ON ERROR BOUNDS FOR MONOTONE APPROXIMATION SCHEMES FOR MULTI-DIMENSIONAL ISAACS EQUATIONS.

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ABSTRACT. Recently, Krylov, Barles, and Jakobsen developed the theory for estimating errors of monotone approximation schemes for the Bellman equation (a convex Isaacs equation). In this paper we consider an extension of this theory to a class of non-convex multidimensional Isaacs equations. This is the first result of this kind for non-convex multidimensional fully non-linear problems.

To get the error bound, a key intermediate step is to introduce a penalization approximation. We conclude by (i) providing new error bounds for penalization approximations extending earlier results e.g. Benssousan and Lions, and (ii) obtaining error bounds for approximation schemes for the penalization equation using very precise a priori bounds and a slight generalization of the recent theory of Krylov, Barles, and Jakobsen.

1. INTRODUCTION

In this paper we will study error bounds for approximation schemes for a class of non-convex multidimensional Isaacs equations. To be precise, we will consider the following (one-sided) obstacle problem

$$(1.1) \qquad \min\{F(x, u, Du, D^2u), u - g\} = 0 \quad \text{in} \quad \mathbb{R}^N,$$

where g is the obstacle ($u \geq g$), and F is given by

$$(1.2) \qquad F(x, r, p, X) = \sup_{\alpha \in \mathcal{A}} \left\{ -\text{tr}[a^\alpha(x)X] + b^\alpha(x)p + f^\alpha(x, r) \right\}.$$

Here $\mathcal{A}$ is a compact metric space, a is a positive semidefinite matrix, f is strictly increasing in r, and the data is a least bounded and uniformly continuous. Precise assumptions will be specified later. This equation is non-convex because of the $\min / \sup$ form, but also because the f term may be non-convex in r. It may also be degenerate since a may vanish at certain x and α. Under the assumptions we will use, this equations can always be rewritten as an Isaacs equation,

$$\inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \left\{ -\text{tr} \left[\bar{a}^{\alpha,\beta}(x)D^2u \right] + \bar{b}^{\alpha,\beta}(x)Du + \bar{c}^{\alpha,\beta}(x)u + \bar{f}^{\alpha,\beta}(x) \right\} = 0$$

in $\mathbb{R}^N$, for suitably defined $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{f}$. It is well-known that such equations do not in general have smooth solutions.

The above problem (1.1) is also called a variational inequality, and such problems occur in many applications and have been studied intensively for a long time.

The classical theory for variational inequalities (see e.g. [18, 6]) studies weak or variational solutions and uses either PDE techniques and Sobolev space theory or probabilistic techniques using the connection with optimal stopping time problems. In this paper we will (mostly) consider viscosity solutions which is a weaker and more recent concept of solutions. We refer to [8, 10] for a general overview of the viscosity solution theory, and to e.g. [25, 1, 2] for analysis and applications of obstacle problems in the viscosity solutions setting. We mention in particular the many applications in finance, like e.g. the pricing problem for American options [26].

In the viscosity solution setting the first results on error bounds for monotone schemes were obtained by Crandall and Lions [9] for first order equations. This case has later been studied by many authors. Only recently did Krylov [19, 20] obtain the first results for second order fully non-linear equations (the convex Bellman-equations), and these results were then extended and improved by Barles and Jakobsen [3, 4, 15]. We refer to the recent paper [4] for the best results available at the present time. All these results concern the convex Bellman equation. In the non-convex fully non-linear case, there are to the best of the author's knowledge no results in the multi-dimensional case. The only non-convex result we know about applies to one dimensional problems [14].

In this paper we will give error bounds for general monotone approximation schemes for the non-convex multi-dimensional problem (1.1). We will use the following abstract notation for such schemes,

$$(1.3) \qquad \min\left\{ S(h, x, u_h(x), [u_h]_x); u_h(x) - g(x) \right\} = 0 \quad \text{in} \quad \mathbb{R}^N,$$

where S is loosely speaking a consistent, monotone, and uniformly continuous approximation of F in (1.1). The approximate solution is u_h, $[u_h]_x$ is a function defined from u_h, and the approximation parameter is h. This notation was introduced by Barles and Souganidis [5] to display clearly the monotonicity of the scheme: S is non-decreasing in u_h and non-increasing in $[u_h]_x$. Typical approximations S that we have in mind are certain finite difference methods (FDMs) [21] and so-called control schemes [7]. In Section 5 we will explain the notation for a concrete FDM.

To get an idea of our results, we will now consider an explicit 1D problem:

$$\min\left\{ \sup_{\alpha \in \mathcal{A}} \left\{ -a^\alpha \Delta_h^2 u'' + f^\alpha(x, u) \right\}; u - g(x) \right\} = 0 \quad \text{in} \quad \mathbb{R}^1.$$

We approximate this problem using a monotone FDM,

$$\min\left\{ \sup_{\alpha \in \mathcal{A}} \left\{ -a^\alpha \Delta_h^2 u_h + f^\alpha(x, u_h) \right\}; u_h - g(x) \right\} = 0 \quad \text{in} \quad \mathbb{R}^1,$$

where

$$\Delta_h^2 \phi(x) = \frac{\phi(x+h) - 2\phi(x) + \phi(x-h)}{h^2}.$$

Under suitable assumptions on f and g our results (cf. Proposition 5.1) give the following error bound,

$$\|u - u_h\|_{L^\infty} \leq C h^{1/6}.$$

If we take slightly stronger assumptions on the obstacle g, we get

$$\|u - u_h\|_{L^\infty} \leq C h^{1/4}.$$

The same results hold for problems in arbitrary space dimensions, see Section 5. In the convex case ($g \equiv -\infty$) the corresponding rate is $1/2$ [15].

In the (much) more difficult case when a also depend on x, the rate in the convex case is a least $1/5$ [4]. In this paper we will not be able to handle FDMs when the coefficient a depends on x. Results in this direction can be obtained by combining the methods of this paper with those of [4]. But since the arguments become much more involved in this case, we have chosen to omit it. However note that the results in this paper applies to the so-called control schemes in the general case (when a depends also on x), see Section 5. For a better discussion of this point we refer to [3].

Let us now try to explain how we get our error bounds. As a key intermediate step we introduce the following penalization problem,

$$(1.4) \qquad F(x, v_\varepsilon, D v_\varepsilon, D^2 v_\varepsilon) = \frac{1}{\varepsilon}(v_\varepsilon - g)^- \quad \text{in} \quad \mathbb{R}^N,$$

where $(\cdot)^- = -\min(\cdot, 0)$. Under suitable assumptions on the data, it is possible to show that the solution of the penalized problem (1.4) converges monotonically to the solution of the obstacle problem (1.1) as $\varepsilon \to 0$ [1]. In this paper we prove new error bounds for this convergence using easy comparison arguments.

The next step is then to consider the approximation scheme associated to (1.4) via (1.3),

$$(1.5) \qquad S\big(h, x, v_{h,\varepsilon}(x), [v_{h,\varepsilon}]_x\big) = \frac{1}{\varepsilon}\left(v_{h,\varepsilon} - g\right)^- \quad \text{in} \quad \mathbb{R}^N.$$

Again we prove that $v_{h,\varepsilon}$ converge to u_h the solution of (1.3) with a given error bound. This argument is completely similar to the one mentioned above in connection with (1.4).

The third and more difficult step is to obtain error bounds for convergence the solution $v_{h,\varepsilon}$ of (1.5) to the solution v_ε of (1.4). To get this result we use a slight extension of the arguments in [3, 15]. What is new here is that the equation need not be convex in the zero-order term (the u term), as is the case for (1.4).

As a finial step we combine the previous steps to get the full error bound via the triangle inequality,

$$\|u - u_h\|_{L^\infty} \leq \|u - v_\varepsilon\|_{L^\infty} + \|v_\varepsilon - v_{h,\varepsilon}\|_{L^\infty} + \|v_{h,\varepsilon} - u_h\|_{L^\infty}.$$

The right hand side will depend on h and ε, and the result follows after a minimization over $\varepsilon > 0$. Warning! This last step is *only* possible to perform if the bound on $\|v_\varepsilon - v_{h,\varepsilon}\|_{L^\infty}$ does not depend on ε in a too singular manner. Note that some coefficients in (1.4) and (1.5) depend on $1/\varepsilon$, and that naive computations would lead to a priori bounds on the solutions that also depend on $1/\varepsilon$. With such bounds, we would not be able to prove any error bounds. For our purpose, we need and prove more precise a priori bounds than can be found in the literature.

Let us now return briefly to the penalization problem (1.4). Usually it is easier to obtain existence of solutions of the penalized problem and than of the corresponding obstacle problem. The limit procedure ($\varepsilon \to 0$) then gives existence of also for the obstacle problem. We refer to Bensoussan and Lions [6] for the classical theory and to Amadori [1] for a viscosity solutions approach. Error bounds exist in the classical case. E.g. in [6, p. 197] the following bound is proved,

$$(1.6) \qquad \|v_\varepsilon - u\|_{W^{1,2}} \leq C \varepsilon^{1/2},$$

in the case when F in (1.1) is linear, uniformly elliptic, and in divergence form. In this paper we prove under suitable assumptions that

$$(1.7) \qquad\qquad \|v_\varepsilon - u\|_{L^\infty} \leq C\varepsilon^{1/2},$$

and under sightly stronger assumptions on g we get $\|v_\varepsilon - u\|_{L^\infty} \leq C\varepsilon$. These results applies to very general equations, see Section 2, to all kinds of weak solutions as long as the comparison principle holds, and even to monotone schemes like (1.3) and (1.5). To the best of the author's knowledge this result is new, even in the linear uniformly elliptic case e.g. under the assumptions leading to (1.6). Also note that (1.7) does *not* follow from (1.6) except in one space dimension (by Sobolev embedding).

Let us now introduce some notation: We will use the following (semi) norms,

$$|f|_0 = \operatorname{ess\,sup}_{x \in \mathbb{R}^N} |f(x)|, \ \ [f]_\mu = \operatorname{ess\,sup}_{x,y \in \mathbb{R}^N} \frac{|f(x) - f(y)|}{|x-y|^\mu}, \ \ |f|_\mu = |f|_0 + [f]_\mu,$$

where $f : \mathbb{R}^N \to \mathbb{R}^M$ is a function and $\mu \in (0,1]$. The same notation will be used for vector and matrix valued functions f, in which case $|f|$ is interpreted as a vector and matrix norm respectively. $L^\infty(\mathbb{R}^N)$, $C(\mathbb{R}^N)$, $C_b(\mathbb{R}^N)$, $C^{0,\mu}(\mathbb{R}^N)$, $\mu \in (0,1]$, $C^k(\mathbb{R}^N)$, $k \in \mathbb{N}$, denote the spaces of functions $f : \mathbb{R}^N \to \mathbb{R}$ that are bounded, continuous, bounded and continuous, have finite norm $|f|_\mu$, and are k-times continuous differentiable respectively. Furthermore, $W^{1,2}$, $W_0^{1,2}$, $W^{2,2}$, and $W^{1,\infty} = C^{0,1}$ are standard Sobolev spaces. The space of real symmetric $N \times N$ matrices are denoted by $\mathbb{S}^N$, and $X \geq Y$ in $\mathbb{S}^N$ will mean that $X - Y$ is positive semi-definite. Finally, by $D^k\phi$ we mean the vector of k-order partial derivatives of a function ϕ.

The outline of the rest of this paper is as follows: In the next section, we treat the penalization problem (1.4). We state and prove a very general error bound and compare it with classical results by Benssousan and Lions. In Section 3 we obtain error bounds for equations that are non-convex in the 0-th order term. These results are of auxiliary nature and are needed in Section 4. In this section we state and prove our main result, an error bound for (1.3). Then in Section 5, we apply our main result to obtain error bounds for a FDM and a control scheme. Finally there is an Appendix containing some technical a priori estimates.

2. The penalization method

In this section we will use comparison arguments to derive new error bounds for the convergence of the solution of the penalization problem (1.4) to the solution of the obstacle problem (1.1). Here we will no longer assume (1.2), in stead we will allow for very general structure of F:

(C1) (Comparison) The equations (1.1) and (1.4) satisfy the comparison principle for the class of weak solutions under consideration.

(C2) (Monotonicity) Let $X, Y \in \mathbb{S}^N, p, x \in \mathbb{R}^N, r, s \in \mathbb{R}$. If $X \geq Y$ and $r \leq s$, then

$$F(x, r, p, X) \leq F(x, s, p, Y).$$

(C3) (Regularity) One of the following statements hold:

(i) $g \in C^{0,1}(\mathbb{R}^N)$, $|D^2 g^-| \leq C$, and for every $x \in \mathbb{R}^N$ and $\phi \in C^2(\mathbb{R}^N)$ satisfying $|\phi|_{0,1} + |D^2 \phi^-|_0 \leq R$,

$$F(x, \phi(x), D\phi(x), D^2\phi(x)) \leq C_R.$$

(ii) $g \in C^{0,1}(\mathbb{R}^N)$, and for every $x \in \mathbb{R}^N$ and $\phi \in C^2(\mathbb{R}^N)$ satisfying $|\phi|_{0,1} \leq R$,

$$F(x, \phi(x), D\phi(x), D^2\phi(x)) \leq C_R(1 + |D^2\phi^-|_0).$$

(iii) $g \in C^{0,\mu}(\mathbb{R}^N)$ for some $\mu \in (0,1)$, and for every $x \in \mathbb{R}^N$ and $\phi \in C^2(\mathbb{R}^N)$ satisfying $|\phi|_0 \leq R$,

$$F(x, \phi(x), D\phi(x), D^2\phi(x)) \leq C_R(1 + |D\phi|_0 + |D^2\phi^-|_0).$$

Assumption (C1) is not very precise. In applications we need to specify both the notion of weak solutions and "boundary conditions" at infinity. Assumption (C2) says that F is "proper" in the terminology of the User's Guide [8], and implies that F is degenerate elliptic. Assumption (C3) gives regularity assumption on the obstacle g and corresponding (local) boundedness assumptions on F. Assumption (C3) can be generalized to allow for super-linear growth in $|X^-|$ and $|p|$. This would affect the rates obtained and will not be considered here.

These assumptions are satisfied by a *very* wide class equations and with different concepts of weak solutions. In the viscosity solutions setting (the weakest notion allowed here), we will just mention that the above assumption hold for the Bellman equations from stochastic control [10] and the Isaacs equations from stochastic differential games [11] under natural assumptions on the data. We refer to the User's Guide [8] for many more viscosity solution examples. Typical "boundary conditions" would be to assume bounded solutions or linear growth at infinity. We can also consider variational solutions [6] whenever it makes sense to do so. In this case all point-wise inequalities have to be interpreted in the almost everywhere sense.

The main result in this section gives both the convergence and the rate of convergence for the penalization problem.

Theorem 2.1. *Assume (C1) – (C3) hold and u and v_ε are solutions of (1.1) and (1.4) (we do not assume (1.2)!). Then if $|D^2 g^-|_0 < \infty$ (case (i))*

$$0 \leq u - v_\varepsilon \leq C\varepsilon \quad in \quad \mathbb{R}^N,$$

otherwise ($g \in C^{0,\mu}(\mathbb{R}^N)$ – cases (ii) and (iii))

$$0 \leq u - v_\varepsilon \leq C\varepsilon^{\mu/2} \quad in \quad \mathbb{R}^N,$$

where the constants C only depend on g and C_R from (C3).

Remark 2.1. The rates depends only on the regularity of the obstacle, and not on the regularity of the solution. Even if the solution u is only Hölder continuous, we still get rate 1 if $|D^2 g^-|_0 < \infty$. For many other types of approximation schemes the rates depends directly on the regularity of the solution, see e.g. [3] (FDMs) and [17] (vanishing viscosity method).

To the best of the our knowledge, this is the first time the penalization error has been estimated for degenerate equations, and the above result seem to be new even in the linear uniformly elliptic case (see below).

Before giving the proof of Theorem 2.1, let us briefly consider the linear uniformly elliptic case. Here $L^2/W^{1,2}$-estimates on the penalization error are classical [6]. We

will state a typical such result, so that the reader can compare it with the one we have obtained. In our notation:

$$(2.1) \qquad \min\{-Au + f(x); u - g(x)\} = 0 \qquad \text{in} \quad \Omega,$$

$$(2.2) \qquad -Av_\varepsilon + f(x) = \frac{1}{\varepsilon}(v_\varepsilon - g(x))^- \qquad \text{in} \quad \Omega,$$

where Ω is a smooth bounded domain and A is a linear elliptic operator in divergence form,

$$A\phi(x) := \partial_{x_i}(a_{ij}(x)\partial_{x_j}\phi) + b_i(x)\partial_i\phi - \lambda\phi.$$

The summation convention is used, $\lambda > 0$, and ellipticity means $\xi_i a_{ij}(x)\xi_j \geq \alpha|\xi|^2$ for some $\alpha > 0$ and every $\xi \in \mathbb{R}^N$. The concept of solutions is that of variational (weak) solutions belonging to $W_0^{1,2}(\Omega)$, see [6] for the exact definitions. Typical assumptions on the data are

$$(D) \quad a_{ij} \in L^\infty(\Omega), b_i \in W^{1,\infty}(\Omega), f \in L^2(\Omega), g \in W^{1,2}(\Omega), \; Ag \in L^2(\Omega),$$

where $a = (a_{ij})_{ij}, b = (b_i)_i$, and the error bound obtained is the following [6, p. 197]:

Proposition 2.2 (Bensoussan & Lions)**.** *Assume (D) holds, $\lambda > 0$ large enough, $g|_{\partial\Omega} \geq 0$, and u and v_ε solve (2.1) and (2.2). Then*

$$\|u - v_\varepsilon\|_{W^{1,2}(\Omega)} \leq C\varepsilon^{1/2}.$$

Note that in this theorem we need control over the second derivatives of the obstacle g ($Ag \in L^2$ essentially means that $g \in W^{2,2}$), while in our result we only need to control the first derivative of g (say $g \in W^{1,\infty} = C^{0,1}$). Furthermore, we may use Theorem 2.1 to get a new error bound in this case. Comparison principles for (2.1) and (2.2) are essentially given by Theorems 1.2 and 1.4 p. 192 and p. 198 in [6], so if $g \in W^{1,\infty}(\Omega)$ we can conclude by Theorem 2.1 that

$$\|u - v_\varepsilon\|_{L^\infty(\Omega)} \leq C\varepsilon^{1/2}.$$

Here u and v_ε are variational solutions of (2.1) and (2.2). This result does not follow from Proposition 2.2 unless Ω is a domain in $\mathbb{R}^1$.

The proof of Theorem 2.1. We give a series of simple lemmas that leads the way to the proof of Theorem 2.1. We start by a preliminary error estimate:

Lemma 2.3. *Assume (C1) and (C2). Let u and v_ε solve (1.1) and (1.4). Then*

$$0 \leq u - v_\varepsilon \leq |(v_\varepsilon - g)^-|_0 \quad in \quad \mathbb{R}^N.$$

Proof. First we check that by monotonicity in r (C2),

$$v_\varepsilon + |(v_\varepsilon - g)^-|_0$$

is a supersolution of (1.1). The comparison principle for (1.1) then yields the second inequality. Similarly, the first inequality follows since v_ε is subsolution of (1.1). $\square$

Now we will estimate $|(v_\varepsilon - g)^-|_0$:

Lemma 2.4. *Assume (C1) – (C3) hold and $g \in C^2(\mathbb{R}^N)$. According to (C3) define:*
 Case (i): $K := C_R$ with $R = |g|_0 + |Dg|_0 + |D^2g^-|_0$.
 Case (ii): $K := C_R(1 + |D^2g^-|_0)$ with $R = |g|_0 + |Dg|_0$.

Case (iii): $K := C_R(1 + |Dg|_0 + |D^2 g^-|_0)$ *with* $R = |g|_0$.

Let v_ε be the solution of (1.4). *Then*

$$-\varepsilon K \le v_\varepsilon - g \quad in \quad \mathbb{R}^N.$$

Proof. The result follows from the comparison principle since

$$g - \varepsilon K$$

is a (classical) subsolution of (1.4). □

Since we did not assume that g is smooth, we need an approximation result. Let ρ_δ be the standard mollifier, $\rho_\delta(x) = 1/\delta^N \rho(x/\delta)$ where ρ is a smooth positive function with mass one and support in the unit ball. Let $g_\delta = g * \rho_\delta$ and denote by u^δ and v_ε^δ the solutions of (1.1) and (1.4) when g_δ has replaced g:

$$(2.3) \qquad \min\{F(x, u^\delta, Du^\delta, D^2 u^\delta); u^\delta - g_\delta\} = 0 \qquad \qquad in \quad \mathbb{R}^N,$$

$$(2.4) \qquad \qquad F(x, v_\varepsilon^\delta, Dv_\varepsilon^\delta, D^2 v_\varepsilon^\delta) = \frac{1}{\varepsilon}(v_\varepsilon^\delta - g_\delta)^- \qquad in \quad \mathbb{R}^N.$$

We have the following bounds on $u - u^\delta$ and $v_\varepsilon - v_\varepsilon^\delta$:

Lemma 2.5. *Assume (C1) and (C2), and let u, u^δ, v_ε and v_ε^δ be solutions of* (1.1), (2.3), (1.4), *and* (2.4). *Then*

$$|u - u^\delta|_0 + |v_\varepsilon - v_\varepsilon^\delta|_0 \le |g - g^\delta|_0.$$

Proof. We only prove the v-result, the proof of the u-result is similar. (If we knew a priori that $v_\varepsilon \to u$, the u-result could be obtained by going to the limit in the v-result.) Let $K := |g - g^\delta|_0$ and define

$$w^\pm(x) = v_\varepsilon^\delta(x) \pm K.$$

The result follows by the comparison principle for (1.4) since w^+ and w^- are super- and subsolutions of (1.4) respectively.

Let us prove that w^+ is a supersolution of (1.4), the subsolution part is similar. First observe that by (C2)

$$F(x, w^+, Dw^+, D^2 w^+) \ge F(x, v_\varepsilon^\delta, Dv_\varepsilon^\delta, D^2 v_\varepsilon^\delta).$$

Then observe that by the definition of K,

$$-(w^+ - g)^- = -(v_\varepsilon^\delta + K - g)^- \ge -(v_\varepsilon^\delta - g_\delta)^-.$$

Since v_ε^δ is a supersolution of (2.4), the above observations show (at least formally) that

$$F(x, w^+, Dw^+, D^2 w^+) - \frac{1}{\varepsilon}(w^+ - g)^-$$

$$\ge F(x, v_\varepsilon^\delta, Dv_\varepsilon^\delta, D^2 v_\varepsilon^\delta) - \frac{1}{\varepsilon}(v_\varepsilon^\delta - g_\delta)^- \ge 0.$$

The proof is complete since all the above computations easily can be seen to hold in the weak/viscosity sense. □

Now we can give the proof of Theorem 2.1:

Proof of Theorem 2.1. First we consider the solutions u^δ and v_ε^δ of (2.3) and (2.4). By Lemmas 2.3 and 2.4 we have

$$u^\delta - v_\varepsilon^\delta \le K\varepsilon \quad \text{in} \quad \mathbb{R}^N,$$

where K is defined in Lemma 2.3. Since

$$u - v_\varepsilon = (u - u^\delta) + (u^\delta - v_\varepsilon^\delta) + (v_\varepsilon^\delta - v_\varepsilon),$$

Lemma 2.5 and Hölder continuity of g lead to

$$u - v_\varepsilon \le 2[g]_\mu \delta^\mu + K\varepsilon.$$

If $|g_{xx}^-|_0 < \infty$ then K is independent of δ and we can send $\delta \to 0$, leading to

$$u - v_\varepsilon \le K\varepsilon.$$

Otherwise, by Hölder continuity of g, $K = C\delta^{\mu-2}$, and minimization w.r.t. δ yields

$$u - v_\varepsilon \le C\varepsilon^{\mu/2}.$$

The lower bound follow from Lemma 2.3.   □

Remark 2.2. The procedure used in the above proof is very general, and works for any problem satisfying the assumptions corresponding to (C1) – (C3). E.g. one could consider boundary value problems where one would find that the estimates in Theorem 2.1 still holds. In the next section we will even see the method applied to an obstacle problem for an approximation scheme (Lemma 4.1).

We end this section by indicating an alternative approach. Remember that the lower bounds in Theorem 2.1 follow from the comparison principle since the solution v_ε of (1.4) is a subsolution of (1.1). To obtain the upper bounds, we need in some sense to show that v_ε is an approximate supersolution of (1.1). Observe that formally

$$\min\{F[v_\varepsilon]; v_\varepsilon - g\} \ge -\varepsilon(F[v_\varepsilon])^+.$$

This follows since

$$0 = F[v_\varepsilon] - \frac{1}{\varepsilon}(v_\varepsilon - g)^- = \min\left\{F[v_\varepsilon]; F[v_\varepsilon] + \frac{1}{\varepsilon}(v_\varepsilon - g)\right\}$$

implies that

$$0 = \min\{F[v_\varepsilon]; \varepsilon F[v_\varepsilon] + v_\varepsilon - g\} \le \min\{F[v_\varepsilon]; v_\varepsilon - g\} + \varepsilon(F[v_\varepsilon])^+.$$

This is vanishing viscosity(!), and we should already guess that the error should be $C\varepsilon^{1/2}$ when solutions are Lipschitz continuous. An easy way to get this result is the continuous dependence approach of [16, 17] which leads

$$u - v_\varepsilon \le C``\varepsilon(F[v_\varepsilon])^+`` = C(1 + |Dv_\varepsilon|_0)\varepsilon^{1/2},$$

where u is the solution of (1.1). The loss of rate is caused by v_ε being only Lipschitz continuous while F is a second order operator. On the other hand, if $|D^2 v_\varepsilon^-|_0 < \infty$ then we would have the full rate:

$$u - v_\varepsilon \le C``\varepsilon(F[v_\varepsilon])^+`` = C(1 + |Dv_\varepsilon|_0 + |D^2 v_\varepsilon^-|_0)\varepsilon.$$

In Theorem 2.1 this last estimate is proved under the much weaker assumption $|D^2 g^-|_0 < \infty$.

3. Monotone Approximation Schemes - Preliminaries.

In this section we will give a slight generalization of the results of [19, 20, 3, 15]. We prove error bounds for monotone approximation schemes for equations that have possibly non-convex dependence on 0-order terms. These results will then be used in Section 4 to obtain rates for the more difficult obstacle problem (1.1) and (1.2).

Consider the following equation:

$$(3.1) \qquad F(x, u, Du, D^2u) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

where F is given by (1.2) in the introduction. We make the following assumptions:

(A1) $a^\alpha = \frac{1}{2}\sigma^\alpha \sigma^{\alpha T}$ for some $N \times P$ matrix σ, and there is a C independent of α such that $|\sigma^\alpha|_1 + |b^\alpha|_1 \leq C$.

(A2) There are $\lambda, \Lambda > 0$ such that for every $x \in \mathbb{R}^N$, $\alpha \in \mathcal{A}$, $r, s \in \mathbb{R}$ satisfying $r \geq s$,

$$\lambda(r - s) \leq f^\alpha(x, r) - f^\alpha(x, s) \leq \Lambda(r - s).$$

Furthermore, $f^\alpha(\cdot, 0)$ is bounded uniformly in α, and for every $x, y \in \mathbb{R}^N$, $r \in \mathbb{R}$, and $\alpha \in \mathcal{A}$

$$|f^\alpha(x, r) - f^\alpha(y, r)| \leq C(1 + |r|)|x - y|.$$

Remark 3.1. The first part of assumption (A2) implies that f is Lipschitz and strictly increasing in r. The second part implies that f is bounded and Lipschitz in x for fixed r. If (A1) and (A2) hold and $\varepsilon > 0$ is fixed, the penalization scheme (1.4) can be rewritten in the form (3.1) by redefining $f^\alpha(x, r)$ to be

$$f^\alpha(x, r) + \frac{1}{\varepsilon} \min \{r - g(x); 0\}.$$

This new function than satisfies (A2) with new constants λ, $\Lambda + \frac{1}{\varepsilon}$, and C.

Existence, uniqueness, and regularity follow from standard viscosity solutions arguments. The results parallels the one mentioned in Section 1, and we state them without proofs:

Lemma 3.1. *Assume (A1) and (A2) hold. Then there is a unique bounded Hölder continuous viscosity solution u of (3.1). Furthermore, if λ is big enough (compared to $[\sigma]_1$ and $[b]_1$), then u is Lipschitz continuous.*

Using notation from the introduction, we may write an approximation scheme for (3.1) in the following way

$$(3.2) \qquad S(h, x, u_h(x), [u_h]_x) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

We require S to satisfy:

(S1) (Monotonicity) For every $h > 0$, $x \in \mathbb{R}^N$, $r \in \mathbb{R}$, $m \geq 0$ and bounded functions u, v such that $u \leq v$ in $\mathbb{R}^N$, the following holds:

$$S(h, x, r + m, [u + m]_x) \geq \lambda m + S(h, x, r, [v]_x),$$

where $\lambda > 0$ is given by (A2).

(S2) (Regularity) For every $h > 0$ and $\phi \in C_b(\mathbb{R}^N)$, $x \mapsto S(h, x, \phi(x), [\phi]_x)$ is bounded and continuous in $\mathbb{R}^N$ and the function $r \mapsto S(h, x, r, [\phi]_x)$ is uniformly continuous for bounded r, uniformly in $x \in \mathbb{R}^N$.

(S3) (Consistency) There exists integers $n, k_i > 0$, constants $K_i \geq 0$, $i = 1, 2, \ldots, n$ such that for every smooth ϕ, $h > 0$, and $x \in \mathbb{R}^N$:

$$\left| F(x, \phi(x), D\phi(x), D^2\phi(x)) - S(h, x, \phi(x), [\phi]_x) \right| \leq \sum_{i=1}^{n} K_i |D^i\phi(x)| h^{k_i}.$$

Condition (S1) and (S2) imply a comparison result for bounded continuous solutions of (3.2) (cf. [3]):

Lemma 3.2. *Assume (S1), (S2), and $u, v \in C_b(\mathbb{R}^N)$. If $S[u] \leq 0$ and $S[v] \geq 0$ in $\mathbb{R}^N$, then $u \leq v$ in $\mathbb{R}^N$.*

We proceed with obtaining an upper bound on the error for the scheme (3.2). In order to do so we will consider the following auxiliary problem:

$$(3.3) \qquad \sup_{|e| \leq \delta} \tilde{F}(x + e, u^\delta(x), Du^\delta(x), D^2u^\delta(x)) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

where $\delta > 0$, and with u being the solution of (3.1),

$$\tilde{F}(x, r, p, X) := \sup_{\alpha \in \mathcal{A}} \left\{ -\text{tr}[a^\alpha(x)X] + b^\alpha(x)p + \lambda r - \lambda u(x) + f^\alpha(x, u(x)) \right\}.$$

Actually this is a problem of the same type as (3.1) so well-posedness follows in the same way. At this point we assume the following:

(A3) Let u and u^δ denote the solutions of (3.1) and (3.3). There is a constant $K > 0$ independent of δ such that

$$|u^\delta|_1 + \frac{1}{\delta}|u - u^\delta|_0 \leq K.$$

Remark 3.2. Assumption (A3) follows from assumptions (A1) and (A2) if λ is big enough. After observing that (3.1) can be written as an Isaacs equation, this follows from Lemmas A.1 and A.2 in the Appendix. In the case that λ is not "big enough" things are a little bit more complicated, we refer to [3] for this case.

Now we are in a position to derive an upper bound on the error for the scheme (3.2).

Theorem 3.3. *Let (A1) – (A3), (S1) – (S3) hold, let u be the viscosity solution of (3.1), and let u_h be a solution of the scheme (3.2). Then if $h > 0$ is sufficiently small,*

$$u - u_h \leq Ch^\gamma \quad \text{in} \quad \mathbb{R}^N,$$

where $\gamma := \min_{i:K_i>0} \left\{ \frac{k_i}{i} \right\}$ and $C \leq \frac{K}{\lambda}(\sum_{i=1}^{n} K_i + 2(2\lambda + \Lambda))$.

Proof. 1) We start by showing that $u_\delta := \rho_\delta * u^\delta$ is a subsolution of

$$(3.4) \qquad \tilde{F}(x, w, Dw, D^2w) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

where ρ_δ is the mollifier defined in Section 2. By (A3)

$$\tilde{F}(x + e, u^\delta(x), Du^\delta(x), D^2u^\delta(x)) \leq 0 \quad \text{in} \quad \mathbb{R}^N$$

for every $|e| \leq \delta$. Hence for every $|e| \leq \delta$, $u^\delta(x - e)$ is a subsolution of (3.4). Then u_δ is also a subsolution of (3.4) since it can be viewed as the limit of convex combinations of subsolutions $u^\delta(x - e)$ of the convex equation (3.4), we refer to the Appendix in [3] for the details.

2) u_δ is an approximate subsolution to the scheme (3.2). By properties of mollifiers and (A3), u_δ is smooth and satisfies

$$\delta^{i-1}|D^i u_\delta|_0 + (2\delta)^{-1}|u - u_\delta|_0 \leq K.$$

So by (A2) and the definition of $\tilde{F}$, for every $x \in \mathbb{R}^N$,

$$F(x, u^\delta(x), Du^\delta(x), D^2 u^\delta(x)) \leq \sup_{\alpha \in \mathcal{A}} |\lambda(u_\delta - u) - f^\alpha(x, u_\delta) + f^\alpha(x, u)|$$
$$\leq 2K(\lambda + \Lambda)\delta.$$

Consistency (S3) then leads to

$$S(h, y, u_\delta(y), [u_\delta]_y) \leq K \sum_{i=1}^n K_i \delta^{1-i} h^{k_i} + 2K(\lambda + \Lambda)\delta =: \overline{C}.$$

3) By (S1), $u_\delta - \overline{C}/\lambda$ is a subsolution to the scheme (3.2). By comparison, Lemma 3.2, we have

$$u_\delta - u_h \leq \overline{C}/\lambda \quad \text{in} \quad \mathbb{R}^N.$$

4) Combining the above estimates yields

$$u - u_h = u - u_\delta + u_\delta - u_h \leq 2K\delta + \overline{C}/\lambda \quad \text{in} \quad \mathbb{R}^N.$$

Now we can conclude by choosing

$$\delta = \max_{i:K_i > 0}\{h^{k_i/i}\}.$$

$\square$

Remark 3.3. If we replace sup by inf in equations (3.1) and (3.3), a similar argument would lead to a lower bound of the error: $-Ch^\gamma \leq u - u_h$.

From this remark it is clear that we have the full result for *semi-linear* equations (see also [15] for the linear case):

Corollary 3.4. *Assume* (3.1) *is semi-linear, i.e. that $\mathcal{A}$ is a singleton. Let (A1) – (A3), (S1) – (S3) hold, let u be the viscosity solution of* (3.1)*, and let u_h be a solution of the scheme* (3.2)*. Then if $h > 0$ is sufficiently small,*

$$|u - u_h|_0 \leq Ch^\gamma,$$

where γ and C are defined in Theorem 3.3.

Following the ideas in [3, 19], we proceed to have obtain the full result in for more general situations. Let $\tilde{S}$ denote the scheme S when it is applied to equation $\tilde{F}[w] = 0$ where $\tilde{F}$ is defined just after (3.3), and consider

(3.5) $$\sup_{|e| \leq \delta} \tilde{S}(h, x + e, u_h^\delta(x), [u_h^\delta]_x) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

and the assumption analogous to (A3):

(S4) Assume u_h and u_h^δ are solutions of (3.2) and (3.5), and there is a constant $K' > 0$ independent of δ such that

$$|u_h^\delta|_1 + \frac{1}{\delta}|u_h - u_h^\delta|_0 \leq K'.$$

In addition we need the following assumptions of S:

(S5) (Convexity) For any $v \in C^{0,1}(\mathbb{R}^N)$, $h > 0$, and $x \in \mathbb{R}^N$

$$\int_{\mathbb{R}^N} S(h, x, v(x - e), [v(\cdot - e)]_x)\rho_\delta(e)\,de \geq S(h, x, (v * \rho_\delta)(x), [v * \rho_\delta]_x).$$

(S6) (Commutation with translations) For any $h > 0$ small enough, $0 \leq \delta \leq 1$, $y \in \mathbb{R}^N$, $t \in \mathbb{R}$, $v \in C_b(\mathbb{R}^N)$ and $|e| \leq \delta$, we have

$$S(h, y, t, [v]_{y-e}^h) = S(h, y, t, [v(\cdot - e)]_y^h).$$

Remark 3.4. While (S5) and (S6) are not very restrictive, (S4) is. This assumption is satisfied for control schemes in general [3] and for FDMs when the coefficients multiplying second order derivatives are constants (Section 5). Note that (S4) is not assumed in Corollary 3.4.

It is clear that by repeating the arguments in the proof of Theorem 3.3, with the schemes (3.2) and (3.5) taking the role of the equations (3.1) and (3.3), we obtain a lower bound on the error $-Ch^\gamma \leq u_h - u$. We refer to [3] for more details. Combining this result with Theorem 3.3 then yields the main result in this section.

Theorem 3.5. *Let (A1) – (A3), (S1) – (S6) hold, let u be the viscosity solution of* (3.1) *, and let u_h be a solution of the scheme* (3.2). *Then if $h > 0$ is sufficiently small,*

$$|u - u_h|_0 \leq Ch^\gamma,$$

where γ is defined in Theorem 3.3 and $C \leq \frac{K \vee K'}{\lambda}(\sum_{i=1}^n K_i + 2(2\lambda + \Lambda))$.

The results in this section generalize slightly the results in [19, 20, 3, 15] which consider pure convex or concave equations. Here we allow non-convexity (non-concavity) in the 0-th order terms.

4. Monotone Approximation Schemes - The Main Result.

In this section we will see how to use the results of the previous two sections to obtain error bounds for monotone schemes (1.3) for the non-convex problem (1.1). In Section 5 we give examples of such schemes. We assume that S in (1.3) satisfies assumptions (S1) – (S3) of Section 3.

First we consider the penalization problem corresponding to (1.3), namely problem (1.5) in the introduction. This scheme is also an approximation scheme for the penalization problem (1.4). Note that (1.3) and (1.5) themselves satisfy assumptions (S1) and (S2) (when S is appropriately redefined) and hence the comparison principle, Lemma 3.2, holds also for these schemes.

We start by obtaining the rate of convergence for $v_h^\varepsilon \to u_h$. It is not difficult to see that this result is a consequence of the procedure given in Section 2 if we can prove that assumptions corresponding to (C1) – (C3) hold for S. Because (S1) and (S2) imply comparison, they already imply assumptions corresponding to (C1)

and (C2). But it turns out that (S3) is not sufficient to have the assumption corresponding to (C3) because it involves derivatives of higher order than two. We need to assume "(C3)":

(S7) (Regularity) One of the following statements hold:
  (i) $g \in C^{0,1}(\mathbb{R}^N)$, $|D^2 g^-|_0 \leq C$, and for every $x \in \mathbb{R}^N$ and $\phi \in C^2(\mathbb{R}^N)$ satisfying $|\phi|_{0,1} + |D^2 \phi^-|_0 \leq R$,
$$S(h, x, \phi(x), \phi) \leq C_R.$$
  (ii) $g \in C^{0,1}(\mathbb{R}^N)$, and for every $x \in \mathbb{R}^N$ and $\phi \in C^2(\mathbb{R}^N)$ satisfying $|\phi|_{0,1} \leq R$,
$$S(h, x, \phi(x), \phi) \leq C_R(1 + |D^2 \phi^-|_0).$$
  (iii) $g \in C^{0,\mu}(\mathbb{R}^N)$ for some $\mu \in (0,1)$, and for every $x \in \mathbb{R}^N$ and $\phi \in C^2(\mathbb{R}^N)$ satisfying $|\phi|_0 \leq R$,
$$S(h, x, \phi(x), \phi) \leq C_R(1 + |D\phi|_0 + |D^2 \phi^-|_0).$$

Remark 4.1. This assumption holds for most reasonable schemes (1.3) when (S3) also holds, e.g. for the finite difference method (5.1) below.

By the method of Section 2, we have the following result:

Lemma 4.1. *Assume (S1), (S2), (S7) hold and u_h and $v_{h,\varepsilon}$ are solutions of (1.3) and (1.5). Then if $|D^2 g^-|_0 < \infty$ (case (i))*
$$0 \leq u_h - v_{h,\varepsilon} \leq C\varepsilon \quad in \quad \mathbb{R}^N,$$
Otherwise ($g \in C^{0,\mu}(\mathbb{R}^N)$ – cases (ii) and (iii))
$$0 \leq u_h - v_{h,\varepsilon} \leq C\varepsilon^{\mu/2} \quad in \quad \mathbb{R}^N,$$
where the constants C only depend on g and C_R from (C3).

Now to obtain results for the scheme (1.3), we may use the following diagram:

$$
\begin{array}{ccc}
\min\{F[u]; u - g\} = 0 & \xleftarrow{\quad\quad ? \quad\quad} & \min\{S[u_h]; u - g\} = 0 \\
\Big\uparrow \substack{0 \leq u - v_\varepsilon \leq C_1(\varepsilon) \\ \text{Theorem 2.1}} & & \Big\uparrow \substack{\text{Lemma 4.1} \\ 0 \leq u_h - v_{h,\varepsilon} \leq C_4(\varepsilon)} \\
F[v_\varepsilon] = \frac{1}{\varepsilon}(v_\varepsilon - g)^- & \xleftrightarrow[\substack{\text{Theorem 3.5} \\ C_2(h,\varepsilon) \leq v_\varepsilon - v_{h,\varepsilon} \leq C_3(h,\varepsilon)}]{} & S[v_{h,\varepsilon}] = \frac{1}{\varepsilon}(v_{h,\varepsilon} - g)^-
\end{array}
$$

The main result of this paper is the following:

Theorem 4.2. *Let (A1), (A2), (S1) – (S7) hold with $\lambda > \sup_\alpha([\sigma^\alpha]_1^2 + [b^\alpha]_1)$ in (A2) and K' independent of ε in (S4), let u be the viscosity solution of (1.1) with F defined in (1.2), and let u_h be a solution of the scheme (1.3). Then if $h > 0$ is sufficiently small,*
$$|u - u_h|_0 \leq Ch^{\gamma/3}.$$
If in addition $|D^2 g^-|_0 < \infty$, then
$$|u - u_h|_0 \leq Ch^{\gamma/2}.$$
Here γ is defined in Theorem 3.3 and the constants C are independent of h.

The assumption on λ may be relaxed to simply requiring $\lambda > 0$. This will influence the rates and complicate the arguments, see [3] for a discussion. See also Remark 3.2.

Outline of proof. 1) By Lemmas A.1 and A.2 in the Appendix (see Remark 3.2), assumption (A3) is satisfied for (1.4) with K independent of ε! Note that K' is assumed independent of ε.

2) By Theorem 3.5 with Λ replaced by $\Lambda + \frac{1}{\varepsilon}$,

$$|v_\varepsilon - v_{h,\varepsilon}|_0 \leq C(1 + \frac{1}{\varepsilon})h^\gamma \quad \text{in} \quad \mathbb{R}^N,$$

where $v_{h,\varepsilon}$ solves (1.5) and C is a constant independent of ε.

3) The result now follows from the triangle inequality, part 2), Theorem 2.1, Lemma 4.1, and a minimization in ε (see the above diagram). $\qquad\square$

If F is concave instead of convex so that the obstacle problem (1.1) is concave, then we obtain better rates using directly Theorem 3.5:

$$|u - u_h|_0 \leq Ch^\gamma.$$

This was essentially the case considered by [3, 15]. Theorem 4.2 is the first result for multi-dimensional non-concave/non-convex equation.

5. Applications

5.1. **A finite difference scheme.** In this section we apply a finite difference scheme proposed by Kushner [21] to the N-dimensional non-convex equation (1.1) where F is given by (3.1) and the coefficient a is independent of x.

We will assume that (A1) and (A2) of Section 3 and that the following assumptions hold:

(A4) a^α is independent of x,

(A5) $a_{ii}^\alpha - \sum_{j \neq i} |a_{ij}^\alpha| \geq 0, \quad i = 1, \ldots, N,$

(A6) $\sum_{i=1}^N \left\{ a_{ii}^\alpha - \sum_{j \neq i} |a_{ij}^\alpha| + |b_i^\alpha(x)| \right\} \leq 1 \quad \text{in } \mathbb{R}^N.$

(A7) $\sup_\alpha \left\{ \inf_x c^\alpha - 2\sqrt{N}[b^\alpha]_1 \right\} =: \lambda_0 > 0.$

(A8)   (i) $g \in C^{0,1}(\mathbb{R}^N)$ or (ii) $g \in C^{0,1}(\mathbb{R}^N)$ and $|D^2 g^-|_0 \leq C.$

Here we need (A4) in order to prove condition (S4) of Section 3, for more on this see [3]. To avoid (A4) we must use the much more difficult methods of [4] or [20]. We will not consider this here. Condition (A5) simply says that a is diagonally dominant. This is a standard condition [21] and implies that the scheme (B.1) below is monotone. Conditions (A6) is a normalization of the coefficients in (A.1). We can always have this assumption satisfied by multiplying equation (A.1) by an appropriate positive constant. Conditions (A7) and (A8) together with (A1) and (A2) assure that the solutions of the various schemes (e.g. (1.3)) belong to $C^{0,1}(\mathbb{R}^N)$. Under these assumptions the solutions of various equations (e.g. (1.1)) will also belong to $C^{0,1}(\mathbb{R}^N)$. We refer to the Appendix for the proof of these facts. Condition (A8) is a regularity condition on g, cf. (C3) and (S7).

The difference operators we use are defined in the following way

$$\Delta_{x_i}^{\pm} w(x) = \pm\frac{1}{h}\{w(x \pm e_i h) - w(x)\},$$

$$\Delta_{x_i}^2 w(x) = \frac{1}{h^2}\{w(x + e_i h) - 2w(x) + w(x - e_i h)\},$$

$$\Delta_{x_i x_j}^+ w(x) = \frac{1}{2h^2}\{2w(x) + w(x + e_i h + e_j h) + w(x - e_i h - e_j h)\}$$
$$- \frac{1}{2h^2}\{w(x + e_i h) + w(x - e_i h) + w(x + e_j h) + w(x - e_j h)\},$$

$$\Delta_{x_i x_j}^- w(x) = \frac{1}{2h^2}\{w(x + e_i h) + w(x - e_i h) + w(x + e_j h) + w(x - e_j h)\}$$
$$- \frac{1}{2h^2}\{2w(x) + w(x + e_i h - e_j h) + w(x - e_i h + e_j h)\}.$$

Let $b^+ = \max\{b, 0\}$ and $b^- = (-b)^+$. Note that $b = b^+ - b^-$. For each x, t, $p_i^\pm$, A_{ii}, $A_{ij}^\pm$, $i, j = 1, \ldots, N$, let

$$\tilde{F}(x, r, p_i^\pm, A_{ii}, A_{ij}^\pm)$$
$$= \min\left\{ \sup_{\alpha \in \mathcal{A}} \left\{ \sum_{i=1}^N \left[-\frac{a_{ii}^\alpha}{2} A_{ii} + \sum_{j \neq i} \left(-\frac{a_{ij}^{\alpha+}}{2} A_{ij}^+ + \frac{a_{ij}^{\alpha-}}{2} A_{ij}^- \right) \right.\right.\right.$$
$$\left.\left.\left. - b_i^{\alpha+}(x) p_i^+ + b_i^{\alpha-}(x) p_i^- \right] + f^\alpha(x, r) \right\}, r - g(x) \right\}.$$

Now we can write the finite difference scheme in the following way,

$$(5.1) \qquad \tilde{F}(x, u_h(x), \Delta_{x_i}^\pm u_h(x), \Delta_{x_i}^2 u_h(x), \Delta_{x_i x_j}^\pm u_h(x)) = 0.$$

This is a consistent and monotone scheme.

In order to get our result, we must define S in (1.3) and prove that conditions (A3), (S1) – (S7) of Sections 3 and 4 hold. We have moved most of the details to Appendix B where a more general problem is considered. To see how S may be defined, see (B.3) in Appendix B. Condition (S1) holds by monotonicity of the scheme, (S2) holds trivially, (S3) holds with following estimate:

$$|F(x, v, Dv, D^2 v) - S(h, x, v(x), [v]_x^h)| \leq \bar{K}(|D^2 v|_0 h + |D^4 v|_0 h^2),$$

for any $v \in C^4(\mathbb{R}^N)$. Condition (S5) holds by "convexity" of the sup-part of the scheme, (S6) holds trivially, and by (A8) we immediately get (S7). The only difficult condition is (S4). To prove it we need very precise a priori estimates on the scheme provided by Lemmas B.1 and B.2 in Appendix B. Note in particular that the bounds in (S4) are independent of the penalization parameter ε.

In view of Theorem 3.5 we have the following result:

Proposition 5.1. *Assume (A1),(A2), (A4) – (A8) of Sections 3 and 5 hold, u is the viscosity solution of (1.1) and u_h is the solution of (5.1). Then if $h > 0$ is sufficiently small,*

$$|u - u_h|_0 \leq Ch^{1/6} \quad in \quad \mathbb{R}^N.$$

If in addition $|D^2 g^-|_0 < \infty$, then

$$|u - u_h|_0 \leq Ch^{1/4} \quad in \quad \mathbb{R}^N.$$

In the convex case under similar assumptions the rate is $1/2$ when a is independent of x [15] and at least $1/5$ in the general case [4]. For one-dimensional non-convex problems the rate is at least $1/5$ [14], and for first order problems the rate is again $1/2$ [9].

5.2. **Control schemes.** In this section, we consider a so-called control schemes introduced in the second order case by Menaldi [24]. The scheme is defined in the following way,

$$(5.2) \qquad u_h(x) = \min_{\vartheta \in \Theta} \left\{ (1 - hc^\vartheta(x))\Pi_h^\vartheta u_h(x) + hf^\vartheta(x) \right\},$$

where Π_h^ϑ is the operator defined by

$$\Pi_h^\vartheta \phi(x) =$$
$$\frac{1}{2N} \sum_{m=1}^N \left(\phi(x + hb^\vartheta(x) + \sqrt{h}\sigma_m^\vartheta(x)) + \phi(x + hb^\vartheta(x) - \sqrt{h}\sigma_m^\vartheta(x)) \right),$$

and σ_m^ϑ is the m-th column of σ^ϑ. In the convex case a fully discrete method is derived from (5.2) and analyzed in [7]. The authors also provide an error bound for the convergence of the solution of the fully discrete method to the solution of the scheme (5.2).

In this case we only need to assume conditions (A1), (A2), (A7), and (A8), in particular a may depend on x. All condition (S1) – (S8) then holds, and the consistency condition (S4) takes the form

$$|F(x, v, Dv, D^2v) - S(h, x, v(x), [v]_x^h)| \le \bar{K}(|D^2v|_0 + |D^3v|_0 + |D^4v|_0)h,$$

for any $v \in C^4(\mathbb{R}^N)$. We refer to [3] for the proof of these conditions and the precise definition of S. The only difficult point is again (S4). To prove this condition one must modify the arguments of [3] in a similar way to what we did in the Appendix for the FDM. We omit the details.

In view of Theorem 3.5, we have the following result:

Proposition 5.2. *Assume (A1),(A2), (A7), and (A8) of Sections 3 and 5 hold, u is the viscosity solution of (1.1) and u_h is the solution of (5.2). Then if $h > 0$ is sufficiently small,*

$$|u - u_h|_0 \le Ch^{1/12} \quad in \quad \mathbb{R}^N.$$

If in addition $|D^2g^-|_0 < \infty$, then

$$|u - u_h|_0 \le Ch^{1/8} \quad in \quad \mathbb{R}^N.$$

In the convex case under similar assumptions the rate is at least $1/4$ [15], if the solution in addition has 3 bounded derivatives then the rate is at least $1/2$ [24]. For one-dimensional non-convex problems the rate is at least $1/10$ [14], and for first order problems the rate is again $1/2$, see e.g. [3].

APPENDIX A. ESTIMATES ON THE ISAACS EQUATION.

In this section we will give well posedness results and very precise a priori bounds for the Isaacs equation

$$(A.1) \qquad \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \left\{ -\mathrm{tr}\left[a^{\alpha,\beta}(x)D^2u\right] - b^{\alpha,\beta}(x)Du + c^{\alpha,\beta}(x)u - f^{\alpha,\beta}(x) \right\} = 0$$

in $\mathbb{R}^N$, where $a = \sigma\sigma^T$ for some matrix (function) σ. We take the following assumption:

(B1) $c > 0$ and there is a constant C independent of α, β such that

$$[\sigma^{\alpha,\beta}]_1 + [b^{\alpha,\beta}]_1 + [c^{\alpha,\beta}]_1 + |f^{\alpha,\beta}|_1 \le C.$$

We start by existence, uniqueness, and L^∞-bounds on the solution and its gradient.

Lemma A.1. *If (B1) holds and* $\sup_{\alpha,\beta}\left\{\inf_x c^{\alpha,\beta} - [\sigma^{\alpha,\beta}]_1^2 - [b^{\alpha,\beta}]_1\right\} > 0$, *then there exists a unique solution u of* (A.1) *satisfying the following bounds:*

$$|u|_0 \le \sup_{\alpha,\beta} \frac{|f^{\alpha,\beta}|_0}{\inf_x c^{\alpha,\beta}}, \quad |Du|_0 \le \sup_{\alpha,\beta} \frac{|u|_0[c^{\alpha,\beta}]_1 + [f^{\alpha,\beta}]_1}{\inf_x c^{\alpha,\beta} - [\sigma^{\alpha,\beta}]_1^2 - [b^{\alpha,\beta}]_1}.$$

Remark A.1. Usually the assumption on c is $c \ge \lambda > \sup_{\alpha,\beta}\left\{[\sigma^{\alpha,\beta}]_1^2 + [b^{\alpha,\beta}]_1\right\}$ and all estimates are given in terms of λ instead of c. For our purpose this is not good enough, since we need to consider limit problems where for some values of α, β, both $|f|$ and $|c|$ blow up, while for others they both remain bounded (cf. the penalization method).

Proof. Existence and uniqueness follows from the (strong) comparison principle and Perron's method [13]. Let

$$M := \sup_{\alpha,\beta} \frac{|f^{\alpha,\beta}|_0}{\inf_x c^{\alpha,\beta}},$$

then the first bound on u follows from the comparison principle after checking that M $(-M)$ is a supersolution (subsolution) of (A.1). To get the bound on the gradient of u, consider

$$m := \sup_{x,y \in \mathbb{R}^N} \left\{u(x) - u(y) - L|x - y|\right\}.$$

If by setting

$$L := \sup_{\alpha,\beta} \frac{|u|_0[c^{\alpha,\beta}]_1 + [f^{\alpha,\beta}]_1}{\inf_x c^{\alpha,\beta} - [\sigma^{\alpha,\beta}]_1^2 - [b^{\alpha,\beta}]_1},$$

we can conclude that $m \le 0$, then we are done. Assume for simplicity that the maximum is attained in $(\bar{x}, \bar{y})$. If $\bar{x} = \bar{y}$ then $m = 0$ and we are done. If not, then $L|x - y|$ is smooth at $(\bar{x}, \bar{y})$ and a standard doubling of variables argument leads to $m \le 0$. Since the maximum need not be attained, we must modify the test function in the standard way. We skip the details. (The interested reader can have a look at the appendix of [12] where the above argument is given for a linear equation.) $\quad\square$

Now we proceed to obtain continuous dependence on the coefficients. Let $\bar{u}$ solve the following equation:

$$(A.2) \qquad \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \left\{-\text{tr}\left[\bar{a}^{\alpha,\beta}(x)D^2\bar{u}\right] - \bar{b}^{\alpha,\beta}(x)D\bar{u} + \bar{c}^{\alpha,\beta}(x)\bar{u} - \bar{f}^{\alpha,\beta}(x)\right\} = 0$$

in $\mathbb{R}^N$, where $\bar{a} = \bar{\sigma}\bar{\sigma}^T$ for some matrix (function) $\bar{\sigma}$.

Lemma A.2. *If u and $\bar{u}$ are bounded Lipschitz continuous solutions of* (A.1) *and* (A.2) *respectively, and that both sets of coefficients satisfy (B1). Then*

$$|u - \bar{u}|_0 \le \sup_{\alpha,\beta} \frac{K}{\inf_x c \vee \inf_x \bar{c}}|\sigma - \bar{\sigma}|_0$$

$$+ \sup_{\alpha,\beta} \frac{1}{\inf_x c \vee \inf_x \bar{c}} \left\{ 2L|b - \bar{b}|_0 + M|c - \bar{c}|_0 + |f - \bar{f}|_0 \right\},$$

where $L = [u]_1 \vee [\bar{u}]_1$, $M = |u|_0 \vee |\bar{u}|_0$, *and*

$$K^2 = 32L \sup_{\alpha,\beta} \left\{ 4L[\sigma]_1^2 \wedge [\bar{\sigma}]_1^2 + 2L[b]_1 \wedge [\bar{b}]_1 + M[c]_1 \vee [\bar{c}]_1 + [f]_1 \wedge [\bar{f}]_1 \right\}.$$

Outline of proof. Define

$$m := \sup_{x,y} \left\{ u(x) - \bar{u}(y) - \frac{1}{\delta}|x - y|^2 - \varepsilon(|x|^2 + |y|^2) \right\}.$$

Then do doubling of variables using the 3 last terms in the above expression as test-function. Using the definition of viscosity solutions and subtracting the resulting inequalities lead to

$$0 \leq \sup_{\alpha,\beta} \Big\{ -\operatorname{tr}[\bar{a}(y)Y] + \operatorname{tr}[a(x)X] - \bar{b}(y)p_x + b(x)p_y$$

$$+ \bar{c}(y)\bar{u}(y) - c(x)u(x) - \bar{f}(y) + f(x) \Big\},$$

where x, y is the maximum point for m and (p_x, X), $(-p_y, Y)$ are the elements in second order semi-jets in for u, $\bar{u}$ given by the maximum principle for semi-continuous functions [8]. Now we note that by using Lipschitz regularity of the solutions, a standard argument yields

$$|x - y| \leq \delta L.$$

So using Ishii's trick [13, pp. 33,34] on the 2nd order terms, and a few other manipulations, we get

$$0 \leq \sup_{\alpha,\beta} \Big\{ \frac{4}{\delta}|\sigma(x) - \bar{\sigma}(y)|^2 + 2L|b(x) - \bar{b}(y)| + C\varepsilon(1 + |x|^2 + |y|^2)$$

$$+ M|c(x) - \bar{c}(y)| - (\inf_x c \vee \inf_x \bar{c})m + |f(x) - \bar{f}(y)| \Big\}.$$

Some easy manipulations now lead to an estimate for m, and using the definition of m, we obtain an estimate for $|u - \bar{u}|_0$ depending on δ and ε. We finish the proof by minimizing this expression w.r.t. δ and sending $\varepsilon \to 0$. $\square$

For more details on such manipulations, we refer to [16, 17].

APPENDIX B. ESTIMATES ON A FINITE DIFFERENCE SCHEME

In this section we apply a finite difference scheme proposed by Kushner [21] to the N-dimensional Isaacs equation (A.1) with coefficient a independent of x.

We will assume that (B1) of Section A and the following assumptions hold:

(B2) a is independent of x.

(B3) $a_{ii}^{\alpha,\beta} - \sum_{j \neq i} |a_{ij}^{\alpha,\beta}| \geq 0, \quad i = 1, \ldots, N,$

(B4) $\sum_{i=1}^{N} \left\{ a_{ii}^{\alpha,\beta} - \sum_{j \neq i} |a_{ij}^{\alpha,\beta}| + |b_i^{\alpha,\beta}(x)| \right\} \leq 1 \quad$ in $\mathbb{R}^N$.

Let us define the scheme. For each x, r, $p_i^{\pm}$, A_{ii}, $A_{ij}^{\pm}$, $i, j = 1, \ldots, N$, let

$$\tilde{F}(x, r, p_i^{\pm}, A_{ii}, A_{ij}^{\pm})$$

$$= \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \Big\{ \sum_{i=1}^{N} \Big[-\frac{a_{ii}^{\alpha,\beta}}{2} A_{ii} + \sum_{j \neq i} \Big(-\frac{a_{ij}^{\alpha,\beta+}}{2} A_{ij}^+ + \frac{a_{ij}^{\alpha,\beta-}}{2} A_{ij}^- \Big)$$

$$- b_i^{\alpha,\beta+}(x)p_i^+ + b_i^{\alpha,\beta-}(x)p_i^- \Big] + c^{\alpha,\beta}(x)r - f^{\alpha,\beta}(x) \Big\}.$$

Using the difference operators $\Delta_{x_i}^{\pm}$, $\Delta_{x_i}^2$, $\Delta_{x_i x_j}^{\pm}$ defined in Section 5 we can now write the finite difference scheme in the following way,

(B.1) $$\tilde{F}(x, u_h(x), \Delta_{x_i}^{\pm} u_h(x), \Delta_{x_i}^2 u_h(x), \Delta_{x_i x_j}^{\pm} u_h(x)) = 0.$$

This is a consistent and monotone scheme. In the following it will be convenient to use an equivalent formulation of this scheme (see [3] for more details):

(B.2) $$u_h(x) = \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \Big\{ \frac{1}{1 + h^2 c^{\alpha,\beta}(x)}$$

$$\cdot \Big(\sum_{z \in h\mathbb{Z}^N} p^{\alpha,\beta}(x, x+z) u_h(x+z) + h^2 f^{\alpha,\beta}(x) \Big) \Big\},$$

where

$$p^{\alpha,\beta}(x,x) = 1 - \sum_{i=1}^{N} \Big\{ a_{ii}^{\alpha,\beta} - \sum_{j \neq i} \frac{|a_{ij}^{\alpha,\beta}|}{2} + h|b_i^{\alpha,\beta}(x)| \Big\},$$

$$p^{\alpha,\beta}(x, x \pm e_i h) = \frac{a_{ii}^{\alpha,\beta}}{2} - \sum_{j \neq i} \frac{|a_{ij}^{\alpha,\beta}|}{2} + hb_i^{\alpha,\beta\pm}(x),$$

$$p^{\alpha,\beta}(x, x + e_i h \pm e_j h) = \frac{a_{ij}^{\alpha,\beta\pm}}{2},$$

$$p^{\alpha,\beta}(x, x - e_i h \pm e_j h) = \frac{a_{ij}^{\alpha,\beta\mp}}{2},$$

and $p^{\alpha,\beta}(x,y) = 0$ for all other y. Let $h \leq 1$. Note that by (B3) and (B4), $0 \leq p^{\alpha,\beta}(x,y) \leq 1$ for all α, β, x, y. Furthermore $\sum_{z \in h\mathbb{Z}^N} p^{\alpha,\beta}(x, x+z) = 1$ for all α, β, x.

For the readers' convenience we will state explicitly the function S (as in (1.3) and (3.2)) corresponding to this scheme: We set $[\phi]_x^h(\cdot) := \phi(x + \cdot)$ and

(B.3) $$S(h, y, r, [\phi]_x^h) :=$$

$$\inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \Big\{ -\frac{1}{h^2} \Big[\sum_{z \in h\mathbb{Z}^N} p^{\alpha,\beta}(y, y+z)[\phi]_x^h(z) - t \Big] + c^{\alpha,\beta}(x)r - f^{\alpha,\beta}(y) \Big\}.$$

We use fix point arguments to prove existence, uniqueness, and a priori bounds for equation (B.1).

Lemma B.1. *Assume (B1) – (B4) hold and*

$$\sup_{\alpha,\beta} \Big\{ \inf_x c^{\alpha,\beta} - 2\sqrt{N}[b^{\alpha,\beta}]_1 \Big\} =: \lambda_0 > 0.$$

Then there exists a unique solution $u_h \in C^{0,1}(\mathbb{R}^N)$ of the scheme (B.1) satisfying the following bounds

$$|u_h|_0 \leq \sup_{\alpha,\beta} \frac{|f^{\alpha,\beta}|_0}{\inf_x c^{\alpha,\beta}}, \quad [u_h]_1 \leq \sup_{\alpha,\beta} \frac{\frac{|u_h|_0 + h^2 |f^{\alpha,\beta}|_0}{1 + h^2 \inf_x c^{\alpha,\beta}}[c^{\alpha,\beta}]_1 + [f^{\alpha,\beta}]_1}{\inf_x c^{\alpha,\beta} - 2\sqrt{N}[b^{\alpha,\beta}]_1}.$$

Proof. Define $T_h : C_b(\mathbb{R}^N) \to C_b(\mathbb{R}^N)$ in the following way:

$$T_h v(x) := \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \left\{ \frac{1}{1 + h^2 c^{\alpha,\beta}(x)} \right.$$
$$\left. \cdot \left(\sum_{z \in h\mathbb{Z}^N} p^{\alpha,\beta}(x, x+z)v(x+z) + h^2 f^{\alpha,\beta}(x) \right) \right\}.$$

For $u, v \in C_b(\mathbb{R}^N)$, we subtract the expressions for $T_h u$ and $T_h v$. After we use the inequality $\inf \sup(\cdots) - \inf \sup(\cdots) \le \sup \sup(\cdots - \cdots)$, the properties of $p^{\alpha,\beta}$, and (B1), we obtain

$$T_h u(x) - T_h v(x)$$
$$\le \sup_{\alpha,\beta} \left\{ \frac{1}{1 + h^2 \inf_x c^{\alpha,\beta}} \sum_{z \in h\mathbb{Z}^N} p^{\alpha,\beta}(x, x+z)|u(x+z) - v(x+z)| \right\}$$
$$\le \frac{1}{1 + \lambda_0 h^2} |u - v|_0.$$

Since we may reverse the roles of u and v, we see that T_h is a contraction in $(C_b(\mathbb{R}^N), |\cdot|_0)$. Banach's fixed point theorem then yields the existence and uniqueness of a $u_h \in C_b(\mathbb{R}^N)$ solving (B.2) (and (B.1)). The estimate on $|u_h|_0$ follows easily from the identity $|u_h|_0 = |T_h u_h|_0$.

We proceed by proving that u_h has a bounded Lipschitz constant assuming for simplicity that $c^{\alpha,\beta}$ is independent of x. Let $v \in C^{0,1}(\mathbb{R}^N)$ and subtract the expressions for $T_h v(x)$ and $T_h v(y)$:

$$T_h v(x) - T_h v(y) \le$$
$$\sup_{\alpha,\beta} \left\{ \frac{1}{1 + h^2 c^{\alpha,\beta}} \left(\sum_{z \in h\mathbb{Z}^N} \left[p^{\alpha,\beta}(x, x+z)(v(x+z) - v(y+z)) \right. \right. \right.$$
$$\left. \left. \left. + v(y+z)\left(p^{\alpha,\beta}(x, x+z) - p^{\alpha,\beta}(y, y+z) \right) \right] + h^2 (f^{\alpha,\beta}(x) - f^{\alpha,\beta}(y)) \right) \right\}.$$

In the right-hand side the first sum is bounded by $[v]_1 |x - y|$, and by using the definition of $p^{\alpha,\beta}$, the second sum is equivalent to

$$h \sum_{i=1}^N \left[(b_i^{\alpha,\beta+}(x) - b_i^{\alpha,\beta+}(y))\Delta_{x_i}^+ v(y) - (b_i^{\alpha,\beta-}(x) - b_i^{\alpha,\beta-}(y))\Delta_{x_i}^- v(y) \right]$$
$$\le 2\sqrt{N} h^2 |b^{\alpha,\beta}(x) - b^{\alpha,\beta}(y)|[v]_1 = 2\sqrt{N} h^2 [b^{\alpha,\beta}]_1 [v]_1 |x - y|.$$

Let $C^{\alpha,\beta} := 2\sqrt{N} h^2 [b^{\alpha,\beta}]_1$. By the above expressions, and by exchanging the roles of x and y, we obtain the following estimate

(B.4) $|T_h v(x) - T_h v(y)| \le$
$$\sup_{\alpha,\beta} \left\{ \frac{1}{1 + h^2 c^{\alpha,\beta}} \left((1 + h^2 C^{\alpha,\beta})[v]_1 + h^2 [f^{\alpha,\beta}]_1 \right) \right\} |x - y|.$$

Hence $T_h v \in C^{0,1}(\mathbb{R}^N)$, and $u_h \in C^{0,1}(\mathbb{R}^N)$ since $u_h = \lim_{i\to\infty} (T_h)^i v_0$ for any $v_0 \in C^{0,1}(\mathbb{R}^N)$. Furthermore, since $c^{\alpha,\beta} \ge C^{\alpha,\beta} + \lambda_0$ the estimate on $[u_h]_1$ follows easily from the identity $[u_h]_1 = [T_h u_h]_1$ and (B.4).

When c depend also on x we obtain an expression like (B.4) with $c^{\alpha,\beta}$ and $\sup_{\alpha,\beta}[f^{\alpha,\beta}]_1$ replaced by $\inf_x c^{\alpha,\beta}$ and $\sup_{\alpha,\beta}\left([f^{\alpha,\beta}]_1 + [c^{\alpha,\beta}]_1\frac{|v|_0+h^2|f^{\alpha,\beta}|_0}{1+h^2\inf_x c^{\alpha,\beta}}\right)$ respectively, and hence the lemma would hold again. $\qquad\square$

Using a standard maximum principle type of argument, we now derive a priori estimates on the continuous dependence on the data.

Lemma B.2. *Assume (B1) – (B4) hold and $u_h, \bar{u}_h \in C^{0,1}(\mathbb{R}^N)$. If u_h solve (B.1) with data (a, b, c, f) and $\bar{u}_h$ solve (B.1) with data $(a, \bar{b}, \bar{c}, \bar{f})$ (same a!), then*

$$|u_h - \bar{u}_h|_0 \leq \sup_{\alpha,\beta}\frac{1}{\inf_x c \vee \inf_x \bar{c}}\Big\{2L|b - \bar{b}|_0 + M|c - \bar{c}|_0 + |f - \bar{f}|_0\Big\},$$

where $L = \sqrt{N}[u_h]_1 \vee [\bar{u}_h]_1,\ M = |u_h|_0 \vee |\bar{u}_h|_0$.

Proof. We will assume that $\sup(u - \bar{u}) = (u - \bar{u})(x) \geq 0$. The general case follows from standard modifications to the proof below.

Since the scheme (B.1) is monotone, at the maximum point x we have

$$\sum_{i=1}^N\Big[-\frac{a_{ii}^{\alpha,\beta}}{2}\Delta_{x_i}^2 + \sum_{j\neq i}\Big(-\frac{a_{ij}^{\alpha,\beta+}}{2}\Delta_{x_ix_j}^+ + \frac{a_{ij}^{\alpha,\beta-}}{2}\Delta_{x_ix_j}^-\Big)\Big](u_h - \bar{u}_h)(x) \leq 0,$$

and

$$\sum_{i=1}^N\Big[b_i^{\alpha,\beta+}(x)\Delta_{x_i}^+ + b_i^{\alpha,\beta-}(x)\Delta_{x_i}^-\Big](u_h - \bar{u}_h)(x) \leq 0.$$

At the point x, we subtract the equations for u_h and $\bar{u}_h$. After some rearranging using monotonicity of the scheme (the above two inequalities) we get

$$0 \leq \sup_{\alpha\in\mathcal{A},\beta\in\mathcal{B}}\Big\{\sum_{i=1}^N\Big[(b_i^+ - \bar{b}_i^+)(x)\Delta_{x_i}^+ + (b_i^- - \bar{b}_i^-)(x)\Delta_{x_i}^-\Big]\bar{u}_h(x)$$

$$- c(x)(u_h - \bar{u}_h)(x) - \bar{u}_h(x)(c - \bar{c})(x) + (f - \bar{f})(x)\Big\}.$$

This (almost) immediately gives the upper bound on $u_h - \bar{u}_h$. Reversing the roles of u_h and $\bar{u}_h$ gives the lower bound and the proof is complete. $\qquad\square$

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