CONVERGENCE OF A FINITE DIFFERENCE SCHEME FOR THE CAMASSA–HOLM EQUATION

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ABSTRACT. We prove that a certain finite difference scheme converges to the weak solution of the Cauchy problem on a finite interval with periodic boundary conditions for the Camassa–Holm equation $u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$ with initial data $u|_{t=0} = u_0 \in H^1([0,1])$. Here it is assumed that $u_0 - u_0'' \geq 0$ and in this case, the solution is unique, globally defined, and energy preserving.

1. INTRODUCTION

The Camassa–Holm equation (CH) [2]

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0 \tag{1.1}$$

has received considerable attention the last decade. With κ positive it models, see [11], propagation of unidirectional gravitational waves in a shallow water approximation, with u representing the fluid velocity. The Camassa–Holm equation possesses many intriguing properties: It is, for instance, completely integrable and experiences wave breaking in finite time for a large class of initial data. Most attention has been given to the case with $\kappa = 0$ on the full line, that is,

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.2}$$

which has so-called peakon solutions, i.e., solutions of the form $u(x,t) = ce^{-|x-ct|}$ for real constants c. Local and global well-posedness results as well as results concerning breakdown are proved in [5, 10, 12, 14].

In this paper we study the Camassa–Holm equation (1.1) on a finite interval with periodic boundary conditions. It is known that certain initial data give global solutions, while other classes of initial data experience wave breaking in the sense that u_x becomes unbounded while the solution itself remains bounded. It suffices to treat the case $\kappa = 0$, since solutions with nonzero κ are obtained from solutions with zero κ by the transformation $v(x,t) = u(x + \kappa t, t) - \kappa$. More precisely, the fundamental existence theorem, due to Constantin and Escher [6], reads as follows: If $u_0 \in H^3([0,1])$ and $m_0 := u_0 - u_0'' \in H^1([0,1])$ is non-negative, then equation (1.2) has a unique global solution $u \in C([0,T), H^3([0,1])) \cap C^1([0,T), H^2([0,1]))$ for any T positive. However, if $m_0 \in H^1([0,1])$, u_0 not identically zero but $\int m_0 \, dx = 0$, then the maximal time interval of existence is finite. Furthermore, if $u_0 \in H^1([0,1])$ and $m_0 = u_0 - u_0''$ is a positive Radon measure on $[0,1]$, then (1.2) has a unique global weak solution. Additional results in the periodic case can be found in [3, 6, 4, 8, 13].

We prove convergence of a particular finite difference scheme for the equation, thereby giving the first constructive approach to the actual determination of the solution. We work in the case where one has global solutions, that is, when $m_0 \geq 0$. The scheme is semi-discrete: Time is not discretized, and we have to solve a system of ordinary differential equations. We reformulate (1.1) to give meaning in $C([0,T]; H^1[0,1])$ to solutions such as peakons, and we prove that our scheme converges in $C([0,T]; H^1[0,1])$.

More precisely, we prove the following: Assume that v^n is a sequence of continuous, periodic and piecewise linear functions on intervals $[(i-1)/n, i/n]$, $i = 1, \ldots, n$, that converges to the

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initial data v in $H^1([0,1])$ as $n \to \infty$. Let $u^n = u^n(x,t)$ be the solution of the following system of equations

$$
\begin{aligned}
m^n_t &= -D_-(m^n u^n) - m^n D u^n \\
m^n &= u^n - D_- D_+ u^n
\end{aligned}
\tag{1.3}
$$

with initial condition $u^n|_{t=0} = v^n$. Here $D_\pm$ denotes forward and backward difference operators relative to the lattice with spacing $1/n$, and $D = (D_+ + D_-)/2$. Extrapolate u^n from its lattice values at points i/n to obtain a continuous, periodic, and piecewise linear function also denoted u^n. Assume that $v^n - D_- D_+ v^n \geq 0$. Then u^n converges in $C([0,T]; H^1([0,1]))$ as $n \to \infty$ to the solution u of the Camassa–Holm equation with initial condition $u|_{t=0} = v$. The result includes the case when the initial data $v \in H^1$ is such that $v - v_{xx}$ is a positive Radon measure, see Corollary 2.5.

The numerical scheme (1.3) is tested on various initial data. In addition, we study experimentally the convergence of other numerical schemes for the Camassa–Holm equation. The numerical results are surprisingly sensitive in the explicit form of the scheme, and, among the various schemes we have implemented, only the scheme (1.3) converges to the unique solution.

2. Convergence of the numerical scheme

We consider periodic boundary conditions and solve the equation on the interval $[0,1]$. We are looking for solutions that belong to $H^1([0,1])$ which is the natural space for the equation. Introduce the partition of $[0,1]$ in points separated by a distance $h = 1/n$ denoted $x_i = hi$ for $i = 0, \dots, n-1$. For any $(u_0, \dots, u_{n-1})$ in $\mathbb{R}^n$, we can define a continuous, periodic, piecewise linear function u as

$$
u(x_i) = u_i. \tag{2.1}
$$

It defines a bijection between $\mathbb{R}^n$ and the set of continuous, periodic, piecewise linear function with possible break points at x_i, and we will use this bijection throughout this paper.

Given $u = (u_0, \dots, u_{n-1})$, the quantity $D_\pm u$ given by

$$
(D_\pm u)_i = \frac{\pm 1}{h}(u_{i\pm 1} - u_i)
$$

gives the right and left derivatives, respectively, of u at x_i. In these expressions, u_{-1} and u_n are derived from the periodicity conditions: $u_{-1} = u_{n-1}$ and $u_n = u_0$. The average Du between the left and right derivative is given by

$$
(Du)_i = \frac{1}{2}\big((D_+ u)_i + (D_- u)_i\big) = \frac{1}{2h}(u_{i+1} - u_{i-1}).
$$

The Camassa–Holm equation preserves the H^1-norm. In order to see that, we rewrite (1.2) in its Hamiltonian form, see [2]

$$
m_t = -(mu)_x - m u_x \tag{2.2}
$$

with

$$
m = u - u_{xx}. \tag{2.3}
$$

Assuming that u is smooth enough so that the integration by parts can be carried out, we get

$$
\begin{aligned}
\frac{d}{dt}\|u\|^2_{H^1} &= 2\int_0^1 (u_t - u_{xxt}) u \, dx = 2\int_0^1 u m_t \, dx \\
&= -2\int_0^1 u (mu)_x \, dx - 2\int_0^1 u m u_x \, dx \\
&= 2\int_0^1 u_x m u \, dx - 2\int_0^1 u m u_x \, dx = 0,
\end{aligned}
$$

and the H^1 norm of u is preserved.

From (2.3) and (2.2), we derive a finite difference approximation scheme for the Camassa–Holm equation and prove that it converges to the right solution. This is our main result.

Theorem 2.1. *Let v^n be a sequence of continuous, periodic and piecewise linear functions on $[0,1]$ that converges to v in $H^1([0,1])$ as $n \to \infty$ and such that $v^n - D_-D_+v^n \geq 0$. Then, for any given $T > 0$, the sequence $u^n = u^n(x,t)$ of continuous, periodic and piecewise linear functions determined by the system of ordinary differential equations*

$$
\begin{aligned}
m_t^n &= -D_-(m^n u^n) - m^n D u^n \\
m^n &= u^n - D_-D_+u^n
\end{aligned}
\tag{2.4}
$$

with initial condition $u^n|_{t=0} = v^n$, converges in $C([0,T];H^1([0,1]))$ as $n \to \infty$ to the solution u of the Camassa–Holm equation (1.2) with initial condition $u|_{t=0} = v$.

If we interpret the functions as vectors in (2.4), cf. (2.1), the multiplications are term-by-term multiplications of vectors. We also have to rewrite equation (1.2) so that it makes sense in the sense of distribution for functions that at least belong to $C([0,T];H^1([0,1]))$, more precisely,

$$
u_t - u_{xxt} = -\frac{3}{2}(u^2)_x - \frac{1}{2}(u_x^2)_x + \frac{1}{2}(u^2)_{xxx}. \tag{2.5}
$$

A function u in $L^\infty([0,T];H^1)$ is said to be solution of the periodic Camassa–Holm equation if it is periodic and satisfies (2.5) in the sense of distributions. In [8], a different definition of weak solutions for the Camassa–Holm equation is presented. After proving our main theorem at the end of this section, we also prove that these two definitions are equivalent.

In order to solve equation (2.4), we need to compute u^n from m^n. It is simpler first to consider sequences that are defined in $\mathbb{R}^\mathbb{Z}$ and then discuss the periodic case. Let L denote the linear operator from $\mathbb{R}^\mathbb{Z}$ to $\mathbb{R}^\mathbb{Z}$ given, for all $u \in \mathbb{R}^\mathbb{Z}$ by

$$
Lu = u - D_-D_+u.
$$

We want to find an expression for L^{-1}. Introduce the Kronecker delta by $\delta_i = 1$ if $i = 0$ and zero otherwise. It is enough to find a solution g of

$$
Lg = \delta
$$

which decays sufficently fast at infinity because $L^{-1}m$ is then given, for any bounded $m \in \mathbb{R}^\mathbb{Z}$, by the discrete convolution product of g and m:

$$
L^{-1}m_i = \sum_{j\in\mathbb{Z}} g_{i-j}m_j.
$$

The function g satisfies for i nonzero

$$
g_i - n^2(g_{i+1} - 2g_i + g_{i-1}) = 0. \tag{2.6}
$$

The general solution of (2.6) for all $i \in \mathbb{Z}$ is given by

$$
g_i = Ae^{\kappa_1 i} + Be^{\kappa_2 i}
$$

where A, B are constants, $\kappa_1 = \ln x_1$, $\kappa_2 = \ln x_2$, and x_1 and x_2 are the solutions of

$$
-n^2x^2 + (1 + 2n^2)x - n^2 = 0.
$$

Here x_1 and x_2 are real and positive, and $x_1x_2 = 1$ implies that $\kappa_2 = -\kappa_1$. We set $\kappa = \kappa_1 = -\kappa_2$. After some calculations, we get

$$
\kappa = \ln\left(\frac{1 + 2n^2 + \sqrt{1 + 4n^2}}{2n^2}\right). \tag{2.7}
$$

We take g of the form

$$
g_i = c\,e^{-\kappa|i|}
$$

so that g satisfies (2.6) for all $i \neq 0$ and decays at infinity. The constant c is determined by the condition that $(Lg)_0 = 1$ which yields

$$
c = \frac{1}{1 + 2n^2(1 - e^{-\kappa})}.
$$

We periodize g in the following manner:

$$g_i^p \equiv \sum_{k\in\mathbb{Z}} g_{i+kn} = c\frac{e^{-\kappa i}+e^{\kappa(i-n)}}{1-e^{-\kappa n}}$$

for $i\in\{0,\dots,n-1\}$ and the inverse of L on the set of periodic sequences is then given by

$$u_i = L^{-1}m_i = \sum_{j=0}^{n-1} g_{i-j}^p m_i = \frac{c}{1-e^{-\kappa n}}\sum_{j=0}^{n-1}(e^{-\kappa(i-j)}+e^{\kappa(i-j-n)})m_j. \tag{2.8}$$

Hence,

$$L\big(\sum_{j=0}^{n-1} g_{i-j}^p m_j\big)_i = L\big(\sum_{l\in\mathbb{Z}} g_{i-l}m_l\big)_i = m_i.$$

For sufficiently smooth initial data ($u_0\in H^3$ and $m_0\in H^1$) which satisfies $m_0\ge 0$, Constantin and Escher [5] proved that there exists a unique global solution of the Camassa–Holm equation belonging to $C(\mathbb{R}_+;H^3)\cap C^1(\mathbb{R}_+;H^2)$. The proof of this result relies heavily on the fact that if m is non-negative at $t=0$, then m remains non-negative for all $t>0$. An important feature of our scheme is that it preserves this property. (For simplicity we have here dropped the superscript n appearing on u and m.)

Lemma 2.2. *Assume that $m_i(0)\ge 0$ for all $i=0,\dots,n-1$. For any solution $u(t)$ of the system* (2.4)*, we have that $m_i(t)\ge 0$ for all $t\ge 0$ and for all $i=0,\dots,n-1$.*

Proof. Let us assume that there exist $t>0$ and $i\in\{0,\dots,n-1\}$ such that

$$m_i(t)<0. \tag{2.9}$$

We consider the time interval F in which m remains positive:

$$F=\{t\ge 0 \mid m_i(\tilde t)\ge 0, \text{for all } \tilde t\le t \text{ and } i\in\{0,\dots,n-1\}\}.$$

Because of assumption (2.9), F is bounded and we define

$$T=\sup F.$$

By definition of T, for any integer $j>0$, there exists a $\tilde t_j$ and an i_j such that $T<\tilde t_j<T+\frac{1}{j}$ and $m_{i_j}(\tilde t_j)<0$. The function $m_{i_j}(t)$ is a continuously differentiable function of t. Hence, $m_{i_j}(T)\ge 0$ and there exists a t_j such that

$$m_{i_j}(t_j)=0,$$

with $T\le t_j<T+\frac{1}{j}$.

Since i_j can only take a finite number of values ($i_j\in\{0,\dots,n-1\}$), there exists a $p\in\{0,\dots,n-1\}$ and a subsequence j_k such that $i_{j_k}=p$. The function $m_p(t)$ belongs to C^1 and, since $t_{j_k}\to T$, we have

$$m_p(T)=0. \tag{2.10}$$

We denote by G the set of indices for which (2.10) holds:

$$G=\{k\in\{0,\dots,n-1\}\mid m_k(T)=0\}.$$

G is non-empty because it contains p. If $G=\{0,\dots,n-1\}$, then $m_k(T)=0$ for all k and m must be the zero solution because we know from Picard's theorem that the solution of (2.4) is unique.

If $G\neq\{0,\dots,n-1\}$, then there exists an $l\in\{0,\dots,n-1\}$ such that

$$m_{l-1}(T)>0,\quad m_l(T)=0,\quad \frac{dm_l}{dt}(T)\le 0. \tag{2.11}$$

The last condition, $\frac{dm_l}{dt}(T)\le 0$, comes from the definition of T that would be contradicted if we had $\frac{dm_l}{dt}(T)>0$. Note that we also use the periodicity of m which in particular means that if $l=0$, then $m_{l-1}(T)=m_{-1}(T)=m_{n-1}(T)$.

In (2.4), for $i=l$ and $t=T$, the terms involving $m_l(T)$ cancel and

$$\frac{dm_l}{dt}(T)=\frac{m_{l-1}(T)u_{l-1}(T)}{h}.$$

The fact that all the $m_i(T)$ are positive with one of them, $m_{l-1}(T)$, strictly positive, implies that u_i is strictly positive for all indices i, see (2.8). Since, in addition, $m_{l-1}(T) > 0$, we get

$$\frac{dm_l}{dt}(T) > 0$$

which contradicts the last inequality in (2.11) and therefore our primary assumption (2.9) does not hold. The lemma is proved. $\square$

We want to establish a uniform bound on the H^1 norm of the sequence u^n. Recall that u^n is a continuous piecewise linear function (with respect to the space variable), and its L^2 norm can be computed exactly. We find

$$\|u^n\|_{L^2}^2 = \frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{3}((u_{i+1}^n)^2 + u_i^n u_{i+1}^n + (u_i^n)^2). \tag{2.12}$$

The derivative u_x^n of u^n is piecewise constant and therefore we have

$$\|u_x^n\|_{L^2}^2 = \frac{1}{n}\sum_{i=0}^{n-1}(D_+u^n)_i^2. \tag{2.13}$$

We define a renormalized norm $\|\cdot\|_{l^2}$ and the corresponding scalar product on $\mathbb{R}^n$ by

$$\|u^n\|_{l^2} = \sqrt{\frac{1}{n}\sum_{i=0}^{n-1}(u_i^n)^2}, \quad \langle u^n, v^n\rangle_{l^2} = \frac{1}{n}\sum_{i=0}^{n-1}u_i^n v_i^n.$$

The following inequalities hold

$$\frac{1}{2}\|u^n\|_{l^2} \le \|u^n\|_{L^2} \le \|u^n\|_{l^2} \tag{2.14}$$

which make the two norms $\|\cdot\|_{l^2}$ and $\|\cdot\|_{L^2}$ uniformly equivalent independently of n. In (2.14), u^n either denotes an element of $\mathbb{R}^n$ or the corresponding continuous piecewise linear function as defined previously. By using the Cauchy–Schwarz inequality and the periodicity of u^n, it is not hard to prove that

$$\|u^n\|_{L^2} \le \|u^n\|_{l^2}.$$

For the other equality, it suffices to see that (2.12) can be rewritten as

$$\|u^n\|_{L^2}^2 = \frac{1}{3n}\sum_{i=0}^{n-1}\left[(u_{i+1}^n + \frac{1}{2}u_i^n)^2 + \frac{3}{4}(u_i^n)^2\right]$$

which implies

$$\frac{1}{2}\|u^n\|_{l^2} \le \|u^n\|_{L^2}.$$

We are now in position to establish a uniform bound on the H^1-norm of u^n. Let $E_n(t)$ denote

$$E_n(t) = \left(\|u^n(t)\|_{l^2}^2 + \|D_+u^n(t)\|_{l^2}^2\right)^{\frac{1}{2}} \tag{2.15}$$

which provides an approximation of the H^1-norm of $u^n(t)$. We have, from (2.14) and (2.13),

$$\frac{1}{2}\|u^n(t)\|_{H^1} \le E_n(t) \le \|u^n(t)\|_{H^1}. \tag{2.16}$$

The derivative of $E_n(t)^2$ reads

$$\begin{aligned}\frac{dE_n(t)^2}{dt} &= \frac{2}{n}\sum_{i=0}^{n-1}\left[u_i^n u_{i,t}^n + D_+u_i^n D_+u_{i,t}^n\right]\\ &= \frac{2}{n}\sum_{i=0}^{n-1}(u_i^n - D_-D_+u_i^n)_t u_i^n \quad \text{(summation by parts)}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{n}\sum_{i=0}^{n-1}\left[D_-(m^n u^n)_i u_i^n + m_i^n D u_i^n u_i^n\right] \quad \text{by (2.4)}\\
&= \frac{2}{n}\sum_{i=0}^{n-1}\left[m_i^n u_i^n (D_+ u_i^n - D u_i^n)\right].
\end{aligned}
$$

Since

$$
D_+ u_i^n - D u_i^n = \frac{1}{2}\left[D_+ u_i^n - D_+ u_{i-1}^n\right] = \frac{1}{2n} D_- D_+ u_i^n,
$$

we get

$$
\frac{dE_n(t)^2}{dt} = \frac{1}{n}\sum_{i=0}^{n-1}\left[m_i^n u_i^n \frac{1}{n} D_- D_+ u_i^n\right] = \frac{1}{n^2}\sum_{i=0}^{n-1}\left[m_i^n u_i^n(-m_i^n + u_i^n)\right], \tag{2.17}
$$

and, because u_i^n is positive (see (2.8)),

$$
\frac{dE_n^2(t)}{dt} \le \frac{1}{n^2}\sum_{i=0}^{n-1} m_i^n (u_i^n)^2. \tag{2.18}
$$

A summation by parts gives us that

$$
\frac{1}{n}\sum_{i=0}^{n-1} m_i^n u_i^n = E_n(t)^2.
$$

Since L^∞ is continuously embedded in H^1, there exists a constant $\mathcal{O}(1)$, independent of n, such that

$$
\max_i u_i^n \le \mathcal{O}(1)\left\|u^n\right\|_{H^1} \le \mathcal{O}(1) E_n(t).
$$

Hence, (2.18) implies

$$
E_n'(t) \le \frac{\mathcal{O}(1)}{n} E_n(t)^2
$$

and, after integration,

$$
\frac{1}{E_n(t)} \ge \frac{1}{E_n(0)} - \frac{\mathcal{O}(1)}{n} t.
$$

Since $u^n(0) = v^n$ tends to v in H^1, $\left\|u^n(0)\right\|_{H^1}$ and therefore $E_n(0)$ are bounded. It implies that $E_n(0)^{-1}$ is bounded from below by a strictly positive constant and, for any given $T > 0$, there exists $N \ge 0$ and constant $C' > 0$ such that for all $n \ge N$ and all $t \in [0, T]$, we have $E_n(0)^{-1} - \mathcal{O}(1)t/n \ge 1/C'$. Hence,

$$
E_n(t) \le C'
$$

and, by (2.16), the H^1-norm of $u^n(t)$ is uniformly bounded in $[0, T]$. This result also guarantees the existence of solutions to (2.4) in $[0, T]$ (at least, for n big enough) because, on $[0, T]$, we have that $\max_i |u_i^n(t)| = \left\|u^n(\,\cdot\,, t)\right\|_{L^\infty} \le \mathcal{O}(1)\left\|u^n(t)\right\|_{H^1}$ remains bounded.

To prove that we can extract a converging subsequence of u^n, we need some estimates on the derivative of u^n.

Lemma 2.3. *We have the following properties:*
(i) u_x^n is uniformly bounded in $L^\infty([0,1])$.
(ii) u_x^n has a uniformly bounded total variation.
(iii) u_t^n is uniformly bounded in $L^2([0,1])$.

Proof. (i) From (2.8), we get

$$
D_+ u_i^n = \frac{c}{1 - e^{-\kappa n}}\sum_{j=0}^{n-1}\left[m_j^n e^{-\kappa(i-j)}\left(\frac{e^{-\kappa} - 1}{h}\right) + m_j^n e^{\kappa(i-j-n)}\left(\frac{e^{\kappa} - 1}{h}\right)\right]
$$

where κ is given by (2.7).

One easily gets the following expansion for κ as h tends to 0

$$
\kappa = h + o(h^2),
$$

which implies that for all $i \in \{0, \dots, n-1\}$,

$$
\begin{aligned}
|D_+ u_i^n| &\le (1 + \mathcal{O}(h)) \frac{c}{1 - e^{-\kappa n}} \sum_{j=0}^{n-1} \left(\left| m_j^n \right| e^{-\kappa(i-j)} + \left| m_j^n \right| e^{\kappa(i-j-n)} \right) \\
&\le (1 + \mathcal{O}(h)) \frac{c}{1 - e^{-\kappa n}} \sum_{j=0}^{n-1} \left(m_j^n e^{-\kappa(i-j)} + m_j^n e^{\kappa(i-j-n)} \right) \\
&\le (1 + \mathcal{O}(h))\, u_i^n, \qquad (2.19)
\end{aligned}
$$

where we have used the positivity of m^n and relation (2.8). Hence, since $\|u^n\|_{L^\infty}$ is uniformly bounded, we get a uniform bound on $\|u_x^n\|_{L^\infty}$.

(ii) For each t the total variation of $u_x^n(\,\cdot\,, t)$ is given by

$$
\mathrm{TV}(u_x^n) = \sup_{\phi \in C^1, \|\phi\|_{L^\infty} \le 1} \int_0^1 u_x^n(x) \phi_x(x)\, dx.
$$

On the interval (x_i, x_{i+1}), the function u_x^n is constant and equal to $D_+ u_i^n$. Therefore,

$$
\begin{aligned}
\int_0^1 u_x^n(x)\phi_x(x)\, dx &= \sum_{i=0}^{n-1} D_+ u_i^n \int_{x_i}^{x_{i+1}} \phi_x(x)\, dx = \sum_{i=0}^{n-1} D_+ u_i^n (\phi(x_{i+1}) - \phi(x_i)) \\
&= \sum_{i=0}^{n-1} \frac{1}{n} D_+ u_i^n D_+ \phi(x_i) = -\sum_{i=0}^{n-1} \frac{1}{n} (D_- D_+ u_i^n) \phi(x_i)
\end{aligned}
$$

and

$$
\mathrm{TV}(u_x^n) \le \frac{1}{n} \sum_{i=0}^{n-1} \left| D_- D_+ u_i^n \right|.
$$

Since m_i^n and u_i^n are positive for all i,

$$
|D_- D_+ u_i^n| = |m_i^n - u_i^n| \le m_i^n + u_i^n \le 2u_i^n - D_- D_+ u_i^n.
$$

When summing over i on the right-hand side of the last inequality, the term $D_- D_+ u_i^n$ disappears and we get

$$
\mathrm{TV}(u_x^n) \le 2 \max_i u_i^n \le \mathcal{O}(1) \|u^n\|_{H^1} \le \mathcal{O}(1)
$$

for all t.

(iii) In order to make the ideas clearer, we first sketch the proof directly on equation (2.2). Assuming that m is positive and u is in H^1, we see how, from (2.2), u_t can be defined as an element of $L^2([0,1])$. This will be useful when we afterwards derive a uniform bound for u_t^n in $L^2([0,1])$.

For all smooth v, we have

$$
\int_0^1 u_t\, v\, dx = \int_0^1 (\mathcal{L}^{-1} m_t)\, v\, dx
$$

where $\mathcal{L}$ denotes the operator $\mathcal{L}u = u - u_{xx}$, which is a self-adjoint homeomorphism from H^2 to L^2. If we let $w = \mathcal{L}^{-1} v$, the continuity of $\mathcal{L}^{-1}$ implies

$$
\|w\|_{H^2} \le \mathcal{O}(1) \|v\|_{L^2} \qquad (2.20)
$$

for some constant $\mathcal{O}(1)$ independent of v.

We find

$$
\begin{aligned}
\int_0^1 u_t\, v\, dx &= \int_0^1 \left(\mathcal{L}^{-1} m_t \right) v\, dx = \int_0^1 m_t\, \mathcal{L}^{-1} v\, dx \quad (\mathcal{L}^{-1} \text{ is self-adjoint}) \\
&= -\int_0^1 ((mu)_x + m u_x) w\, dx = \int_0^1 (m u w_x - m u_x w)\, dx.
\end{aligned}
$$

The integrals here must be understood as distributions. Even so, some terms (like mu_x) are not well-defined as distributions. However, we get the same results rigorously by considering the equation written as a distribution (2.5). We have:

$$\begin{aligned}\left|\int_0^1 u_t\, v\, dx\right| &\leq \int_0^1 (|muw_x| + |mu_x w|)\, dx \\ &\leq (\|u\|_{L^\infty} \|w_x\|_{L^\infty} + \|u_x\|_{L^\infty} \|w\|_{L^\infty}) \int_0^1 |m|\ dx.\end{aligned}$$

Recall that $\|u\|_{L^\infty}$ and $\|u_x\|_{L^\infty}$ are uniformly bounded. Furthermore, m positive implies $\int_0^1 |m| = \int_0^1 m = \int_0^1 u \leq \|u\|_{L^\infty}$ and therefore m is also uniformly bounded. From (2.20) and the fact that H^1 is continuously embedded in L^∞, we get

$$\|w_x\|_{L^\infty} \leq \mathcal{O}(1) \|w_x\|_{H^1} \leq \mathcal{O}(1) \|w\|_{H^2} \leq \mathcal{O}(1) \|v\|_{L^2},$$

and similarly

$$\|w\|_{L^\infty} \leq \mathcal{O}(1) \|v\|_{L^2}.$$

Finally,

$$\left|\int_0^1 u_t\, v\, dx\right| \leq \mathcal{O}(1) \|v\|_{L^2}$$

which implies, by Riesz's representation theorem, that u_t is in L^2 and

$$\|u_t\|_{L^2} \leq \mathcal{O}(1).$$

We now turn to the analogous derivations in the discrete case. Consider the sequence u^n. The aim is to derive a uniform bound for u_t^n in L^2. We take a continuous piecewise linear function v^n,

$$\langle u_t^n, v^n\rangle_{l^2} = \left\langle L^{-1} m_t^n, v^n\right\rangle_{l^2} = \left\langle m_t^n, L^{-1} v^n\right\rangle_{l^2} \tag{2.21}$$

because L and therefore L^{-1} are self-adjoint.

Let w^n denote

$$w^n = L^{-1} v^n.$$

We have

$$\langle v^n, w^n\rangle_{l^2} = \langle Lw^n, w^n\rangle_{l^2} = \frac{1}{n}\sum_{i=0}^{n-1} (w_i^n - D_- D_+ w_i^n)\, w_i^n = \frac{1}{n}\sum_{i=0}^{n-1} \left[(w_i^n)^2 + (D_+ w_i^n)^2\right].$$

Then, after using (2.16) and Cauchy–Schwarz, we get

$$\|w^n\|_{H^1}^2 \leq 4 \|v^n\|_{l^2} \|w^n\|_{l^2}.$$

By (2.14), (2.16) we find

$$\|w^n\|_{H^1}^2 \leq \mathcal{O}(1) \|v^n\|_{l^2} \|w^n\|_{H^1}$$

and

$$\|w^n\|_{H^1} \leq \mathcal{O}(1) \|v^n\|_{l^2} \tag{2.22}$$

where $\mathcal{O}(1)$ is a constant independent of n. Since H^1 is continuously embedded in L^∞, we get

$$\max_i |w_i^n| \leq \mathcal{O}(1) \|v^n\|_{l^2}. \tag{2.23}$$

Let us define y^n as follows

$$y_i^n = (D_+ w^n)_{i-1}.$$

We want to find a bound on y^n. From (2.14) and (2.22), we get

$$\|y^n\|_{l^2} \leq \|w^n\|_{H^1} \leq \mathcal{O}(1) \|v^n\|_{l^2}. \tag{2.24}$$

We also have, using the definition of y^n and w^n,

$$D_+ y^n = D_- D_+ w^n = w^n - v^n$$

which gives

$$\|D_+ y^n\|_{l^2} \leq \mathcal{O}(1) \|v^n\|_{l^2} \tag{2.25}$$

because, by (2.22),

$$\|w^n\|_{l^2} \leq \mathcal{O}(1)\,\|v^n\|_{l^2}\,.$$

Equations (2.24), (2.25), and (2.16) give us a uniform bound on the H^1 norm of y^n:

$$\|y^n\|_{H^1} \leq \mathcal{O}(1)\,\|v^n\|_{l^2}\,.$$

Since H^1 is continuously embedded in L^∞, we get

$$\max_i |D_+w_i^n| = \max_i |y_i^n| = \|y^n\|_{L^\infty} \leq \mathcal{O}(1)\,\|v^n\|_{l^2}\,. \tag{2.26}$$

Going back to (2.21), we have

$$\begin{aligned}\langle u_t^n, v^n\rangle_{l^2} &= \langle m_t^n, w^n\rangle_{l^2} = \langle -D_-(m^n u^n) - m^n D u^n, w^n\rangle_{l^2}\\ &= \langle m^n u^n, D_+ w^n\rangle_{l^2} - \langle m^n D u^n, w^n\rangle_{l^2}\,.\end{aligned}$$

Hence,

$$|\langle u_t^n, v^n\rangle_{l^2}| \leq \frac{1}{n}\left(\max_i |u_i^n| \max_i |D_+w_i^n| + \max_i |D_+u_i^n| \max_i |w_i^n|\right)\sum_{i=0}^{n-1} |m_i^n|\,.$$

The functions u_i^n and $D_+u_i^n$ are uniformly bounded with respect to n. and

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1} |m_i^n| &= \frac{1}{n}\sum_{i=0}^{n-1} m_i^n \quad (m^n \text{ is positive})\\ &= \frac{1}{n}\sum_{i=0}^{n-1} u_i^n \quad \Big(\text{cancellation of } \sum_{i=0}^{n-1} D_-D_+u_i^n\Big)\\ &\leq \mathcal{O}(1). \qquad (u_i^n \text{ is bounded})\end{aligned}$$

Finally, using the bounds we have derived on w^n, see (2.23), and D_+w^n, see (2.26), we get

$$|\langle u_t^n, v^n\rangle_{l^2}| \leq \mathcal{O}(1)\,\|v^n\|_{l^2}\,.$$

Taking $v^n = u_t^n$ yields

$$\|u_t^n\|_{l^2} \leq \mathcal{O}(1)$$

which, since the l^2 and L^2 norm are uniformly equivalent, gives us a uniform bound on $\|u_t^n\|_{L^2}$. □

To prove the existence of a converging subsequence of u^n in $C([0,T],H^1)$ we recall the following compactness theorem given by Simon [15, Corollary 4].

Theorem 2.4 (Simon)**.** *Let X, B, Y be three continuously embedded Banach spaces*

$$X \subset B \subset Y$$

with the first inclusion, $X \subset B$, compact. We consider a set $\mathcal{F}$ of functions mapping $[0,T]$ into X. If $\mathcal{F}$ is bounded in $L^\infty([0,T],X)$ and $\frac{\partial \mathcal{F}}{\partial t} = \left\{\frac{\partial f}{\partial t} \mid f \in \mathcal{F}\right\}$ is bounded in $L^r([0,T],Y)$ where $r > 1$, then $\mathcal{F}$ is relatively compact in $C([0,T],B)$.

We now turn to the proof of our main theorem.

Proof of Theorem 2.1. (i) First we establish that there exists a subsequence of u^n that converges in $C([0,T],H^1)$ to an element $u \in H^1$. To apply Theorem 2.4, we have to determine the Banach spaces with the required properties. In our case, we take X as the set of functions of H^1 which have derivatives of bounded variation:

$$X = \left\{v \in H^1 \mid v_x \in \mathrm{BV}\right\}.$$

X endowed with the norm

$$\|v\|_X = \|v\|_{H^1} + \|v_x\|_{BV} = \|v\|_{H^1} + \|v_x\|_{L^\infty} + \mathrm{TV}(v_x)$$

is a Banach space. Let us prove that the injection $X \subset H^1$ is compact. We consider a sequence v_n which is bounded in X. Since $\|v_n\|_{L^\infty}$ is bounded ($H^1 \subset L^\infty$ continuously), there exists a point x_0 such that $v_n(x_0)$ is bounded and we can extract a subsequence (that we still denote v_n) such that $v_n(x_0)$ converges to some $l \in \mathbb{R}$. By Helly's theorem, we can also extract a subsequence such that

$$v_{n,x} \to w \quad \text{a.e.} \tag{2.27}$$

for some $w \in L^\infty$. By Lebesgue's dominated convergence theorem, it implies that $v_{n,x} \to w$ in L^2. We set

$$v(x) = l + \int_{x_0}^{x} w(s)\,ds.$$

We have that $v_x = w$ almost everywhere. We also have

$$v_n(x) = v_n(x_0) + \int_{x_0}^{x} v_{n,x}(s)\,ds$$

which together with (2.27) implies that v_n converges to v in L^∞. Therefore v_n converges to v in H^1 and X is compactly embedded in H^1.

The estimates we have derived previously give us that u^n and u^n_t are uniformly bounded in $L^\infty([0,T],X)$ and $L^\infty([0,T],L^2)$, respectively. Since $X \subset H^1 \subset L^2$ with the first inclusion compact, Simon's theorem gives us the existence of a subsequence of u^n that converges in $C([0,T],H^1)$ to some $u \in H^1$.

(ii) Next we show that the limit we get is a solution of the Camassa–Holm equation (1.2).

Let us now take φ in $C^\infty([0,1]\times[0,T])$ and multiply, for each i, the first equation in (2.4) by $h\varphi(x_i,t)$. We denote φ^n the continuous piecewise linear function given by $\varphi^n(x_i,t) = \varphi(x_i,t)$. We sum over i and get, after one summation by parts,

$$\begin{aligned}\sum_{i=0}^{n-1} h\left(u^n_{i,t} - (D_-D_+u^n_i)_t\right)\varphi^n_i &= \underbrace{\sum_{i=0}^{n-1} h(u^n_i)^2 D_+\varphi_i}_{A} - \underbrace{\sum_{i=0}^{n-1} hu^n_i D_-D_+u^n_i D_+\varphi^n_i}_{B}\\ &\quad - \underbrace{\sum_{i=0}^{n-1} hu^n_i Du^n_i\varphi^n_i}_{C} + \underbrace{\sum_{i=0}^{n-1} hD_-D_+u^n_i Du^n_i\varphi^n_i}_{D}.\end{aligned} \tag{2.28}$$

We are now going to prove that each term in this equality converges to the corresponding terms in (2.5).

Term A: We want to prove that

$$\left\langle (u^n)^2 D_+\varphi^n\right\rangle \to \int_0^1 u^2\varphi_x\,dx, \tag{2.29}$$

where we have introduced the following notation

$$\langle u\rangle = h\sum_{i=0}^{n-1} u_i$$

to denote the average of a quantity u. We have

$$\begin{aligned}\left|\int_0^1 u^2\varphi_x\,dx - \left\langle (u^n)^2 D_+\varphi^n\right\rangle\right| &\le \left|\int_0^1 (u^2-(u^n)^2)\varphi_x\,dx\right|\\ &\quad + \left|\int_0^1 (u^n)^2(\varphi_x - D_+\varphi^n)\,dx\right|\\ &\quad + \left|\int_0^1 (u^n)^2 D_+\varphi^n\,dx - \left\langle (u^n)^2 D_+\varphi^n\right\rangle\right|.\end{aligned}$$

The first term tends to zero because $u^n \to u$ in L^2 for all $t \in [0,T]$. The second tends to zero by Lebesgue's dominated convergence theorem. It remains to prove that the last term tends to zero.

The integral of a product between two continuous piecewise linear function, v and w, and a piecewise constant function z can be computed explicitly. We skip the details of the calculation and give directly the result:

$$\int_0^1 zvw\,dx = \frac{1}{3}\langle zS_+vS_+w\rangle + \frac{1}{6}\langle zS_+vw\rangle + \frac{1}{6}\langle zvS_+w\rangle + \frac{1}{3}\langle zvw\rangle\,. \tag{2.30}$$

Here S_+ and S_- denote shift operators

$$(S_\pm u)_i = u_{i\pm 1}.$$

After using (2.30) with $v = w = u^n$ and $z = D_+\varphi^n$, we get

$$\begin{aligned}\int_0^1 (u^n)^2 D_+\varphi^n - \big\langle (u^n)^2 D_+\varphi^n\big\rangle &= \frac{1}{3}\langle (S_+u^n - u^n)D_+\varphi^n u^n\rangle \\ &\quad + \frac{1}{3}\big\langle (u^n)^2 D_+(S_-\varphi^n - \varphi^n)\big\rangle\,.\end{aligned}$$

We use the uniform equivalence of the l^2 and L^2 norm to get the following estimate

$$\begin{aligned}\langle (S_+u^n - u^n)D_+\varphi^n u^n\rangle &\leq \|S_+u^n - u^n\|_{l^2}\,\|D_+\varphi^n u^n\|_{l^2} \quad \text{(Cauchy–Schwarz)} \\ &\leq \mathcal{O}(1)\,\|u^n(\,\cdot\,+h) - u^n(\,\cdot\,)\|_{L^2}\,.\end{aligned} \tag{2.31}$$

Since $u_n \in H^1$, we have (see, for example, [1]):

$$\|u^n(\,\cdot\,+h) - u^n(\,\cdot\,)\|_{L^2} \leq h\,\|u_x^n\|_{L^2} \leq \mathcal{O}(1)h$$

because $\|u_x^n\|_{L^\infty}$ is uniformly bounded. Hence $|\langle (S_+u^n - u^n)D_+\varphi^n u^n\rangle|$ tends to zero. The quantity $\big\langle (u^n)^2 D_+(S_-\varphi^n - \varphi^n)\big\rangle$ tends to zero because φ is C^∞ and u^n uniformly bounded. We have proved (2.29).

Term B: We want to prove

$$\langle u^n D_-D_+u^n D_+\varphi^n\rangle \to \frac{1}{2}\int_0^1 u^2\varphi_{xxx}\,dx - \int_0^1 u_x^2\varphi_x. \tag{2.32}$$

We rewrite $u^n D_-D_+u^n$ in such a way that the discrete double derivative D_-D_+ does not appear in a product (so that we can later sum by parts). We have

$$u^n D_-D_+u^n = \frac{1}{2}(D_-D_+((u^n)^2) - D_+u^nD_+u^n - D_-u^nD_-u^n).$$

We can prove in the same way as we did for term A that

$$\begin{aligned}\big\langle D_-D_+((u^n)^2)D_+\varphi^n\big\rangle &= \big\langle (u^n)^2 D_-D_+D_+\varphi^n\big\rangle \quad \text{(summation by parts)} \\ &\to \int_0^1 u^2\varphi_{xxx}\,dx.\end{aligned}$$

The quantity $(u_x^n)^2\varphi_x^n$ is a piecewise constant function. Therefore,

$$\int_0^1 (u_x^n)^2\varphi_x^n\,dx = \langle D_+u^nD_+u^nD_+\varphi^n\rangle\,.$$

Since $u_x^n \to$ in L^2 for all $t \in [0,T]$, and

$$\int_0^1 u_x^2\varphi_x\,dx - \langle D_+u^nD_+u^nD_+\varphi^n\rangle = \int_0^1 (u_x^2 - (u_x^n)^2)\varphi_x\,dx + \int_0^1 (u_x^n)^2(\varphi_x - \varphi_x^n)\,dx,$$

we have

$$\langle D_+u^nD_+u^nD_+\varphi^n\rangle \to \int_0^1 u_x^2\varphi_x\,dx.$$

In the same way, we get

$$\langle D_-u_i^nD_-u_i^nD_+\varphi^n\rangle \to \int_0^1 u_x^2\varphi_x$$

and (2.32) is proved.

Term C: We want to prove

$$\langle u^n D u^n \varphi^n \rangle \to \int_0^1 u u_x \varphi \, dx. \tag{2.33}$$

We have

$$\begin{aligned}\int_0^1 u u_x \varphi \, dx - \langle u^n D_+ u^n \varphi^n \rangle &= \int_0^1 (u - u^n) u_x \varphi \, dx + \int_0^1 u^n (u_x - u_x^n) \varphi \, dx \\ &\quad + \int_0^1 u^n u_x^n (\varphi - \varphi^n) \, dx + \int_0^1 u^n u_x^n \varphi^n \, dx \\ &\quad - \langle u^n D_+ u^n \varphi^n \rangle .\end{aligned}$$

The first two terms converge to zero because $u^n \to u$ in H^1 for all $t \in [0,T]$. The third term converges to zero by Lebesgue's dominated convergence theorem. We use formula (2.30) to evaluate the last integral:

$$\begin{aligned}\int_0^1 u^n u_x^n \varphi^n \, dx &= \frac{1}{3} \langle D_+ u^n S_+ u^n S_+ \varphi^n \rangle + \frac{1}{6} \langle D_+ u^n S_+ u^n \varphi^n \rangle \\ &\quad + \frac{1}{6} \langle D_+ u^n u^n S_+ \varphi^n \rangle + \frac{1}{3} \langle D_+ u^n u^n \varphi^n \rangle .\end{aligned}$$

Using the same type of arguments as those we have just used for term A, one can show that

$$\int_0^1 u^n u_x^n \varphi^n \, dx \to \langle D_+ u^n u^n \varphi^n \rangle .$$

Thus, in order to prove (2.33), it remains to prove that

$$\langle D_+ u^n u^n \varphi^n \rangle - \langle D u^n u^n \varphi^n \rangle \to 0. \tag{2.34}$$

Since $D = \frac{1}{2}(D_+ + D_-)$, we have:

$$\langle D_+ u^n u^n \varphi^n \rangle - \langle D u^n u^n \varphi^n \rangle = \frac{1}{2} \langle (D_+ u^n - D_- u^n) u^n \varphi^n \rangle$$

and

$$\begin{aligned}|\langle (D_+ u^n - D_- u^n) u^n \varphi^n \rangle| &\le C \sum_{i=0}^{n-1} h \left| D_+ u_i^n - D_+ u_{i-1}^n \right| \\ &\le \mathcal{O}(1) \int_0^1 |u_x^n(x) - u_x^n(x-h)| \, dx \\ &\le \mathcal{O}(1) h \ \mathrm{TV}(u_x^n).\end{aligned}$$

Since $\mathrm{TV}(u_x^n)$ is uniformly bounded, (2.34) holds and we have proved (2.33).

Term D: We want to prove that

$$\langle D_- D_+ u^n D u^n \varphi^n \rangle \to -\frac{1}{2} \int_0^1 u_x^2 \varphi_x \, dx. \tag{2.35}$$

We have

$$\frac{1}{2} \int_0^1 u_x^2 \varphi_x \, dx + \langle D_- D_+ u^n D u^n \varphi^n \rangle \tag{2.36}$$

$$= \frac{1}{2} \int_0^1 (u_x^2 - (u_x^n)^2) \varphi_x \, dx + \frac{1}{2} \int_0^1 (u_x^n)^2 (\varphi_x - D_- \varphi^n) \, dx \tag{2.37}$$

$$- \frac{1}{2} \langle D_+ (D_+ u^n D_+ u^n) \varphi^n \rangle + \langle D_- D_+ u^n D u^n \varphi^n \rangle . \tag{2.38}$$

The two first terms on the right-hand side tend to zero. After using the following identity

$$D_+ (D_+ u^n D_+ u^n) = D_+ D_+ u^n D_+ u^n + D_+ D_+ u^n D_+ S_+ u^n,$$

we can rewrite the two last terms in (2.36) as

$$
\begin{aligned}
&-\frac{1}{2}\left\langle D_+(D_+u^nD_+u^n)\varphi^n\right\rangle + \left\langle D_-D_+u^nDu^n\varphi^n\right\rangle\\
&\quad= -\frac{1}{2}\left\langle D_-D_+S_+u^nD_+u^n\varphi^n\right\rangle - \frac{1}{2}\left\langle D_-D_+S_+u^nD_+S_+u^n\varphi^n\right\rangle\\
&\qquad\quad+\frac{1}{2}\left\langle D_-D_+u^nD_+S_-u^n\varphi^n\right\rangle + \frac{1}{2}\left\langle D_-D_+u^nD_+u^n\varphi^n\right\rangle\\
&\quad= \frac{1}{2}\left\langle D_-D_+u^nD_-u^n(\varphi^n - S_-\varphi^n)\right\rangle + \frac{1}{2}\left\langle D_-D_+u^nD_+u^n(\varphi^n - S_-\varphi^n)\right\rangle
\end{aligned}
$$

which tends to zero because, as we have seen before, due to the positivity of m, $\langle |D_-D_+u_i^nD_+u_i^n|\rangle$ is uniformly bounded. We have proved (2.35).

Up to now we have not really considered the time variable. We integrate (2.28) with respect to time and integrate by part the left-hand side:

$$
\begin{aligned}
\int_0^T\sum_{i=0}^{n-1} h\left(u_{i,t}^n - D_-D_+u_{i,t}^n\right)\varphi(x_i,t)\,dt = &-\int_0^T\sum_{i=0}^{n-1} h\left(u_i^n - D_-D_+u_i^n\right)\varphi_t(x_i,t)\,dt\\
&+\left[\sum_{i=0}^{n-1} h\left(u_i^n - D_-D_+u_i^n\right)\varphi(x_i,t)\right]_{t=0}^{t=T}
\end{aligned}
$$

and, after summing by parts, the limit of this expression is (we use Lebesgue's dominated convergence theorem with respect to x and t)

$$
-\int_0^T\int_0^1 u(\varphi_t - \varphi_{txx})\,dxdt + \left[\int_0^1 u(\varphi - \varphi_{xx})\,dx\right]_{t=0}^{t=T}.
$$

It is not hard to see that the right-hand side of (2.28) is uniformly bounded by a constant and we can integrate over time and use the Lebesgue dominated convergence theorem to conclude that u is indeed a solution of (2.5) in the sense of distribution.

The analysis in [8] shows that the weak solution of the Camassa–Holm with initial conditions satisfying $m(x,0)\geq 0$ is unique. This implies that in our algorithm not only a subsequence but the whole sequence u^n converges to the solution. However, in [8], a solution of the Camassa–Holm equation is defined as an element u of H^1 satisfying

$$
u_t + uu_x + \left[\int_{-\infty}^{\infty} p(x-y)[u^2(y,t) + \frac{1}{2}u_x^2(y,t)]\,dy\right]_x = 0 \tag{2.39}
$$

where p is the solution of

$$
\mathcal{A}p \equiv (I - \partial_x^2)p = \delta.
$$

We want to prove that weak solutions of (2.39) and (2.5) are the same. Periodic distributions belong to the class of tempered distribution $\mathcal{S}'$ (see for example [9]). The operator $\mathcal{A}$ defines a homeomorphism on the Schwartz class $\mathcal{S}$ (or class of rapidly decreasing function): The Fourier transform is a homeomorphism on $\mathcal{S}$ and $\mathcal{A}$ restricted to $\mathcal{S}$ can be written as

$$
\mathcal{A} = \mathcal{F}^{-1}(1+\xi^2)\mathcal{F} \tag{2.40}
$$

where ξ denotes the frequency variable. It is clear from (2.40) that the inverse of $\mathcal{A}$ in $\mathcal{S}$ is

$$
\mathcal{A}^{-1} = \mathcal{F}^{-1}\frac{1}{1+\xi^2}\mathcal{F}.
$$

Hence $\mathcal{A}$ is a homeomorphism on $\mathcal{S}$.

We can now define the inverse $\mathcal{A}^{-1}$ of $\mathcal{A}$ in $\mathcal{S}'$. Given T in $\mathcal{S}'$, $\mathcal{A}^{-1}T$ is given by

$$
\left\langle \mathcal{A}^{-1}T,\phi\right\rangle = \left\langle T,\mathcal{A}^{-1}\phi\right\rangle,\quad \phi\in\mathcal{S}.
$$

It is easy to check that $\mathcal{A}^{-1}$ indeed satisfies

$$
\mathcal{A}^{-1}\mathcal{A} = \mathcal{A}\mathcal{A}^{-1} = \mathrm{Id},
$$

and that $\mathcal{A}^{-1}$ is continuous on $\mathcal{S}'$. The operator $\mathcal{A}$ is therefore a homeomorphism on $\mathcal{S}'$.

Let u be a solution of (2.39). Then we have

$$u_t + \partial_x(\frac{u^2}{2}) + \partial_x\mathcal{A}^{-1}[u^2 + \frac{1}{2}u_x^2] = 0. \tag{2.41}$$

The operators ∂_x and $\mathcal{A}^{-1}$ commute because ∂_x and $\mathcal{A}$ commute. We apply $\mathcal{A}$ on both sides of (2.41) and get:

$$u_t - u_{xxt} + \mathcal{A}\partial_x(\frac{1}{2}u^2) + \partial_x[u^2 + \frac{1}{2}u_x^2] = 0, \tag{2.42}$$

which is exactly (2.5). Since $\mathcal{A}$ is a bijection, (2.42) also implies (2.41) and we have proved that the weak solutions of (2.5) are the same as the weak solutions given by (2.39). □

In Theorem 2.1, some restrictions on the initial data v are implicitly imposed by the condition $v^n - D_-D_+v^n \geq 0$. We are going to prove that if $v \in H^1([0,1])$ is periodic with $v - v_{xx} \in \mathcal{M}^+$, where $\mathcal{M}^+$ denotes the space of positive Radon measures, then there exists a sequence of piecewise linear, continuous, periodic functions v^n that converges to v in H^1 and satisfies $v^n - D_-D_+v^n \geq 0$ for all n.

We can then apply Theorem 2.1 and get the existence result contained in the following corollary which coincides with results obtained in [8] by a different method.

Corollary 2.5. *If $u_0 \in H^1$ is such that $u_0 - u_{0,xx} \in \mathcal{M}^+$ then the Camassa–Holm equation has a global solution in $C(\mathbb{R}_+, H^1)$. The solution is obtained as a limit of the numerical scheme defined by* (2.4).

To apply Theorem 2.1, we need to prove that, given $u \in H^1([0,1])$ such that $u - u_{xx} \in \mathcal{M}^+$, there exists a sequence u^n of piecewise linear, continuous and periodic functions such that

$$\begin{aligned} &u^n \to u \text{ in } H^1,\\ &u^n - D_-D_+u^n \geq 0. \end{aligned}$$

Let $\{\psi_i^n\}$ be a partition of unity associated with the covering $\cup_{i=0}^{n-1}(x_{i-1}, x_{i+1})$. For all $i \in \{0,,\dots n-1\}$, the functions ψ_i^n are non-negative with supp $\psi_i^n \subset (x_{i-1}, x_{i+1})$, and $\sum_{i=0}^{n-1}\psi_i^n = 1$. Define

$$v_i^n = \frac{1}{h}\left\langle u - u_{xx}, \psi_i^n\right\rangle$$

and

$$u_i^n - D_-D_+u_i^n = v_i^n. \tag{2.43}$$

Recall that the operator $u^n - D_-D_+u^n$ is invertible, see (2.8), so that u^n is well-defined by (2.43). Since $u - u_{xx}$ belongs to $\mathcal{M}^+$ and $\psi_i^n \geq 0$, we have $v_i^n = u_i^n - D_-D_+u_i^n \geq 0$ and it only remains to prove that u^n converges to u in H^1. Since the application $\mathcal{L} : H^1 \to H^{-1}$ given by $\mathcal{L}u = u - u_{xx}$ is an homeomorphism, it is equivalent to prove that

$$u^n - u_{xx}^n \to u - u_{xx} \ \text{ in } H^{-1}.$$

The homeomorphism $\mathcal{L}$ is also an isometry, so that

$$\left\|\mathcal{L}u\right\|_{H^{-1}} = \left\|u\right\|_{H^1}.$$

We can find a bound on $\left\|u^n\right\|_{H^1}$. Let E_n be defined, as before, by

$$E_n = \left(h\sum_{i=0}^{n-1}\left[(u_i^n)^2 + (D_+u^n)_i^2\right]\right)^{\frac{1}{2}}.$$

The inequality (2.16) still holds. We have

$$\begin{aligned} E_n^2 &= h\sum_{i=0}^{n-1}(u_i^n - D_-D_+u_i^n)u_i^n\\ &= h\sum_{i=0}^{n-1}v_i^n u_i^n \end{aligned}$$

$$
\begin{aligned}
&\leq \|u^n\|_{L^\infty} \sum_{i=0}^{n-1} hv_i^n \\
&\leq \|u^n\|_{L^\infty} \langle u - u_{xx}, \sum_{i=0}^{n-1} \psi_i^n \rangle \\
&\leq \|u^n\|_{L^\infty} \|u - u_{xx}\|_{\mathcal{M}^+} \quad (\text{since } \sum_{i=0}^{n-1} \psi_i^n = 1).
\end{aligned}
$$

Hence, since L^∞ is continuously embedded in H^1, there exists a constant C (independent of n) such that

$$E_n^2 \leq C \|u^n\|_{H^1} \|u - u_{xx}\|_{\mathcal{M}^+}.$$

We use inequality (2.16) to get the bound on $\|u^n\|_{H^1}$ we were looking for:

$$\|u^n\|_{H^1} \leq 4C \|u - u_{xx}\|_{\mathcal{M}^+}.$$

To prove that $u^n - u_{xx}^n \to u - u_{xx}$ in H^{-1}, since $\|u^n - u_{xx}^n\|_{H^{-1}} = \|u^n\|_{H^1}$ is uniformly bounded, we just need to prove that

$$\langle u^n - u_{xx}^n, \varphi \rangle \to \langle u - u_{xx}, \varphi \rangle$$

for all φ belonging to a dense subset of H^1 (for example C^∞).

The function u^n is continuous and piecewise linear. Its second derivative u_{xx}^n is therefore a sum of Dirac functions:

$$u_{xx}^n = \sum_{i=0}^{n-1} hD_-D_+u_i^n \delta_{x_i}$$

and, for any φ in C^∞, we have

$$
\begin{aligned}
\langle u^n - u_{xx}^n, \varphi \rangle &= \int_0^1 u^n(x)\varphi(x)\,dx - h\sum_{i=0}^{n-1} D_-D_+u_i^n \varphi(x_i) \\
&= \int_0^1 u^n(x)(\varphi(x) - \varphi^n(x))\,dx + \int_0^1 u^n(x)\varphi^n(x)\,dx \\
&\quad - h\sum_{i=0}^{n-1} u_i^n\varphi_i^n + h\sum_{i=0}^{n-1} v_i\varphi_i^n
\end{aligned} \tag{2.44}
$$

where φ^n denotes the piecewise linear, continuous function that coincides with φ on x_i, $i = 0, \ldots, n-1$.

The first integral in (2.44) tends to zero by the Lebesgue dominated convergence theorem. We use formula (2.30) to compute the second integral:

$$\int_0^1 u^n(x)\varphi^n(x)\,dx = \frac{2}{3}\langle u^n\varphi^n \rangle + \frac{1}{6}\langle S_+u^n\varphi^n \rangle + \frac{1}{6}\langle u^n S^+\varphi^n \rangle.$$

One can prove that this term tends to $\langle u^n\varphi^n \rangle$ (see the proof of the convergence of term A in the proof of Theorem 2.1). The last sum equals

$$\sum_{i=0}^{n-1} hv_i^n\varphi(x_i) = \left\langle u - u_{xx}, \sum_{i=0}^{n-1} \varphi_i^n\psi_i^n(x) \right\rangle.$$

For all $x \in [0, 1]$, there exists a k such that $x \in [x_k, x_{k+1}]$. Then,

$$
\begin{aligned}
\left| \varphi(x) - \sum_{i=0}^{n-1} \varphi_i^n\psi_i^n(x) \right| &= \left| \sum_{i=0}^{n-1} (\varphi(x) - \varphi(x_i))\psi_i^n(x) \right| \\
&\leq |\varphi(x) - \varphi(x_k)| + |\varphi(x) - \varphi(x_{k+1})| \\
&\leq 2 \sup_{|z-y|\leq h} |\varphi(y) - \varphi(z)|
\end{aligned}
$$

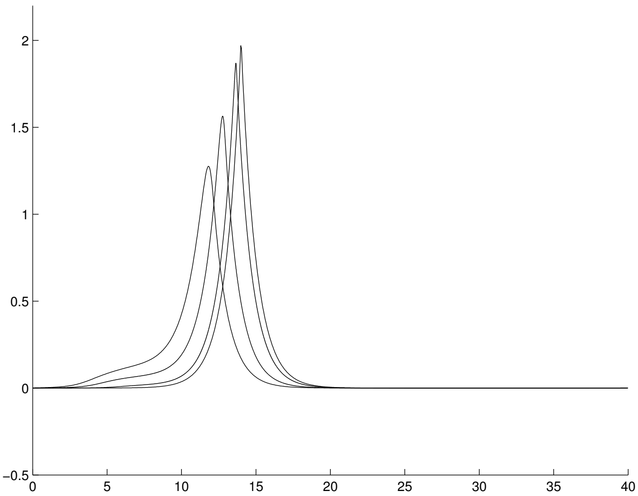

FIGURE 1. Periodic single peakon. The initial condition is given by $u(x,0) = 2e^{-|x|}$ and period $a = 40$. The computed solutions are shown at time $t = 6$ for (from left to right) $n = 2^{10}$, $n = 2^{12}$, $n = 2^{14}$ together with the exact solution (at the far right).

and therefore, by the uniform continuity of φ,

$$\sum_{i=0}^{n-1} \varphi(x_i)\psi_i^n(x) \to \varphi(x) \text{ in } L^\infty.$$

Thus,

$$\sum_{i=0}^{n-1} hv_i^n\varphi(x_i) = \left\langle u - u_{xx}, \sum_{i=0}^{n-1} \varphi(x_i)\psi_i^n \right\rangle \to \langle u - u_{xx}, \varphi\rangle$$

and, from (2.44), we get

$$\langle u^n - u^n_{xx}, \varphi\rangle \to \langle u - u_{xx}, \varphi\rangle\,.$$

As already explained, it implies that

$$u^n \to u \text{ in } H^1.$$

3. Numerical results

The numerical scheme (2.4) is semi-discrete: The time derivative has not been discretized and we have to deal with an ordinary differential equation. We integrate in time by using an explicit Euler method. Given a positive time step Δt, we compute u_i^j, the approximated value of u_i at time $t = j\Delta t$, by taking

$$m_i^{j+1} = m_i^j + \Delta t\left(-D_-(mu)_i - m_iDu_i\right). \tag{3.1}$$

A first important consequence of taking finite time steps is that the positivity of m is no longer automatically preserved (Lemma 2.2 does not apply anymore), and for that reason we are not able to prove convergence of the fully discrete scheme in the same way as we did for the semi-discrete scheme. However, for all cases we have tested, the algorithm (3.1) appears to converge.

To compute the discrete spatial derivative, we need at each step to compute u from m. The function u is given by a discrete convolution product

$$u_i = h\sum_{j=0}^{n-1} g^p_{i-j}m_j.$$

It is advantageous to apply the Fast Fourier Transform (FFT), see [9]. In the frequency space, a convolution product becomes a multiplication which is cheap to evaluate. Going back and forth

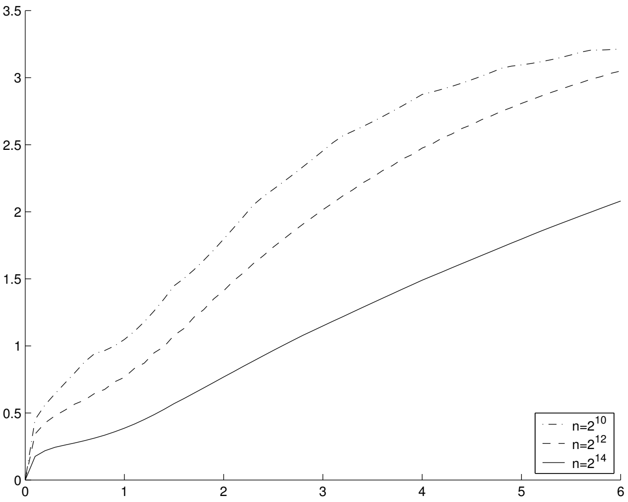

FIGURE 2. Plot of $\|u(t)-u^n(t)\|_{H^1} \,/\, \|u(t)\|_{H^1}$ in the one peakon case of Figure 1.

to the frequency space is not very expensive due to the efficiency of the FFT. We use a formula of the form (see [9] for more details):

$$u = \mathcal{F}_N^{-1}(\mathcal{F}_N[g]\cdot\mathcal{F}_N[m])$$

where $\mathcal{F}_N$ denotes the FFT.

We have tested algorithm (3.1) with single and double peakons. In the single peakon case, the initial condition is given by

$$u(x,0) = c\frac{\cosh(d-\frac{a}{2})}{\sinh\frac{a}{2}}, \tag{3.2}$$

which is the periodized version of $u(x,0) = ce^{-|x|}$. The period is denoted by a and $d = \min(x, a-x)$ is the distance from x to the boundary of the interval $[0,a]$. The peakons travel at a speed equal to their height, that is

$$u(x,t) = ce^{-|x-ct|}.$$

If u satisfies the initial condition $u(x,0) = e^{-|x|}$, then $m = 2\delta$ at $t = 0$ and we take

$$m_i(0) = \begin{cases} \frac{2}{h} & \text{if } i = 0,\\ 0 & \text{otherwise,}\end{cases} \tag{3.3}$$

as initial discrete condition. The function m_i gives a discrete approximation of 2δ. Figure 1 shows the result of the computation for different refinements. Figure 2 indicates that the computed solution converges to the exact solution.

The sharp increase of the error $\|u(t)-u^n(t)\|_{H^1}$ at time $t = 0$ can be predicted by looking at (2.17) which gives a first-order approximation of the time derivative of $\|u(t)\|_{H^1}^2$:

$$\frac{dE_n(t)^2}{dt} = -\sum_{i=0}^{n-1} u_i(hm_i)^2 + \mathcal{O}(h).$$

Hence,

$$\frac{d\|u\|_{H^1}^2}{dt} \approx \frac{dE_n(t)^2}{dt} \approx -4 \quad\text{at } t = 0.$$

At the beginning of the computation, we can therefore expect a sharp decrease of the H^1 norm. To get convergence in H^1, it is therefore necessary that the solution becomes smooth enough so that $\frac{d\|u\|_{H^1}^2}{dt} \to 0$. In any case, we cannot hope for high accuracy and convergence rate in this case. Figure 3 shows the same plots in the two peakon case.

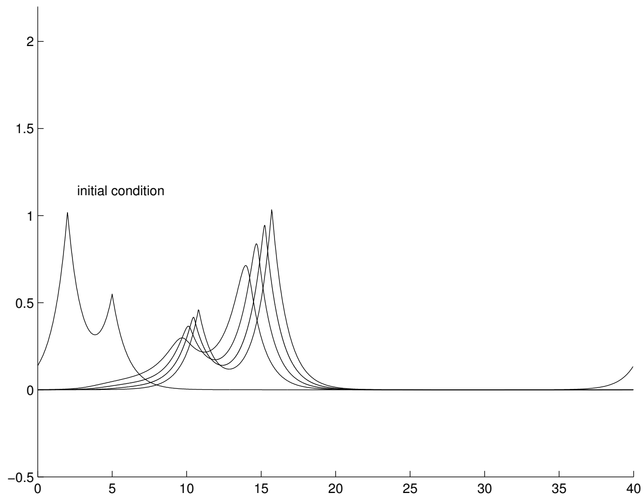

FIGURE 3. Two peakon case. The initial condition is the periodized version of $2e^{-|x-2|} + e^{-|x-5|}$. The computed solutions are shown at time $t = 12$ for (from left to right) $n = 2^{10}$, $n = 2^{12}$, $n = 2^{14}$ together with the exact solution (at the far right).

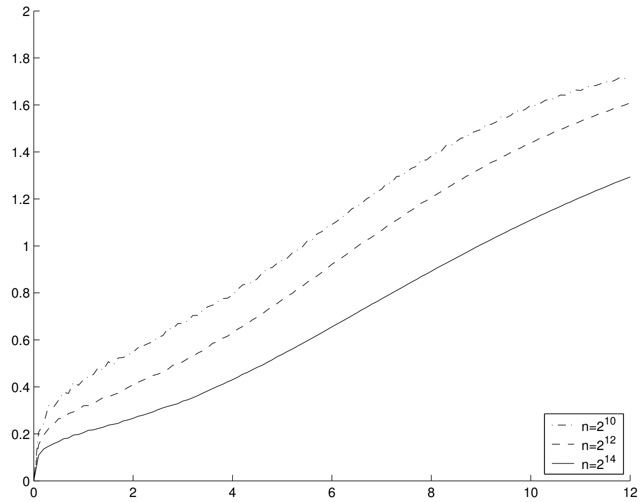

FIGURE 4. Plot of $\|u(t) - u^n(t)\|_{H^1} / \|u(t)\|_{H^1}$ in the two peakon case of Figure 3.

We have tested our algorithm with smooth initial conditions. In this case, the H^1 norm remains constant in a much more accurate manner. The convergence is probably much better but we have no analytical solution to compare with.

Other time integration methods (second-order Runge–Kutta method, variable order Adams–Bashforth–Moulton) have also been tried and the results do not differ significantly from those given by (3.1). It follows that the CH equation is not very sensitive to the way time is discretized. But the situation is completely different when we consider different space discretizations. The following schemes

$$m_t = -D_-(mu)_i - m_iD_+u_i, \tag{3.4}$$

$$m_t = -D(mu)_i - m_iDu_i, \tag{3.5}$$

$$m_t = -D_+(mu)_i - m_iD_-u_i \tag{3.6}$$

are all at first glance good candidates for solving the CH equation. They preserve the H^1 norm, are finite difference approximations of (2.2) and finally look very similar to (2.4). But, tested on a single peakon, (3.4) produces a peakon that grows, (3.5) produces oscillations, and (3.6) behaves in a completely unexpected manner (at the first time step, m becomes a negative Dirac function and starts traveling backward!).

Let us have a closer look at the scheme (3.4). We compute $\frac{dE_n^2}{dt}$:

$$\frac{1}{2}\frac{dE_n^2}{dt} = \sum_{i=0}^{n-1} m_{i,t}^n u_i^n = \sum_{i=0}^{n-1} \big(-D_-(m^n u^n)_i u_i - m_i^n D_+ u_i u_i \big) = 0.$$

Thus, E_n is exactly preserved. Lemma 2.2 still holds since the same proof applies to (3.4). It allows us to derive the bounds of Lemma 2.3 and, after applying Simon's theorem, we get the existence of a converging subsequence. The problem is that, in general, this subsequence *does not* converge to the solution of the Camassa–Holm equation. In order to see that, we compare how our original algorithm (3.4) and algorithm (3.5) handle a peakon solution $u = ce^{-|x-ct|}$. The only terms that differ are $m^n Du^n$ and $m^n D_+ u^n$. We have proved earlier that, for any smooth function φ,

$$\sum_{i=0}^{n-1} m_i^n Du_i^n \varphi(x_i) \to \frac{1}{2}\int_0^1 (u^2 - u_x^2)\varphi(x)\,dx$$

as $n \to \infty$. In the peakon case, $u^2 = u_x^2$ and this term tends to zero. Roughly speaking, we can say that m^n converges to a Dirac function, see (3.3), but at the same time it is multiplied by Du^n which is the average of the left and right derivatives and which tends to zero at the top of the peak. Eventually the whole product $m^n Du^n$ tends to zero. We follow the same heuristic approach with the term $m^n D_+ u^n$ in (3.5). This time, m^n is multiplied by the right derivative $D_+ u^n$ of u^n which tends, at the top of the peak, to $-c$. Hence, $-m^n D_+ u^n$ tends to $c\delta$ and not zero as it would if (3.5) converged to the correct solution. This example shows how sensitive the numerical approximation is, regarding the explicit form of the finite difference scheme, for the Camassa–Holm equation.

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