ON THE OPTIMALITY OF VELOCITY AVERAGING LEMMAS

CAMILLO DE LELLIS, MICHAEL WESTDICKENBERG

Abstract. Studying weak solutions of Burgers' equation with finite entropy dissipation we show the sharpness of recent results of Jabin and Perthame on velocity averaging. Similar arguments give bounds on the regularity of asymptotic finite–energy states for some variational problems of Ginzburg–Landau type.

1. Introduction

It is well–known that entropy solutions of Burgers' equation

$$\partial_t u + \tfrac{1}{2}\partial_x u^2 = 0 \quad \text{on } \Omega := (0,1)\times\mathbf{R} \tag{1}$$

are locally in BV, see [8]. This is no longer true for a general weak solution, which can have wild oscillations. Thus an improved regularity can only be expected if additional assumptions are imposed. We will consider here the following class of functions

Definition 1.1. *Let $\mathcal{S}$ be the set of functions $u \in L^\infty(\Omega)$ with*

- *u is a weak solution of (1),*
- *the entropy dissipation $-\mu_\eta := \partial_t\eta(u) + \partial_x q(u)$ is a Radon measure for every convex entropy–entropy flux pair (η, q).*

We stress that it is quite natural to consider this class of solutions. First, $\mathcal{S}$ contains all entropy solutions. For these, μ_q is non–negative. Second, for $u \in \mathcal{S}$ the conservation law allows a *kinetic formulation*, see [5]. Using this remark we will show the sharpness of some recent results in velocity averaging lemmas contained in [9].

To explain this point we recall the kinetic formulation for scalar conservation laws, first introduced in [7]. Assume $u \in L^\infty(\mathbf{R}^+\times\mathbf{R}^n)$ is an entropy solution of some scalar conservation law $\partial_t u + \operatorname{div}_x A(u) = 0$ in several space dimensions. One introduces the local Maxwellian

$$\chi(v,u) = \left\{ \begin{array}{rl} +1 & \text{for } 0 < v \le u \\ -1 & \text{for } u \le v < 0 \\ 0 & \text{otherwise} \end{array} \right\}.$$

Then there exists a non–negative Radon measure m in $(t,x) \in \mathbf{R}^+\times\mathbf{R}^n$ and $v \in \mathbf{R}$ such that $\chi(v, u(t,x))$ satisfies the transport equation

$$\partial_t\chi(v,u) + A'(v)\cdot\nabla_x\chi(v,u) = \partial_v m \quad \text{in } \mathcal{D}'. \tag{2}$$

Vice versa, if there exist a bounded function u and a non–negative measure m such that (2) holds, then u is the unique entropy solution of the conservation law. For entropy solutions of Burgers' equation we obtain the transport equation

$$\partial_t\chi(v,u) + v\,\partial_x\chi(v,u) = \partial_v m \quad \text{in } \mathcal{D}'. \tag{3}$$

One can show that also for functions u as in Definition 1.1, $\chi(v,u)$ satisfies a transport equation (2), see [5]. In that case, however, the measure m changes sign in general.

Since $u = \int \chi(v,u)\,dv$, velocity averaging lemmas can be applied to (2). Under certain assumptions on the non–degeneracy of the flux A, they yield regularity results for u, see [7, 9]. We stress the fact that the velocity averaging techniques known so far do not use the non–negativity of m: they give exactly the same regularity also if m in (2) is just a Radon measure. For $u \in \mathcal{S}$, the results in [9] yield

$$u \in W^{\alpha,3/2}_{loc} \quad \text{for all } \alpha < \tfrac{1}{3}.$$

We prove here that this is essentially sharp.

Let us fix some notation.

For $\alpha \in \mathbf{R}$ and $1 \le p, q \le \infty$ let $B^\alpha_{p,q}$ be the usual Besov space (see [10] Section 10f).

Definition 1.2. *Let $\sigma \ge 1$ and consider the 1–parameter family of spaces*

$$\left\{ B^{1/\sigma}_{r,r} \quad \textit{for } 1 \le r < \sigma, \qquad B^{1/r}_{r,\infty} \quad \textit{for } r \ge \sigma \right\}. \tag{4}$$

We say that a set of functions $\mathcal{S} \subset L^\infty(\Omega)$ is not better than (4) locally if

$$\textit{for every triple } (\alpha,p,q) \textit{ with } \left\{ \begin{array}{llll} & \alpha > 1/\max\{\sigma,p\} & & \\ \textit{or} & \alpha = 1/\sigma & p < \sigma & q < p \\ \textit{or} & \alpha = 1/p & p \ge \sigma & q < \infty \end{array} \right\}$$

there exist $u \in \mathcal{S}$ and a test function $\psi \in \mathcal{D}(\Omega)$ such that $\psi u \notin B^\alpha_{p,q}$.

We can prove the following result

Proposition 1.3. *Let $\mathcal{S}$ be the set of Definition 1.1. Then we have*

$\mathcal{S}$ is not better than (4) locally with $\sigma = 3$.

Thanks to the embedding $W^{\alpha,p} \subset B^\alpha_{p,q}$ for $q \ge 2$ (see [10] Section 11.4) we conclude that $u \notin W^{\alpha,p}_{loc}$ if $\alpha > 1/3$, or if $\alpha \le 1/3$ and $p \ge 1/\alpha$. Thus the results in [9] are essentially sharp in the number of derivatives, whereas the integrability could still be improved.

We prove Proposition 1.3 by constructing weak solutions of (1) with low regularity. The argument can be generalized to more general fluxes. It is also possible to derive upper bounds for the regularity of entropy solutions. We refer to Section 3 for further discussion.

We mention that the starting point of our investigation was a class of variational problems of Ginzburg–Landau type, see [2, 6]. Here one considers a family of functionals F_ε defined on curl free planar vector fields (see (15)), and tries to characterize the possible limits w of sequences $\{w_\varepsilon\}_\varepsilon$ with $\limsup_\varepsilon F_\varepsilon(w_\varepsilon) < \infty$. Combining an example of [1] with a construction of [4] one can show that there exist limits which do not belong to BV. On the other hand, the results in [9] can again be applied and give $w \in W^{\alpha,3/2}$ for all $\alpha < 1/3$. Using estimates similar to those in Section 2 we can show that generically w is not better than (4) with $\sigma = 3$, see Proposition 3.3.

2. Proof of Proposition 1.3

We proceed in three steps.

Step 1 A family of weak solutions of (1).

Assume that $\{\Delta_k\}_k \in \ell^1$ and $\{c_k\}_k \in \ell^3$ are two nonincreasing sequences of positive numbers which will be determined later on. Let

$$x_1^- := \sum_{k=1}^\infty \Delta_k.$$

Inductively, we now define numbers

$$x_{k+1}^- := x_k^- - \Delta_k \quad \text{and} \quad x_k^+ := x_k^- - \Delta_k/2,$$

and intervals $I_k := [x_k^+, x_k^-]$ for $k \in \mathbf{N}$. We prescribe initial data

$$u(0,\cdot) = \sum_{k=1}^\infty c_k \, \mathbf{1}_{I_k} \tag{5}$$

and construct a weak solution u of (1) with (5), consisting of shocks only. For a jump connecting 0 and c_k, the Rankine-Hugoniot condition gives a shock speed $s_k := \frac{1}{2}c_k$. Define vectors $X_k := (1, s_k)$. Then

$$u = \sum_{k=1}^\infty c_k \, \mathbf{1}_{A_k} \quad \text{where } A_k := \big(\{0\} \times I_k\big) + \mathbf{R}^+ X_k.$$

This u is a weak solution of (1), but it is entropy violating. We compute $\partial_t\chi(v,u)+v\partial_x\chi(v,u)$ and show that it is the v–derivative of a Radon measure $m \in \mathcal{M}_{loc}(\mathbf{R}_v \times \mathbf{R}_t^+ \times \mathbf{R}_x)$. We restrict our attention on the set $\Omega := [0,1]_t \times \mathbf{R}_x$. Define

$$J_k^\pm := \big(\{0\} \times x_k^\pm\big) + [0,1]X_k \quad \text{for } j \in \mathbf{N}.$$

The unit normal to J_k is given by $(1+s_k^2)^{-1/2}(s_k, -1)$. Then

$$\partial_t\chi(v,u) + v\partial_x\chi(v,u) = -\sum_{k=1}^\infty h_k(v)\big(\mathcal{H}^1 \llcorner J_k^+ - \mathcal{H}^1 \llcorner J_k^-\big) \qquad \text{in } \mathcal{D}'(\Omega) \tag{6}$$

where the function h_k is defined as

$$h_k(v) := (1,v) \cdot \frac{(s_k,-1)}{\sqrt{1+s_k^2}} = \frac{\frac{1}{2}c_k - v}{\sqrt{1+c_k^2/4}} \quad \text{for } v \in (0, c_k]$$

and $h_k(v) = 0$ otherwise. Then

$$m = -\sum_{k=1}^\infty H_k(v)\big(\mathcal{H}^1 \llcorner J_k^+ - \mathcal{H}^1 \llcorner J_k^-\big),$$

where H_k is a v–primitive of (6). For example we can choose

$$H_k(v) := \int_0^v h_k(s)\,ds = \left\{ \begin{array}{cl} \frac{1}{2}\frac{c_k v - v^2}{\sqrt{1+c_k^2/4}} & \text{for } v \in [0,c_k] \\ 0 & \text{otherwise} \end{array} \right\}.$$

The function H_k is non–negative. Its $L^1(\mathbf{R})$–norm equals

$$\int_{\mathbf{R}} H_k(v)\,dv = \frac{1}{12}\frac{c_k^3}{\sqrt{1+c_k^2/4}}, \tag{7}$$

i.e. is essentially cubic in the shock strength. Then we have

$$\|m\|_{\mathcal{M}(\mathbf{R}\times\Omega)} = \sum_{k=1}^{\infty} 2\cdot\sqrt{1+c_k^2/4}\cdot\frac{1}{12}\frac{c_k^3}{\sqrt{1+c_k^2/4}} = \frac{1}{6}\sum_{k=1}^{\infty} c_k^3. \tag{8}$$

Since $\{c_k\}_k \in \ell^3$ by assumption, m is a finite measure.

Step 2 Besov norms and main estimate.

We will work with Besov spaces $B^{\alpha}_{p,q}$ defined on $\mathbf{R}^2$. Several equivalent norms are known for these spaces. A particularly useful one is the "local means" norm. Let $\Phi, \varphi \in \mathcal{D}$ be mollifiers and assume that φ satisfies a certain moment condition (we refer to [10] Section 11 for details). Define a scaling $\varphi_s(t,x) := s^{-2}\varphi(t/s, x/s)$ for $(t,x)\in\mathbf{R}^2$ and $s>0$. Then

$$\|u\|_{B^{\alpha}_{p,q}} \approx \|u\star\Phi\|_{L^p} + \left(\int_0^{\varepsilon} s^{-\alpha q}\|u\star\varphi_s\|_{L^p}^q \frac{ds}{s}\right)^{1/q} \quad \text{if } q<\infty \tag{9}$$

for some $\varepsilon>0$. If $q=\infty$ the integral in s must be replaced by a sup. We also use that

$$\|u\|_{B^{\alpha}_{p,q}} \approx \|u\|_{B^{\alpha-1}_{p,q}} + \|\partial_t u\|_{B^{\alpha-1}_{p,q}} + \|\partial_x u\|_{B^{\alpha-1}_{p,q}},$$

and will concentrate on the $\partial_x u$–part with $\alpha<1$. For negative regularity φ may be chosen to be a standard mollifier (satisfying no moment conditions). Moreover, the first term in (9) can be dropped, see [11] Remark 6. Thus we only have to consider

$$\|\partial_x u\|_{B^{\alpha-1}_{p,q}} \approx \left(\int_0^{\varepsilon} s^{(1-\alpha)q}\big\|(\partial_x u)\star\varphi_s\big\|_{L^p}^q \frac{ds}{s}\right)^{1/q} \quad \text{if } q<\infty$$

with the obvious modification for $q=\infty$. We remark in passing that

$$\|Du\|_{\mathcal{M}} \approx \sup_{0<s<\varepsilon}\|Du\star\varphi_s\|_{L^1} \quad \text{for all } u\in BV. \tag{10}$$

This follows easily from the lower semi–continuity of the variation. Hence the space BV fits very neatly into the Besov space framework. (In fact, in his first work on Caccioppoli sets [3] De Giorgi defined the space BV as the set of all $u\in L^\infty$ such that (10) stays finite, where φ_s is the heat kernel.)

To fix ideas, we let φ be a non–negative, radially symmetric test function with

$$\varphi(t,x) = \left\{\begin{array}{ll} 1 & \text{for } (t,x)\in B_1(0) \\ 0 & \text{for } (t,x)\in\mathbf{R}^2\setminus B_2(0)\end{array}\right\}.$$

We now fix some test function $\psi\in\mathcal{D}(\Omega)$ which equals 1 on a sufficiently large open set, say $[1/4,3/4]\times[-1,3]$. Note that $\partial_x(\psi u) = u\partial_x\psi + \psi\partial_x u$. Since $u\partial_x\psi\in L^\infty$, clearly $u\partial_x\psi\in B^{\alpha-1}_{p,q}$ for any $\alpha<1$. Thus to show that $\partial_x(\psi u)\notin B^{\alpha-1}_{p,q}$ for some $\alpha<1$, it is sufficient to show that $\psi\partial_x u\notin B^{\alpha-1}_{p,1}$.

We proceed as follows: The partial derivative of u is a measure

$$\partial_x u = \sum_{k=1}^{\infty}\frac{c_k}{\sqrt{1+c_k^2/4}}\big(\mathcal{H}^1\llcorner J_k^+ - \mathcal{H}^1\llcorner J_k^-\big) =: \sum_{k=1}^{\infty}\big(\mu_k^+ - \mu_k^-\big).$$

By construction, the distance between $J_k^{\pm}$ and its neighbors is not less than $\Delta_k/(2\sqrt{1+s_k^2})$. We choose $c_k\le 2$ so to ensure that this distance is bigger than $\Delta_k/4$. Hence, for all k with

$4s < \Delta_k/4$, only one line $J_k^\pm$ contributes to the convolution $(\partial_x u) \star \varphi_s$ (recall that φ_s is supported in a ball of radius $2s$). Thus

$$\big|(\partial_x u) \star \varphi_s\big|(t,x) \;\; \geq \sum_{4s<\Delta_k/4} |\mu_k^\pm \star \varphi_s|(t,x) \tag{11}$$

where the functions appearing in the sum have pairwise disjoint supports. Since φ_s is supported in a ball of radius $2s$ and ψ is 1 on $[1/4, 3/4] \times [-1, 3]$ we have

$$\big|(\partial_x u) \star \varphi_s\big|(t,x) = \big|(\psi \partial_x u) \star \varphi_s\big|(t,x) \qquad \text{for } t \in [1/4+2s, 3/4-2s]\,.$$

Moreover, since φ is non–negative, each summand of (11) can be estimated from below

$$\big|\mu_k^\pm \star \varphi_s\big|(t,x) \geq \left\{ \begin{array}{cc} 2c_k s^{-2} \frac{\sqrt{s^2-y^2}}{\sqrt{1+c_k^2/4}} & \text{for } y \in [-s,s] \\ 0 & \text{otherwise} \end{array} \right\}$$

$$\text{where } y := \frac{x-(x_k^\pm + t\, c_k/2)}{\sqrt{1+c_k^2/4}} \text{ for } x \in \mathbf{R} \text{ and } t \in [s, 1-s],$$

see Figure 1. These remarks allow us to give the following estimate:

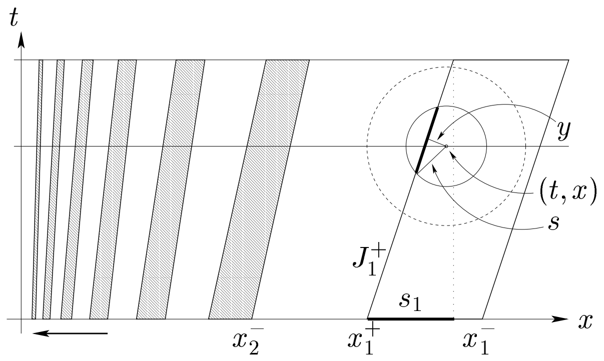

FIGURE 1. Local means $(\partial_x u) \star \varphi_s$.

$$\int_0^1 \int_0^{2x_1^-} \big|(\psi\partial_x u) \star \varphi_s\big|^p(t,x)\,dx\,dt \;\; \geq \sum_{4s<\Delta_k/4} 2^p c_k^p \left(1+\frac{c_k^2}{4}\right)^{\frac{1-p}{2}} s^{1-p} \int_{\frac14+2s}^{\frac34-2s} \int_{-1}^1 (1-z^2)^{\frac{p}{2}}\,dz\,dt$$

$$\geq \; c\, s^{1-p} \sum_{4s<\Delta_k/4} c_k^p \quad \text{for some constant } c = c(p,\varepsilon).$$

Then we have

$$\big\|(\psi\partial_x u) \star \varphi_s\big\|_{L^p} \;\geq\; c\, s^{\frac1p -1} \left(\sum_{4s<\Delta_k/4} c_k^p \right)^{1/p}. \tag{12}$$

Step 3 Conclusion.
Now we consider three cases:

CASE I: $p \geq 3$. $\left\{ \begin{array}{ll} \text{If } \alpha > 1/p & \text{then } u \notin B^\alpha_{p,q}. \\ \text{If } q < \infty & \text{then } u \notin B^{1/p}_{p,q}. \end{array} \right\}$

Since $\ell^3 \hookrightarrow \ell^p$ if $p \geq 3$, we have

$$\sum_{4s<\Delta_k/4} c_k^p = \mathcal{O}(1) \quad \text{as } s \downarrow 0.$$

Therefore, if $q < \infty$ and $\alpha \geq 1/p$, (12) yields

$$\|\psi \partial_x u\|_{B^{\alpha-1}_{p,q}} \geq c \left(\int_0^\varepsilon s^{(1-\alpha)q}\, s^{(\frac{1}{p}-1)q} \frac{ds}{s} \right)^{1/q} = c \left(\int_0^\varepsilon \frac{ds}{s^{1+(\alpha-\frac{1}{p})q}} \right)^{1/q} = \infty.$$

Similarly, if $q = \infty$ and $\alpha > 1/p$, then

$$\|\psi \partial_x u\|_{B^{\alpha-1}_{p,\infty}} \geq c \sup_{0<s<\varepsilon} s^{1-\alpha}\, s^{\frac{1}{p}-1} = \infty.$$

This proves that u cannot be better than $B^{1/p}_{p,\infty}$ for $p \geq 3$.

Case II: $p < 3$, $\alpha > 1/3$. Then $u \notin B^\alpha_{p,q}$.

We consider sequences of the form

$$\Delta_k = k^{-\beta} \quad \text{and} \quad c_k = k^{-\gamma} \quad \text{for } k \in \mathbf{N}$$

with suitable numbers β, γ. Since we assume that $\{\Delta_k\}_k \in \ell^1$ and $\{c_k\}_k \in \ell^3$, we require $\beta > 1$ and $\gamma > 1/3$. Then we have

$$4s < \Delta_k/2 \quad \Longleftrightarrow \quad k < (16s)^{-1/\beta}.$$

As a consequence, we obtain the following estimate:

$$\text{If } \gamma < 1/p, \text{ then} \qquad \sum_{4s<\Delta_k/4} c_k^p = \mathcal{O}\left(s^{-(1-\gamma p)/\beta}\right) \quad \text{as } s \downarrow 0.$$

Since $\beta > 1$ and $\gamma > 1/3$, we have

$$0 < \frac{1-\gamma p}{\beta} < 1 - \gamma p < 1 - \frac{p}{3} = \frac{3-p}{3},$$

and thus

$$\frac{1}{p} > \frac{1}{p}\left[1 - \frac{1-\gamma p}{\beta}\right] > \frac{1}{3}.$$

Assume now that $\alpha > 1/3$. Then we can find $\beta > 1$ and $1/p > \gamma > 1/3$ such that

$$-\delta := (1-\alpha) + \left(\frac{1}{p} - 1\right) - \frac{1}{p}(1-\gamma p)/\beta < 0,$$

and therefore (12) yields

$$\|\psi \partial_x u\|_{B^{\alpha-1}_{p,q}} \;\geq\; c \left(\int_0^\varepsilon s^{(1-\alpha)q}\, s^{(\frac{1}{p}-1)q} s^{-\frac{q}{p}(1-\gamma p)/\beta} \frac{ds}{s} \right)^{1/q} \;=\; c \left(\int_0^\varepsilon s^{-\delta} \frac{ds}{s} \right)^{1/q} \;=\; \infty.$$

A similar estimate holds in case $q = \infty$.

Case III: $q < p < 3$. Then $u \notin B^{1/3}_{p,q}$.

To achieve this last step, we need a more refined estimate. Let $\varphi(z) := z\,(\log z)^\sigma$ for some $\sigma > 1$. This φ is strictly monotone increasing for $z \geq 1$ and unbounded. Hence, φ is invertible and the inverse function φ^{-1} is increasing and unbounded, too. We define

$$\Delta_k := \big(\varphi(k+2)\big)^{-1} \quad \text{and} \quad c_k := (\varphi(k+2))^{-1/3} \quad \text{for } k \in \mathbf{N}.$$

Then $\{\Delta_k\}_k \in \ell^1$ and $\{c_k\}_k \in \ell^3$ as needed. We estimate

$$\sum_{4s<\Delta_k/4} c_k^p \;\geq\; \int_1^{\varphi^{-1}(1/(16s))-2} \big(\varphi(z+2)\big)^{-p/3}\,dz \;=\; \int_{\varphi(3)}^{1/(16s)} y^{-p/3}\,\frac{dy}{\varphi'\big(\varphi^{-1}(y)\big)}$$

with a substitution of variables $y(z) := \varphi(z+2)$. Now we claim that

$$\varphi'\big(\varphi^{-1}(y)\big) \leq c\,(\log y)^\sigma \qquad \text{for } y \text{ large}, \tag{13}$$

with $c > 0$ some constant. If (13) holds we can estimate

$$\sum_{4s<\Delta_k/4} c_k^p \;\geq\; c\big(-\log 16s\big)^{-\sigma}\int_{\varphi(3)}^{1/(16s)} y^{-p/3}\,dy \;\geq\; c\big(-\log 16s\big)^{-\sigma} s^{-(3-p)/3} \qquad \text{for } 0 < s < \varepsilon$$

for suitable ε and $c = c(\varepsilon, \sigma)$. Since $q < p$ we can choose $\sigma > 1$ small enough such that $\sigma q/p \leq 1$. Then (12) yields

$$\|\psi\partial_x u\|_{B^{-2/3}_{p,q}} \geq c\left(\int_0^\varepsilon \big(-\log 16s\big)^{-\sigma q/p}\,\frac{ds}{s}\right)^{1/q} = \infty.$$

This proves that u does not belong to $B^{1/3}_{p,q}$.

It remains to show (13). Since φ^{-1} is increasing and unbounded, (13) is equivalent to

$$\varphi'(z) \leq c\,\big(\log\varphi(z)\big)^\sigma \quad \text{for } z \text{ large}.$$

To establish this, we compute

$$\varphi'(z) \;=\; (\log z)^\sigma\big[1 + \sigma\,(\log z)^{-1}\big] \;\leq\; c\,(\log z)^\sigma \;\leq\; c\,\big(\log(z\,(\log z)^\sigma)\big)^\sigma \;=\; c\,\big(\log\varphi(z)\big)^\sigma$$

for some $c > 0$ and z large. This proves our claim.

3. Concluding remarks

3.1. **Weak solutions.** If we have better regularity in the right hand side of (3), then u can be smoother. Refining the calculations of Section 2 we can prove

Proposition 3.1. *Let $\sigma \in [1,3]$, and consider the family $\mathcal{S}$ of bounded weak solutions of* (1) *for which the distribution*

$$\partial_v^{2-\sigma}\Big\{\partial_t\chi(v,u) + v\,\partial_x\chi(v,u)\Big\} \quad \textit{is a Radon measure.}$$

Then $\mathcal{S}$ is not better than (4) *locally.*

Assume for example that $\partial_v m$ in (3) itself is a measure (hence $\sigma = 2$). Then, instead of estimating the norm of H_k in (7)&(8), we consider

$$\|\partial_v m\|_{\mathcal{M}(\mathbf{R}\times\Omega)} = \sum_{k=1}^{\infty} 2 \cdot \sqrt{1 + c_k^2/4} \cdot \int_{\mathbf{R}} |h_k(v)|\, dv = \frac{1}{2}\sum_{k=1}^{\infty} c_k^2.$$

Then the arguments in Section 2 can be repeated in a straightforward way and they give the result above, i.e. that u can have up to 1/2 derivatives. When even $\partial_v^2 m$ is a measure, then u can have a full derivative. We conjecture that in this case u is in BV locally.

Our construction can be adapted for different fluxes.

Proposition 3.2. *Let $\sigma \geq 3$, and consider the family $\mathcal{S}$ of bounded weak solutions of $\partial_t u + \frac{1}{\sigma-1}\,\partial_x u^{\sigma-1} = 0$ on $(0,1)\times\mathbf{R}$ such that*

$$\partial_t \chi(v,u) + v^{\sigma-2}\,\partial_x\chi(v,u) = \partial_v m \quad \text{in } \mathcal{D}', \tag{14}$$

for some Radon measure m. Then $\mathcal{S}$ is not better than (4) locally.

This suggests that the regularity of u depends on the non–degeneracy of the flux which is also what we expect from the velocity averaging arguments, see [7].

We conclude by giving an exact statement of what examples can be constructed for the variational problems cited in the introduction. Let $\Omega \subset \mathbf{R}^2$ be bounded and define

$$F_\varepsilon(w) := \int_\Omega \frac{(1-|w|^2)^2}{\varepsilon} + \varepsilon|\nabla w|^2 \quad \text{for } w \in H^1(\Omega,\mathbf{R}^2). \tag{15}$$

Proposition 3.3. *There exists $\{w_\varepsilon\}_{\varepsilon\downarrow 0} \subset L^\infty(\Omega,\mathbf{R}^2)$ such that*

- $\nabla \times w_\varepsilon = 0$ *and* $\limsup_\varepsilon F_\varepsilon(w_\varepsilon) < \infty$;
- w_ε *converges (strongly in L^p for every $p < \infty$) to a w which locally is not better than (4) with $\sigma = 3$.*

3.2. **Entropy solutions.** Similar arguments can also be applied for entropy solutions. Let us consider the flux $f(u) := \frac{1}{\sigma+1}u^{\sigma+1}$ for $\sigma \geq 1$. Since f is convex, only decreasing jumps satisfy the entropy condition. Increasing jumps must be replaced by rarefaction waves.

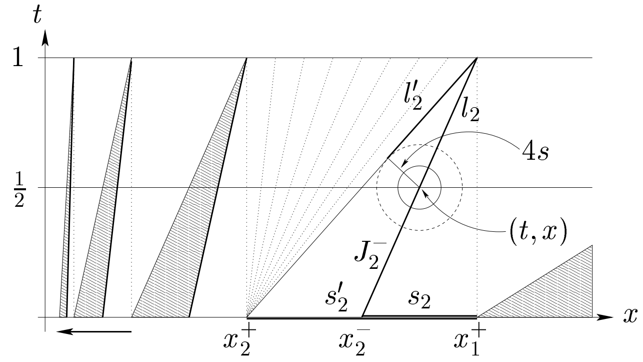

FIGURE 2. Entropy solutions.

We want to use a construction as in Step 1 of Section 2, i.e. we define suitable piecewise constant initial data. We start with sequences $\{\Delta_k\}_k \in \ell^1$ and $\{c_k\}_k$ of positive numbers, and put inductively

$$x_1^+ := \sum_{k=1}^{\infty} \Delta_k \quad \text{and} \quad x_{k+1}^+ := x_k^+ - \Delta_{k+1}$$

for $k \in \mathbf{N}$. An initial jump connecting 0 to c_k evolves into a rarefaction wave whose leading edge moves with speed $s_k' := f'(c_k) = c_k^\sigma$. We want to ensure that waves do not interact in the time interval $[0,1]$. Then Δ_k and c_k cannot be chosen independently. Instead, we are forced to assume $s_k' \leq \Delta_k$ for $k \in \mathbf{N}$, i.e. $\{c_k\}_k \in \ell^\sigma$. We emphasize that $\{c_k\}_k \in \ell^{\sigma+2}$ would be sufficient to make m in the kinetic equation (16) a bounded measure. But since rarefaction waves need more space, better summability of $\{c_k\}_k$ is required. To have the densest possible packing of shocks, we put $\Delta_k := c_k^\sigma$. We define

$$x_{k+1}^- := x_k^+ - s_{k+1} \quad \text{with} \quad s_k := \frac{f(c_k) - f(0)}{c_k - 0} = \frac{1}{\sigma+1} c_k^\sigma$$

for $k \in \mathbf{N}$. Then the shock starting at x_k^- and connecting c_k to 0 hits the rarefaction wave at time $t = 1$, see Figure 2. As in Section 2 we can write down a kinetic equation (16). We observe that only the jumps contribute to the non–negative measure m which again is bounded. We note that for all k with $\sqrt{|l_k|^2 - |l_k'|^2} \geq 4s$, only one jump (and no rarefaction wave) is involved in the convolution $(\partial_x u) \star \varphi_s(t,x)$ if $t \leq \frac{1}{2}$, see Figure 2. This amounts to

$$16s \leq \frac{\sigma}{\sigma+1} \frac{c_k^\sigma}{\sqrt{1 + c_k^{2\sigma}}} = \frac{\sigma}{\sigma+1} \frac{\Delta_k}{\sqrt{1 + \Delta_k^2}}.$$

Arguing as in Step 3 of Section 2, we can then prove

Proposition 3.4. *Let $\sigma \geq 1$, and consider the family $\mathcal{S}$ of bounded entropy solutions of $\partial_t u + \frac{1}{\sigma+1}\,\partial_x u^{\sigma+1} = 0$ on $(0,1) \times \mathbf{R}$ such that*

$$\partial_t \chi(v,u) + v^\sigma\, \partial_x \chi(v,u) = \partial_v m \quad \text{in } \mathcal{D}', \tag{16}$$

for some Radon measure m. Then $\mathcal{S}$ is not better than (4) *locally.*

This corresponds to the regularity conjectured in [7].

Acknowledgments Camillo De Lellis acknowledges partial support by the EU Network *Hyperbolic and kinetic equations* HPRN-CT-2002-00282. Michael Westdickenberg acknowledges partial support by the Deutsche Forschungsgemeinschaft through SFB 611 *Singularities and scaling in mathematical models* of the University of Bonn.

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Camillo De Lellis, Max–Planck Institute for Mathematics in the Sciences, Inselstr. 22, D-04103 Leipzig, Germany

E-mail address: `delellis@mis.mpg.de`

Michael Westdickenberg, Institute for Applied Mathematics, University of Bonn, Wegelerstr. 10, D-53115 Bonn, Germany

E-mail address: `mwest@iam.uni-bonn.de`