EXISTENCE OF SOLUTIONS FOR A CLASS OF HYPERBOLIC SYSTEMS OF CONSERVATION LAWS IN SEVERAL SPACE DIMENSIONS

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1. Introduction

In a recent paper [3] Bressan has shown that the Cauchy problem for the system of conservation laws

$$\begin{cases} \partial_t u_i + \sum_{\alpha=1}^n \partial_{x_\alpha}(f_\alpha(|u|)u_i) \;=\; 0 \\ \\ u_i(0,\cdot) \;=\; \overline{u}_i(\cdot) \end{cases} \tag{1}$$

can be ill posed for suitable Lipschitz flux functions f and L^∞ initial data $\overline{u}$ which are bounded away from 0. His analysis is based on the scalar conservation law associated to (1) (formally giving the absolute value ρ of the solution u), namely

$$\begin{cases} \partial_t \rho + \sum_{\alpha=1}^n \partial_{x_\alpha}\big(f_\alpha(\rho)\rho\big) \;=\; 0 \\ \\ \rho(0,\cdot) \;=\; \overline{\rho}(\cdot) \end{cases} \tag{2}$$

and on the analysis of the ODE $\dot{x}(t) = f(\rho(t, x(t)))$, which formally gives, via the method of characteristics, the angular part $\theta = u/\rho$ of the solution. In the final part of his paper Bressan points out that the Cauchy problem could be well posed for BV initial data, looking for suitable compactness properties of the Cauchy fluxes associated to BV vector fields, on the same line of the theory developed for Sobolev spaces by DiPerna and Lions in [5].

In a recent paper [1] the first author extended the Di Perna–Lions theory to BV vector fields satisfying natural L^∞ bounds, as in [5], on the distributional divergence. The theory developed in [1] is not directly applicable to the vector field $f(\rho(t,x))$ appearing in the Cauchy problem (1) because its (spatial) divergence is formally given by

$$-\frac{1}{\rho}\left[\rho_t + \sum_{\alpha=1}^n f_\alpha(\rho)\cdot\partial_{x_\alpha}\rho\right],$$

neither bounded nor absolutely continuous with respect to the Lebesgue measure in general.

Lifting the ODE considered by Bressan to an higher dimensional one and using the special structure of the Cauchy problem (1), we are able however to reduce ourselves to the case of divergence-free vectorfields, where the theory of [1] is fully applicable. Our approach is indeed based on the analysis of the autonomous ODE

$$\big(\dot{\omega}(s), \dot{\Phi}(s)\big) \;=\; \Big(\rho\big(\omega(s),\Phi(s)\big)\,,\, f\big(\rho\big(\omega(s),\Phi(s)\big)\big)\,\rho\big(\omega(s),\Phi(s)\big)\Big) \tag{3}$$

that we use, through a reparameterization, to recover solutions of the ODE $\dot{x}(t) = f(\rho(t, x(t)))$ (here ρ is extended to negative times considering the backward Cauchy problem (11) associated to (2)).

In particular we give a positive anwer to Bressan's conjecture, obtaining in Theorem 2.6 a general existence result for bounded weak solutions of (1) assuming that $f \in W^{1,\infty}_{loc}$ and that $\overline{u} \in L^\infty$ with $|\overline{u}| \geq c > 0$ $\mathscr{L}^n$-a.e. and $|\overline{u}| \in BV_{loc}$. By bounded weak solution we mean, as usual, a map $u \in L^\infty(\mathbf{R}_t \times \mathbf{R}^n_x, \mathbf{R}^k)$ such that for every test function $\varphi \in C^\infty_c(\mathbf{R}_t \times \mathbf{R}^n_x)$ and any $i = 1, \ldots, k$ we have

$$\int_0^\infty \int_{\mathbf{R}^n_x} \left(\partial_t \varphi + \sum_{\alpha=1}^n f_\alpha(|u|) \partial_{x_\alpha} \varphi \right) u_i \, dx\, dt + \int_{\mathbf{R}^n_x} \varphi(0, x) \overline{u}_i(x)\, dx \;=\; 0\ .$$

The solution is built as follows: denote by $(\Phi_{x,t}(s), \omega_{x,t}(s))$ the solution of the ODE (3) having $(x, t) \in \mathbf{R}^n \times \mathbf{R}$ as initial data, provided by [1], and set

$$\Psi(t, x) := \Phi_{x,t}\left(\omega^{-1}_{x,t}(0)\right).$$

Our solution of (1) is defined by

$$u(t, x) := \rho(t, x)\overline{\theta}\left(\Psi(t, x)\right)\ .$$

This construction also provides entropy conditions for a quite rich family of entropy-entropy flux pairs, therefore it is natural to investigate whether these entropy conditions are sufficiently strong to enforce uniqueness of solutions.

We are not able to give here a definite answer to this problem, but a careful analysis of our construction shows some necessary conditions for uniqueness which play also a role in the stability problem with respect to approximation of the initial data (see Theorem 4.5). These conditions involve a family of measures μ_N built from the transport map Ψ as follows:

$$\mu_N(A) := \mathscr{L}^{n+1}\left(([0, N] \times \mathbf{R}^n) \cap \Psi^{-1}(A) \right) \qquad \text{for any Borel set } A \subset \mathbf{R}^n.$$

We show in Proposition 4.4 that the absolute continuity with respect to $\mathscr{L}^n$ of all measures μ_N is a necessary condition for uniqueness of entropy solutions. However, we are not presently able to show that this condition is sufficient, or to exhibit examples where this absolute continuity property fails.

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2. Preliminaries and statement of the result

Before stating the main theorem, we recall the notion of entropy solution of a scalar conservation law and the classical theorem of Kruzhkov, which provides existence, stability and uniqueness of entropy solutions to the Cauchy problem for scalar laws.

Definition 2.1. *Let $g \in W^{1,\infty}_{loc}(\mathbf{R}, \mathbf{R}^n)$. A pair (η, q) of functions $\eta \in W^{1,\infty}_{loc}(\mathbf{R}, \mathbf{R})$, $q \in W^{1,\infty}_{loc}(\mathbf{R}, \mathbf{R}^n)$ is called an* entropy–entropy flux pair *relative to g if*

$$q' \;=\; \eta' g' \qquad \mathscr{L}^1\text{–almost everywhere on } \mathbf{R}. \tag{4}$$

If, in addition, η is a convex function, then we say that (η, q) is a convex entropy–entropy flux *pair. A weak solution $u \in L^\infty(\mathbf{R}_t^+ \times \mathbf{R}_x^n)$ of*

$$\begin{cases} \partial_t u + \mathrm{div}_x[g(u)] \;=\; 0 \\ \\ u(0,\cdot) \;=\; \overline{u} \end{cases} \tag{5}$$

is called an entropy solution if $\partial_t[\eta(u)] + \mathrm{div}_x[q(u)] \leq 0$ in the sense of distributions for every convex entropy–entropy flux pair (η, q).

Theorem 2.2 ([6] Kruzhkov)**.** *Let $g \in W^{1,\infty}_{loc}(\mathbf{R}, \mathbf{R}^n)$ and $\overline{u} \in L^\infty$. Then there exists a* unique *entropy solution u of (5). If in addition $\overline{u} \in BV_{loc}(\mathbf{R}^n)$, then, for every open set $A \subset\subset \mathbf{R}^n$ and for every $T \in]0, \infty[$, there exists an open set $A' \subset\subset \mathbf{R}^n$ (whose diameter depends only on A, T, g and $\|\overline{u}\|_\infty$) such that*

$$\|u\|_{BV(]0,T[\times A)} \;\leq\; \|\overline{u}\|_{BV(A')}\,. \tag{6}$$

We recall now the notion of *entropy–entropy flux pair* for systems.

Definition 2.3. *A pair of Lipschitz functions $\eta : \mathbf{R}^k \to \mathbf{R}$, $q : \mathbf{R}^k \to \mathbf{R}^n$ is called an* entropy–entropy flux pair *for the system (1) if for every open set $\Omega \subset \mathbf{R}_t \times \mathbf{R}_x^n$ and for every $u \in C^1(\Omega, \mathbf{R}^k)$ which solves the system $\partial_t u_i + \sum_\alpha \partial_{x_\alpha}(f_\alpha(|u|)u_i) = 0$ pointwise, we have*

$$\partial_t[\eta(u)] + \mathrm{div}_x[q(u)] \;=\; 0 \quad \text{on } \Omega \text{ in the sense of distributions.} \tag{7}$$

We denote by $\mathcal{R}$ the set of all entropy–entropy flux pairs (η, q) such that η is convex and both η and q are radially symmetric (that is $\eta(x) = \eta(y)$ and $q(x) = q(y)$ whenever $|x| = |y|$).

Remark 2.4. It is easy to check that the couple $\eta(y) := |y|$, $q_\alpha(y) := f_\alpha(|y|)|y|$ is an entropy–entropy flux pair for the system (1). Moreover, if (E, Q) is an entropy–entropy flux pair for the scalar law $\partial_t\rho + \mathrm{div}_x[f(\rho)\rho] = 0$, then $\big(E(|y|), Q(|y|)\big)$ is an entropy–entropy flux pair for (1). We remark that all couples $(\eta, q) \in \mathcal{R}$ can be generated with the procedure above.

In addition to the notion of entropy, we introduce that of *companion radial system.*

Definition 2.5. *Assume that a map $S \in W^{1,\infty}_{loc}(\mathbf{R}^k, \mathbf{R}^k)$ satisfies:*

- *$|S(y)| = |y|$ for every $y \in \mathbf{R}^k$;*
- *$S(y) = |y| S\left(\dfrac{y}{|y|}\right)$ for every $y \neq 0$.*

Then we say that the system of equations

$$\begin{cases} \partial_t [S(u)]_i + \sum\limits_{\alpha=1}^{n} \partial_{x_\alpha}\big(f_\alpha(|u|)[S(u)]_i\big) \;=\; 0 \\ \\ \big[S(u(0,\cdot))\big]_i = \; [S(\overline{u}(\cdot))]_i \end{cases} \tag{8}$$

is a companion radial system *of* (1).

We are now in the position of stating the main theorem of this paper:

Theorem 2.6. *Let $f \in W^{1,\infty}_{loc}(\mathbf{R}, \mathbf{R}^k)$ and $\overline{u} \in L^\infty$. Assume that $|\overline{u}| \in BV_{loc}(\mathbf{R}^n)$ and $|\overline{u}| \geq c > 0$ $\mathscr{L}^n$-a.e. Then there exists a bounded weak solution u of (1) such that*

- *u solves in the sense of distributions every companion radial system;*
- *For any $(\eta, q) \in \mathcal{R}$ the distribution $\partial_t[\eta(u)] + \mathrm{div}_x[q(u)]$ is a nonpositive measure.*

The main tool for proving this theorem is the following consequence of the theory developed by the first author in [1] for ODEs $\dot{x} = b(x)$ with BV coefficients b having absolutely continuous and bounded divergence (extending the theory developed for Sobolev spaces in [5]). The theorem stated below is a particular case of this theory (as the conditions on the divergence could be relaxed and also the non-autonomous case could be considered), but it is sufficient to our purposes.

Theorem 2.7 ([1], Theorem 6.5)**.** *Assume $b \in BV_{loc} \cap L^\infty(\mathbf{R}^m, \mathbf{R}^m)$ and $\mathrm{div}\, b = 0$ in the sense of distributions. Then there exists a unique locally bounded map $\Phi : \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m$ such that:*

(i) *$\Phi(\cdot, x) \in W^{1,\infty}(\mathbf{R}, \mathbf{R}^m)$ and $\Phi(0, x) = x$ for $\mathscr{L}^m$–a.e. $x \in \mathbf{R}^m$.*
(ii) *$\frac{d}{ds}\Phi(s, x) = b(\Phi(s, x))$ for $\mathscr{L}^{m+1}$–a.e. $(s, x) \in \mathbf{R} \times \mathbf{R}^m$.*
(iii) *Let $\{b^j\} \subset W^{1,\infty}_{loc} \cap L^\infty(\mathbf{R}^m, \mathbf{R}^m)$ with $\mathrm{div}\, b^j = 0$ and denote by Φ^j the unique solutions of*

$$\begin{cases} \dot{\Phi}^j(t, x) &= b(\Phi^j(t, x)) \\ \\ \Phi^j(0, x) &= x \,. \end{cases}$$

If $\{\|b^j\|_\infty\}$ is bounded and $b^j \to b$ in L^1_{loc}, then

$$\lim_{j\to\infty} \int_{B_R} \sup_{t\in[-T,T]} \left|\Phi^j(t, x) - \Phi(t, x)\right| \, dx \;=\; 0 \qquad \forall R,\, T > 0.$$

Using the terminology of [1], we say that this Φ is the *regular lagrangian flow* generated by b. The regular Lagrangian flow Φ can also be characterized, as in [5], by conditions (a), (b) and replacing the stability property (c) by the following one: for any $T > 0$ and any bounded open set $A \subset \mathbf{R}^n$ there exists a constant C such that the measures

$$\int_{\mathbf{R}^n} \varphi\, \mu_t := \int_A \varphi\left(\Phi(t, x)\right) \, dx \qquad \varphi \in C_c(\mathbf{R}^n),\; t \in [-T, T]$$

satisfy

$$\mu_t \leq C\mathscr{L}^n.$$

Remark 2.8. An easy consequence of the previous theorem and of a diagonal argument is the following stability property: assume that $\{b^j\} \subset BV_{loc}$, $\mathrm{div}\, b^j = 0$ in the sense of distributions, $\{\|b^j\|_\infty\}$ is bounded and $b^j \to b$ in L^1_{loc}. Let Φ^j be the regular lagrangian flows generated by b^j and let Φ be the regular lagrangian flow generated by b. Then

$$\lim_{j\to\infty} \int_{B_R} \sup_{t\in[-T,T]} \left|\Phi^j(t, x) - \Phi(t, x)\right| \, dx \;=\; 0 \qquad \forall R,\, T > 0.$$

A further diagonal argument also provides a subsequence $j(r)$ such that $\Phi^{j(r)}(\cdot, x)$ converge to $\Phi(\cdot, x)$ locally uniformly in $\mathbf{R}$ for $\mathscr{L}^m$-a.e. $x \in \mathbf{R}^m$.

3. Proof of Theorem 2.6

Before coming to the proof of Theorem 2.6 we need the following elementary lemma.

Lemma 3.1. *Let μ be a finite measure on $\mathbf{R}^n$ and let $\theta : \mathbf{R}^n \to \mathbf{S}^k$. Then there exists a sequence of continuous maps $\theta^j : \mathbf{R}^n \to \mathbf{S}^k$ such that $\theta^j \to \theta$ in $L^1(\mu)$.*

Proof. Note that for some $y \in \mathbf{S}^k$, we have $\mu(\theta^{-1}(y)) = 0$. Fix such a y and take a smooth diffemorphism $\varphi : \mathbf{S}^k \setminus \{y\} \to \mathbf{R}^k$. Moreover, for every $\varepsilon > 0$, denote by R_ε the map $R_\varepsilon : \mathbf{S}^k \to \mathbf{S}^k$ such that

- R_ε is the identity on $\mathbf{S}^k \setminus B_\varepsilon(y)$, where $B_\varepsilon(y)$ is the geodesic ball of $\mathbf{S}^k$ centered on y.
- R_ε maps radially $B_\varepsilon(y)$ on the geodesic sphere $\partial B_\varepsilon(y)$ (thus R_ε is not defined and discontinuous on y).

The maps $\theta^\varepsilon = R_\varepsilon \circ \theta$ are well defined because $\mu(\theta^{-1}(y)) = 0$. Moreover $\theta^\varepsilon \to \theta$ in $L^1(\mu)$. For every ε we can find a sequence of continuous maps $\{\theta^{\varepsilon,j}\}^j$ such that $\theta^{\varepsilon,j} \to \theta^\varepsilon$ in $L^1(\mu)$. Indeed, consider the map $\varphi \circ \theta^\varepsilon$. This map takes values in $\mathbf{R}^k$, is bounded and in L^1. Set a system of standard coordinates $x_1, \dots x_k$. For each $[\varphi(\theta^\varepsilon)]_i$, $i \in \{1, \dots, k\}$, standard measure theory gives a sequence of continuous maps $\tilde{\theta}^{\varepsilon,j}_i$ which converges to $[\varphi(\theta^\varepsilon)]_i$ in $L^1(\mu)$ and such that the sequence $\{\|\tilde{\theta}^{\varepsilon,j}_i\|_\infty\}^j$ is bounded. Thus $\tilde{\theta}^{\varepsilon,j} \to \varphi(\theta^\varepsilon)$ in $L^1(\mu)$ and the sequence $\{\|\tilde{\theta}^{\varepsilon,j}\|_\infty\}$ is bounded. Since φ^{-1} is bounded and continuous, the maps $\theta^{\varepsilon,j} = \varphi^{-1}(\tilde{\theta}^{\varepsilon,j})$ are all continuous and converge to θ^ε in $L^1(\mu)$. A standard diagonal argument gives two sequences $\varepsilon_r \downarrow 0$, $j(\varepsilon_r) \uparrow \infty$ such that $\theta^{\varepsilon_r, j(\varepsilon_r)} \to \theta$ in $L^1(\mu)$. □

Proof of Theorem 2.6. Throughout this proof, for every map $u : \Omega \to \mathbf{R}^k$ we set $\rho := |u|$ and $\theta := u/|u|$. Since we will consider only functions u which are bounded away from the origin, θ is well defined. Moreover, we set $\overline{\rho} = |\overline{u}|$ and $\overline{\theta} = \overline{u}/|\overline{u}|$. In the first three steps we prove the theorem under the assumption that $\overline{\theta}$ is continuous. In the fourth step we pass to the general case.

First Step. Scalar equation.

We define the function ρ as the unique entropy solution of the Cauchy problem for the scalar conservation law

$$\begin{cases} \partial_t \rho + \sum_{\alpha=1}^n \partial_{x_\alpha}\big(f_\alpha(\rho)\rho\big) \;=\; 0 \\ \\ \rho(0,\cdot) \;=\; \overline{\rho}(\cdot)\,. \end{cases} \tag{9}$$

Thanks to Theorem 2.2, $\rho \in BV_{loc}$ and satisfies the entropy inequality for every convex entropy–entropy flux pair (η, q) related to the scalar law (9). Moreover, by Remark 2.4, we have

$$\text{If } (E(|x|), Q(|x|)) \in \mathcal{R}, \text{ then } (E, Q) \text{ is a convex entropy–entropy flux pair for (9).} \tag{10}$$

For technical reasons, it will be convenient to extend ρ to a function defined on the *whole* $\mathbf{R}_t \times \mathbf{R}^n_x$ (that is to define ρ even for negative times). In order to do this we adopt the following elementary procedure: we define ρ^- as the unique entropy solution of the Cauchy problem

$$\begin{cases} \partial_t \rho^- - \sum_{\alpha=1}^n \partial_{x_\alpha}\big(f_\alpha(\rho^-)\rho^-\big) \;=\; 0 \\ \\ \rho^-(0,\cdot) \;=\; \overline{\rho}(\cdot)\,. \end{cases} \tag{11}$$

Then, we extend ρ to $\mathbf{R}_t^- \times \mathbf{R}_x^n$ setting $\rho(t,x) = \rho^-(-t,x)$. Theorem 2.2 implies that $\rho \in BV_{\rm loc}(\mathbf{R}_t \times \mathbf{R}_x^n)$. Moreover, it is immediate to check that

$$\partial_t \rho + \sum_{\alpha=1}^n \partial_{x_\alpha}(f_\alpha(\rho)\rho) = 0 \qquad \text{on } \mathbf{R}_t \times \mathbf{R}_x^n \tag{12}$$

and that $\rho|_{\mathbf{R}_t^+ \times \mathbf{R}_x^n}$ is the unique entropy solution of the Cauchy problem (9).

Second Step. Smooth approximation of the transport equation.

Let ξ^ε be a standard convolution kernel and define $\rho^\varepsilon := \rho * \xi^\varepsilon$ and $g^\varepsilon := (f(\rho)\rho) * \xi^\varepsilon$. Since $\rho \geq c > 0$ $\mathscr{L}^n$-a.e. the same inequality is true everywhere for ρ^ε. Then, let θ^ε be the unique solution of the Cauchy problem for the transport equation

$$\begin{cases} \partial_t \theta^\varepsilon + \dfrac{g^\varepsilon}{\rho^\varepsilon} \cdot \nabla_x \theta^\varepsilon \;=\; 0 \\ \theta^\varepsilon(0,\cdot) \;=\; \overline{\theta}(\cdot)\,. \end{cases} \tag{13}$$

Thus we have that $\rho^\varepsilon \partial_t \theta^\varepsilon + g^\varepsilon \cdot \nabla_x \theta^\varepsilon = 0$. Since $\partial_t \rho^\varepsilon + {\rm div}_x g^\varepsilon = [\partial_t \rho + {\rm div}_x(f(\rho)\rho)] * \xi^\varepsilon = 0$, we have

$$\partial_t(\rho^\varepsilon \theta^\varepsilon) + {\rm div}_x(g^\varepsilon \theta^\varepsilon) \;=\; 0\,. \tag{14}$$

Clearly $\|\theta^\varepsilon\|_\infty$ is uniformly bounded. Thus, there is subsequence $\{\theta^{\varepsilon_j}\}$ which converges weakly* in L^∞ to a function θ. Since $(\rho^\varepsilon, g^\varepsilon) \to (\rho, f(\rho)\rho)$ strongly in L^1_{loc}, we have that $\rho^{\varepsilon_j}\theta^{\varepsilon_j} \to \rho\theta$ and $g^{\varepsilon_j}\theta^{\varepsilon_j} \to f(\rho)\rho\theta$ in the sense of distributions. Hence, setting $u = \rho\theta$, the function u satisfies

$$\begin{cases} \partial_t u + \sum_{\alpha=1}^n \partial_{x_\alpha}(f_\alpha(\rho)u) \;=\; 0 \\ \\ u(0,\cdot) \;=\; \overline{\rho}(\cdot)\overline{\theta}(\cdot) \end{cases} \tag{15}$$

in the sense of distributions. Thus, if we could prove that $|\theta| = 1$ $\mathscr{L}^{n+1}$–a.e. on $\mathbf{R}_t^+ \times \mathbf{R}_x^n$, the function u would be a solution of (1) in the sense of distributions. Moreover, in view of (10), it would satisfy the entropy inequality $\partial_t[\eta(u)] + {\rm div}_x[q(u)] \leq 0$ for every $(\eta, q) \in \mathcal{R}$.

Now, let S be as in Definition 2.5. Since ρ^ε, θ^ε and g^ε are smooth, we have

$$\begin{cases} \partial_t[\rho^\varepsilon S(\theta^\varepsilon)] + {\rm div}_x\big[g(\rho^\varepsilon)S(\theta^\varepsilon)\big] \;=\; 0 \\ \\ \rho^\varepsilon S\big(\theta^\varepsilon(0,\cdot)\big) \;=\; \overline{\rho}(\cdot)S(\overline{\theta}(\cdot))\,. \end{cases}$$

If $|\theta| = 1$ $\mathscr{L}^{n+1}$–a.e., we would have that $\theta^{\varepsilon_j} \to \theta$ strongly in L^1_{loc} and hence $\rho^{\varepsilon_j} S(\theta^{\varepsilon_j}) \to \rho S(\theta)$ in L^1_{loc}. By definition, $\rho S(\theta) = S(\rho\theta) = S(u)$. Thus u would satisfy

$$\begin{cases} \partial_t[S(u)] + \mathrm{div}_x\big[f(|u|)S(u)\big] \;=\; 0 \\ \\ S\big(u(0,\cdot)\big) \;=\; S(\overline{u}(\cdot)) \end{cases}$$

Summarizing, the theorem would follow if we could prove that $|\theta| = 1$ $\mathscr{L}^{n+1}$–a.e. .

Third Step. Strong convergence of θ^ε when $\overline{\theta} \in C(\mathbf{R}^n)$.

In this step we assume that $\overline{\theta} \in C(\mathbf{R}^n)$ and we prove that under this assumption $\theta^\varepsilon(T,x) \to \theta(T,x)$ for $\mathscr{L}^{n+1}$–a.e. $(T,x) \in \mathbf{R}^+_t \times \mathbf{R}^n_x$. In view of the previous step, this proves the theorem when $\overline{\theta}$ is continuous.

We start by defining the following autonomous system of ODEs:

$$\begin{cases} \dfrac{d}{ds}\Phi^\varepsilon_{x,y}(s) \;=\; g^\varepsilon\big(\omega^\varepsilon_{x,y}(s), \Phi^\varepsilon_{x,y}(s)\big) \\ \\ \dfrac{d}{ds}\omega^\varepsilon_{x,y}(s) \;=\; \rho^\varepsilon\big(\omega^\varepsilon_{x,y}(s), \Phi^\varepsilon_{x,y}(s)\big) \\ \\ \omega^\varepsilon_{x,y}(0) = y \in \mathbf{R}, \qquad \Phi^\varepsilon_{x,y}(0) = x \in \mathbf{R}^n\,. \end{cases} \tag{16}$$

We stress on the fact that we solve this system of ODEs for all times (that is, even when s is negative) thus finding trajectories $\big(\Phi^\varepsilon_{x,y}, \omega^\varepsilon_{x,y}\big) : \mathbf{R} \to \mathbf{R}^n \times \mathbf{R}$.

Since $\rho^\varepsilon \geq c$ the map $\omega^\varepsilon_{x,y} : \mathbf{R} \to \mathbf{R}$ is invertible with Lipschitz inverse. We define $\zeta^\varepsilon_{x,y}$ as the inverse of $\omega^\varepsilon_{x,y}$ and we set

$$\Gamma^\varepsilon_{x,T}(t) \;:=\; \Phi^\varepsilon_{x,T}(\zeta^\varepsilon_{x,T}(t))\,.$$

Clearly

$$\begin{aligned} \frac{d}{dt}\Gamma^\varepsilon_{x,T}(t) \;&=\; \frac{d\Phi^\varepsilon_{x,T}}{ds}\big(\zeta^\varepsilon_{x,T}(t)\big)\frac{d\zeta^\varepsilon_{x,T}}{dt}(t) \\ &=\; g^\varepsilon\left[\omega^\varepsilon_{x,T}\big(\zeta^\varepsilon_{x,T}(t)\big), \Phi^\varepsilon_{x,T}\big(\zeta^\varepsilon_{x,T}(t)\big)\right]\left\{\frac{1}{\rho^\varepsilon\left[\omega^\varepsilon_{x,T}\big(\zeta^\varepsilon_{x,T}(t)\big), \Phi^\varepsilon_{x,T}\big(\zeta^\varepsilon_{x,T}(t)\big)\right]}\right\} \\ &=\; \frac{g^\varepsilon\big(t, \Gamma^\varepsilon_{x,T}(t)\big)}{\rho^\varepsilon\big(t, \Gamma^\varepsilon_{x,T}(t)\big)}\,. \end{aligned}$$

Moreover note that $\omega^\varepsilon_{x,T}(0) = T$ and hence $\zeta^\varepsilon_{x,T}(T) = 0$. Note also that, since $\frac{d}{ds}\zeta^\varepsilon_{x,T} > 0$ and $\zeta^\varepsilon_{x,T}(T) = 0$, then $\zeta^\varepsilon_{x,T}(0) < 0$: this is why we solved the ODEs (16) even for negative times.

Thus $\Gamma^\varepsilon_{x,T}(T) = \Phi^\varepsilon_{x,T}(\zeta^\varepsilon_{x,T}(T)) = \Phi^\varepsilon_{x,T}(0) = x$. We conclude that the trajectory $\Gamma^\varepsilon_{x,T}(t)$ is the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\Gamma^\varepsilon_{x,T}(t) \;=\; \dfrac{g^\varepsilon\,\big(t, \Gamma^\varepsilon_{x,T}(t)\big)}{\rho^\varepsilon\,\big(t, \Gamma^\varepsilon_{x,T}(t)\big)} \\ \Gamma^\varepsilon_{x,T}(T) \;=\; x\,. \end{cases} \tag{17}$$

Since θ^ε solves the Cauchy problem (13), we have that

$$\theta^\varepsilon(T,x) \;=\; \theta^\varepsilon(0,\Gamma^\varepsilon_{x,T}(0)) \;=\; \overline{\theta}\big(\Phi^\varepsilon_{x,T}(\zeta^\varepsilon_{x,T}(0))\big)\,. \tag{18}$$

Set $g := f(\rho)\rho$ and consider the ODE

$$\begin{cases} \dfrac{d}{ds}\Phi_{x,y}(s) \;=\; g\big(\omega_{x,y}(s),\Phi_{x,y}(s)\big) \\[2ex] \dfrac{d}{ds}\omega_{x,y}(s) \;=\; \rho\big(\omega_{x,y}(s),\Phi_{x,y}(s)\big) \\[2ex] \omega_{x,y}(0)=y, \qquad \Phi_{x,y}(0)=x\,. \end{cases} \tag{19}$$

Since the flux $(\rho,g)\in BV_{loc}\cap L^\infty$ is divergence free, we can apply Theorem 2.7 to obtain the existence of a unique regular Lagrangian flow $(\Phi_{x,y},\omega_{x,y})$. Theorem 2.7 and Remark 2.8 imply that possibly extracting a subsequence (not relabelled) from ε_j we have

- For $\mathscr{L}^{n+1}$–a.e. (x,y), the maps $\Phi_{x,y}:\mathbf{R}\to\mathbf{R}^n$, $\omega_{x,y}:\mathbf{R}\to\mathbf{R}$ solve (19);
- For $\mathscr{L}^{n+1}$–a.e. (x,y), the maps $(\Phi^{\varepsilon_j}_{x,y},\omega^{\varepsilon_j}_{x,y})$ converge locally uniformly in $\mathbf{R}$ to $(\Phi_{x,y},\omega_{x,y})$ as $n\to\infty$.

Since $\frac{d}{ds}\omega^\varepsilon_{x,y}\geq c>0$, we conclude that $\left|\frac{d}{ds}\zeta^\varepsilon_{x,y}\right|\leq\frac{1}{c}$. Thus, for every (x,y), the family of real functions $\{\zeta^\varepsilon_{x,y}\}^\varepsilon$ is precompact in the topology of locally uniform convergence. Recall that for $\mathscr{L}^{n+1}$–a.e. (x,y) the functions $\omega^{\varepsilon_j}_{x,y}$ converge locally uniformly to $\omega_{x,y}$. Hence for $\mathscr{L}^{n+1}$–a.e. (x,y), $\zeta^{\varepsilon_j}_{x,y}$ converge uniformly to a Lipschitz map $\zeta_{x,y}$, which is the inverse of $\omega_{x,y}$.

Thus, for a.e. (x,T) we have that $\Phi^{\varepsilon_j}_{x,T}(\zeta^{\varepsilon_j}_{x,T}(0))$ converge to $\Phi_{x,T}(\zeta_{x,T}(0))$. By the continuity of $\overline{\theta}$ and (18), we have that $\theta^{\varepsilon_j}(T,x)\to\overline{\theta}[\Phi_{x,T}(\zeta_{x,T}(0))]$ for $\mathscr{L}^{n+1}$–a.e. $(T,x)\in\mathbf{R}^+_t\times\mathbf{R}^n_x$. This yields that the θ constructed in the previous step is given by

$$\theta(T,x) \;=\; \overline{\theta}\left[\Phi_{x,T}\big(\zeta_{x,T}(0)\big)\right] \tag{20}$$

and hence takes its values in $\mathbf{S}^{k-1}$ almost everywhere. This completes the proof of Theorem 2.6 when $\overline{\theta}$ is continuous.

Fourth Step. The general case $\theta\in L^\infty$.

Now fix a general $\overline{u}$ satisfying the assumptions of the Theorem. Define the map $\Psi:\mathbf{R}^+_t\times\mathbf{R}^n_x\to\mathbf{R}^n$ as

$$\Psi(T,x) \;:=\; \Phi_{x,T}\big(\zeta_{x,T}(0)\big)\,. \tag{21}$$

For every $N\in\mathbb{N}$, consider the measure μ_N defined on $\mathbf{R}^n$ in the following way: μ_N is the pushforward, via Ψ, of the Lebesgue measure $\mathscr{L}^{n+1}$ restricted to the set $[0,N]\times\mathbf{R}^n_x$. Note that for every N there exists a constant $C(N)$ such that $|\Psi(T,x)-x|\leq C(N)$ for every $T\in[0,N]$. Thus the measure μ_N is locally finite.

We fix a map $\tilde{\theta}$ in the equivalence class of $\overline{\theta}$ (our construction might be sensitive to the choice of this representative, see the next section). By applying Lemma 3.1 to the map $\tilde{\theta}$ and to the measure

$$\mu \;:=\; \sum_{R=1}^\infty 2^{-R}\chi_{B_R}\left(\mathscr{L}^n+\sum_{N=1}^\infty\frac{1}{2^N\mu_N(B_R)}\mu_N\right)$$

we find a sequence of continuous maps $\tilde{\theta}^j : \mathbf{R}^n \to \mathbf{S}^{k-1}$ such that $\tilde{\theta}^j(x) \to \tilde{\theta}(x)$ in $L^1(\mu)$. Thus a subsequence (not relabeled) $\tilde{\theta}^j$ converges to $\tilde{\theta}$ μ–almost everywhere. This means that $\tilde{\theta}^j(x) \to \theta$ for $\mathscr{L}^{n+1}$–a.e. x and for μ_N–a.e. x for all N.

Using the construction of the previous steps, we can find functions $\rho : \mathbf{R}^+_t \times \mathbf{R}^n_x \to \mathbf{R}$ and $\theta^j : \mathbf{R}^+_t \times \mathbf{R}^n_x \to \mathbf{S}^{k-1}$ such that ρ is the entropy solution of (9) and θ^j solves, in the sense of distributions, the Cauchy problem

$$\begin{cases} \partial_t(\rho\theta^j) + \sum_{\alpha=1}^n \partial_{x_\alpha}\big(f_\alpha(\rho)\rho\theta^j\big) \;=\; 0\,. \\ \\ \rho\theta^j(0,\cdot) \;=\; \overline{\rho}(\cdot)\tilde{\theta}^j(\cdot)\,. \end{cases} \tag{22}$$

Up to subsequences, (θ^j) converge weakly* in L^∞ to a map θ which solves

$$\begin{cases} \partial_t(\rho\theta) + \sum_{\alpha=1}^n \partial_{x_\alpha}\big(f_\alpha(\rho)\rho\theta\big) \;=\; 0\,. \\ \\ \rho\theta(0,\cdot) \;=\; \overline{\rho}(\cdot)\overline{\theta}(\cdot)\,. \end{cases} \tag{23}$$

Arguing as in the second step, to complete the proof we only need to show that $\theta(t,x) \in \mathbf{S}^{k-1}$ for $\mathscr{L}^{n+1}$–a.e. (t,x). Let $G \subset \mathbf{R}^n_x$ be the set

$$G \;:=\; \{x : \;\tilde{\theta}^j(x) \to \tilde{\theta}(x)\}.$$

Thanks to our assumptions, we have $\mu_N(\mathbf{R}^n \setminus G) = 0$ for every N. This means that

$$\text{for } \mathscr{L}^{n+1}\text{–a.e. } (T,x) \text{ we have } \Psi(T,x) \in G. \tag{24}$$

Since $\theta^j(T,x) = \tilde{\theta}^j(\Psi(T,x))$, clearly

$$\theta^j(T,x) \to \tilde{\theta}(\Psi(T,x)) \text{ for every } (T,x) \text{ such that } \Psi(T,x) \in G.$$

Thanks to (24) we conclude that $\theta^j(T,x) \to \tilde{\theta}(\Psi(T,x))$ for $\mathscr{L}^{n+1}$–a.e. (T,x). Since the weak limit has to coincide with the pointwise limit, we obtain that $\theta(T,x) = \tilde{\theta}(\Psi(T,x)) \in \mathbf{S}^{k-1}$ for $\mathscr{L}^{n+1}$–a.e. $(T,x) \in \mathbf{R}^+_t \times \mathbf{R}^n_x$. This completes the proof. □

4. Some remarks about uniqueness and stability

In analogy with the terminology of scalar conservation laws, we say that a weak solution $u \in L^\infty(\mathbf{R}^+ \times \mathbf{R}^n_x, \mathbf{R}^k)$ of (1) is an *entropy* solution if

- u is a weak solution of any companion radial system;
- $\partial_t[\eta(u)] + \mathrm{div}_x[q(u)] \le 0$ for any couple $(\eta, q) \in \mathcal{R}$.

The following conjecture is quite natural:

Conjecture 4.1. *If u^1 and u^2 are two entropy solutions of the same Cauchy problem* (1) *then $u^1 = u^2$ $\mathscr{L}^{n+1}$–a.e. in $\mathbf{R}^+_t \times \mathbf{R}^n_x$.*

Notice that the entropy condition generated by pairs $(\eta, q) \in \mathcal{R}$ ensures that $|u^1| = |u^2| = \rho$ $\mathscr{L}^{n+1}$–a.e. and one can guess that the fact that u is a weak solution of all companion radial systems should imply that the $u^1/\rho = u^2/\rho$ $\mathscr{L}^{n+1}$–a.e.

We will show in the next subsection how this conjecture is related to the map Ψ defined by (21) and to the measures μ_N (see Definition 4.2 below and the Fourth Step of Proof of

Theorem 2.6). In the last subsection we will discuss the relations of these measures with the stability of entropy solutions.

4.1. **The transport map Ψ.** Thanks to Theorem 2.7 and Remark 2.8 the map $\Psi \in L^\infty(\mathbf{R}^+_t \times \mathbf{R}^n, \mathbf{R}^n)$ defined on (21) only depends on the initial data $\overline{\rho}$ (and obviously on f) but it *does not depend* on the construction of the Third Step. In particular Ψ does not depend on the choice of the convolution kernel ξ and of the vanishing sequence $\{\varepsilon_j\}$. We call Ψ the *transport map generated* by the initial data $\overline{\rho}$ and we introduce the following notation:

Definition 4.2. *Let Ψ be as above and for every $N \in \mathbb{N}$ define the measure μ_N as the push–forward, via Ψ, of the Lebesgue measure $\mathscr{L}^{n+1}$ restricted to the set $[0,N]\times\mathbf{R}^n$. A Borel set S is called* a singular set *for Ψ if $\mathscr{L}^n(S)=0$ and $\mu_N(S)>0$ for some N.*

Notice that if all measures μ_N are absolutely continuous with respect to $\mathscr{L}^n$ there is no singular set. Using the language just introduced, the construction of the proof of Theorem 2.6 can be summarized in the following

Theorem 4.3. *Let $\overline{\rho} \in BV_{loc}(\mathbf{R}^n)$ with $0 < c \le \overline{\rho} \le C$ and let $\overline{\theta} \in L^\infty(\mathbf{R}^n, \mathbf{S}^{k-1})$.*

1. *Define ρ as in the first step of the proof of Theorem 2.6. Denote by Ψ be the transport map generated by $\overline{\rho}$.*
2. *Let $\tilde\theta : \mathbf{R}^n \to \mathbf{S}^{k-1}$ be any Borel map such that $\tilde\theta = \overline{\theta}$ $\mathscr{L}^n$–a.e. on $\mathbf{R}^n$.*
3. *Define $u \in L^\infty(\mathbf{R}^+_t \times \mathbf{R}^n_x)$ as $u(t,x) = \rho(t,x)\tilde\theta\big(\Psi(t,x)\big)$.*

Then u is an entropy solution of

$$\left\{\begin{array}{l} \partial_t u + \sum\limits_{\alpha=1}^{n} \partial_{x_\alpha}(f_\alpha(|u|)u) \;=\; 0 \\ \\ u(0,\cdot) \;=\; \overline{\rho}(\cdot)\overline{\theta}(\cdot)\ . \end{array}\right. \tag{25}$$

The following proposition shows that the absolute continuity of all measures μ_N is necessary for the validity of Conjecture 4.1.

Proposition 4.4. *If Conjecture 4.1 holds true, then for every $\overline{\rho}$ the transport map Ψ generated by $\overline{\rho}$ has no singular set. That is, the measures μ_N of Definition 4.2 are all absolutely continuous with respect to $\mathscr{L}^n$.*

Proof. Assume that for some $\overline{\rho}$ as above the transport map Ψ generated by $\overline{\rho}$ has a singular set S. Then, for every $\overline{\theta} \in L^\infty(\mathbf{R}^n, \mathbf{S}^{k-1})$ we can find Borel maps $\tilde\theta^1$, $\tilde\theta^2 : \mathbf{R}^n \to \mathbf{S}^{k-1}$ such that

- $\tilde\theta^1 = \overline{\theta} = \tilde\theta^2$ on $\mathbf{R}^n \setminus S$.
- $\tilde\theta^1(x) \neq \tilde\theta^2(x)$ for every $x \in S$.

If we define $u^j(t,x) = \rho(t,x)\tilde\theta^j\big(\Psi(t,x)\big)$, then both u^1, u^2 solve (25) by Theorem 4.3. Moreover $u^1(t,x) \neq u^2(t,x)$ for every $(t,x) \in \Psi^{-1}(S)$. According to the definition of S, we have

$$\mathscr{L}^{n+1}\left(\Psi^{-1}(S) \cap [0,N]\times\mathbf{R}^n\right) > 0$$

for some $N > 0$. □

4.2. **Stability.** We have seen in the previous subsection that the absolute continuity of all measures μ_N is necessary for uniqueness. Here we show that the natural extension of this property to the case when the initial data are given by a sequence of maps $\overline{\rho}^j$, namely the equi-integrability of the measures μ_N^j associated to $\overline{\rho}^j$, leads to a stability result for the solutions built as in Theorem 2.6.

Let $\overline{\rho}$, $\overline{\rho}^j \in BV_{loc} \cap L^\infty(\mathbf{R}^n)$, $\overline{\theta}$, $\overline{\theta}^j \in L^\infty(\mathbf{R}^n, \mathbf{S}^{k-1})$ and assume that

- $0 < c \leq \overline{\rho}^j \leq C$ for every j;
- $\overline{\theta}^j \to \overline{\theta}$ in L^1_{loc};
- $\overline{\rho}^j \to \overline{\rho}$ in L^1_{loc} and $\sup_j \|\overline{\rho}^j\|_{BV(A)} < +\infty$ for any bounded open set $A \subset \mathbf{R}^n$.

Denote by Ψ, Ψ^j the transport maps generated by $\overline{\rho}$, $\overline{\rho}^j$. For any $N, j \in \mathbb{N}$ denote by μ_N^j (resp. μ_N) the measures which are the pushforward via Ψ^j (resp. Ψ) of the Lebesgue measure $\mathscr{L}^{n+1}$ restricted to the set $[0,N] \times \mathbf{R}^n$. Define the maps $\rho, \rho^j \in BV_{loc}\big(\mathbf{R}_t \times \mathbf{R}^n_x\big)$ as in Theorem 4.3 (i.e. as constructed in the first Step of the proof of Theorem 2.6).

We recall that a sequence of locally integrable functions g^j is said to be locally equiintegrable if for any $R, \varepsilon > 0$ there exists $\delta = \delta(R, \varepsilon) > 0$ such that $\int_A |g^j|\, dx < \varepsilon$ for any Borel set $A \subset B_R$ with $\mathscr{L}^n(A) < \delta$.

Theorem 4.5. *Define*

$$u^j(t,x) \;:=\; \rho^j(t,x)\overline{\theta}^j\big(\Psi^j(t,x)\big), \qquad u(t,x) \;:=\; \rho(t,x)\overline{\theta}\big(\Psi(t,x)\big)$$

and assume that

$$\mu_N^j = f_N^j \mathscr{L}^n \text{ and the sequence } \{f_N^j\}^j \text{ is locally equiintegrable for any } N. \tag{26}$$

Then $u^j \to u$ strongly in L^1_{loc}.

Proof. Theorem 4.3 gives that the maps u^j, u satisfy

$$\begin{cases} \partial_t u^j + \sum_{\alpha=1}^n \partial_{x_\alpha}(f_\alpha(|u^j|)u^j) \;=\; 0 \\ \\ u^j(0,\cdot) \;=\; \overline{\rho}^j(\cdot)\overline{\theta}^j(\cdot) \end{cases} \qquad \begin{cases} \partial_t u + \sum_{\alpha=1}^n \partial_{x_\alpha}(f_\alpha(|u|)u) \;=\; 0 \\ \\ u(0,\cdot) \;=\; \overline{\rho}(\cdot)\overline{\theta}(\cdot)\,. \end{cases}$$

Fix an open set $\Omega \subset\subset \mathbf{R}_t^+ \times \mathbf{R}_x^n$. We will prove that

(P) For *every* sequence $\{j(r)\}_r \subset \mathbb{N}$ going to infinity there exists a further subsequence $\{j(r(l))\}$ such that $u^{j(r(l))}$ converges to u in $L^1(\Omega)$.

This implies that the *whole* initial sequence $\{u^j\}$ converges to u in $L^1(\Omega)$. The arbitrariness of Ω gives the claim. We now come to the proof of (P). Thus let us fix any subsequence $\{u^{j(r)}\}$ and to simplify the notation let us drop the index r.

Theorem 2.2 and the compactness of the embedding of BV in L^1_{loc} imply that $\rho^j \to \rho$ in L^1_{loc}. Using arguments similar to those of the Third Step, we can see that Theorem 2.7 and Remark 2.8 imply that (possibly passing to a subsequence) $\Psi^j \to \Psi$ in L^1_{loc} and pointwise $\mathscr{L}^{n+1}$ almost everywhere. Note that there exists a compact set $K \subset \mathbf{R}^n$ such that $\Psi(\Omega), \Psi^j(\Omega) \subset K$ (cf. Fourth Step of Proof of Theorem 2.6). Since

$$\lim_{j\to\infty} \int_{\mathbf{R}^n} \varphi\, d\mu_N^j = \lim_{j\to\infty} \int_{[0,N]\times\mathbf{R}^n} \varphi \circ \Psi^j \, dtdx = \int_{[0,N]\times\mathbf{R}^n} \varphi \circ \Psi \, dtdx = \int_{\mathbf{R}^n} \varphi\, d\mu_N$$

for any continuous function φ with compact support in $\mathbf{R}^n$, we have the weak convergence as measures of μ_N^j to μ_N. As a consequence, the equiintegrability gives

$$\mu_N \text{ is absolutely continuous with respect to } \mathscr{L}^n \text{ for any } N. \tag{27}$$

By Egorov Theorem, there exists a sequence of compact sets $\{K_i\}$ such that $\lim_i \mathscr{L}^n(K \setminus K_i) = 0$ and the sequence $\{\overline{\theta}^j\}$ is equicontinuous on every K_i and converges to $\overline{\theta}$ uniformly on K_i. Condition (27) implies that $\lim_i \mathscr{L}^{n+1}(\Omega \setminus \Psi^{-1}(K_i)) = 0$. For each i, consider a sequence of compact sets $\{K_i^l\}$ such that

- $K_i^l \subset \Psi^{-1}(K_i)$;
- $\lim_l \mathscr{L}^{n+1}\big([\Psi^{-1}(K_i)] \setminus K_i^l\big) = 0$;
- $\Psi^j \to \Psi$ uniformly on every K_i^l.

We will prove that for each i, l there exists a subsequence $\{u^j\}$, not relabeled, which converges to u in $L^1(K_i^l)$. A diagonal argument yields a subsequence $\{u^j\}$ (again not relabeled) which converges strongly in $L^1(\Omega)$.

Since $\Psi^j \to \Psi$ uniformly on K_i^l and $\Psi(K_i^l) \subset \Psi\big(\Psi^{-1}(K_i)\big) = K_i$, we have that

$$\lim_{j\to\infty} \mathscr{L}^n\left(\Psi^j(K_i^l) \setminus K_i\right) \;=\; 0\,. \tag{28}$$

Assumption (26) and (28) imply that

$$\lim_{j\to\infty} \mathscr{L}^{n+1}\Big(K_i^l \setminus \big[\big(\Psi^j\big)^{-1}(K_i)\big]\Big) \;=\; 0\,. \tag{29}$$

Fix $M \in \mathbb{N}$ and for each r let $j(r) \geq r$ be such that

$$\mathscr{L}^{n+1}\Big(K_i^l \setminus \big[\big(\Psi^{j(r)}\big)^{-1}(K_i)\big]\Big) \;\leq\; \frac{2^{-r}}{M}\,. \tag{30}$$

Thus if we define the set

$$K'_M \;:=\; K_i^l \cap \left\{\bigcap_r \left[\big(\Psi^{j(r)}\big)^{-1}(K_i)\right]\right\}$$

we get $\mathscr{L}^{n+1}(K_i^l \setminus K'_M) \leq \frac{1}{M}$. We will show $u^{j(r)} \to u$ in $L^1(K'_M)$. Note that $\Psi^{j(r)}(K'_M) \subset K_i$. Since

- $\Psi^{j(r)} \to \Psi$ uniformly on K'_M,
- $\overline{\theta}^{j(r)}$ is equicontinuous on K_i,
- $\overline{\theta}^{j(r)} \to \overline{\theta}$ uniformly on K_i,

we conclude that $u^{j(r)} = \overline{\theta}^{j(r)} \circ \Psi^{j(r)}$ converges uniformly to $u = \overline{\theta} \circ \Psi$ on K'_M. Hence $u^{j(r)}$ converges to u in $L^1(K'_M)$. A diagonal argument yields a subsequence which converges to u in $L^1(K_i^l)$. This completes the proof of (P) and hence the proof of the theorem. $\square$

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