A_i(\varepsilon) + \sum_{i=3}^{4} A_i(\varepsilon) + \frac{2\varepsilon}{p}e^{\lambda T}(1+|x|)^p.
\end{aligned}$$

This bound holds for every $\varepsilon > 0$. Next we find a bound independent of ε. Let ε_i be the minimum point of

$$A_i(\varepsilon) + \varepsilon C(1+|x|)^p$$

for $i = 0, \dots, 4$, i.e.,

$$\varepsilon_0 = CM_0(1+|x|)^{p_0-p}, \qquad \varepsilon_i = CT^{\frac{p-p_i}{p}}\eta_i(1+|x|)^{p_i-p}, \ i = 1, 2,$$

and

$$\varepsilon_i = CT^{\frac{p-2p_i}{2p}}\eta_i(1+|x|)^{p_i+p_s-p},\ i=3,4.$$

Now we set

$$\bar{\varepsilon} = \max(\varepsilon_1,\dots,\varepsilon_5).$$

With this value of ε, since $\bar{\varepsilon}_i \le \varepsilon$ for $i=0,\dots,4$, we have

$$\begin{aligned} u(t,x)-\bar{u}(t,x) &\le A_0(\bar{\varepsilon}) + \sum_{i=1}^{2} A_i(\bar{\varepsilon}) + \sum_{i=3}^{4} A_i(\bar{\varepsilon}) + \frac{2\bar{\varepsilon}}{p}e^{\lambda T}(1+|x|)^p \\ &\le A_0(\varepsilon_0) + \sum_{i=1}^{2} A_i(\varepsilon_i) + \sum_{i=3}^{4} A_i(\varepsilon_i) + \frac{2\bar{\varepsilon}}{p}e^{\lambda T}(1+|x|)^p, \end{aligned}$$

which is (3.2) and the proof is complete. □

4. The Bellman/Isaacs equation

In this section we consider the Bellman/Isaacs equation (1.4) with initial values

$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}^N. \tag{4.1}$$

We will state assumptions that are natural and standard in view of the connections to the theory of stochastic control and differential games, see [25, 38, 26, 51, 44]. Under these assumptions we then derive continuous dependence results for sub- and supersolutions that are bounded or have polynomial growth at infinity.

We assume that there are constants $K_1,\dots,K_5,K_{t,x} \ge 0$, $\lambda \in \mathbb{R}$, $p \ge 2$, and a function $\rho \ge 0$ such that the following statements hold for every $t \in [0,T]$, $x,y \in \mathbb{R}^N$, $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, and $z \in \mathbb{R}^M \setminus \{0\}$:

(A0) σ, b, c, f, j are continuous w.r.t. t, x, α, β and Borel measurable w.r.t. z; $\mathcal{A}, \mathcal{B}$ are compact metric spaces; π is a positive σ-finite Radon measure on $\mathbb{R}^M \setminus \{0\}$ satisfying $\pi(\{0\}) = 0$ and

$$K_0 := \int_{B(0,1)\setminus\{0\}} \rho(z)^2\pi(dz) + \int_{\mathbb{R}^M\setminus B(0,1)} (1+\rho(z))^p\pi(dz) < \infty,$$

(A1) $|f^{\alpha,\beta}(t,x) - f^{\alpha,\beta}(t,y)| + |u_0(x)-u_0(y)| \le K_1|x-y|$,

(A2) $c^{\alpha,\beta} \ge \lambda$ and $|c^{\alpha,\beta}(t,x) - c^{\alpha,\beta}(t,y)| \le K_2|x-y|$,

(A3) $|\sigma^{\alpha,\beta}(t,x) - \sigma^{\alpha,\beta}(t,y)| + |b^{\alpha,\beta}(t,x) - b^{\alpha,\beta}(t,y)| \le K_3|x-y|$,

(A4) $|j^{\alpha,\beta}(t,x,z)| \le K_4\rho(z)(1+|x|)$, $|j^{\alpha,\beta}(t,x,z)|\chi_{B(0,1)}(z) \le K_{t,x}$, and $|j^{\alpha,\beta}(t,x,z) - j^{\alpha,\beta}(t,y,z)| \le K_5\rho(z)|x-y|$.

The Lévy measure $\pi(dz)$ may have a singularity at $z=0$. As an example in $\mathbb{R}^1$, take $\rho(z) = |z|$ and $\pi(dz) = z^{-\delta}\chi_{B(0,1)}(z)$ where $\delta \in (0,3)$. Furthermore, it integrates functions growing like $(1+\rho(x))^p$ at infinity. If assumption (A0) and (A4) hold, then the integral part of the Bellman/Isaacs equation (1.4) is well defined for functions in $C_p^{1,2}(Q_T)$, see, e.g., Garroni and Menaldi [29]. Assumptions (A0) – (A4) were used in Pham [44] to obtain comparison results for second order integro-PDEs, see also [36].

Note that (A1) – (A3) imply

$$|\sigma^{\alpha,\beta}(t,x)| + |b^{\alpha,\beta}(t,x)| + |c^{\alpha,\beta}(t,x)| + |f^{\alpha,\beta}(t,x)| + |u_0(x)| \le C(1+|x|),$$

for some constant $C > 0$. The growth at infinity of the solutions of (1.4) is equal to the growth of the fastest growing function among the initial data u_0 and the "source term" f. Hence, assumption (A1) leads to at most linear growth.

We will now state the continuous dependence results. For $i = 1, 2$ we consider a sub- or supersolution u^i of

$$\begin{aligned} u_t^i + \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \left\{ -\mathcal{L}_i^{\alpha,\beta} u^i - \mathcal{B}_i^{\alpha,\beta} u^i + f_i^{\alpha,\beta} \right\} = 0 \qquad &\text{in} \quad Q_T, \\ u^i(0,x) = u_0^i(x) \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{4.2}$$

where $\mathcal{L}_i^{\alpha,\beta}$ and $\mathcal{B}_i^{\alpha,\beta}$ are the operators defined in (1.5) corresponding to $\sigma_i, b_i, c_i, j_i, \pi_i$.

Theorem 4.1 (Bounded Case I)**.** *Assume σ_i, b_i, c_i, f_i, u_0^i, j_i, π_i, $i = 1, 2$, satisfy (A0) – (A4), $u^1 \in USC_0(Q_T)$ is a viscosity subsolution of* (4.2) *with $i = 1$, and $u^2 \in LSC_0(Q_T)$ is a viscosity supersolution of* (4.2) *with $i = 2$. Then the following pointwise estimate holds:*

$$\begin{aligned} u^1(t,x) - u^2(t,x) \le{}& \left|(u_0^1 - u_0^2)^+\right|_0 \\ &+ T \sup_{\alpha,\beta} \left\{ |f_1 - f_2|_0 + |u^1|_0 \vee |u^2|_0 |c_1 - c_2|_0 \right\} \\ &+ CT^{1/2} \Big(\sup_{\alpha,\beta} |\sigma_1 - \sigma_2|_0 + \sup_{\alpha,\beta} |b_1 - b_2|_0 \Big) \\ &+ CT^{1/2} \sup_{\alpha,\beta} \Big| \int_{\mathbb{R}^N\setminus\{0\}} |j_1 - j_2|^2 \pi(dz) \Big|_0^{1/2} \\ &+ CT^{1/2}(1 + |x|) \sup_{\alpha,\beta} \Big| \int_{\mathbb{R}^N\setminus\{0\}} j^2 |\pi_1 - \pi_2|(dz) \Big|_{0,2}^{1/2}, \end{aligned}$$

where $\pi = \max\{\pi_1, \pi_2\}$ and $j = \max\{j_1, j_2\}$.

We can get better results when u_1 and u_2 are more regular. We will only state one such result.

Theorem 4.2 (Bounded Case II)**.** *Assume σ_i, b_i, c_i, f_i, u_0^i, j_i, π_i, $i = 1, 2$, satisfy (A0) – (A4), $u^1 \in C(\overline{Q}_T)$ is a viscosity subsolution of* (4.2) *with $i = 1$, $u^2 \in C(\overline{Q}_T)$ is a viscosity supersolution of* (4.2) *with $i = 2$, and*

$$|Du^1|_0 + |Du^2|_0 < \infty.$$

Then the following pointwise estimate holds:

$$\begin{aligned} u^1(t,x) - u^2(t,x) \le{}& \left|(u_0^1 - u_0^2)^+\right|_0 \\ &+ t \sup_{\alpha,\beta} \left\{ |f_1 - f_2|_0 + |u^1|_0 \vee |u^2|_0 |c_1 - c_2|_0 + |Du^1|_0 \vee |Du^2|_0 |b_1 - b_2|_0 \right\} \\ &+ Ct^{1/2} \sup_{\alpha,\beta} \left\{ |\sigma_1 - \sigma_2|_0 + \Big| \int_{\mathbb{R}^N\setminus\{0\}} |j_1 - j_2|^2 \pi(dz) \Big|_0^{1/2} \right\} \\ &+ Ct^{1/2}(1 + |x|) \sup_{\alpha,\beta} \Big| \int_{\mathbb{R}^N\setminus\{0\}} j^2 |\pi_1 - \pi_2|(dz) \Big|_{0,2}^{1/2}, \end{aligned}$$

where $\pi = \max\{\pi_1, \pi_2\}$ and $j = \max\{j_1, j_2\}$.

In the case of sub- and supersolutions with polynomial growth, we will relax assumption (A1) and strengthen assumption (A2) in the following way:

(A1') There is a real number $p_s \ge 0$ such that

$$|f^{\alpha,\beta}(t,x) - f^{\alpha,\beta}(t,y)| + |u_0(x) - u_0(y)| \le K_1(1+|x|+|y|)^{p_s}|x-y|.$$

(A2') $\quad c^{\alpha,\beta} \ge \lambda$ and $c^{\alpha,\beta}$ is constant for each $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$.

These assumption have been used in Krylov [38] (but see Remark 4.2), where the convex Bellman equation without an integro operator is considered. See also [25]. Note that (A1') implies the following bound on f and u_0

$$|f^{\alpha,\beta}(t,x)| + |u_0(x)| \le C(1+|x|)^{1+p_s}.$$

In view of earlier remarks, such a bound also applies to the solutions of (1.4). In particular, if $p_s = 0$, the solutions have (at most) linear growth at infinity.

Theorem 4.3 (Polynomial growth)**.** *Assume σ_i, b_i, c_i, f_i, u_0^i, j_i, π_i, $i = 1,2$, satisfy (A0), (A1'), (A2'), (A3), and (A4), $u^1 \in USC_{1+p_s}(Q_T)$ is a viscosity subsolution of* (4.2) *with $i = 1$, and $u^2 \in LSC_{1+p_s}(Q_T)$ is a viscosity supersolution of* (4.2) *with $i = 2$. Let $R, r \ge 0$. If $p > 2\max(R, r, 1+p_s)$, then the following pointwise estimate holds:*

$$\begin{aligned}
&u^1(t,x) - u^2(t,x) \\
&\quad \le C(1+|x|)^R\Big(\big|(u_0^1 - u_0^2)^+\big|_{0,R} + T^{1-\frac{R}{p}} \sup_{\alpha,\beta} |f_1 - f_2|_{0,R} \Big) \\
&\qquad + CT^{1-\frac{1+p_s}{p}}(1+|x|)^{1+p_s} \sup_{\alpha,\beta} |c_1 - c_2| \\
&\qquad + CT^{\frac{1}{2}-\frac{r}{p}}(1+|x|)^{r+p_s} \\
&\qquad\quad \times \sup_{\alpha,\beta} \Big(|\sigma_1 - \sigma_2|_{0,r} + |b_1 - b_2|_{0,r} + \Big| \int_{\mathbb{R}^N\setminus\{0\}} |j_1 - j_2|^2 \pi(dz) \Big|_{0,r}^{1/2} \Big) \\
&\qquad + CT^{\frac{1}{2}-\frac{1}{p}}(1+|x|)^{1+p_s} \sup_{\alpha,\beta} \Big| \int_{\mathbb{R}^N\setminus\{0\}} j^2 |\pi_1 - \pi_2|(dz) \Big|_{0,2}^{1/2},
\end{aligned}$$

where $\pi = \max\{\pi_1, \pi_2\}$ and $j = \max\{j_1, j_2\}$.

Remark 4.1. The various constants C in the above two theorems depend on integrability and Lipschitz bounds and growth at infinity of the data/initial values of two problems, and also on the constant λ defined in (A2)/(A2'). In other words, the various constants and exponents defined in (A0) – (A4), (A1'), and (A2').

We also remark that all constants C in the two theorems above, except the ones in front of the $|\pi_1 - \pi_2|$ terms, can be chosen to depend only on one of the data-sets. Either the u_1-data or the u_2-data. This fact is written out explicitly in [34].

In applications, the constants R and r appearing in Theorem 4.3 are to be chosen such that the weighted norms are finite. In the next section, we will see examples where (i) $R = p_s$ and $r = 0$ and (ii) $R = 1 + p_s$ and $r = 1$. Note that one could let all the weighted norms above be different (have different R's and r's), but we have omitted this case for simplicity.

Remark 4.2. The restrictive assumption (A2') was introduced to simplify the estimates. With this assumption the structure of the equation is respected in the sense that the coefficients of the i-th order term is $\mathcal{O}(x^i)$ for $i = 0, 1, 2$. We could, however, use a more general assumption like the following used by Krylov [38]:

$$|c^{\alpha,\beta}(t,x) - c^{\alpha,\beta}(t,y)| \le K_2(1+|x|+|y|)^{p_c}|x-y|$$

for some $p_c \ge 0$. In addition to modifications to the c-term, the effect on Theorem 4.3 would be to replace p_s by $1 + p_s + p_c$ in the last two terms.

Remark 4.3. Due to the complexity of the problems considered here, it is not possible to give one continuous dependence result that is well suited for every special case. We have given some results that are good for problems with order two integro operators and the specified regularity of the sub- and supersolutions. By varying the assumptions, many other (mostly easier) results can be obtained from Theorem 3.1. Let us mention a few possible modifications:

- Better estimates can be had for integral operators of order 0 and 1, at least when the solutions are, e.g., Lipschitz continuous.
- Estimates for locally Hölder continuous u_0, f, c can be obtained by adapting the arguments in [34] for the global Hölder case.
- When jump-vectors j_1, j_2 are x-bounded, the estimate of Theorems 4.1 and 4.2 have no growth.

Proofs of Theorems 4.1 – 4.3. The theorems will be proved by invoking Theorem 3.1 (see also Remark 3.1 and Corollary 3.3), so we have to define the appropriate functions F, F_κ and check that they satisfy assumptions (C1) – (C4) and (F0) – (F6). We set

$$\begin{aligned}
&F(t,x,r,q,X,\phi(t,\cdot))\\
&= \inf_{\beta\in\mathcal{B}}\sup_{\alpha\in\mathcal{A}}\Big\{ -\operatorname{tr}[a^{\alpha,\beta}(t,x)\,X] - b^{\alpha,\beta}(t,x)\,q + c^{\alpha,\beta}(t,x)\,r + f^{\alpha,\beta}(t,x)\\
&\quad - \int_{\mathbb{R}^M\setminus\{0\}} \big[\phi(t,x+j^{\alpha,\beta}(t,x,z)) - \phi(t,x) - j^{\alpha,\beta}(t,x,z)\,q\big]\,\pi(dz)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&F_\kappa(t,x,r,q,X,v(t,\cdot),\phi(t,\cdot))\\
&= \inf_{\beta\in\mathcal{B}}\sup_{\alpha\in\mathcal{A}}\Big\{ -\operatorname{tr}[a^{\alpha,\beta}(t,x)\,X] - b^{\alpha,\beta}(t,x)\,q + c^{\alpha,\beta}(t,x)\,r\\
&\quad + f^{\alpha,\beta}(t,x) - B_\kappa^{\alpha,\beta}(t,x,q,\phi(t,\cdot)) - B^{\alpha,\kappa}(t,x,q,v(t,\cdot))\Big\}
\end{aligned}$$

where $a^{\alpha,\beta}$ is defined in (1.5) and

$$\begin{aligned}
&B_\kappa^{\alpha,\beta}(t,x,q,\phi(t,\cdot))\\
&\quad = \int_{B(0,\kappa)\setminus\{0\}} \big[\phi(t,x+j^{\alpha,\beta}(t,x,z)) - \phi(t,x) - j^{\alpha,\beta}(t,x,z)\,q\big]\,\pi(dz),\\
&B^{\alpha,\kappa}(t,x,q,v(t,\cdot))\\
&\quad = \int_{\mathbb{R}^M\setminus B(0,\kappa)} \big[v(t,x+j^{\alpha,\beta}(t,x,z)) - v(t,x) - j^{\alpha,\beta}(t,x,z)\,q\big]\,\pi(dz).
\end{aligned}$$

Note that p is defined in (A0). By (A0) and (A4), F satisfies (C1) – (C4) and F_κ satisfies (F0) – (F5).

The main difficulty is assumption (F6). For Theorem 4.1 to be true, the constants in (F6) must be the following: $p_1 = p_2 = p_3 = 0$, $p_4 = 1$, $p_s = 0$,

$$\begin{aligned}
&\eta_1 = \sup_{\alpha,\beta}\big\{|f_1 - f_2|_0 + |u^1|_0 \vee |u^2|_0 |c_1 - c_2|_0\big\},\\
&\eta_2 = 0,\\
&\eta_3^2 = C\sup_{\alpha,\beta}\Big\{|\sigma_1-\sigma_2|_0^2 + |b_1-b_2|_0^2 + \Big|\int_{\mathbb{R}^N\setminus\{0\}} |j_1-j_2|^2\pi(dz)\Big|_0\Big\},
\end{aligned}$$

$$\eta_4^2 = C \sup_{\alpha,\beta} \Big| \int_{\mathbb{R}^N\setminus\{0\}} j^2 |\pi_1 - \pi_2|(dz) \Big|_{0,2}.$$

Theorem 4.2 corresponds to (F6) being satisfied with the p's and η_2, η_4 defined as above and

$$\eta_1 = \sup_{\alpha,\beta} \left\{ |f_1 - f_2|_0 + |u^1|_0 \vee |u^2|_0 |c_1 - c_2|_0 + |Du^1|_0 \vee |Du^2|_0 |b_1 - b_2|_0^2 \right\},$$

$$\eta_3^2 = C \sup_{\alpha,\beta} \Big\{ |\sigma_1 - \sigma_2|_0^2 + \Big| \int_{\mathbb{R}^N\setminus\{0\}} |j_1 - j_2|^2 \pi(dz) \Big|_0 \Big\}.$$

Theorem 4.3 corresponds to (F6) being satisfied with $p_1 = R, p_2 = 1 + p_s, p_3 = r, p_4 = 1, p_s = p_s$,

$$\eta_1 = \sup_{\alpha,\beta} |f_1 - f_2|_{0,R},$$

$$\eta_2 = C \sup_{\alpha,\beta} |c_1 - c_2|,$$

$$\eta_3^2 = C \sup_{\alpha,\beta} \Big\{ |\sigma_1 - \sigma_2|_{0,r}^2 + |b_1 - b_2|_{0,r}^2 + \Big| \int_{\mathbb{R}^N\setminus\{0\}} |j_1 - j_2|^2 \pi(dz) \Big|_{0,r} \Big\},$$

and η_4 defined as above.

In [34, 35], an assumption like (F6) was shown to hold for the pure PDE version of (1.4) when sub- and supersolution are bounded. So the first case above of p's and η's have been verified when $\pi \equiv 0$. The difficulty was the second order term which was handled by a standard trick due to Ishii [31]. Therefore, here we will only consider the case where $\sigma_i = b_i = c_i = 0$, which means that for $i = 1, 2$,

$$\begin{aligned} &F_\kappa^i(t, x, r, q, X, v(t,\cdot), \phi(t,\cdot)) \\ &= \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \Big\{ f_i^{\alpha,\beta}(t,x) - B_{i,\kappa}^{\alpha,\beta}(t,x,q,\phi(t,\cdot)) - B_i^{\alpha,\beta,\kappa}(t,x,q,v(t,\cdot)) \Big\}. \end{aligned}$$

The general result easily follows from combining the argument given below with the ones given in [34, 35], where any modification due to growth should be clear from the argument below. Furthermore, we only detail the proof of Theorem 4.3 since Theorems 4.1 and 4.2 can be proved in similar but easier ways.

Starting with the r.h.s. of the inequality in (F6) we have

$$\begin{aligned} F_\kappa^2(\dots) - F_\kappa^1(\dots) \le \sup_{\alpha\in\mathcal{A},\beta\in\mathcal{B}} \Big\{ & f_2^{\alpha,\beta}(t,y) - f_1^{\alpha,\beta}(t,x) \\ &+ B_{1,\kappa}^{\alpha,\beta}(t,x,D_x\phi(t,x,y),\phi(t,\cdot,y)) - B_{2,\kappa}^{\alpha,\beta}(t,y,-D_y\phi(t,x,y),-\phi(t,x,\cdot)) \\ &+ B_1^{\alpha,\beta,\kappa}(t,x,D_x\phi(t,x,y),u(t,\cdot)) - B_2^{\alpha,\beta,\kappa}(t,x,-D_y\phi(t,x,y),v(t,\cdot)) \Big\}. \end{aligned} \tag{4.3}$$

By (A1') we have

$$f_2^{\alpha,\beta}(t,y) - f_1^{\alpha,\beta}(t,x) \le (1+|x|)^R \left| \frac{f_1^{\alpha,\beta} - f_2^{\alpha,\beta}}{(1+|\cdot|)^R} \right|_0 + K_1(1+|x|+|y|)^{p_s}|x-y|,$$

and hence η_1 becomes what we announced above. Furthermore, the difference of the $B_{1,\kappa}^{\alpha,\beta}$ and $B_{2,\kappa}^{\alpha,\beta}$ terms is bounded by some modulus $\omega_{\alpha,\varepsilon}(\kappa)$, as can be seen from (A0) and (A4) (see also [44]). So we are left with the difference of the $B_1^{\alpha,\beta,\kappa}$ and $B_2^{\alpha,\beta,\kappa}$ terms.

Here we will distinguish between the set on which the signed measure $\pi_1 - \pi_2$ is a positive measure and the set on which it is a negative measure. We denote these sets

by $D_\pm$, and remark that by the Hahn decomposition theorem $D_+ \cup D_- = \mathbb{R}^M \setminus \{0\}$ and $D_+ \cap D_- = \emptyset$. Note that

$$(\pi_1 - \pi_2)|_{D_\pm} = (\pi_1 - \pi_2)^\pm.$$

Let $D_\pm^\kappa := D_\pm \cap \{|z| \geq \kappa\}$ and observe that

$$B_i^{\alpha,\beta,\kappa}(t,x,q,v) = B_{i,\mathbb{R}^M\cap\{|z|\geq\kappa\}}^{\alpha,\beta,\kappa}(t,x,q,v) = B_{i,D_+^\kappa}^{\alpha,\beta,\kappa}(t,x,q,v) + B_{i,D_-^\kappa}^{\alpha,\beta,\kappa}(t,x,q,v),$$

where $i = 1,2$ and the extra subscript denotes the domain of integration and v is a semicontinuous function. We have

$$\begin{aligned}
(4.4)\quad & B_{1,D_+^\kappa}^{\alpha,\beta,\kappa}(t,x,D_x\phi(t,x,y),u(t,\cdot)) - B_{2,D_+^\kappa}^{\alpha,\beta,\kappa}(t,y,-D_y\phi(t,x,y),v(t,\cdot)) \\
&= \int_{D_+^\kappa} \Big[u(t,x+j_1^{\alpha,\beta}(t,x,z)) - u(t,x) - \Big(v(t,y+j_2^{\alpha,\beta}(t,y,z)) - v(t,y)\Big) \\
&\qquad - j_1^{\alpha,\beta}(t,x,z)D_x\phi(t,x,y) - j_2^{\alpha,\beta}(t,y,z)D_y\phi(t,x,y)\Big]\pi_1(dz) \\
&\quad + \int_{D_+^\kappa} \Big[v(t,y+j_2^{\alpha,\beta}(t,y,z)) - v(t,y) + j_2^{\alpha,\beta}(t,y,z)D_y\phi(t,x,y)\Big] \\
&\qquad\qquad \times (\pi_1 - \pi_2)(dz).
\end{aligned}$$

Let ψ be as in the proof of Theorem 3.1 with v replacing $\bar{u}$. Since

$$\psi(t,x,y) \geq \psi(t,x+j_1^{\alpha,\beta}(t,x,z), y+j_2^{\alpha,\beta}(t,y,z)),$$

the first integrand is bounded by

$$e^{\lambda t}\frac{\alpha}{2}|j_1^{\alpha,\beta}(t,x,z) - j_2^{\alpha,\beta}(t,y,z)|^2 + Ce^{\lambda t}\varepsilon\rho(z)^2(1+\rho(z)^{p-2})(1+|x|^p+|y|^p),$$

where the last term follows from (A4) and a Taylor expansion in x and y of the ε-terms. Furthermore, since we $\pi_1 - \pi_2 = (\pi_1 - \pi_2)^+$ on D_+, a similar argument considering $\psi(t,x,y) \geq \psi(t,x,y+j_2^{\alpha,\beta}(t,y,z))$ leads to the following upper bound on the second integral

$$\int_{D_+^\kappa} \Big(\frac{\alpha}{2}e^{\lambda t}|j_2^{\alpha,\beta}(t,y,z)|^2 + Ce^{\lambda t}\varepsilon\rho(z)^2(1+\rho(z)^{p-2})(1+|y|^p)\Big)(\pi_1-\pi_2)^+(dz).$$

Note that to obtain the last estimate, it was crucial to have v and not u in the second integral in (4.4). Combining the above estimates and using the Lipschitz regularity of $j_1^{\alpha,\beta}, j_2^{\alpha,\beta}$, and the integrability conditions (A0) and (A4), we get

$$\begin{aligned}
& B_{1,D_+^\kappa}^{\alpha,\beta,\kappa}(t,x,D_x\phi(t,x,y),u(t,\cdot)) - B_{2,D_+^\kappa}^{\alpha,\beta,\kappa}(t,y,-D_y\phi(t,x,y),v(t,\cdot)) \\
&\leq \frac{\alpha}{2}e^{\lambda t}\int_{D_+^\kappa} |j_1^{\alpha,\beta}(t,x,z) - j_2^{\alpha,\beta}(t,x,z)|^2\pi_1(dz) \\
&\quad + \frac{\alpha}{2}e^{\lambda t}\int_{D_+^\kappa} |j_2^{\alpha,\beta}(t,y,z)|^2(\pi_1-\pi_2)^+(dz) \\
&\quad + Ce^{\lambda t}\alpha|x-y|^2 + Ce^{\lambda t}\varepsilon(1+|x|^p+|y|^p).
\end{aligned}$$

Note that the constants C are independent of κ since by (A4), there is a factor $\rho(z)^2$ in all relevant integrands above. In a similar way, but by interchanging the roles of v and $(\pi-\pi)^+$ with u and $(\pi-\pi)^-$, we get

$$B_{1,D_-^\kappa}^{\alpha,\beta,\kappa}(t,x,D_x\phi(t,x,y),u(t,\cdot)) - B_{2,D_-^\kappa}^{\alpha,\beta,\kappa}(t,y,-D_y\phi(t,x,y),v(t,\cdot))$$

$$
\begin{aligned}
&\le \frac{\alpha}{2} e^{\lambda t} \int_{D^\kappa_-} |j_1^{\alpha,\beta}(t,x,z) - j_2^{\alpha,\beta}(t,x,z)|^2 \pi_2(dz) \\
&\quad + \frac{\alpha}{2} e^{\lambda t} \int_{D^\kappa_-} |j_1^{\alpha,\beta}(t,y,z)|^2 (\pi_1 - \pi_2)^-(dz) \\
&\quad + C e^{\lambda t} \alpha |x-y|^2 + C e^{\lambda t} \varepsilon (1 + |x|^p + |y|^p).
\end{aligned}
$$

Remember that $|\pi_1 - \pi_2| = (\pi_1 - \pi_2)^+ + (\pi_1 - \pi_2)^-$. By the above estimates and the linear growth at infinity of j_1, j_2, see (A4), we can conclude that

$$
\begin{aligned}
&F^2_\kappa(\dots) - F^1_\kappa(\dots) \le (1+|x|)^R \eta_1 + \alpha (1+|x|)^{2r} \eta_3^2 + \alpha (1+|x|)^2 \eta_4^2 \\
&+ C\Big((1+|x|+|y|)^{p_s} |x-y| + e^{\lambda t} \alpha |x-y|^2 + e^{\lambda t} \varepsilon (1 + |x|^p + |y|^p) \Big),
\end{aligned}
$$

where η_1, η_3, η_4 were defined above. This completes the proof of condition (F6) when $\sigma_i, b_i, c_i = 0$ for $i = 1, 2$. □

4.1. **The obstacle problem.** We will now state continuous dependence results for bounded sub- and supersolutions of the obstacle problem corresponding to the Bellman/Isaacs equation (1.4). For $i = 1, 2$ we consider

$$
\begin{aligned}
&\max\Big\{ u^i_t + \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \Big\{ -\mathcal{L}^{\alpha,\beta}_i u^i - \mathcal{B}^{\alpha,\beta}_i u^i + f^{\alpha,\beta}_i \Big\}, u^i - g_i \Big\} = 0 \qquad \text{in } Q_T, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u^i(0,x) = u^i_0(x) \text{ in } \mathbb{R}^N.
\end{aligned} \tag{4.5}
$$

The operators $\mathcal{L}^{\alpha,\beta}_i$ and $\mathcal{B}^{\alpha,\beta}_i$ are the operators defined in (1.5) corresponding to $\sigma_i, b_i, c_i, j_i, \pi_i$. Now we replace assumptions (A0) and (A1) by the following

(A0') Assumption (A0) holds and g is continuous and compatible with u_0, i.e. $u_0(x) \le g(0,x)$ for all $x \in \mathbb{R}^N$.

(A1") $|f^{\alpha,\beta}(t,x) - f^{\alpha,\beta}(t,y)| + |g(t,x) - g(t,y)| + |u_0(x) - u_0(y)| \le K_3 |x-y|$.

Theorem 4.4 (Obstacle problem)**.** *Assume σ_i, b_i, c_i, f_i, u^i_0, j_i, π_i, g_i, $i = 1, 2$, satisfy (A0'), (A1"), (A2), (A3), and (A4), $u^1 \in USC_0(Q_T)$ is a viscosity subsolution of* (4.5) *with $i = 1$, and $u^2 \in LSC_0(Q_T)$ is a viscosity supersolution of* (4.5) *with $i = 2$. Then the following pointwise estimate holds:*

$$
\begin{aligned}
u^1(t,x) - u^2(t,x) &\le \big|(u^1_0 - u^2_0)^+\big|_0 + |g_1 - g_2|_0 \\
&\quad + T \sup_{\alpha,\beta} \big\{ |f_1 - f_2|_0 + |u^1|_0 \vee |u^2|_0 |c_1 - c_2|_0 \big\} \\
&\quad + C T^{1/2} \Big(\sup_{\alpha,\beta} |\sigma_1 - \sigma_2|_0 + \sup_{\alpha,\beta} |b_1 - b_2|_0 \Big) \\
&\quad + C T^{1/2} \sup_{\alpha,\beta} \Big| \int_{\mathbb{R}^N \setminus \{0\}} |j_1 - j_2|^2 \pi(dz) \Big|_0^{1/2} \\
&\quad + C T^{1/2} (1 + |x|) \sup_{\alpha,\beta} \Big| \int_{\mathbb{R}^N \setminus \{0\}} j^2 |\pi_1 - \pi_2|(dz) \Big|_{0,2}^{1/2},
\end{aligned}
$$

where $\pi = \max\{\pi_1, \pi_2\}$ and $j = \max\{j_1, j_2\}$.

The proof of this result relies on an obstacle version of Theorem 3.1 (see also Remark 3.1) and follows along the lines of the proof of Theorem 4.1. The modifications are easy and will be omitted here. See [33] for a proof in the case of no

integral term. We mention that the results corresponding to Theorems 4.2 and 4.3 also hold for the obstacle problem.

In the next sections the above result will be used in the American option problem and in a singular perturbation problem by J.-L. Lions and S. Koike.

5. Applications

5.1. **Regularity of solutions.** In this subsection we will use the results of the previous section to obtain Lipschitz estimates for the viscosity solution u of the Bellman/Isaacs equation (1.4). We remark that the procedure given below have essentially been used in [34, 33] (bounded solutions) and in [14] (solutions with sub-quadratic growth) to obtain x-regularity of solutions and in the two last papers also to obtain the t-regularity. While it is not the most general approach for obtaining x-regularity, it seems to be a natural approach for t-regularity.

The estimates below will be derived under natural assumptions on the data. In fact, we will use the same assumptions on the coefficients as Krylov [38] (but see Remark 4.2), and all results given below will be consistent with those obtained in Chapter 4.1 in [38]. Note however that as opposed to Krylov, we consider also non-convex equations and equations with integro terms. Furthermore, we do not use stochastic control theory, but pure PDE methods.

Let us start by giving an estimate of the Lipschitz regularity in x. We assume that (A0), (A1'), (A2'), (A3), and (A4) hold. Theorem 4.3 yields directly the next result, as can be seen by choosing $u^1(t,x) = u(t,x+h)$, $u^2(t,x) = u(t,x)$, $R = p_s$, and $r = 0$.

Proposition 5.1. *Under the assumptions given above, there is a constant C depending only on T and the data, such that for every $t \in [0,T], x, h \in \mathbb{R}^N$,*

$$|u(t,x+h) - u(t,x)| \le C(1+|x|)^{p_s}|h|.$$

We will now show how one can obtain regularity in time – at least when the initial condition has suitable growth restrictions on its two first derivatives. We proceed in three steps. First we estimate the difference $|u(t+h,x) - u(t,x)|$ using Theorem 4.3 with $u^1(t,x) = u(t+h,x)$ and $u^2(t,x) = u(t,x)$ and the following natural assumptions on the time regularity of the data: There are constants $C_1, \dots, C_3$ such that for every $t, s \in [0,T]$, $x \in \mathbb{R}^N$, and $z \in \mathbb{R}^M \setminus \{0\}$,

(B1) $\quad |f^{\alpha,\beta}(t,x) - f^{\alpha,\beta}(s,x)| \le C_1(1+|x|)^{1+p_s}|t-s| \quad$ (p_s defined in (A1'))),

(B2) $\quad |\sigma^{\alpha,\beta}(t,x) - \sigma^{\alpha,\beta}(s,x)| + |b^{\alpha,\beta}(t,x) - b^{\alpha,\beta}(s,x)| \le C_2(1+|x|)|t-s|,$

(B3) $\quad |j^{\alpha,\beta}(t,x,z) - j^{\alpha,\beta}(s,x,z)| \le C_3\rho(z)(1+|x|)|t-s|.$

The result is (with $R = 1+p_s$ and $r = 1$):

Lemma 5.2. *Under the assumption given above, there is a constant C depending only on T and the data, such that for every $x \in \mathbb{R}^N$, and t, h such that $h \ge 0$ and $t, t+h \in (0,T]$,*

$$|u(t+h,x) - u(t,x)| \le C(1+|x|)^{1+p_s}|u(h,\cdot) - u_0|_{0,1+p_s} + C(1+|x|)^{1+p_s}h.$$

The second step is to estimate the weighted norm above. We want to show that

$$\left|\frac{u(h,x) - u_0(x)}{(1+|x|)^{1+p_s}}\right| \le Ch. \tag{5.1}$$

Here we make the following simplifying assumption on the initial data:

(B4) $u_0 \in C^2(\mathbb{R}^N)$ and $|D^i u_0(x)| \le \overline{C}_i(1+|x|)^{1+p_s-i}$ for $i = 0,1,2$. (p_s is defined in (A1')).

It is not difficult to see that if C is large and t is small then

$$w^{\pm}(t,x) := u_0(x) \pm Ct(1+|x|^2)^{(1+p_s)/2}$$

is a subsolution of (1.4) when the sign is minus and a supersolution when it is plus. By the comparison principle we have $w^- \le u \le w^+$ which implies (5.1) for small h.

Combining step 1 and step 2 we have

$$|u(t+h,x) - u(t,x)| \le C(1+|x|)^{1+p_s} h$$

for small h. The third step is to obtain an estimate for any h. Pick an arbitrary h and let M be an integer such that h/M is small enough for the above estimate to apply. Then we have

$$\begin{aligned} |u(t+h,x) - u(t,x)| &\le \sum_{i=1}^{M} |u(t+ih/M,x) - u(t+(i-1)h/M,x)| \\ &\le \sum_{i=1}^{M} C(1+|x|)^{1+p_s} h/M = C(1+|x|)^{1+p_s} h, \end{aligned}$$

and we are done. What we have proved is the following proposition:

Proposition 5.3. *Under the assumption given above, there is a constant C depending only on T and the data, such that for every $x \in \mathbb{R}^N$, and t, h such that $h \ge 0$ and $t, t+h \in (0,T]$,*

$$|u(t+h,x) - u(t,x)| \le C(1+|x|)^{1+p_s} h.$$

We remark that assumption (B4) can be relaxed to requiring that u_0 belongs to $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ and the growth restrictions on the derivatives hold a.e. This follows from Theorem 4.3 after a mollification of u_0, see [33] p. 14 for a similar argument (see also [14]). However, except for the case where all coefficients are bounded, it is not straightforward to use this procedure to obtain Hölder 1/2 regularity estimates in time when u_0 is only Lipschitz continuous. Such estimates have been obtained by probabilistic arguments, at least for convex Bellman equations. We refer to Pham [44] for the case where $p_s = 0$ and solutions have linear growth at infinity, and to Krylov [38] Exercise 4.1.2 for the pure PDE case where solutions have polynomial growth at infinity.

5.2. **The vanishing viscosity method.** In this section we will study the vanishing viscosity problem for the Bellman/Isaacs equation,

$$\begin{aligned} u^\varepsilon_t + \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \left\{ -\mathcal{L}^{\alpha,\beta} u^\varepsilon - \mathcal{B}^{\alpha,\beta} u^\varepsilon + f^{\alpha,\beta} \right\} &= \varepsilon^2 \Delta u^\varepsilon \quad \text{in} \quad Q_T, \\ u^\varepsilon(x,0) &= u_0(x) \quad \text{in} \quad \mathbb{R}^N, \end{aligned}$$

where the operators $\mathcal{L}$ and $\mathcal{B}$ have been defined in (1.5). The idea is to obtain the rate of convergence of $u^\varepsilon \to u^0$ as $\varepsilon \to 0$. The vanishing viscosity method have been widely used to obtain both existence and uniqueness of solutions of first order non-linear equations, see ,e.g., [41, 20, 52] for Hamilton-Jacobi equations and

[21] for conservation laws. Note that this construction procedure is in general not reasonable for non-convex second order equations, since now the "viscous problem" need not have smooth solutions.

We will assume that (A0), (A1'), (A2'), (A3), and (A4) hold, and use Theorem 4.3 to compare u^ε and u^0. Note that all coefficients coincide, except for the diffusion coefficients. In vanishing viscosity equation it is $\sigma\sigma^T+\varepsilon^2 I$, and in the limit equation it is $\sigma\sigma^T$. It is not difficult to see that

$$\sqrt{\sigma\sigma^T+\varepsilon^2 I}-\sqrt{\sigma\sigma^T}\le \varepsilon I,$$

so we immediately have:

Proposition 5.4. *Under the assumptions given above, there is a constant C depending only on T and the data, such that for every $\varepsilon>0$ and $(t,x)\in\overline{Q}_T$,*

$$|u(t,x)-u^\varepsilon(t,x)|\le Ct^{1/2}(1+|x|)^{p_s}\varepsilon.$$

Such estimates have been known from stochastic control theory, at least for the convex Bellman equation without an integro operator and no growth in the solutions (cf. Fleming and Soner [25] p. 181). From a PDE point of view, similar results have been given for first order Hamilton-Jacobi equation in [19] and recently for second order equations in [16, 34, 35]. However, the above result is valid under more general assumptions (polynomial growth and integro terms), and it also gives the dependence on time t as opposed to earlier results.

5.3. **The vanishing jump viscosity method.** Now we propose a new limit procedure which we call the vanishing jump viscosity method in analogy with the vanishing viscosity method considered above. Consider

$$\begin{aligned} u^\varepsilon_t+\inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\left\{-\mathcal{L}^{\alpha,\beta}u^\varepsilon+f^{\alpha,\beta}\right\}&=\mathcal{B}_\varepsilon u^\varepsilon \quad \text{in} \quad Q_T,\\ u^\varepsilon(x,0)&=u_0(x) \quad \text{in} \quad \mathbb{R}^N, \end{aligned} \tag{5.2}$$

where the operator $\mathcal{L}$ is as above and $\mathcal{B}$ is defined as

$$\mathcal{B}_\varepsilon\phi=\int_{\mathbb{R}^M\setminus\{0\}}\Big[\phi(\cdot,\cdot+\varepsilon z)-\phi-\varepsilon z D\phi\Big]\pi(dz),$$

for any smooth function ϕ and $z\in\mathbb{R}^N$. We may write

$$\mathcal{B}_\varepsilon\phi(t,x)=\varepsilon^2\int_{\mathbb{R}^M\setminus\{0\}}(\int_0^1\int_0^s\left[z^TD^2\phi(t,x+zr)z\right]dr\,ds)\pi(dz)$$

to see that this term is non-local second order term with "ellipticity" constant ε^2. Assume that (A0) – (A3) hold with $\rho(z)=|z|$, and note that the jump vector $j=\varepsilon z$ is bounded in x. Then by Theorem 4.1 and Remark 4.3, the following result holds:

Proposition 5.5. *Under the assumptions given above, there is a constant C depending only on T and the data, such that for every $\varepsilon>0$ and $(t,x)\in\overline{Q}_T$,*

$$|u(t,x)-u^\varepsilon(t,x)|\le Ct^{1/2}\varepsilon.$$

Compared with Proposition 5.4, $p_s=0$ and solutions are bounded. We remark that for $\varepsilon>0$ the underlying stochastic process is a jump-diffusion process, while in the limit $\varepsilon=0$ the jump term is zero and the underlying process is a pure diffusion. We also remark that this result can be generalized to general jump-vectors j and

a nonlinear dependence on the integro operator. Finally, we mention that it is not clear if the vanishing jump viscosity method is useful in practice, since it is not known in general if equation (5.2) has smooth solutions for $\varepsilon > 0$.

5.4. **A singular perturbation problem by J.-L. Lions and S. Koike.** In this subsection we study a generalization to integro-PDEs of a singular perturbation problem studied in Koike [37]. This problem is a generalization of the following problem proposed and analyzed by J. L. Lions [40]:

$$\begin{aligned} \max\{-\varepsilon^2\Delta u^\varepsilon + u^\varepsilon, u^\varepsilon - g\} = 0 \quad &\text{a.e. in } \Omega, \\ u^\varepsilon = 0 \quad &\text{on } \partial\Omega, \end{aligned}$$

where Ω is a smooth bounded domain and g is any given smooth function such that $g = 0$ on $\partial\Omega$. Using the classical theory of variational inequalities, Lions proves that

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \le C\varepsilon,$$

for some constant C. Armed with viscosity solution techniques, Koike studies the following generalization:

$$\begin{aligned} \max\{\max_k\{-L^{k,\varepsilon}u^\varepsilon - f^k\}, u^\varepsilon - g\} = 0 \quad &\text{in } \Omega, \\ u^\varepsilon = 0 \quad &\text{on } \partial\Omega, \end{aligned}$$

where Ω is smooth and bounded, $\varepsilon > 0$, $k \in \mathbb{N}$,

$$L^{k,\varepsilon}\phi = \varepsilon^2 \mathrm{tr}[a^k D^2\phi] + \varepsilon b^k D\phi - c^k\phi,$$

and the data belong to $C^2(\Omega)$. Furthermore, he assumes that there are $\alpha, \theta > 0$ such that $c^k \ge \alpha$ and $\xi^T A^k \xi \ge \theta|\xi|^2$ for every k and $\xi \in \mathbb{R}^N$. Under the compatibility assumption that $u^0 \equiv \min\{\min_k\{f^k/c^k\}, g\} = 0$ on $\partial\Omega$, he proves that

$$\|u^\varepsilon - u^0\|_{L^\infty(\Omega)} \le C\varepsilon.$$

We will study the following parabolic generalization of the above problems to integro-PDEs,

$$\begin{aligned} \max\{\max_{\alpha\in\mathcal{A}}\{u_t - \mathcal{L}^{\alpha,\varepsilon}u^\varepsilon - \mathcal{B}^\alpha u^\varepsilon - f^\alpha\}, u^\varepsilon - g\} = 0 \quad &\text{in} \quad Q_T, \\ u^\varepsilon(0,x) = 0 \quad &\text{in} \quad \mathbb{R}^N, \end{aligned}$$

where $\varepsilon > 0$, $\mathcal{B}^\alpha$ is defined in (1.5) (let e.g. $\beta \in \{0\}$), and

$$\mathcal{L}^{\alpha,\varepsilon}\phi = \varepsilon^2\frac{1}{2}\mathrm{tr}\left[\sigma^\alpha\sigma^{\alpha T}D^2\phi\right] + \varepsilon b^\alpha D\phi - c^\alpha\phi.$$

We will assume that all coefficients are bounded and satisfy (A0'), (A1"), (A2) – (A4). The boundedness of f (and the 0 initial condition) implies that that both u^ε and u^0 are bounded, so using Theorem 4.4 to compare u^ε and u^0 yields:

Proposition 5.6. *Under the assumptions given above, there is a constant C depending only on T and the data, such that for every $\varepsilon > 0$ and $(t,x) \in \overline{Q}_T$,*

$$\|u^\varepsilon - u^0\|_{L^\infty(\overline{Q}_T)} \le C\varepsilon.$$

6. Continuous dependence in the Black-Scholes model

6.1. **Introduction.** The standard model for describing the evolution of stock prices is the geometric Brownian motion, and this model assumes that the stock returns are normally distributed. However, the normal distribution poorly fits the stock returns. Indeed, it is well known that returns distributions are, for example, leptokurtic and have longer and fatter tails than the normal distribution (see, e.g., [49, 22, 10, 45]). To improve on this unfortunate situation, many Lévy, or jump-diffusion models, have been suggested in the literature over the years (we say a little bit more about this at the end of this section). For a general introduction to the theory of pricing contingent claims in diffusion as well as jump-diffusion markets, we refer to [49]. We also refer to [43, 15] (there are many more) for some particular papers studying option pricing problems in the context of Lévy processes.

In this section we illustrate our continuous dependence results on some integro-PDEs for pricing European/American options in a financial market model driven by a geometric Lévy process for the stock price. In this context solutions are not bounded, and even in the pure PDE case our previous results [34] cannot be applied.

For the sake of clarity and simplicity of presentation, we will in this section restrict ourselves to a model consisting of one risky asset (stock) and hence one-dimensional integro-PDEs. In view of the previous sections in this paper, we can certainly do this without loss of generality.

6.2. **Option pricing in Lévy markets.** We consider a financial market where the underlying uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual assumptions [49]. The financial market consists of a bond (bank account) whose price process evolves according to $dB_t = rB_t\,dt$, where $r > 0$ is a constant interest rate, and a risky asset (stock) with price dynamics denoted by X_t. Under the no-arbitrage assumption there exists a measure equivalent to P that turns $X_t e^{-rt}$ into a martingale. In a complete financial market the unique arbitrage free price of a contingent claim is given as a discounted conditional expectation value with respect to the unique equivalent martingale measure, which in turn solves the Black-Scholes PDE. In an incomplete market, however, there exist infinitely many equivalent martingale measures and corresponding arbitrage free prices. Consequently, to price a contingent claim, one needs to select an appropriate equivalent martingale measure. Lévy markets are indeed incomplete, but we are not interested in any particular choice of an equivalent martingale measures. Instead, without loss of generality, we assume that P is a given martingale measure.

The (risk-neutral) price dynamics X_t under martingale measure P is here given by the geometric Lévy model

$$X_t = X_0 \exp(rt + L_t), \quad t > 0, \qquad X_0 := x > 0, \tag{6.1}$$

where L_t is a Lévy process. The Lévy-Khintchine decomposition of L_t reads [49]

$$L_t = \mu t + \sigma W_t + \int_0^t \int_{0<|z|<1} z\,\tilde{N}(dz, ds) + \int_0^t \int_{|z|\geq 1} z\,N(dz, ds),$$

where $\mu \in \mathbb{R}$; $\sigma \geq 0$; W_t is a Brownian motion; $N(dt, dz)$ is the jump measure of L_t with a compensator $\pi(dz) \times dt$; and the so-called Lévy measure $\pi(dz)$ is a positive

Radon measure on $\mathbb{R}\setminus\{0\}$ satisfying

$$\pi\left(\{0\}\right)=0, \qquad \int_{\mathbb{R}\setminus\{0\}}\left(|z|^2\wedge 1\right)\,\pi(dz)<\infty.$$

The triplet $\left(\mu,\sigma^2,\pi(dz)\right)$ is called the characteristic triplet of the Lévy process L_t. We assume that the Lévy measure $\pi(dz)$ satisfies the integrability condition

$$\int_{|z|\geq 1}\left(e^z-1\right)\,\pi(dz)<\infty, \tag{6.2}$$

which is a necessary and sufficient condition for the stock price given by (6.1) to possess first moments.

The condition that X_te^{-rt} should be a martingale puts some restrictions on the characteristic triplet $\left(\mu,\sigma^2,\pi(dz)\right)$. Namely,

$$\mu=-\frac{1}{2}\sigma^2-\int_{\mathbb{R}\setminus\{0\}}\left(e^z-1-z\mathbf{1}_{|z|<1}\right)\pi(dz).$$

Hence, under (6.2), we can use Itô's formula (see, e.g., [49]) for semimartingales to write (6.1) as

$$dX_t=rX_t\,dt+\sigma X_t\,dW_t+X_{t-}\int_{\mathbb{R}\setminus\{0\}}\left(e^z-1\right)\tilde{N}(dz,dt). \tag{6.3}$$

For a European option $g(X_t)$ with maturity T, given P, the corresponding arbitrage-free price at time t is

$$c_t=\mathbb{E}\left[e^{-r(T-t)}g(X_T)\,|\,\mathcal{F}_t\right],$$

while for an American option with payoff $\{g(X_t)\}_{0\leq t\leq T}$ it is given by

$$C_t=\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}\left[e^{-r(\tau-t)}g(X_\tau)\,|\,\mathcal{F}_t\right],$$

where $\mathcal{T}_{t,T}$ denotes all the stopping times between t and T.

Introducing the change of variables $t\mapsto T-t$, the arbitrage-free price of the European option can be stated as

$$c_t=c(T-t,X_t), \qquad c(t,x)=\mathbb{E}\left[e^{-rT}g(X_T(x))\,|\,X_t=x\right].$$

Similarly, the arbitrage-free price of the American option is can be stated as

$$C_t=C(T-t,X_t), \qquad C(t,x)=\sup_{\tau\in\mathcal{T}_{0,t}}\mathbb{E}\left[e^{-r\tau}g(X_\tau(x))\,|\,X_t=x\right].$$

6.3. **Integro-PDEs and continuous dependence.** It is well know that $c(t,x):[0,T]\times(0,\infty)\to\mathbb{R}$ and $C(t,x):[0,T]\times(0,\infty)\to\mathbb{R}$ solve uniquely in the viscosity solution sense (see, e.g., [44]) the following integro-PDE problems:

$$\begin{cases} u_t-\dfrac{1}{2}\sigma^2x^2u_{xx}-rxu_x+ru \\ -\displaystyle\int_{\mathbb{R}\setminus\{0\}}\left[u(t,x+x(e^z-1))-u(t,x)-x(e^z-1)u_x(t,x)\right]\,\pi^p(dz)=0, \\ u(0,x)=g(x). \end{cases} \tag{6.4}$$

and

$$
(6.5)\quad \begin{cases} \min\Big\{ u-g, u_t - \frac{1}{2}\sigma^2 x^2 u_{xx} - rxu_x + ru \\ \quad - \int_{\mathbb{R}\setminus\{0\}} \left[u(t,x+x(e^z-1)) - u(t,x) - x(e^z-1)u_x(t,x)\right] \pi^p(dz) \Big\} = 0, \\ u(0,x) = g(x). \end{cases}
$$

Typical examples of payoff functions are

$$
g(x) = (x-K)^+ \text{ (call) and } g(x) = (K-x)^+ \text{ (put)},
$$

where $K > 0$ is the exercise price. Instead of being specific, we shall here simply assume that $g(x)$ is some function satisfying, for any $x, y \in (0,\infty)$,

$$
(6.6)\qquad |g(x) - g(y)| \le C|x-y|, \qquad |g(x)| \le C(1+|x|).
$$

Moreover, we shall assume that the Lévy measure $\pi(dz)$ admits a density (see below for an example):

$$
(6.7)\qquad \pi(dz) = m(z)\,dx, \qquad \text{for some function } m : \mathbb{R}\setminus\{0\} \to \mathbb{R}.
$$

The following theorem is a consequence of our previous more general results.

Theorem 6.1. *Assume* (6.6) *and* (6.7) *hold. For $i = 1,2$, assume $\sigma_i > 0$, $r_i > 0$ are constants, and $\pi_i(dz) = m_i(z)\,dz$ are Lévy measures admitting densities. For $i = 1,2$, let u_i be a viscosity solution of* (6.4)*, or* (6.5)*, with the "data" $\sigma, r, \pi(dz), g$ replaced by*

$$
\sigma_i, \quad r_i, \quad \pi_i^{p_i}(dz) = m_i(z)\,dz, \quad g_i.
$$

Then for any $p > 2$ and any $(t,x) \in [0,T) \times (0,\infty)$

$$
\begin{aligned}
&|u_1(t,x) - u_2(t,x)| \\
&\qquad \le C(1+x)\left(|g_1 - g_2|_{0,1} + |r_1 - r_2| + |\sigma_1 - \sigma_2|\right) \\
&\qquad\quad + C(1+x)\sqrt{\int_{\mathbb{R}\setminus\{0\}} |z|^2 \Big| m_1(z) - m_2(z)\Big|\, dz m}
\end{aligned}
$$

for some constant C that depends on the data of the two problems and T.

Remark 6.1. In the previous sections we have considered equations set in the domain $(0,T)\times\mathbb{R}^N$, while here we are on the domain $(0,T)\times(0,\infty)$. So strictly speaking we can not use our previous results "directly" to obtain Theorem 6.1. But it is easy to overcome this, simply extend the function g by symmetry to all of $(0,T)\times\mathbb{R}$, and consider equations (6.4) and (6.5) on the new domain $(0,T)\times\mathbb{R}$. Now we may use Theorem 3.1. Theorem 6.1 then follows since the solutions of these new problems will coincide with u_1 and u_2 (defined in Theorem 6.1) on $[0,T)\times(0,\infty)$. The reason for this is that the equations degenerate at $x = 0$, so there is no "communication" between the two domains $(0,T)\times(-\infty,0)$ and $(0,T)\times(0,\infty)$.

We will now display some applications of Theorem 6.1.

(i) Different Lévy measures. Note that different choices of the Lévy measure correspond to different geometric Lévy models for the stock price dynamics. In particular, as an application of Theorem 6.1, we have an explicit estimate on the

difference between the unique arbitrage free European/American option (Black-Scholes) price in a complete diffusion market, call it $v_{\mathrm{com}}(t,x)$ and the arbitrage free European/American option price in our Lévy market, call it $v_{\mathrm{incom}}(t,x)$:

$$|v_{\mathrm{com}}(t,x) - v_{\mathrm{incom}}(t,x)| \le C(1+x)\sqrt{\int_{\mathbb{R}\setminus\{0\}} |z|^2\, m(z)\, dz}.$$

(ii) Truncation of domain of integration I. When attempting to solve integro-PDEs like (6.4) or (6.5) by numerical methods, one needs to reduce the integration domain $\mathbb{R}\setminus\{0\}$ to a bounded domain. One way to achieve this is to replace the original Lévy process L_t with characteristic triplet $\big(\mu,\sigma^2,\pi(dz)\big)$ by another Lévy process L_t^ε with characteristic triplet $\big(\mu_\varepsilon,\sigma^2,\pi_\varepsilon(dz)\big)$, where $\varepsilon>0$ is small and

$$\begin{aligned}
\mu_\varepsilon &= -\frac{1}{2}\sigma^2 - \int_{\mathbb{R}\setminus\{0\}} \big(e^z - 1 - z\mathbf{1}_{|z|<1}\big)\,\pi_\varepsilon(dz),\\
\pi_\varepsilon(dz) &= \mathbf{1}_{|z|<1/\varepsilon}\,\pi(dz).
\end{aligned}$$

Let c and c_ε denote the prices of the European option corresponding to the Lévy process L_t and L_t^ε, respectively. Then c_ε solves (6.4) with the integral $\int_{\mathbb{R}\setminus\{0\}}$ replaced by $\int_{\mathbb{R}\setminus\{0\}\cap|z|<1/\varepsilon}$. Theorem 6.1 provides us with the following pointwise error estimate for the truncation of the Lévy measure:

$$|c(t,x) - c_\varepsilon(t,x)| \le C(1+x)\sqrt{\int_{\mathbb{R}\setminus\{0\}\cap|z|\ge 1/\varepsilon} |z|^2\, m(z)\, dz}.$$

For example, if the Lévy measure has enough exponential decay towards infinity, in the sense that

$$\int_{|z|>1} |z|^2 e^{2K|z|}\,\pi(dx) < \infty,$$

for some constant $K>0$. Then we obtain from the above estimate

$$|c(t,x) - c_\varepsilon(t,x)| \le \widetilde{C}(1+x)e^{-K/\varepsilon},$$

which shows that the truncation error decays to zero exponentially fast as ε tends to zero. The same type of estimate holds for the American option value C and its approximation C_ε.

(iii) Truncation of domain of integration II. For numerical purposes, one needs to remove also the small jumps (infinite activity region) from the integro operator. The small jumps acts like a diffusion term, and one way to account for this is to replace the original Lévy process L_t by another Lévy process L_t^ε with characteristic triplet $\big(\mu_\varepsilon,\sigma^2,\pi_\varepsilon(dz)\big)$, where $\varepsilon>0$ is small and

$$\begin{aligned}
\mu_\varepsilon &= -\frac{1}{2}\sigma_\varepsilon^2 - \int_{\mathbb{R}\setminus\{0\}} \big(e^z - 1 - z\mathbf{1}_{|z|<1}\big)\,\pi_\varepsilon(dz),\\
\sigma_\varepsilon^2 &= \sigma^2 + \int_{|z|\le\varepsilon} |z|^2\,\pi(dz),\\
\pi_\varepsilon(dz) &= \mathbf{1}_{|z|>\varepsilon}\,\pi(dz).
\end{aligned}$$

This approach was used in [23, 24], see also the discussion in the introduction of [11]. Corresponding to the Lévy process L_t and L_t^ε, let c and c_ε denote the respective prices of the European option. Then c_ε solves (6.4) with σ^2 replaced by σ_ε^2 and

the integral $\int_{\mathbb{R}\setminus\{0\}}$ replaced by $\int_{\mathbb{R}\setminus\{0\}\cap|z|>\varepsilon}$. In other words, we have removed the singularity at the origin by introducing a new diffusion term in the integro-PDE.

Theorem 6.1 provides us with the following pointwise error estimate for this procedure:

$$|c(t,x) - c_\varepsilon(t,x)| \le C(1+x)\sqrt{\int_{|z|\le\varepsilon} |z|^2\, m(z)\, dz}.$$

The same type of estimate holds for the American option value C and its approximation C_ε. Suppose $\pi(dz) = m(z)\, dz$ for some density $m(z)$ that satisfies for some constant $C > 0$

$$m(z) \le C/|z|^{1+\alpha}, \qquad \alpha \in [1,2),$$

for all z sufficiently close to the origin. Then the above estimate yields

$$|c(t,x) - c_\varepsilon(t,x)| \le \widetilde{C}(1+x)\varepsilon^{1-\frac{\alpha}{2}}, \qquad \alpha \in [1,2).$$

In the specific Lévy measures mentioned below, we have $\alpha = 1$.

It is interesting to notice that even if we did not insert the removed small jumps as an additional diffusion term in the integro-PDE, the rate of convergence would still be $\varepsilon^{1-\frac{\alpha}{2}}$.

Remark 6.2. The above estimates are probably not optimal in the case of an European option, as the solution to (6.4) is classical away from $x = 0$. One should however keep in mind that with techniques developed in this paper these estimates hold also for the American option value and in fact for general fully nonlinear degenerate integro-PDEs for which classical solutions do not exist.

6.4. **Examples of Lévy models.** As already mentioned before, many Lévy models have been suggested in the literature over the years. As an example, let us mention the HYP (hyperbolic) Lévy model, which is proposed in [22] as a model for German stock prices, and it is shown to give an extremely good fit. In [10] the NIG (normal inverse Gaussian) Lévy model is suggested, and in [45] it is shown to perform well in modeling German stock prices. The last two models belong to the class of GH (generalized hyperbolic) Lévy models. These models are characterized by independent increments which belong to the class of GH distributions. This class of distributions, and in particular its two corresponding subclasses, NIG distributions and HYP distributions, has proved to provide an excellent fit to empirically observed log-return distributions.

The class of GH distributions, introduced by [9], can be characterized as normal variance-mean mixtures, where the mixing distribution is a GIG distribution. This class of distributions includes many interesting subclasses, and limiting cases like the NIG, HYP, VG, Student-t, and normal distributions. All of them have been used to model financial returns.

The density of a GH distribution depends on five parameters $(\lambda, \alpha, \beta, \delta, \mu)$, with domain of variation

$$\lambda \in \mathbb{R}, \quad \alpha > 0, \quad \beta \in (-\alpha, \alpha), \quad \delta > 0, \quad \mu \in \mathbb{R},$$

and with the following interpretation: α is a steepness parameter (the larger α, the steeper density), β is a parameter of asymmetry (if $\beta = 0$ the density is symmetric around the mean), δ is a scale parameter, and μ is a location parameter. The special case of $\lambda = -\frac{1}{2}$ gives a NIG distribution. For $\lambda = \frac{1}{2}$ we get the HYP distribution.

The Lévy measure $\pi^{\mathrm{GH}}(dz)$ is absolutely continuous with respect to the Lebesgue measure dz, and its density $m^{\mathrm{GH}}(z)$ is given by a fairly complicated representation. As an example, we display the density of the NIG distribution (a subclass of the GH distributions):

$$m^{\mathrm{NIG}}(z) = \frac{\alpha\delta}{\pi} \exp\Big(\delta\sqrt{\alpha^2-\beta^2} + \beta(z-\mu)\Big) \frac{K_1\big(\alpha\sqrt{\delta^2+(z-\mu)^2}\big)}{\sqrt{\delta^2+(z-\mu)^2}},$$

where K_1 is the modified Bessel function of the third kind and index 1, i.e.,

$$K_1(y) = \frac{1}{2}\int_0^\infty \exp\Big(-\tfrac{1}{2}y(s+s^{-1})\Big)\,ds, \qquad \text{for } y>0,$$

where $z \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\delta > 0$, and $0 \le |\beta| \le \alpha$. The parameters have the following meaning: α is the steepness of the distribution, β the asymmetry, μ the location and δ the scale. If $\beta = 0$ then the distribution is symmetric. The Lévy-Khintchine representation for the normal inverse Gaussian Lévy process takes the form

$$L_t = \xi t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\,\tilde{N}(dt,dz), \qquad \xi = \mu + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}.$$

In empirical studies one usually centers the data and let $\mu = 0$. In this case the Lévy measure takes the form

$$\pi^{\mathrm{NIG}}(dz) = m^{\mathrm{NIG}}(z)\,dz = \frac{\alpha\delta}{\pi|z|} e^{\beta z} K_1(\alpha|z|)\,dz.$$

Finally, notice that due to the properties of the Bessel function, the density $m^{\mathrm{NIG}}(z)$ behaves like $1/|z|^2$ near the origin.

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