Instability of Travelling Wave Profiles for the Lax-Friedrichs scheme

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Abstract

We study travelling wave profiles for discrete approximations to hyperbolic systems of conservation laws. A detailed example is constructed, showing that for the Lax-Friedrichs scheme the travelling profiles do not depend continuously on the wave speed, in the BV norm. Namely, taking a sequence of wave speeds $\lambda_n \to \lambda$, the corresponding profiles Ψ_n converge to a limit Ψ uniformly on the real line, but $\text{Tot.Var.}\{\Psi_n - \Psi\} \geq c_0 > 0$ for all n.

1 Introduction

Consider a strictly hyperbolic $N \times N$ system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0. \tag{1}$$

For initial data with small total variation, the existence of a globally defined entropy weak solution is well known [10]. The uniqueness of solutions obtained as limits of Glimm or front tracking approximations has now also been established [6], [4], together with their continuous dependence on the initial data. For vanishing viscosity approximations

$$u_t + f(u)_x = \varepsilon\, u_{xx}\,, \tag{2}$$

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uniform BV bounds, stability and convergence as $\varepsilon \to 0$ were recently established in [3]. Assuming that all the eigenvalues of the Jacobian matrix $Df(u)$ are strictly positive, similar results are proved in [2] for solutions of the semidiscrete (upwind) Godunov scheme

$$\frac{d}{dt}\,u_j(t) + \frac{1}{\Delta x}\Big[f\big(u_j(t)\big) - f\big(u_{j-1}(t)\big)\Big] = 0\,, \qquad u_j(t) = u(t, j\,\Delta x)\,.$$

A major remaining open question is whether the entropy weak solutions of (1) can also be recovered as the unique limits of fully discrete schemes, where the derivatives w.r.t. both time and space are replaced by finite differences. We recall that, for the 2×2 system of isentropic gas dynamics, the convergence of Lax-Friedrichs and Godunov approximations was proved in [8], within the framework of compensated compactness. Further results have been obtained for *straight line systems*, where all the Rankine-Hugoniot curves are straight lines. For straight line systems of two equations, LeVeque and Temple [12] used the existence of a coordinate system of Riemann invariants to prove stability and convergence of the Godunov scheme. For $N \times N$-systems in the same class, uniform BV bounds, stability and convergence of a relaxation scheme, Godunov and Lax-Friedrichs approximations were established in [7], [5], [17], respectively. The analysis relies on the fact that, due to the very particular geometry, the interaction of waves of the same family does not generate additional oscillations.

A key ingredient of the proofs in [3] and [2] is the local decomposition of approximate solutions in terms of travelling waves. This approach strongly relies on the existence of a center manifold of travelling wave profiles having a sufficient degree of smoothness. In principle, one could try to use this same strategy to prove uniform BV bounds and stability also for discrete numerical approximations. In this case, the existence of travelling wave profiles has been established in [14], [13]. However, an argument put forward by D. Serre [15] indicates that the continuous dependence on parameters of discrete travelling wave profiles meets some basic obstructions.

In the present paper, after reviewing Serre's argument, we elucidate this crucial point, constructing an explicit example where the discrete shock profiles for the Lax-Friedrichs scheme do not depend continuously on the speed λ, in the BV norm. Our example is a 2×2 system in triangular form

$$u_t + f(u)_x = 0, \tag{3}$$

$$v_t + g(u)_x = 0. \tag{4}$$

The characteristic speeds are 0 and $f'(u)$ and the system is strictly hyperbolic provided $f'(u) > 0$. The Lax-Friedrichs scheme for (3)-(4)

with $\Delta x = \Delta t$ takes the form

$$u_{n+1,j} = \frac{1}{2}\big(u_{n,j+1} + u_{n,j-1}\big) - \frac{1}{2}\big(f(u_{n,j+1}) - f(u_{n,j-1})\big), \quad (5)$$

$$v_{n+1,j} = \frac{1}{2}\big(v_{n,j+1} + v_{n,j-1}\big) - \frac{1}{2}\big(g(u_{n,j+1}) - g(u_{n,j-1})\big). \quad (6)$$

For the rest of the paper we fix a flux function $f(u)$ which satisfies $f'(u) > 1/8$ together with the CFL condition $|f'(u)| < 1$, for all $u \in \mathbb{R}$. The second flux function g will be constant to the right and to the left of a given interval.

A *discrete shock profile* (DSP) with speed λ for (5)-(6) is a pair of functions

$$\big(U(x), V(x)\big) = \big(U^{(\lambda)}(x), V^{(\lambda)}(x)\big)$$

satisfying

$$U(x-\lambda) = \frac{U(x+1) + U(x-1)}{2} - \frac{f(U(x+1)) - f(U(x-1))}{2}, \quad (7)$$

$$V(x-\lambda) = \frac{V(x+1) + V(x-1)}{2} - \frac{g(U(x+1)) - g(U(x-1))}{2}. \quad (8)$$

We consider a rational speed $\lambda = p/q$ and a sequence of perturbations $\lambda + \varepsilon_n$ with $\varepsilon_n \to 0$. For a particular choice of the flux function g, we show that the sequence $V^{(\lambda+\varepsilon_n)}$ cannot converge to $V^{(\lambda)}$ in the BV norm. Indeed, for every n the difference $V^{(\lambda)} - V^{(\lambda+\varepsilon_n)}$ has uniformly positive total variation. This non-zero amount of oscillation occurs far downstream, within an interval of the form $[-a\varepsilon_n^{-2}, -b\varepsilon_n^{-2}]$, for some fixed $a > b > 0$. As a consequence, there cannot exist any smooth center manifold of travelling wave profiles in the space BV (in fact, not even a continuous one).

The present analysis brings to light a basic mechanism which generates spurious oscillations in numerically computed solutions. Roughly speaking, these oscillations are due to resonances between the speed of a shock and the ratio $\Delta x/\Delta t$ in the grid size. By further developing these ideas, in a forthcoming paper [1] we will show how these resonances can amplify the initial total variation by an arbitrarily large factor. In particular, no a-priori BV bounds can hold for finite difference approximations of general systems of conservation laws.

2 Obstructions to the continuous dependence of discrete profiles

In this section we give a precise formulation of an argument originally due to D. Serre [15], in the context of solutions to the Lax-Friedrichs scheme. The main result can be stated as follows.

Theorem 2.1. *Consider a strictly hyperbolic $N \times N$ system of conservation laws of the form (1). Assume that the eigenvalues of the Jacobian matrix $Df(u)$ satisfy*

$$-1 < \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_N(u) \approx 0$$

and that the N-th characteristic field is genuinely nonlinear. Then, for a generic flux function $f \in \mathcal{C}^2(\mathbb{R}^N;\mathbb{R}^N)$, the discrete travelling profiles corresponding to shocks of the N-th family do not depend continuously on their speed, in the BV norm.

Proof. Set

$$u_{n,j} \doteq u(n\Delta t, j\Delta x), \qquad \Delta t = \Delta x = 1\,,$$

and consider the Lax-Friedrichs approximation scheme

$$u_{n+1,j} = \frac{u_{n,j+1} + u_{n,j-1}}{2} - \frac{f(u_{n,j+1}) - f(u_{n,j-1})}{2}\,.$$

This can be written in conservation form as

$$u_{n+1,j} = u_{n,j} + \Big[F(u_{n,j-1}, u_{n,j}) - F(u_{n,j}, u_{n,j+1})\Big], \tag{9}$$

with

$$F(u,v) = \frac{u-v}{2} + \frac{f(u)+f(v)}{2}\,. \tag{10}$$

Let $\psi = \psi(y)$ be a discrete shock profile of the N-th family, having bounded variation and connecting the left and right states u^-, u^+. Calling σ the speed of the shock profile, by definition one has

$$\psi(y-\sigma) = \psi(y) + \Big[F\big(\psi(y-1), \psi(y)\big) - F\big(\psi(y), \psi(y+1)\big)\Big]\,. \tag{11}$$

Since $\psi(\pm\infty) = u^\pm$ and ψ has bounded variation, for every $x_0 \in \mathbb{R}$ and every integer $k \in \mathbb{Z}$ there holds

$$W(k) \doteq \sum_j \big[\psi(x_0 + j + k) - \psi(x_0 + j)\big] = k(u^+ - u^-)\,. \tag{12}$$

Moreover, using (11) with $y = x_0 + j + \sigma$, for every $x_0 \in \mathbb{R}$ we obtain

$$\begin{aligned}
W(\sigma) &\doteq \sum_j \big[\psi(x_0+j+\sigma) - \psi(x_0+j)\big] \\
&= \sum_j \Big[F\big(\psi(x_0+j+\sigma), \psi(x_0+j+\sigma+1)\big) - \\
&\qquad\quad - F\big(\psi(x_0+j+\sigma-1), \psi(x_0+j+\sigma)\big)\Big] \\
&= \sum_j \left[\psi(x_0+j+\sigma) - \frac{\psi(x_0+j+\sigma+1) + \psi(x_0+j+\sigma-1)}{2}\right] + \\
&\qquad + \sum_j \left[\frac{f\big(\psi(x_0+j+\sigma+1)\big) - f\big(\psi(x_0+j+\sigma-1)\big)}{2}\right] \\
&= f(u^+) - f(u^-) \\
&= \sigma(u^+ - u^-)\,. \qquad (13)
\end{aligned}$$

For every discrete profile $\psi \in BV$ travelling with speed σ, (12) and (13) together imply

$$W(s) = s(u^+ - u^-) \qquad \text{for all } s \in \mathbb{Z} + \sigma\mathbb{Z}. \qquad (14)$$

If σ is irrational, from (14) by continuity we obtain

$$W(s) = \sum_j \big[\psi(x_0+j+s) - \psi(x_0+j)\big] = s(u^+ - u^-), \qquad (15)$$

for all $s \in \mathbb{R}$.

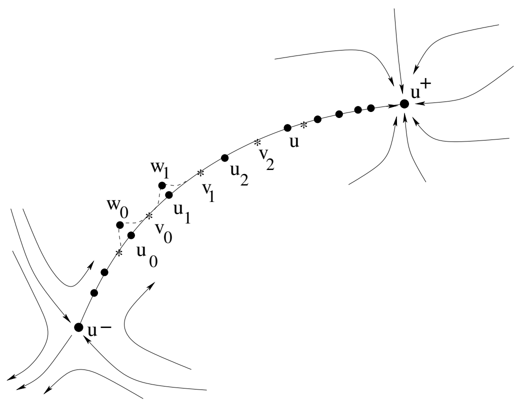

Figure 1

Now assume that there exists a family of discrete wave profiles $\psi^{(\sigma)}$ corresponding to irrational speeds σ such that $\psi^{(\sigma)} \to \psi$ in BV, as $\sigma \to 0$. Defining $u^\pm \doteq \psi(\pm\infty)$, by continuity one finds

$$W(s) = \sum_j \big[\psi(x_0 + j + s) - \psi(x_0 + j)\big] = s(u^+ - u^-)\,, \tag{16}$$

for every $x_0, s \in \mathbb{R}$. In particular, the limiting profile with zero speed will satisfy

$$\psi(y) = \psi(y) + F\big(\psi(y-1), \psi(y)\big) - F\big(\psi(y), \psi(y+1)\big),$$

hence

$$F\big(\psi(y), \psi(y+1)\big) = f_\infty \doteq f(u^+) = f(u^-)\,.$$

Recalling (10), this yields

$$\psi(y+1) - \psi(y) + f\big(\psi(y+1)\big) + f\big(\psi(y)\big) = 2f_\infty\,.$$

Since all the eigenvalues of Df have size strictly smaller than 1, by the implicit function theorem we obtain a solution in the form

$$\psi(y+1) = \Phi\big(\psi(y)\big). \tag{17}$$

The matrix $Df(u^+)$ has N negative eigenvalues, while $Df(u^-)$ has 1 positive and $N-1$ negative eigenvalues. This geometry implies that the discrete travelling profiles joining u^- with u^+ are precisely the orbits contained in the 1-dimensional unstable manifold at u^-. Therefore, any discrete travelling profile with speed 0 must be formed by a continuous family of these orbits (Figure 1). Let

$$\ldots, u_{-1}, u_0, u_1, u_2, \ldots$$

$$\ldots, v_{-1}, v_0, v_1, v_2, \ldots$$

any two distinct orbits, with

$$u_{j+1} = \Phi(u_j), \qquad v_{j+1} = \Phi(v_j), \qquad \text{for all } j\,,$$

$$u_{-\infty} = v_{-\infty} = u^-, \qquad\qquad u_\infty = v_\infty = u^+.$$

By (16) we must have

$$\sum_j (u_j - v_j) = \kappa(u^+ - u^-) \tag{18}$$

for some constant $\kappa \neq 0$. In particular, the right hand side of (18) must be parallel to $u^+ - u^-$. We claim that, for a generic flux function f, there exists a couple of distinct orbits for which the relation (18) fails. Indeed, if such a couple of orbits exists for f, by continuity the relation (18) still fails for al nearby fluxes $\tilde f$ suitably close to f in the $\mathcal{C}^2$ norm.

Otherwise, consider two consecutive points u_0, u_1 on the first orbit. We can slightly perturb the function f and obtain a new flux function g such that $g = f$ outside two small neighborhoods of u_0 and u_1, not containing any other point u_j, v_j. We claim that it is possible to choose this perturbation so that the new orbit corresponding to g is

$$\ldots, u_{-1}, w_0, w_1, u_2, u_3, \ldots$$

i.e., only the two points u_0, u_1 are changed. To achieve this, we need the relations

$$w_0 - u_{-1} + f(u_{-1}) + g(w_0) = 2f_\infty\,, \tag{19}$$

$$w_1 - w_0 + g(w_0) + g(w_1) = 2f_\infty\,, \tag{20}$$

$$u_2 - w_1 + g(w_1) + f(u_2) = 2f_\infty\,. \tag{21}$$

This is a set of 3 equations for 4 variables. We can use (19) to obtain w_0 and (21) to obtain w_1. Then, summing all three equations we get the compatibility condition

$$2\big[g(w_0) + g(w_1)\big] = 6f_\infty - \big[f(u_{-1}) + f(u_2)\big] + (u_{-1} - u_2)\,. \tag{22}$$

Comparing (19) and (21) with the corresponding equations satisfied by u_0 and u_2 we obtain

$$w_0 - u_0 + g(w_0) - f(u_0) = 0\,,$$

$$u_1 - w_1 + g(w_1) - f(u_1) = 0\,.$$

Hence

$$w_0 + w_1 - u_0 - u_1 = g(w_1) - g(w_0) - f(u_1) + f(u_0)\,. \tag{23}$$

It is now clear that one can choose $g(w_0)$, $g(w_1)$ so that their sum satisfies (22), but the quantity on the right hand side of (23) is not parallel to $u^+ - u^-$. This shows that the relation (18) is generically false, hence the convergence $\psi^{(\sigma)} \to \psi$ in BV cannot hold. □

3 The heart of the matter

We begin by observing that the system (3)-(4) is in triangular form. Let (U, V) be a discrete travelling wave profile with speed λ. Then U must be a travelling profile for the scalar difference equation (5), while V should be a travelling profile for the linear difference equation with sources

$$v_{n+1,j} = \frac{1}{2}\big(v_{n,j+1} + v_{n,j-1}\big) + g_{n,j}\,, \tag{24}$$

where

$$g_{n,j} = \frac{1}{2}\Big[g\big(U(j-1-\lambda n)\big) - g\big(U(j+1-\lambda n)\big)\Big]\,.$$

We shall choose flux function $g = g(u)$ in such a way that

$$g'(U(\xi)) = 0 \qquad \text{if} \qquad \xi \notin]-\rho, \rho[\,.$$

As a consequence, we have

$$g_{n,j} \neq 0 \qquad \text{only if} \qquad |j - n\lambda| < 1 + \rho\,.$$

All the source terms in (24) are thus located within a small strip around the line $x = \lambda t$.

Over large time intervals, the equation (24) is well approximated in terms of a heat equation with almost periodic sources. To understand how much oscillations can be produced in the solution $v_{n,j}$, in this section we first study in detail a "toy problem", consisting of the heat equation with point sources. We claim that this simple P.D.E. actually captures the heart of the matter. Indeed, by careful estimates of the various approximation errors, we will later prove that the same amount of oscillations occurs also for solutions of (24). For clarity of exposition, we consider three cases of increasing difficulty.

CASE 1. Let us begin with the simplest case of a heat equation, where the source acts continuously in time and is concentrated along the line $x = \sigma t$ with $\sigma > 0$, namely

$$v_t - v_{xx} = \delta_{t,\sigma t}\,. \tag{25}$$

In this case the travelling wave solution is found explicitly:

$$v(t, x) = \phi(x - \sigma t) \tag{26}$$

with

$$\phi(y) = \int_0^\infty G(t, y + \sigma t)\, dt = \begin{cases} \sigma^{-1} e^{-\sigma y} & \text{if } y \geq 0, \\ \sigma^{-1} & \text{if } y \leq 0. \end{cases} \tag{27}$$

Here $G(t, x) := e^{-x^2/4t}/2\sqrt{\pi t}$ is the standard heat kernel. Notice that the travelling profile can also be obtained as the value at time $t = 0$ of a solution of (25) defined for $t \in]-\infty, 0]$. We also have

$$\phi'(y) = \int_0^\infty G_x(t, y + \sigma t)\, dt = \begin{cases} -e^{-\sigma y} & \text{if } y > 0, \\ 0 & \text{if } y < 0. \end{cases} \tag{28}$$

CASE 2. Next, we consider the case where the sources are located on a discrete set of points $P_n = (n, \sigma n)$ with $n \in \mathbb{Z}$ (the white circles in Figure 2)

$$v_t - v_{xx} = \delta_{n,\sigma n}\,. \tag{29}$$

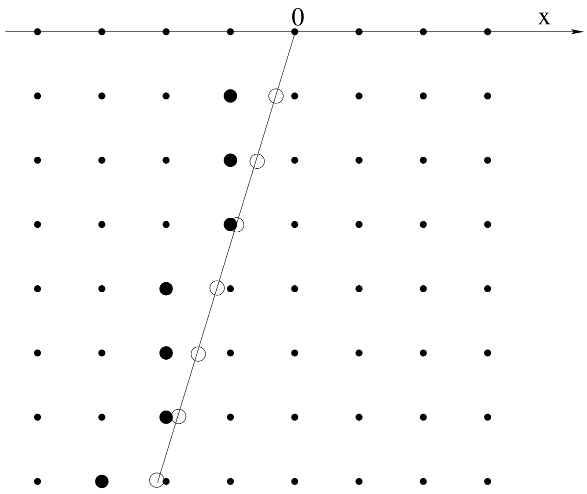

Figure 2

We again assume that $\sigma > 0$ and consider a solution of (29) defined for $t \in]-\infty, 0]$. Its value at time $t = 0$ is now computed as

$$v(0, y) = \Phi(y) \doteq \sum_{n \geq 1} G(n, y + \sigma n) \,. \tag{30}$$

For $y \to \infty$, it is clear that $\Phi(y)$ tends to zero exponentially fast, together with all derivatives. We wish to understand how the oscillations decay for $y \to -\infty$, i.e. far downstream from the shock. For $y < 0$, the sum (30) can be expressed as an integral

$$\Phi(y) \doteq \sum_{n \geq 1} G(n, y + \sigma n) = \int_0^\infty G(t, y + \sigma t) \big(1 + h_1'(t)\big) \, dt, \tag{31}$$

where

$$h_1(t) \doteq [\![t]\!] - t + 1/2 \,.$$

By induction, we can find a sequence of periodic functions h_m such that

$$h_m(t) = h_m(t+1) \,, \qquad \int_0^1 h_m(t) \, dt = 0 \,, \qquad \frac{d}{dt} h_m(t) = h_{m-1}(t) \,.$$

Integrating by parts and recalling (27), from (31) we obtain

$$\Phi(y) = \int_0^\infty G(t, y + \sigma t) \Big(1 + \frac{d^m}{dt^m} h_m(t)\Big) \, dt \tag{32}$$

$$= \frac{1}{\sigma} + (-1)^m \int_0^\infty \frac{d^m}{dt^m} G(t, y + \sigma t) \, h_m(t) \, dt \,. \tag{33}$$

Since

$$h_m(t) = \int_{\xi_m}^{t} h_{m-1}(s)\,ds \qquad\qquad t \in [0,1]$$

for some point $\xi_m \in [0,1]$, by induction on m we find

$$\big|h_m(t)\big| \le 1$$

for all $m \ge 1$ and $t \in [0,1]$. The identities

$$G(t,x) = t^{-1/2} G(1, x/\sqrt{t})\,, \qquad\qquad G_t = G_{xx}\,,$$

imply

$$\frac{\partial^m}{\partial x^m} G(t,x) = t^{-(m+1)/2} \cdot \frac{\partial^m}{\partial x^m} G(1, x/\sqrt{t})\,, \tag{34}$$

$$\frac{\partial^m}{\partial t^m} G(t,x) = t^{-(2m+1)/2} \cdot \frac{\partial^m}{\partial t^m} G(1, x/\sqrt{t})\,. \tag{35}$$

We recall here some basic estimates for the heat kernel and its derivatives, that will be used throughout the paper. For $m = 0, 1, \ldots$ we have

$$\left\| \frac{\partial^m}{\partial x^m} G(t,\cdot) \right\|_{\mathbf{L}^\infty} = O(1) \cdot \frac{1}{t^{(m+1)/2}}, \tag{36}$$

$$\left\| \frac{\partial^m}{\partial t^m} G(t,\cdot) \right\|_{\mathbf{L}^\infty} = O(1) \cdot \frac{1}{t^{(2m+1)/2}}. \tag{37}$$

In addition we observe that, as $y \to -\infty$, the function $t \mapsto G(t, y+\sigma t)$ becomes exponentially small together with all its derivatives, outside the interval centered at $|y|/\sigma$ with width $|y|^{\delta+1/2}$, for any $\delta > 0$. More precisely

$$\sup_{|t+y/\sigma| < |y|^{\delta+1/2}} \left| \frac{d^m}{dt^m} G(t, y+\sigma t) \right| = O(1) \cdot e^{c_\delta y} \qquad \text{as } y \to -\infty\,, \tag{38}$$

for some constant $c_\delta > 0$.

Letting $y \to -\infty$, for every $m \ge 1$ the above estimates imply

$$\left| \Phi(y) - \frac{1}{\sigma} \right| \le \int_0^\infty \left| \frac{d^m}{dt^m} G(t, y+\sigma t) \right| dt = O(1) \cdot y^{-m/2}. \tag{39}$$

Similarly,

$$\big|\Phi'(y)\big| \le \int_0^\infty \left| \frac{d^{m+1}}{dt^{m+1}} G(t, y+\sigma t) \right| dt = O(1) \cdot y^{-(m+1)/2}. \tag{40}$$

Since $m \ge 1$ is arbitrary, this shows that the function Φ' is rapidly decreasing as $y \to -\infty$. In particular, taking $m = 2$ in (40) we obtain the integrability of Φ', hence a bound on the total variation of Φ.

CASE 3. Finally, assume that the impulses are located not at the points $P_n = (n, \sigma n)$ but at the points with integer coordinates $Q_n \doteq \big(n, [\![\sigma n]\!]\big)$ (the black circles in Figure 2)

$$v_t - v_{xx} = \delta_{n,[\![\sigma n]\!]}\,, \tag{41}$$

where $[\![a]\!]$ denotes the integer part of a real number a.

Again we consider a solution defined for $t \in]-\infty, 0]$ and study its profile at the terminal time $t = 0$. Assuming $\sigma > 0$, a direct computation yields

$$v(0, y-1) = \Psi(y) := \sum_{n\geq 1} G\big(n, y + [\![\sigma n]\!]\big).$$

Because of (39), to determine the asymptotic behavior as $y \to -\infty$, it suffices to estimate the difference

$$K(y) \doteq \Psi(y) - \Phi(y) = -\sum_{n\geq 1}\Big[G\big(n, y+\sigma n\big) - G\big(n, y + [\![\sigma n]\!]\big)\Big]\,.$$

It is here that, if the speed σ is close to a rational, a resonance is observed. To see a simple case, let $\sigma = 1 + \varepsilon$, with $\varepsilon > 0$ small. Then we can approximate

$$\begin{aligned} K(y) &\approx -\sum_{n\geq 1} G_x(n, y+\sigma n)\big(\sigma n - [\![\sigma n]\!]\big) \\ &\approx -\int_0^\infty G_x(t, y+\sigma t)\,\big(\varepsilon t - [\![\varepsilon t]\!]\big)\,dt\,. \end{aligned} \tag{42}$$

The functions appearing in the above integration are shown in Figure 3.

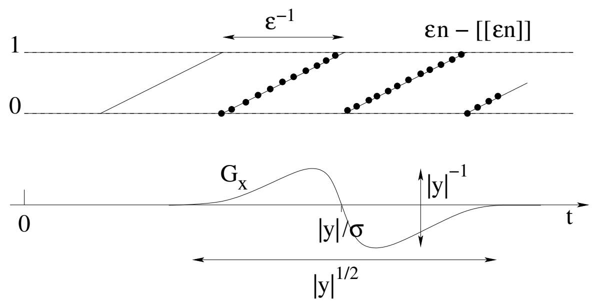

Figure 3

We recall that

$$\int_0^\infty G_x(t, y+\sigma t)\,dt = -\int_0^\infty \frac{y+\sigma t}{4t\sqrt{\pi t}}\,\exp\left\{-\frac{(y+\sigma t)^2}{4t}\right\}\,dt = 0$$

for every $y < 0$. Set $y_\varepsilon \doteq -\varepsilon^{-2}$. When y ranges within the interval

$$I_\varepsilon \doteq [y_\varepsilon, y_\varepsilon/2] = [-\varepsilon^{-2}, -\varepsilon^{-2}/2]\,,$$

the integral in (42) can be of the same order of magnitude as

$$\int_0^\infty \big|G_x(t, y_\varepsilon + \sigma t)\big|\, dt \geq c_0\, y_\varepsilon^{-1/2} = c_0\, \varepsilon\,.$$

Moreover, each time that y increases by an amount $\Delta y = \varepsilon^{-1}$, the phase of the fractional part $[\![\varepsilon y]\!] - \varepsilon y$ goes through a full cycle, hence the map

$$y \mapsto \int_0^\infty G_x(t, y + \sigma t)\,\big(\varepsilon t - [\![\varepsilon t]\!]\big)\, dt$$

oscillates by an amount $\geq c_1\varepsilon$. In all, we have approximately $1/2\varepsilon$ cycles within the interval I_ε. Hence the total variation of the discrete profile $\Psi^{(1+\varepsilon)}$ on I_ε can be estimated as

$$T.V.\big\{\Psi^{(1+\varepsilon)}\ ;\ I_\varepsilon\big\} \geq c_2 \tag{43}$$

for some constant $c_2 > 0$ independent of ε. We write here $\Psi = \Psi^{(1+\varepsilon)}$ to emphasize that the profile depends on the speed $\sigma = 1 + \varepsilon$. By (43) it is clear that, as $\varepsilon \to 0+$, the functions $\Psi^{(1+\varepsilon)}$ do not form a Cauchy sequence and cannot converge in the space BV.

4 Construction of discrete shock profiles

Since (3) is a decoupled scalar equation, the existence of discrete shock profiles (DSP) for the Lax-Friedrichs scheme (5) follows from [11], [16]. For given left and right states u^-, u^+ we denote by $U^\lambda(x)$ the DSP connecting u^- to u^+ and moving with speed

$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}\,.$$

We proceed to construct the second component of a discrete travelling profile with speed λ by using the discrete Green kernel for (6) and the Duhamel principle. Consider the Lax-Friedrichs scheme for the equation $z_t = 0$,

$$z_{n+1,j} = \frac{z_{n,j+1} + z_{n,j-1}}{2}. \tag{44}$$

We observe that the discrete Green's function $K_{n,k}$ for (44) is given by

$$K_{n,k} = \begin{cases} \left(\frac{1}{2}\right)^n \binom{n}{(n-k)/2} & \text{for } k = -n, -n+2, \ldots, n-2, n, \\ 0 & \text{otherwise.} \end{cases} \tag{45}$$

Given any DSP for the first equation, a DSP for (6) is obtained by prescribing vanishing v-data at time $n = -\infty$, and then letting the u-terms act as a source in (6) from $n = -\infty$ to $n = 0$. Consider first the difference equation

$$v_{n+1,j} = \frac{1}{2}\big(v_{n,j+1} + v_{n,j-1}\big) + g_{n,j}, \tag{46}$$

where the sources $g_{n,j}$ are assumed given. By Duhamel's principle we have that if vanishing data are given at time step $-N$, then the solution of (46) at time step $n \geq -N$ is

$$v_{n,j} = \sum_{m=-N-1}^{n-1} \sum_{k \in \mathbb{Z}} g_{m,j-k} K_{n-1-m,k}. \tag{47}$$

To apply this in our situation we introduce the functions $\psi,\ H : \mathbb{R} \to \mathbb{R}$ by

$$\psi(s) := \frac{d}{ds} g(U(s)),$$

and

$$H(x) := -\frac{1}{2}\big[g(U(x+1)) - g(U(x-1))\big] = -\frac{1}{2}\int_{x-1}^{x+1} \psi(s)\, ds,$$

where $U = U^{(\lambda)}$ is a scalar DSP for the first equation. In this case (6) may be written in the form (46) with the sources $g_{n,j}$ given by

$$g_{n,j} = H(j - \lambda n).$$

From now on we make the assumption that $g(u)$ is such that ψ and hence also H have compact support.

Proposition 4.1. *The pair of functions $(U^{(\lambda)}, V^{(\lambda)})$ where $U^{(\lambda)}$ is a DSP for (5) and $V^{(\lambda)}$ is defined by*

$$V^{(\lambda)}(x) := \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} H(x - k + \lambda n) K_{n-1,k},$$

is a DSP for the system (5)-(6).

$\triangleleft$ **Remark 4.1:** The proof of this proposition is immediate once it is verified that the double sum converges. Since H has compact support the convergence follows. $\triangleright$

We next give a useful integral representation of $V^{(\lambda)}(x)$. For a fixed ξ we define the function $v^{(\lambda)}(\cdot;\xi) : \mathbb{R} \to \mathbb{R}$ by

$$v^{(\lambda)}(x;\xi) := \sum_{n=1}^{\infty} \left(K_{n-1,[\![x+\lambda n-\xi]\!]} + K_{n-1,[\![x+\lambda n-\xi]\!]+1}\right).$$

Proposition 4.2. *The function $V^{(\lambda)}(x)$ is given by*

$$V^{(\lambda)}(x) = -\frac{1}{2}\int_{-\infty}^{\infty} \psi(\xi) v^{(\lambda)}(x;\xi)\, d\xi. \tag{48}$$

Proof.

$$\begin{aligned}
V^{(\lambda)}(x) &= \sum_{n=1}^{\infty}\sum_{k\in\mathbb{Z}} H(x-k+\lambda n) K_{n-1,k} \\
&= -\frac{1}{2}\sum_{n=1}^{\infty}\sum_{k\in\mathbb{Z}}\left[\int_{x-k+\lambda n}^{x-k+\lambda n+1} \psi(\xi)\, d\xi K_{n-1,k} + \int_{x-k+\lambda n-1}^{x-k+\lambda n} \psi(\xi)\, d\xi K_{n-1,k}\right] \\
&= -\frac{1}{2}\sum_{n=1}^{\infty}\sum_{k\in\mathbb{Z}}\left[\int_{x-k+\lambda n}^{x-k+\lambda n+1} \psi(\xi) K_{n-1,[\![x+\lambda n-\xi]\!]+1}\, d\xi + \right. \\
&\qquad\qquad \left. + \int_{x-k+\lambda n-1}^{x-k+\lambda n} \psi(\xi) K_{n-1,[\![x+\lambda n-\xi]\!]}\, d\xi\right] \\
&= -\frac{1}{2}\sum_{n=1}^{\infty}\left[\int_{-\infty}^{\infty} \psi(\xi) K_{n-1,[\![x+\lambda n-\xi]\!]+1}\, d\xi + \int_{-\infty}^{\infty} \psi(\xi) K_{n-1,[\![x+\lambda n-\xi]\!]}\, d\xi\right] \\
&= -\frac{1}{2}\int_{-\infty}^{\infty} \psi(\xi) \sum_{n=1}^{\infty}\left(K_{n-1,[\![x+\lambda n-\xi]\!]+1} + K_{n-1,[\![x+\lambda n-\xi]\!]}\right)\, d\xi \\
&= -\frac{1}{2}\int_{-\infty}^{\infty} \psi(\xi) v^{(\lambda)}(x;\xi)\, d\xi.
\end{aligned}$$

□

5 Approximation by the heat kernel

We will compare solutions of the Lax-Friedrichs scheme with certain solutions of the heat equation. As a first step we approximate the discrete Green's function $K_{n,k}$ using the heat kernel

$$G(t,x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}.$$

We use the following notation (see [9]):

$$a_k(\nu) := \left(\frac{1}{2}\right)^{2\nu} \binom{2\nu}{\nu+k}.$$

By Stirling's formula we have

$$a_k(\nu) = h\mathcal{N}(hk)\cdot \exp(\varepsilon_1 - \varepsilon_2),$$

where

$$h = \sqrt{\frac{2}{\nu}}, \qquad \mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

and the errors $\varepsilon_1, \varepsilon_2$ satisfy

$$-\frac{3k^2}{4\nu^2} < \varepsilon_1 < \frac{k^4}{4\nu^3}, \qquad \text{provided } |k| < \nu/3, \text{ and } \varepsilon_2 = O(1/\nu).$$

There are two cases to consider depending on whether both n and k are even or both are odd.

- Case 1: $n = 2m$, $k = 2l$. In this case we have,

$$K_{n,k} = a_l(m) = 2G\left(\frac{n}{2}, k\right) \cdot e^{\varepsilon_1 - \varepsilon_2}.$$

- Case 2: $n = 2m+1$, $k = 2l+1$. In this case we have,

$$\begin{aligned} K_{n,k} &= \frac{n}{n+k} a_l(m) \\ &= 2\frac{n}{n+k} G\left(\frac{n-1}{2}, k-1\right) \cdot e^{\varepsilon_1 - \varepsilon_2} \\ &= 2G\left(\frac{n}{2}, k\right) \cdot e^{\varepsilon_1 - \varepsilon_2} \frac{n}{n+k} \sqrt{\frac{n}{n-1}} e^{\frac{k^2}{2n} - \frac{(k-1)^2}{2(n-1)}}. \end{aligned}$$

We will use this approximation only in the case when $|k| \leq O(1) n^{1/2+\delta}$ where $0 < \delta \ll 1$. It follows that in either case we have

$$e^{\varepsilon_1 - \varepsilon_2} = 1 + O(1) n^{-1+4\delta}.$$

An easy calculation shows that under the same condition on k,

$$\frac{n}{n+k} \sqrt{\frac{n}{n-1}} e^{\frac{k^2}{2n} - \frac{(k-1)^2}{2(n-1)}} = 1 + O(1) n^{-1+\delta}.$$

Recalling (36), summing up we have the following.

Proposition 5.1. *For $n \geq 1$ and for $|k| \leq O(1) n^{1/2+\delta}$, with $0 < \delta \ll 1$, we have*

$$K_{n,k} = 2G\left(\frac{n}{2}, k\right) + O(1) n^{-3/2+\delta}. \tag{49}$$

For a given speed $\lambda > 0$ and for $\delta \in]0, 1/2[$ we define the time interval

$$I(y; \lambda, \delta) = \left[\frac{y}{\lambda} - y^{1/2+\delta}, \frac{y}{\lambda} + y^{1/2+\delta}\right]. \tag{50}$$

We will make repeated use of the fact that, for $y < 0$, the indices outside $I(|y|; \lambda, \delta)$ contribute exponentially little to the sum

$$S(y) := \sum_{n=1}^{\infty} K_{n, [\![y + \lambda n]\!]}.$$

Proposition 5.2. *For $y<0$, $\lambda>0$ and $\delta\in]0,1/2[$ we have*

$$\sum_{n\notin I(|y|;\lambda,\delta)} K_{n,[\![y+\lambda n]\!]} \le O(1)e^{-C(\delta,\lambda)|y|^{c(\lambda,\delta)}}, \tag{51}$$

for some positive constants $C(\delta,\lambda)$ and $c(\lambda,\delta)$.

Proof. For notational convenience assume that $\lambda=1$, the case $\lambda<1$ being similar. First of all we have $K_{n,[\![y+n]\!]}\neq 0$ iff $n\ge\frac{|y|}{2}$. We divide the above sum in four parts, where n ranges over $I_1=\big]\frac{|y|}{2},\frac{|y|+1}{2}\big]$, $I_2=\big]\frac{|y|+1}{2},|y|-|y|^{1/2+\delta}\big[$, $I_3=\big]|y|+|y|^{1/2+\delta},2|y|\big[$ and $I_4=\big[2|y|,+\infty\big[$, respectively. If $n\in I_1$ (and there is at most one such n) then $[\![y+n]\!]=-n$ and $n=O(|y|)$, so that $K_{n,[\![y+n]\!]}=\frac{1}{2^n}=2^{-C|y|}$ is transcendentally small. For the remaining indexes we can use the following estimate obtained by Stirling's formula

$$\binom{n}{k}\le\frac{Cn^{n+1/2}}{k^{k+1/2}(n-k)^{n-k+1/2}},\qquad \text{for } 0<k<n. \tag{52}$$

If $n\in I_2$, from (45) and (52) it follows

$$K_{n,[\![y+n]\!]}\le C\sqrt{\frac{n^{2n}}{y^y(2n-y)^{2n-y}}}=:\sqrt{F(n,y)}.$$

A calculation shows that $n\mapsto F(n,y)$ is increasing on I_2 and that

$$F_1(y):=\ln\left(\max_{n\in I_2}F(n,y)\right)=\ln F(|y|-|y|^{\frac12+\delta})=-|y|^{2\delta}+O(1)|y|^{3\delta-\frac12}.$$

The case $n\in I_3$ is treated in a similar way. It follows that

$$\sum_{n\in I_2\cup I_3}K_{n,[\![y+n]\!]}\le O(1)|y|e^{-C(\delta)|y|^{2\delta}}.$$

Finally let $n\in I_4$. Since the map $y\mapsto F(n,y)$ is increasing when $|y|\le n$, then $F(n,y)\le F(n,n/2)\le(2/3)^{n/2}$ for $n\in I_4$. Thus

$$\sum_{n\in I_4}K_{n,[\![y+n]\!]}=O(1)\sum_{n\ge 2|y|}\left(\frac{2}{3}\right)^{n/4}=O(1)\left(\frac{2}{3}\right)^{|y|/2}.$$

This completes the proof. ∎

In the following we will also need an analogous result for the heat kernel G.

Proposition 5.3. *Let $y < 0$, $\lambda > 0$ and $\delta \in]0, 1/2[$. Then, outside the interval $I(|y|; \lambda, \delta)$ the integral of G as well as any of its derivatives is transcendentally small, i.e. for any $m \geq 0$, we have*

$$\int_{\mathbb{R}^+ \setminus I(|y|;\lambda,\delta)} \left| \frac{\partial^m}{\partial x^m} G(t, y + \lambda t) \right| dt \leq O(1) e^{-C(\delta,\lambda)|y|^{c(\lambda,\delta)}}, \tag{53}$$

for some positive constants $C(\delta, \lambda)$ and $c(\lambda, \delta)$. Moreover

$$\left| \sum_{n \notin I(|y|;\lambda,\delta)} \frac{\partial^m}{\partial x^m} G(n, y + \lambda n) \right| \leq O(1) e^{-C(\delta,\lambda)|y|^{c(\lambda,\delta)}}, \tag{54}$$

$$\left| \sum_{n \notin I(|y|;\lambda,\delta)} \frac{\partial^m}{\partial x^m} G\big(n, [\![y + \lambda n]\!]\big) \right| \leq O(1) e^{-C(\delta,\lambda)|y|^{c(\lambda,\delta)}}. \tag{55}$$

Proof. For simplicity, assume $\lambda = 1$. Concerning the integral, we have

$$\begin{aligned} \int_0^{|y|-|y|^{1/2+\delta}} \left| \frac{\partial^m}{\partial x^m} G(t, y + t) \right| dt &= O(1) \int_0^{|y|-|y|^{1/2+\delta}} e^{-|y|^{2\delta}/5}\, dt \\ &= O(1) \cdot |y|\, e^{-|y|^{2\delta}/5}, \end{aligned}$$

$$\begin{aligned} \int_{|y|+|y|^{1/2+\delta}}^{\infty} \left| \frac{\partial^m}{\partial x^m} G(t, y + t) \right| dt &= O(1) \cdot \left\{ \int_{|y|^{1/2+\delta}}^{|y|} + \int_{|y|}^{\infty} \right\} e^{-\frac{\tau^2}{4(\tau+|y|)}}\, d\tau \\ &\leq C|y| e^{-|y|^{2\delta}/8} + \int_{|y|}^{\infty} e^{-\tau/8}\, d\tau = O(1) e^{-C|y|^{2\delta}}. \end{aligned}$$

Hence (53) follows. The other two estimates can be obtained from (53), (39) and (40). ☐

As a consequence of the previous propositions, we are authorized to add or subtract the tails of the integrals/sums, introducing an error which is exponentially decreasing with y. This will be frequently and tacitly used in the following.

5.1 Estimates on the approximation errors

In order to estimate the variation of the second component of the DSPs we will need that $v^{(\lambda)}(x; \xi)$ is close, within acceptable errors, to the corresponding function defined in terms of the heat kernel. This function is given as

$$w^{(\lambda)}(x; \xi) := 2 \sum_{n=1}^{\infty} G\left(\frac{n}{2}, [\![z_n]\!] \right),$$

where

$$z_n = z_n(x,\lambda) := x + \lambda(n+1) - \xi. \tag{56}$$

We will need to carefully keep track of the dependence of $w^{(\lambda)}$ on λ.

We proceed estimate $w^{(\lambda)}$. Using (31), (39), (36) and (37), for all $m \geq 1$, $x \ll 0$ and ξ in the support of ψ, writing $z = x - \xi + \lambda$, we have

$$\begin{aligned}
w^{(\lambda)}(x;\xi) &= 2\sum_{n=1}^{\infty} G\left(\frac{n}{2}, [\![z_n]\!]\right) \\
&= 2\sum_{n=1}^{\infty} G\left(\frac{n}{2}, z_n\right) - 2\sum_{n=1}^{\infty}\left\{G\left(\frac{n}{2}, z_n\right) - G\left(\frac{n}{2}, [\![z_n]\!]\right)\right\} \\
&= \frac{2}{\lambda} + O(1)|x|^{-m} + \\
&\quad - 2\sum_{n\in I(|z|;\lambda,\delta)}\left\{G_x\left(\frac{n}{2}, z_n\right)\cdot (\!(z_n)\!) + O(1)n^{-3/2}(\!(z_n)\!)^2\right\} \\
&= \frac{2}{\lambda} - 2\sum_{n\in I(|z|;\lambda,\delta)} G_x\left(\frac{n}{2}, z_n\right)\cdot (\!(z_n)\!) \;+\; O(1)|x|^{-1+\delta},
\end{aligned} \tag{57}$$

where $(\!(a)\!) := a - [\![a]\!]$ is the fractional part of a real number a. In this calculation we have used that $|I(|z|;\lambda,\delta)| = O(1)|x|^{1/2+\delta}$ and that $n = O(1)|x|$ when $n \in I(|z|;\lambda,\delta)$.

The case of rational speed. The estimate (57) is valid for any speed λ. Now suppose that λ is a rational speed,

$$\lambda = \frac{p}{q}, \qquad p, q \in \mathbb{N},$$

with p and q relatively prime, and consider the sum on the right-hand side of (57). By writing $n = mq + j$ with $m \geq 0$, $j \in \{0,\ldots,q-1\}$, Taylor expanding about the points (t_m, x_m) where

$$t_m = mq/2, \qquad x_m = x_m(x,\lambda) := x + \lambda mq - \xi,$$

and using the formula

$$\sum_{j=1}^{q}\left(\!\!\left(z + \frac{pj}{q}\right)\!\!\right) = (\!(qz)\!) + \frac{q-1}{2}, \tag{58}$$

from (36) and (37), we obtain, for $z_n = z_n(x,\lambda)$ and $z = x - \xi + \lambda$,

$$\begin{aligned}
&\sum_{n\in I(|z|;\lambda,\delta)} G_x\left(\frac{n}{2}, z_n\right)(\!(z_n)\!) \\
&= \underbrace{\sum_{m\geq 0}\sum_{j=0}^{q-1}}_{mq+j\in I(|z|;\lambda,\delta)} G_x\left(\frac{mq+j}{2}, x+\lambda(mq+j+1)-\xi\right)(\!(z_{mq+j})\!) \\
&= \underbrace{\sum_{m\geq 0}\sum_{j=0}^{q-1}}_{mq+j\in I(|z|;\lambda,\delta)} \left\{G_x\left(\frac{mq}{2}, x_m\right) + \sup_{t,x}\left(|G_{xt}|+|G_{xx}|\right)O(q)\right\}(\!(z_{mq+j})\!) \\
&= \sum_{m\geq 1} G_x\left(\frac{mq}{2}, x_m\right)\sum_{j=0}^{q-1}(\!(z_{mq+j})\!) + \\
&\qquad\qquad + O(q)|x|^{1/2+\delta}\left(|x|^{-2}+|x|^{-3/2}\right) + e^{-C|x|} \qquad (59) \\
&= \left\{(\!(q(x-\xi))\!) + \frac{q-1}{2}\right\}\sum_{m\geq 1} G_x\left(\frac{mq}{2}, x_m\right) + O(q)|x|^{-1+\delta}.
\end{aligned}$$

In this calculation we have used that $|I(|z|;\lambda,\delta)| = O(1)|x|^{1/2+\delta}$ and that $n = O(1)|x|$ when $n \in I(|z|;\lambda,\delta)$. Analogously to (40), one can prove that

$$\sum_{m\geq 1} G_x\left(\frac{mq}{2}, x_m\right) = O(q^{M-1})|x|^{-\frac{M+1}{2}+\delta},$$

for every integer M. We thus have that

$$\begin{aligned}
&\sum_{n\in I(|z|;\lambda,\delta)} G_x\left(\frac{n}{2}, z_n\right)\cdot(\!(z_n)\!) \\
&\qquad = \left\{(\!(q(x-\xi))\!) + \frac{q-1}{2}\right\}\sum_{m\geq 1} G_x\left(\frac{mq}{2}, x_m\right) + O(q)|x|^{-1+\delta} \\
&\qquad = O(q^M)|x|^{-\frac{M+1}{2}+\delta} + O(q)|x|^{-1+\delta}. \qquad (60)
\end{aligned}$$

From (57) and (60) we conclude that when the speed $\lambda = p/q$ is a rational we have

$$w^{(\lambda)}(x;\xi) = \frac{2}{\lambda} + O(q^M)|x|^{-\frac{M+1}{2}+\delta} + O(q)|x|^{-1+\delta}. \qquad (61)$$

Due to the dependence on the denominator q, equation (61) is not useful for computing the variation of differences $w^{(\lambda)} - w^{(\tilde{\lambda})}$ as $\tilde{\lambda}\to\lambda$. Instead, (61) will be used for a fixed reference speed λ, while we need an alternative analysis to estimate $w^{(\tilde\lambda)}$, where $\tilde\lambda = \lambda + \varepsilon$ is a small perturbation of λ.

For this we return to the right hand side of (59), with $x_m = x_m(x, \tilde\lambda)$ and $z_n = z_n(x, \tilde\lambda)$. We establish the following technical result.

Proposition 5.4. *Let $\lambda = p/q$ and $\tilde\lambda = \lambda + \varepsilon$, with $0 < \varepsilon \ll 1$. Then for $z = x - \xi$, where ξ lies in the support of ψ, there holds*

$$\begin{aligned}\sum_{m\ge 0} G_x\left(\frac{mq}{2}, z + \tilde\lambda mq\right) \cdot \sum_{j=1}^{q} (\!(z + mq\varepsilon + \tilde\lambda j)\!) \\ = \int_0^\infty G_x\left(\frac{sq}{2}, z + \tilde\lambda sq\right) \cdot (\!(q(z + sq\varepsilon))\!)\, ds + \\ + O(q)|x|^{-1+\delta} + O(\varepsilon q^2)|x|^{-1/2+\delta}. \quad (62)\end{aligned}$$

Proof. We compare both the sum and the integral to the same integral and show that in each case the error is $O(q)|x|^{-1+\delta} + O(\varepsilon q^2)|x|^{-1/2+\delta}$. Let $a_j = z + \tilde\lambda j$ and consider the following difference

$$\begin{aligned}&\Bigg|\sum_{m\ge 0} G_x\left(\frac{mq}{2}, z + \tilde\lambda mq\right) \cdot \sum_{j=1}^{q} (\!(z + mq\varepsilon + \tilde\lambda j)\!) - \\ &\qquad - \int_0^\infty G_x\left(\frac{sq}{2}, z + \tilde\lambda sq\right) \cdot \sum_{j=1}^{q} (\!(a_j + sq\varepsilon)\!)\, ds\Bigg| \\ &\le \Bigg|\sum_{m\ge 0}\int_m^{m+1} \Big\{G_x\left(\frac{mq}{2}, z + \tilde\lambda mq\right) - \\ &\qquad - G_x\left(\frac{sq}{2}, z + \tilde\lambda sq\right)\Big\} \sum_{j=1}^{q} (\!(a_j + mq\varepsilon)\!)\, ds\Bigg| + \\ &+ \Bigg|\sum_{m\ge 0}\int_m^{m+1} G_x\left(\frac{sq}{2}, z + \tilde\lambda sq\right)\left\{\sum_{j=1}^{q} (\!(a_j + mq\varepsilon)\!) - (\!(a_j + sq\varepsilon)\!)\right\} ds\Bigg| \\ &\le \underbrace{\sum_{m\ge 0}\sum_{j=1}^{q}}_{mq+j\in I(|z|;\tilde\lambda,\delta)} \left\{\sup_{t,x}\left(|G_{xt}| + |G_{xx}|\right)O(q)\right\} (\!(a_j + mq\varepsilon)\!) + e^{-C|x|} + \\ &\quad + \frac{O(1)}{|x|}\sum_{j=1}^{q}\sum_{mq\in I(|z|;\tilde\lambda,\delta)}\int_m^{m+1} \left|(\!(a_j + mq\varepsilon)\!) - (\!(a_j + sq\varepsilon)\!)\right| ds. \quad (63)\end{aligned}$$

The first sum on the right hand side of (63) is bounded by $O(q)|x|^{-1+\delta}$. To estimate the second sum, we divide the interval $I(|z|;\tilde\lambda,\delta)$ into

$r = O\big(\varepsilon q\,|I(|z|;\tilde\lambda,\delta)|\big) = O\big(\varepsilon q\big)|x|^{1/2+\delta}$ intervals $J_1,\dots,J_r$ of equal length $1/(\varepsilon q)$, and re-write the sum over m as

$$\sum_{k=1}^{r}\int_{J_k}\big|((a_j+[\![s]\!]q\varepsilon))-((a_j+sq\varepsilon))\big|\,ds.$$

The integrand is bounded by εq except on a sub-interval of length 1 where it is $O(1)$. Hence, for each k, the integral over J_k is bounded by order one. It follows that the second sum on the right hand side of (63) is bounded by $O(\varepsilon q^2)|x|^{-1/2+\delta}$.

On the other hand, using (58), (28) and (36), we have

$$\begin{aligned}
&\bigg|\int_0^\infty G_x\left(\frac{sq}{2},z+\tilde\lambda sq\right)\cdot((q(z+sq\varepsilon)))\,ds-\\
&\qquad-\int_0^\infty G_x\left(\frac{sq}{2},z+\tilde\lambda sq\right)\cdot\sum_{j=1}^{q}((a_j+sq\varepsilon))\,ds\bigg|\\
&=\left|\int_0^\infty G_x\left(\frac{sq}{2},z+\tilde\lambda sq\right)\cdot\left\{((q(z+sq\varepsilon)))-\sum_{j=1}^{q}((a_j+sq\varepsilon))\right\}ds\right|\\
&=\bigg|\int_0^\infty G_x\left(\frac{sq}{2},z+\tilde\lambda sq\right)\cdot\\
&\qquad\cdot\left\{\sum_{j=1}^{q}\Big(((z+\lambda j+sq\varepsilon))-((a_j+sq\varepsilon))\Big)-\frac{q-1}{2}\right\}ds\bigg|\\
&=\left|\int_0^\infty G_x\left(\frac{sq}{2},z+\tilde\lambda sq\right)\cdot\sum_{j=1}^{q}\Big(((z+\lambda j+sq\varepsilon))-((a_j+sq\varepsilon))\Big)\,ds\right|\\
&\le\frac{O(1)}{q|z|}\sum_{j=1}^{q}\int_{I(|z|;2\tilde\lambda,\delta)}\Big|((z+\lambda j+2\varepsilon t))-((a_j+2\varepsilon t))\Big|\,dt+O(1)|x|^{-M}\\
&=\frac{O(1)}{\varepsilon q|z|}\sum_{j=1}^{q}\int_{J(|z|;j,\varepsilon,\tilde\lambda,\delta)}\big|((\eta))-((\eta+\varepsilon j))\big|\,d\eta+O(1)|x|^{-M},
\end{aligned}\tag{64}$$

where $J(|z|;j,\varepsilon,\tilde\lambda,\delta)=z+\lambda j+2\varepsilon I(|z|;2\tilde\lambda,\delta)$. Each of the integrals in the sum is therefore of order $O(\varepsilon^2 q)|z|^{1/2+\delta}$. Finally, using that $z=x-\xi$ and that ξ varies in a compact interval, from (63) and (64) we thus obtain (62). □

Using (59) and (57) for speed $\tilde\lambda$ together with Proposition 5.4 yields

$$\begin{aligned} w^{(\tilde\lambda)}(x;\xi) &= \frac{2}{\tilde\lambda} - 2\sum_{m\geq 0} G_x\left(\frac{mq}{2}, z+\tilde\lambda mq\right)\cdot\sum_{j=1}^{q}((z+mq\varepsilon+\tilde\lambda j))+ \\ &\qquad + O(q)|x|^{-1+\delta} \\ &= \frac{2}{\tilde\lambda} - 2\int_0^\infty G_x\left(\frac{sq}{2}, z+\tilde\lambda sq\right)\cdot((q(z+sq\varepsilon)))\,ds \\ &\qquad + O(q)|x|^{-1+\delta} + O(\varepsilon q^2)|x|^{-1/2+\delta} \\ &= \frac{2}{\tilde\lambda} - \frac{2}{q}\int_0^\infty G_x\left(\frac{t}{2}, z+\tilde\lambda t\right)\cdot((qz+q\varepsilon t))\,dt + E(|x|;\varepsilon,q), \end{aligned}$$

where $z = x-\xi$ and

$$E(y;\varepsilon,q) = \frac{O(q)}{y^{1-\delta}} + \frac{O(\varepsilon q^2)}{y^{1/2-\delta}}. \tag{65}$$

We proceed by simplifying this expression. Put $y=-x\gg 0$, such that $z+\tilde\lambda t = -y-\xi+\tilde\lambda t$, and introduce the coordinate τ by

$$\tau\sqrt{\frac{y}{\tilde\lambda}} = y - \tilde\lambda t.$$

Thus τ-values outside an interval $[-Cy^\delta, Cy^\delta]$ corresponds to t-values for which the contribution to the integral is exponentially small. A simple calculation now shows that

$$G_x\left(\frac{t}{2}, z+\tilde\lambda t\right) = \frac{1}{\sqrt{2\pi}}\frac{\tilde\lambda}{y}\tau e^{-\tau^2/2} + O(1)y^{-3/2+\delta}.$$

Substitution into the expression above yields

$$\begin{aligned} w^{(\tilde\lambda)}(-y;\xi) &= \frac{2}{\tilde\lambda} + E(y;\varepsilon,q) - \\ &- \frac{1}{q}\sqrt{\frac{2}{\pi\tilde\lambda y}}\int\limits_{|\tau|<Cy^\delta}\tau e^{-\tau^2/2}\cdot\left(\!\!\left(\frac{q\varepsilon}{\tilde\lambda}\left(y-\tau\sqrt{\frac{y}{\tilde\lambda}}\right) - qy - q\xi\right)\!\!\right)d\tau. \end{aligned} \tag{66}$$

Recalling (61) we get that for a rational speed $\lambda = p/q$ and a perturbation $\tilde\lambda = \lambda+\varepsilon$, there holds

$$\begin{aligned} w^{(\lambda)}(x;\xi) - w^{(\tilde\lambda)}(x;\xi) &= \frac{2}{\lambda} - \frac{2}{\tilde\lambda} + E(y,\varepsilon,q) - \\ &- \frac{C_1}{qy^{1/2}}\int\limits_{|\tau|<Cy^\delta}\tau e^{-\tau^2/2}\cdot\left(\!\!\left(q\xi+qy-\frac{q\varepsilon}{\tilde\lambda}\left(y-\tau\sqrt{\frac{y}{\tilde\lambda}}\right)\right)\!\!\right)d\tau, \end{aligned} \tag{67}$$

where C_1 is a positive constant. We will use this estimate for rational speeds $\tilde{\lambda}$ converging to a fixed rational speed λ. The relevance of the estimate is that it does not depend explicitly on the denominator of the approximating speeds.

5.2 Estimating $v^{(\lambda)} - w^{(\lambda)}$

In this subsection we show that $w^{(\lambda)}$ is a sufficiently good approximation of $v^{(\lambda)}$. We have the following estimate which actually holds for any real speed λ.

Proposition 5.5. *Provided ξ is in the compact support of ψ, the functions $v^{(\lambda)}(x;\xi)$ and $w^{(\lambda)}(x;\xi)$ satisfy*

$$v^{(\lambda)}(x;\xi) - w^{(\lambda)}(x;\xi) = \sum_{\substack{n\in I(|z|;\lambda,\delta)\\ n-[\![z_n]\!]\text{odd}}} 2G_x\left(\frac{n}{2}, z_n\right) + O(1)|x|^{-1+2\delta}, \quad (68)$$

when $x < 0$, where $z = x - \xi + \lambda$ and $z_n = z_n(x,\lambda)$ is given by (56).

Proof. Observe that

$$G\left(\frac{n}{2}, [\![z_n]\!] + 1\right) = G\left(\frac{n}{2}, [\![z_n]\!]\right) + G_x\left(\frac{n}{2}, z_n\right) + O(1)n^{-3/2}.$$

By Proposition 5.2 only indices $n \in I(|z|;\lambda,\delta)$ in the sums defining $v^{(\lambda)}$ and $w^{(\lambda)}$ are significant. If $n \in I(|z|;\lambda,\delta)$ then $n = O(1)|x|$ while $[\![z_n]\!]$ is $O(1)|x|^{1/2+\delta}$, so that we are within the validity of the approximation (49). Since exactly one of $n - [\![z_n]\!]$, $n - ([\![z_n]\!] + 1)$ is even, we have

$$\begin{aligned}
v^{(\lambda)}(x;\xi) &= \sum_{n\in I(|z|;\lambda,\delta)} \left(K_{n,[\![z_n]\!]} + K_{n,[\![z_n]\!]+1}\right) + O(1)e^{-C|x|} \\
&= \sum_{\substack{n\in I(|z|;\lambda,\delta)\\ n-[\![z_n]\!]\text{even}}} 2G\left(\frac{n}{2}, [\![z_n]\!]\right) + \sum_{\substack{n\in I(|z|;\lambda,\delta)\\ n-[\![z_n]\!]\text{odd}}} 2G\left(\frac{n}{2}, [\![z_n]\!] + 1\right) + \\
&\quad + \sum_{n\in I(|z|;\lambda,\delta)} O(1)n^{-3/2+\delta} + O(1)e^{-C|x|} \\
&= \sum_{n\in I(|z|;\lambda,\delta)} 2G\left(\frac{n}{2}, [\![z_n]\!]\right) + \sum_{\substack{n\in I(|z|;\lambda,\delta)\\ n-[\![z_n]\!]\text{odd}}} 2G_x\left(\frac{n}{2}, z_n\right) + \\
&\quad + \sum_{n\in I(|z|;\lambda,\delta)} O(1)n^{-3/2+\delta} + O(1)e^{-C|x|} \\
&= w^{(\lambda)}(x;\xi) + \sum_{\substack{n\in I(|z|;\lambda,\delta)\\ n-[\![z_n]\!]\text{odd}}} 2G_x\left(\frac{n}{2}, z_n\right) + O(1)|x|^{-1+2\delta},
\end{aligned}$$

since $I(|z|;\lambda,\delta)$ has length $O(1)|x|^{1/2+\delta}$. ∎

To estimate the sum on the right-hand side of (68) we will make particular choices for the speeds λ and $\tilde{\lambda}$. In what follows we will fix

$$\lambda = \frac{1}{4}, \qquad \tilde{\lambda} = \frac{k}{4k-1}, \tag{69}$$

such that $\varepsilon = \frac{1}{16k-4}$. In order to satisfy the non-resonance condition of Majda and Ralston [14] we will let k be an even integer. We have the following key estimates that are independent of k (and ε).

Proposition 5.6. *With λ and $\tilde{\lambda}$ given by (69) we have*

$$\sum_{\substack{n \in I(|z|;\lambda,\delta) \\ n - [\![z_n(x,\lambda)]\!] \text{odd}}} G_x\left(\frac{n}{2}, z_n(x,\lambda)\right) = \frac{O(1)}{|x|^{1-\delta}}, \tag{70}$$

and

$$\sum_{\substack{n \in I(|z|;\tilde{\lambda},\delta) \\ n - [\![z_n(x,\tilde{\lambda})]\!] \text{odd}}} G_x\left(\frac{n}{2}, z_n(x,\tilde{\lambda})\right) = \frac{O(1)}{|x|^{1-\delta}}. \tag{71}$$

Proof. We give the proof for (71), the proof of (70) being similar. Given ξ and x, we set $z = x - \xi + \tilde{\lambda}$ such that $z_n(x, \tilde{\lambda}) = z + \tilde{\lambda} n$. We denote by $\mathcal{N}$ the set of the integers n contributing to the sum (71). Let a k-block denote a half-open interval of length $4k-1$ consisting of k consecutive subintervals of equal length $1/\tilde{\lambda} = 4 - 1/k$, each on which the function $B(s) := [\![z + \tilde{\lambda} s]\!]$ is constant. Without loss of generality we can assume that $I(|z|; \tilde{\lambda}, \delta)$ is exactly partitioned into finitely many k-blocks. A k-block contains $4k-1$ integers and each subinterval has length $4-1/k$. It follows that in each k-block there are $k-1$ subintervals containing exactly four integers, and one subinterval containing three integers. Since $B(s)$ is constant on each of the subintervals, then the function $n \mapsto n - [\![z + \tilde{\lambda} n]\!]$ takes on two even and two odd value on each subinterval that contains exactly four integers. As k is even two consecutive k-blocks will contain exactly $4k-1$ integers n in $\mathcal{N}$. Thus, of two consecutive k-blocks, one contains $2k$, and the other $2k-1$, of integers n in $\mathcal{N}$. We observe that in the sequence of these indices n, the elements are at most a distance 3 apart from each other. Finally, since k is an even integer, in all subsequent unions of two consecutive k-blocks, the distribution of the indices $n \in \mathcal{N}$ is the same. We can therefore define a map μ that maps $\mathcal{N}$ bijectively onto the *regular* grid of even integers in $I(|z|; \tilde{\lambda}, \delta)$, in such a way that $|n - \mu(n)| \leq 3$. The sum in (71) can therefore be estimated as follows,

$$
\begin{aligned}
\sum_{n\in\mathcal{N}} G_x\left(\frac{n}{2}, z_n(x,\tilde{\lambda})\right) &= \sum_{n\in\mathcal{N}} G_x\left(\frac{\mu(n)}{2}, z_{\mu(n)}(x,\tilde{\lambda})\right) + \\
&\quad + \sum_{n\in\mathcal{N}}\left[G_x\left(\frac{n}{2}, z_n(x,\tilde{\lambda})\right) - G_x\left(\frac{\mu(n)}{2}, z_{\mu(n)}(x,\tilde{\lambda})\right)\right] \\
&= \sum_{n\in 2\mathbb{Z}\cap I(|z|;\tilde{\lambda},\delta)} G_x\left(\frac{n}{2}, z_n(x,\tilde{\lambda})\right) + \\
&\quad + O(1)\sup(|G_{xt}|+|G_{xx}|)\cdot|I(|z|;\tilde{\lambda},\delta)| \\
&= \frac{O(1)}{|x|^{1-\delta}},
\end{aligned}
$$

where we have used (28), (36) and (37). □

◁ **Remark 5.1:** We note that the complexity of the preceding arguments is essentially due to the fact that we are working with the Lax-Friedrichs scheme. The same computations would be significantly simpler for e.g. the upwind scheme, in which case the complication of even and odd terms do not occur. ▷

Combining Proposition 5.5 and Proposition 5.6 we conclude that, if the velocities λ and $\tilde{\lambda}$ are given by (69), then

$$
v^{(\lambda)}(x;\xi) - v^{(\tilde{\lambda})}(x;\xi) = w^{(\lambda)}(x;\xi) - w^{(\tilde{\lambda})}(x;\xi) + \frac{O(1)}{|x|^{1-\delta}}. \tag{72}
$$

6 Estimates on the amount of oscillation

In the remaining part of the paper the velocities λ and $\tilde{\lambda} = \lambda + \varepsilon$ are given by (69). Recalling that ψ has compact support we conclude from (48), (72) and (67) that

$$
\begin{aligned}
V^{(\lambda)}(x) - V^{(\tilde{\lambda})}(x) &= -\frac{\varepsilon}{\lambda\tilde{\lambda}}\int_{-\infty}^{\infty}\psi(\xi)\,d\xi + E(y,\varepsilon) + \\
&+ \frac{C_1}{y^{1/2}}\int_{-\infty}^{\infty}\int_{|\tau|<Cy^\delta}\psi(\xi)\tau e^{-\tau^2/2}\left(\left(4\xi + 4y - \frac{4\varepsilon}{\tilde{\lambda}}\left(y - \tau\sqrt{\frac{y}{\tilde{\lambda}}}\right)\right)\right)\,d\tau\,d\xi, \\
&= A(\varepsilon) + E(y,\varepsilon) + \frac{C_1}{y^{1/2}}\int_{|\tau|<Cy^\delta}\tau e^{-\tau^2/2}h(\tau;y,\varepsilon)\,d\tau,
\end{aligned} \tag{73}
$$

where $y = -x$ and where we have introduced

$$E(y,\varepsilon) := E(y,\varepsilon,4)\int_{-\infty}^{\infty}\psi(\xi)\,d\xi,$$

$$A(\varepsilon) := -\frac{\varepsilon}{\lambda\tilde{\lambda}}\int_{-\infty}^{\infty}\psi(\xi)\,d\xi,$$

$$h(\tau;y,\varepsilon) := \int_{-\infty}^{\infty}\psi(\xi)\left(\!\!\left(4\xi+4y-\frac{4\varepsilon}{\tilde{\lambda}}\left(y-\tau\sqrt{\frac{y}{\tilde{\lambda}}}\right)\right)\!\!\right)d\xi.$$

Making the change of variables $\eta = 4\xi$ and denoting $\phi(\eta) = \psi(\eta/4)$, $\beta = 4\lambda/\tilde{\lambda}$, and $\gamma = 4/\tilde{\lambda}^{3/2}$, we have that

$$h(\tau;y,\varepsilon) = \frac{1}{4}\int_{-\infty}^{\infty}\phi(\eta)\big(\!\big(\eta+\beta y+\gamma\varepsilon\sqrt{y}\tau\big)\!\big)\,d\eta. \tag{74}$$

We proceed to use (73) to show that the function $V^{(\lambda)} - V^{(\tilde{\lambda})}$ contains an $O(1)$ amount of variation on an interval of the form

$$J(\varepsilon) := [-C\varepsilon^{-2/(1+2\delta)}, -c\varepsilon^{-2/(1+2\delta)}].$$

Recalling (65) we see that $E(y,\varepsilon)$ is of order $O(\varepsilon^{2(1-\delta)/(1+2\delta)})$ on $J(\varepsilon)$.

As a first step we consider the limiting case where $\phi(\eta)/4$ is the Dirac delta function centered at $\eta = 0$. In this case

$$h(\tau;y,\varepsilon) = h_0(\tau;y,\varepsilon) := \big(\!\big(\beta y+\gamma\varepsilon\sqrt{y}\tau\big)\!\big),$$

such that

$$V^{(\lambda)}(x) - V^{(\tilde{\lambda})}(x) = A(\varepsilon) + E(y,\varepsilon) + \frac{C_1}{y^{1/2}}H_0(y;\varepsilon), \tag{75}$$

where

$$H_0(y;\varepsilon) := \int_{|\tau|<Cy^{\delta}}\tau e^{-\tau^2/2}\big(\!\big(\beta y+\gamma\varepsilon\sqrt{y}\tau\big)\!\big)\,d\tau. \tag{76}$$

Lemma 6.1. *There exist an $O(\varepsilon^{-1/(1+2\delta)})$ number of points $y_1,\dots,y_L$ in $-J(\varepsilon)$ with*

$$H_0(y_n;\varepsilon) = \begin{cases} -O(1) & \text{for } n \text{ odd,} \\ O(1)\varepsilon^{2\delta/(1+2\delta)} & \text{for } n \text{ even.} \end{cases} \tag{77}$$

Proof. Set

$$y_1 := [\![\varepsilon^{-2/(1+2\delta)}]\!]/\beta,$$

and let

$$y_n := y_1 + \frac{1}{\beta}\left(n[\![\varepsilon^{-1/(1+2\delta)}]\!] + \frac{n+1}{2}\right),$$

for $n = 2, \ldots, [\![\varepsilon^{-1/(1+2\delta)}]\!]$. These choices imply that

$$\big(\!\big(\beta y_n + \gamma\varepsilon\sqrt{y_n}\tau\big)\!\big)\big|_{\tau=0} = (\!(\beta y_n)\!) = \begin{cases} 0 & \text{for } n \text{ odd,} \\ 1/2 & \text{for } n \text{ even.} \end{cases}$$

We observe that an $O(1)$ change in the constant C in (76) induces an exponentially small error (with respect to y). In order to simplify the computations we make the following choices for the constant C in (76),

$$C = C_n := \begin{cases} \frac{1}{\gamma\varepsilon y_n^{\delta+1/2}} = O(1) & \text{for } n \text{ odd,} \\ \frac{1}{2\gamma\varepsilon y_n^{\delta+1/2}} = O(1) & \text{for } n \text{ even.} \end{cases}$$

Thus, up to an exponentially small error, we have for n odd

$$\begin{aligned} H_0(y_n;\varepsilon) &= \int_{|\tau|<C_n y_n^\delta} \tau e^{-\tau^2/2}\big(\!\big(\beta y_n + \gamma\varepsilon\sqrt{y_n}\tau\big)\!\big)\, d\tau \\ &= \int_{-\infty}^{0} \tau e^{-\tau^2/2} = -O(1), \end{aligned} \tag{78}$$

while for n even a similar argument yields

$$H_0(y_n;\varepsilon) = O(1)\varepsilon^{2\delta/(1+2\delta)}.$$

□

For a fixed, small ε it now follows from Lemma 6.1 and (75) that the function $V^{(\lambda)}(x) - V^{(\tilde{\lambda})}(x)$ alternates between values

$$A(\varepsilon) + O(1)\varepsilon^{2(1-\delta)/(1+2\delta)} - O(1)\varepsilon^{1/(1+2\delta)},$$

for $x = -y_n$, n odd, and

$$A(\varepsilon) + O(1)\varepsilon^{2(1-\delta)/(1+2\delta)} + O(1)\varepsilon,$$

for $x = -y_n$, n even. Provided ε is small enough, this implies that in every interval $[-y_{n+1}, -y_n]$, the function $V^{(\lambda)}(x) - V^{(\tilde{\lambda})}(x)$ contains an $O(1)\varepsilon^{1/(1+2\delta)}$ amount of variation. Since there are an $O(1)\varepsilon^{-1/(1+2\delta)}$ number of such intervals, this argument shows that, in the limiting case where $\phi(\eta)/4$ in (74) is a Dirac delta, the function $V^{(\lambda)}(x) - V^{(\tilde{\lambda})}(x)$ contains at least an $O(1)$ amount of variation on $J(\varepsilon)$.

It remains to argue that the same result holds whenever $\phi(\eta)/4$ is close to a Dirac delta function. For this it is sufficient to show that the difference $|H(y;\varepsilon) - H_0(y;\varepsilon)|$ can be made arbitrarily small independently of the $O(1)$-estimate in (78). This will be accomplished by imposing that the support of ϕ (hence ψ) be sufficiently small. We have

$$|H(y;\varepsilon) - H_0(y;\varepsilon)|$$
$$\leq \int_{-\infty}^{+\infty} \frac{\phi(\eta)}{4} \int_{|\tau|<Cy^\delta} \Big| K(\tau) \Big\{ \big(\!\big(F(y;\varepsilon,\tau) - \eta\big)\!\big) - \big(\!\big(F(y;\varepsilon,\tau)\big)\!\big) \Big\} \Big| \, d\tau \, d\eta,$$

where $F(y;\varepsilon,\tau) = \beta y + \gamma\varepsilon\sqrt{y}\tau$. Since $\varepsilon\sqrt{y}$ is $O(1)$ it follows that there is a constant $\bar{C}$ such that

$$\int_{|\tau|<Cy^\delta} \Big| K(\tau) \Big\{ \big(\!\big(F(y;\varepsilon,\tau) - \eta\big)\!\big) - \big(\!\big(F(y;\varepsilon,\tau)\big)\!\big) \Big\} \Big| \, d\tau \leq \bar{C}\eta.$$

We thus have that

$$|H(y;\varepsilon) - H_0(y;\varepsilon)| \leq C \int_{-\infty}^{+\infty} \phi(\eta)\eta \, d\eta,$$

which can be made arbitrarily small, compared to the $O(1)$-estimate in (78), by choosing ϕ sufficiently close to a Dirac delta.

We have thus proved the following theorem.

Theorem 6.2. *Discrete shock profiles for the Lax-Friedrichs scheme for strictly hyperbolic systems of two conservation laws of the form (3)-(4) do not depend continuously in BV on their speeds. More precisely, one can find a sequence of rational speeds λ_n converging to $\lambda \in \mathbb{Q}$, for which there are discrete shock profiles Ψ_n and Ψ of speeds λ_n and λ, respectively, and such that*

$$\mathrm{Tot.Var.}\{\Psi_n - \Psi\} \geq c_0$$

for some constant $c_0 > 0$ independent of n.

$\triangleleft$ **Remark 6.1:** From (48), (68), (70) and (61), we observe that with our particular choices of speeds we have

$$V^{(\lambda)}(x + \Delta x) - V^{(\lambda)}(x) = \frac{O(1)}{|x|^{1-\delta}},$$

for $x \ll 0$ and $\Delta x = O(1)$. It follows that an $O(1)$ translation of $V^{(\lambda)}$ relative to $V^{(\tilde{\lambda})}$ changes the total variation of their difference only by $O(\varepsilon)$ in the region $J(\varepsilon)$. $\triangleright$

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