MINIMAL ENTROPY CONDITIONS FOR BURGERS EQUATION

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Abstract. We consider strictly convex, 1-d scalar conservation laws. We show that a single strictly convex entropy is sufficient to characterize a Kruzhkov solution. The proof uses the concept of viscosity solution for the related Hamilton-Jacobi equation.

1. Introduction

We consider a scalar conservation law in one space dimension with strictly convex flux $f \in C^2(\mathbf{R})$

$$u_t + f(u)_x = 0 \quad \text{for } (t,x) \in [0,\infty) \times \mathbf{R} \tag{1}$$

and initial data $u(0,\cdot) = u_0 \in L^\infty(\mathbf{R})$. It is well–known that, even for smooth initial data, the solution of (1) ceases to exist in the classical sense due to the formation of shocks in finite time (cf. Riemann 1859 [10]). Therefore one has to consider weak solutions, i.e. functions with suitable integrability satisfying (1) in distributional sense. Since weak solutions lack uniqueness, additional assumptions must be imposed to select the (physically) relevant ones.

In 1957, Oleinik [8] proved existence and uniqueness of bounded weak solutions of (1) which satisfy her *"condition E"*: She proved that the one–sided Lipschitz condition

$$\|(u_x)_+(t,\cdot)\|_{L^\infty(\mathbf{R})} \;\leq\; \frac{1}{ct} \quad \text{for all } t > 0, \tag{2}$$

where $c := \inf f''$, singles out one particular among all possible weak solutions. Note that (2) only allows for decreasing jumps. As an immediate consequence of (2) we obtain a smoothing effect: Initial data in L^∞ are instantaneously regularized to BV_{loc}. Additional nice properties follow from (2) such as: regularity of solutions, rates of convergence to limiting profiles as $t \to \infty$, convergence of approximation schemes etc. We refer to Section 11 of [4].

A more flexible approach to scalar conservation laws, feasible also in higher space dimensions and for nonconvex fluxes, was given by Kruzhkov [5] in 1970, following earlier work by Conway&Smoller and Vol'pert. For initial data in L^∞, Kruzhkov proved existence and uniqueness of bounded weak solutions satisfying the following *entropy condition*: He considers the family of convex entropy–entropy flux pairs $(\eta_k, q_k)_{k\in\mathbf{R}}$, where

$$\eta_k(u) \;:=\; |u-k| \quad \text{and} \quad q_k(u) \;:=\; \operatorname{sgn}(u-k)\big(f(u) - f(k)\big).$$

Then an entropy solution is a bounded function u satisfying (1) in distributional sense and

$$\eta(u)_t + q(u)_x \;\leq\; 0 \quad \text{in } \mathcal{D}' \tag{3}$$

for all $(\eta, q) \in (\eta_k, q_k)_{k\in\mathbf{R}}$. Equivalently, we may assume (3) for all convex functions (entropies) η, with corresponding entropy flux q defined by $q' = \eta' f'$. Kruzhkov's proof is based on the observation that the solution operator of (1) is, in fact, an L^1–contraction. As a consequence of this, if initial data are in BV_{loc}, then $u \in BV_{loc}$ for all later times.

One can show that for convex flux, the two entropy conditions (2) and (3) are equivalent. Hence Oleinik's "condition E"– and Kruzhkov's entropy solutions coincide. It has been an important open question whether a restricted entropy condition, i.e. assuming (3) only for a subset of convex entropy–entropy flux pairs, would enforce uniqueness of the solution (and hence provide us with all the nice features of Oleinik's solutions). This question is also quite interesting in view of the fact that for most systems of conservation laws only few convex entropies—or even just one (the physical) entropy—are known to exist.

The problem has been solved first in a paper by Panov [9] who proves that, if f is strictly convex, then assumption (3) for *one* single strictly convex entropy–entropy flux pair (η, q) is sufficient to establish Oleinik's E–condition, and thereby characterize entropy solutions among all bounded weak solutions of (1). Thus it is not necessary to consider the whole family of convex entropies, or all Kruzhkov entropies. One single pair (η, q), with η strictly convex, gives all the information.

In this paper, we give a new proof of this result, see Theorem 2.3. Like Panov, we use the theory of Hamilton-Jacobi equations. But our argument is different from his, and more general for three reasons: First, our approach is not restricted to the Cauchy problem on the real line. Second, we can allow a right hand side in the entropy inequality. Third, our proof also works for unbounded functions.

The last generalization is important since it allows for the derivation of new estimates for the Kuramoto-Shivashinsky equation, see [7]. We prove that for Burgers' flux $f(u) = \frac{1}{2}u^2$, and for the special convex entropy $\eta(u) := \frac{1}{2}u^2$ (with corresponding entropy flux $q(u) := \frac{1}{3}u^3$), it is sufficient to require $u \in L^4_{loc}$ only, instead of $u \in L^\infty$. Then still a restricted entropy information is sufficient to single out entropy solutions of Burgers equation, see Theorem 2.4 below. We believe that this result can be generalized, allowing for different strictly convex fluxes and entropies. Then the optimal integrability of u should depend on the growth rates of f and η at infinity.

As mentioned above, our proof is based on the relation between scalar conservation laws in one space dimension and Hamilton–Jacobi equations. Integrating (1) in x we obtain

$$h_t + f(h_x) \;=\; 0 \quad \text{for } (t,x) \in [0,\infty) \times \mathbf{R}, \tag{4}$$

where h is the x–primitive of u, i.e. $h_x = u$. Therefore, (1) and (4) are formally equivalent. There exists a rather complete existence and uniqueness theory for (4) based on the concept of *viscosity solutions*, first introduced by Crandall and Lions in [3] (see also [6]). It is well known that, when f is strictly convex, h is a viscosity solution of (4) if and only if $u = h_x$ is an entropy solution of (1). In Theorems 2.3 and 2.4 we prove that the assumptions on u are sufficient to guarantee that h is a viscosity solution of (4).

Our proof has been inspired by a recent result of Ambrosio, Lecumberry and Rivière [2]. There the authors proved that, if $h \in W^{1,\infty}_{loc}(\mathbf{R}^2)$ solves the eikonal equation, then a certain one–parameter family of entropy conditions on ∇h are sufficient to ensure that h is a viscosity solution. As in that paper, our proofs rely on commutator estimates which are similar in spirit to the arguments used in compensated compactness theory.

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2. Main result

We begin by giving the definitions of entropy solutions of scalar conservation laws and viscosity solutions of Hamilton–Jacobi equations.

Definition 2.1. *Let $\Omega \subset \mathbf{R}^2$ and $f \in W^{1,\infty}_{loc}(\mathbf{R})$. We say that $\eta, q \in W^{1,\infty}_{loc}(\mathbf{R})$ is a* convex entropy–entropy flux pair *if η is convex and $q' = f'\eta'$ almost everywhere. A function $u \in L^1_{loc}(\Omega)$ is called an* entropy solution *of $u_t + f(u)_x = 0$ if*

$$
\begin{aligned}
u_t + f(u)_x &= 0 \quad \text{in } \mathcal{D}'(\Omega), && (5)\\
\eta(u)_t + q(u)_x &\leq 0 \quad \text{in } \mathcal{D}'(\Omega) && (6)
\end{aligned}
$$

for every convex entropy–entropy flux pair (η, q).

Definition 2.2. *A function $h \in C(\Omega)$ is called a* viscosity solution *of $h_t + f(h_x) = 0$ if for any $(t, x) \in \Omega$ the following hold:*

(1) *If ζ is any smooth function such that $h - \zeta$ has a local maximum at (t, x), then*

$$\zeta_t(t,x) + f\big(\zeta_x(t,x)\big) \leq 0\,.$$

(2) *If ζ is any smooth function such that $h - \zeta$ has a local minimum at (t, x), then*

$$\zeta_t(t,x) + f\big(\zeta_x(t,x)\big) \geq 0\,.$$

A function h for which (1) resp. (2) holds, is called a viscosity sub/supersolution.

We are now ready to state our results:

Theorem 2.3. *Let $f \in C^2(\mathbf{R})$ be strictly convex and fix an entropy–entropy flux pair (η, q) with η strictly convex. Assume that $\Omega \subset \mathbf{R}^2$ is an open set and $u \in L^\infty(\Omega)$ satisfies*

$$
\begin{aligned}
u_t + f(u)_x &= 0 \quad \text{in } \mathcal{D}'(\Omega), && (7)\\
\eta(u)_t + q(u)_x &\leq 0 \quad \text{in } \mathcal{D}'(\Omega)\,. && (8)
\end{aligned}
$$

If h is a function with $h_t = -f(u)$ and $h_x = u$, then h is a viscosity solution of $h_t + f(h_x) = 0$.

For Burgers equation we even have

Theorem 2.4. *Let $\Omega \subset \mathbf{R}^2$ be open, and assume that the function $u \in L^4_{loc}(\Omega)$ satisfies*

$$
\begin{aligned}
u_t + (\tfrac{u^2}{2})_x &= 0 \quad \text{in } \mathcal{D}'(\Omega), && (9)\\
(\tfrac{u^2}{2})_t + (\tfrac{u^3}{3})_x &\leq \mu \quad \text{in } \mathcal{D}'(\Omega) && (10)
\end{aligned}
$$

for some non-negative Radon measure μ with

$$\lim_{r \downarrow 0} \frac{\mu\big(B_r(t,x)\big)}{r} = 0 \quad \text{for every } (t,x) \in \Omega. \tag{11}$$

If h is a function with $h_t = -\frac{u^2}{2}$ and $h_x = u$, then h is a viscosity solution of $h_t + (\frac{h_x^2}{2}) = 0$.

We observe that condition (11) on the measure μ is optimal. It is just enough to rule out the presence of undercompressive shocks in (t,x), in which case

$$\liminf_{r\downarrow 0}\frac{\mu\big(B_r(t,x)\big)}{r}>0\,.$$

The first theorem is proved in Section 4 and the second is proved in Section 5. They both rely on some elementary inequalities on averages of functions, which are proved in Section 3. As a consequence of the theorems above we obtain:

Corollary 2.5. *The function u of Theorem 2.3 is an entropy solution of* (7). *Similarly, the function u of Theorem 2.4 is an entropy solution of* (9).

Proof. As already mentioned in the introduction, it is a well–known fact that h is a viscosity solution of $h_t+f(h_x)=0$ if and only if $u=h_x$ is an entropy solution of $u_t+f(u)_t=0$. To prove this we can use, for instance, the Hopf–Lax formula, which gives explicitly the viscosity solution to $h_t+f(h_x)=0$ in terms of $h(0,\cdot)$. We here give an alternative proof, which uses the properties of viscosity solutions to show that $u=h_x\in BV_{loc}$; for BV functions it can be shown directly that u meets the shock–admissibility criterion which distinguishes entropy solutions.

First Step BV regularity.

In the case of Theorem 2.3, since h is locally Lipschitz and is a viscosity solution of $h_t+f(h_x)$, Corollary 9.2 of [6] gives that for every open $U\subset\subset\Omega$, there exists a constant C such that $h_{xx}\le C$ in the sense of distributions. Differentiating $h_t+f(h_x)=0$ with respect to x we get that $h_{xt}=h_{tx}=-(f(h_x))_x$ is also a measure. Hence $u=h_x\in BV_{loc}(\mathbf{R}^2)$.

In the case of Theorem 2.4, a standard comparison argument with supersolutions of type $k\psi(t)|x-x_0|$ gives that h_x is locally bounded. Thus $h_t=-\frac{(h_x)^2}{2}$ is locally bounded as well and hence $h\in W^{1,\infty}_{loc}$. Thus we can apply the argument above to show that $h_x\in BV_{loc}$.

Second Step Shock admissibility.

By Vol'pert chain rule (see [1]), the condition that the entropy dissipation must be non-positive translates into the following shock admissibility condition:

$$-s[\eta(u^+)-\eta(u^-)]+q(u^+)-q(u^-)\le 0 \tag{12}$$

for all convex entropy–entropy flux pairs (η,q) and for all points in the jump (shock) set J_u of u. Here $u^\pm$ are the right resp. left trace of u on J_u, and s is shock speed, i.e. the slope of the tangent to J_u in (t,x). We refer to Section 8.5 of [4]. Condition (12) follows from the fact that

$$u^+\le u^-\ \mathcal{H}^1\text{–a.e. on } J_u \text{ (only decreasing jumps)} \tag{13}$$

which itself is a consequence of h being a viscosity solution. For simplicity, we check that (13) holds when h is smooth outside a differentiable curve $\gamma:=\{(t,x(t))|\ t\ge 0\}$. Indeed, by a standard blow–up argument, it is sufficient to prove (13) when u is a piecewise constant function which jumps along a line.

Suppose that h is given on the right resp. left hand side of γ by smooth functions $h^\pm$, satisfying $h^\pm_t+f(h^\pm_x)=0$. Assume h is continuous along the curve, i.e. $h^+(t,x(t))=h^-(t,x(t))$ for all $t\ge 0$. From $\frac{d}{dt}h^+(t,x(t))=\frac{d}{dt}h^-(t,x(t))$, after a straightforward calculation we obtain

$$-x'(t)\big[h_x^+(t,x(t)) - h_x^-(t,x(t))\big] + f\big(h_x^+(t,x(t))\big) - f\big(h_x^-(t,x(t))\big) \;=\; 0 \tag{14}$$

for all $t \geq 0$ (Rankine–Hugoniot). Now fix some point $(T, X = x(T)) \in \gamma$. Then we claim

$$h_x^+(T,X) \;\leq\; h_x^-(T,X).$$

In fact, assume the opposite and fix some $c \in \big(h_x^-(T,X), h_x^+(T,X)\big)$. Consider the function

$$\zeta(t,x) \;=\; h^-(t,x(t)) + c(x - x(t)).$$

Then $h - \zeta$ has a local minimum at (T,X). We compute

$$\begin{aligned}\zeta_t(T,X) + f\big(\zeta_x(T,X)\big) \;&=\; h_t^-(T,X) + x'(T)h_x^-(T,X) - cx'(T) + f(c)\\ &=\; -x'(T)\big[c - h_x^-(T,X)\big] + f(c) - f\big(h_x^-(T,X)\big).\end{aligned}$$

The r.h.s. is a strictly convex function in c which vanishes in $h_x^\pm(T,X)$ because of (14). Since $h_x^-(T,X) < c < h_x^+(T,X)$ by assumption, we have $\zeta_t(T,X) + f(\zeta_x(T,X)) < 0$. But that is a contradiction: h is a viscosity solution. This gives the claim. □

3. Preliminary lemmas on averages of functions

To prove the theorems we need a result on averages of functions.

Definition 3.1. *Let μ be a probability measure on $\mathbf{R}$. For every vector–valued map $V \in L^1(\mathbf{R},\mu)$ we set $\langle V\rangle := \int_\Omega V(u)\,d\mu(u)$. Let $f,\eta \in W^{1,\infty}_{loc}(\mathbf{R})$ and $q(v) := \int_0^v f'(\tau)\eta'(\tau)\,d\tau$. If μ is compactly supported then we define the bilinear form*

$$\begin{aligned} B(f,\eta) \;&:=\; \left\langle \begin{pmatrix}\eta\\ q\end{pmatrix} \cdot \begin{pmatrix}-f\\ u\end{pmatrix}\right\rangle - \left\langle \begin{pmatrix}\eta\\ q\end{pmatrix}\right\rangle \cdot \left\langle \begin{pmatrix}-f\\ u\end{pmatrix}\right\rangle \\ &=\; \langle uq\rangle - \langle \eta f\rangle + \langle f\rangle\langle\eta\rangle - \langle u\rangle\langle q\rangle\,. \end{aligned} \tag{15}$$

When μ has noncompact support we define $B(f,\eta)$ whenever all the functions appearing in (15) *are μ–summable.*

In the definition above and in what follows, u will always denote the independent real variable or the identity function on $\mathbf{R}$. For example, $\langle u\rangle$ and $\langle u^2\rangle$ denote $\int_{\mathbf{R}} u\,d\mu(u)$ and $\int_{\mathbf{R}} u^2\,d\mu(u)$.

Proposition 3.2.

(a) *If u^2 is μ–summable, then $3B(u^2,u^2) \geq \langle (u-\langle u\rangle)^4\rangle$.*

(b) *If μ has compact support and f and η are both convex, then $B(f,\eta) \geq 0$.*

(c) *If μ has compact support and $f'', \eta'' \geq 2c$, then $3B(f,\eta) \geq c^2\langle (u-\langle u\rangle)^4\rangle$.*

Proof. **(a)** Since $f(u) = \eta(u) = u^2$ and $q(u) = \frac{4}{3}u^3$, (15) implies

$$B(u^2,u^2) \;=\; \frac{4}{3}\left\langle u^4\right\rangle - \left\langle u^4\right\rangle + \left\langle u^2\right\rangle^2 - \frac{4}{3}\left\langle u\right\rangle\left\langle u^3\right\rangle \;=\; \frac{1}{3}\left\langle u^4\right\rangle + \left\langle u^2\right\rangle^2 - \frac{4}{3}\left\langle u\right\rangle\left\langle u^3\right\rangle\,. \tag{16}$$

Recall that $\langle u\rangle^2 \leq \langle u^2\rangle$ by Jensen and thus

$$0 \;\leq\; u^2 - 2u\langle u\rangle + \langle u\rangle^2 \;\leq\; u^2 - 2u\langle u\rangle + \left\langle u^2\right\rangle.$$

Hence we obtain

$$\begin{aligned}\Big\langle\big(u-\langle u\rangle\big)^4\Big\rangle &= \Big\langle\big(u^2-2u\langle u\rangle+\langle u\rangle^2\big)^2\Big\rangle \;\le\; \Big\langle\big(u^2-2u\langle u\rangle+\langle u^2\rangle\big)^2\Big\rangle\\ &= \langle u^4\rangle-4\langle u\rangle\langle u^3\rangle+2\langle u^2\rangle^2+4\langle u^2\rangle\langle u\rangle^2-4\langle u\rangle^2\langle u^2\rangle+\langle u^2\rangle^2\\ &= \langle u^4\rangle-4\langle u\rangle\langle u^3\rangle+3\langle u^2\rangle^2 \overset{(16)}{=} 3B(u^2,u^2)\,. \end{aligned}\tag{17}$$

(b) W.l.o.g. we may assume that η is smooth and η'' has compact support. Then we have

$$\eta(u) \;=\; \int_{-\infty}^{+\infty}\frac{1}{2}\eta''(k)|u-k|\,dk+\text{const.}$$

Hence we may assume that η is of Kruzhkov's form

$$\eta(u) \;=\; |u-k| \quad\text{and thus}\quad q(u) \;=\; \operatorname{sign}(u-k)\big(f(u)-f(k)\big).$$

We now have

$$\begin{aligned}B(f,\eta) &= \langle(u-k)q\rangle-\big\langle\eta\big(f(u)-f(k)\big)\big\rangle\\ &\quad+\langle f(u)-f(k)\rangle\,\langle\eta\rangle-\langle u-k\rangle\langle q\rangle\\ &= \langle f(u)-f(k)\rangle\,\langle|u-k|\rangle\\ &\quad-\langle u-k\rangle\,\big\langle\operatorname{sign}(u-k)\big(f(u)-f(k)\big)\big\rangle\\ &= \langle f(u)-f(k)-f'(k)(u-k)\rangle\,\langle|u-k|\rangle\\ &\quad-\langle u-k\rangle\,\big\langle\operatorname{sign}(u-k)\big(f(u)-f(k)-f'(k)(u-k)\big)\big\rangle\,.\end{aligned}$$

The latter expression is non-negative, since by convexity of f

$$f(u)-f(k)-f'(k)(u-k) \;\ge\; 0\,.$$

(c) Consider the convex functions $f_1(v)=f(v)-cv^2$ and $\eta_1(v)=\eta(v)-cv^2$. Thanks to (b), $B(f_1,\eta_1)$, $B(f_1,v^2)$ and $B(v^2,\eta_1)$ are all non-negative. Hence (a) completes the proof. ☐

4. Proof of Theorem 2.3

Note that h is a Lipschitz function. The proof of the Theorem is split into two parts. In the first one we prove that h is a viscosity subsolution of $h_t+f(h_x)=0$. In the second one we prove that h is a viscosity supersolution.

4.1. **Viscosity subsolution.**

We have by construction of h:

$$h_t=-f(u)=-f(h_x) \quad \text{a.e. in } \Omega. \tag{18}$$

Let $\xi\in C_c^\infty(\mathbf{R}^2)$ be non-negative with $\int_{\mathbf{R}^2}\xi\,dt\,dx=1$. Set $\xi_\varepsilon(t,x):=\frac{1}{\varepsilon^2}\xi(\frac{t}{\varepsilon},\frac{x}{\varepsilon})$. By Jensen

$$0 \;=\; \big(h_t+f(h_x)\big)\star\xi_\varepsilon \;\;\ge\;\; h_t\star\xi_\varepsilon+f\left(h_x\star\xi_\varepsilon\right) \;=\; (h\star\xi_\varepsilon)_t+f\big((h\star\xi_\varepsilon)_x\big)\,.$$

Hence $h\star\xi_\varepsilon$ is a classical subsolution and thus also a viscosity subsolution (see Corollary I.6 of [3]). Since h is continuous, $h\star\xi_\varepsilon$ converges locally uniformly to h as $\varepsilon\downarrow 0$. Thus also h is a viscosity subsolution, by the stability result in Theorem I.1 of [3].

4.2. **Viscosity supersolution.** To prove that h is a viscosity supersolution we have to show the following fact: If ζ is a smooth function such that $h-\zeta$ has a minimum at some $(t,x)\in\Omega$, then $[\zeta_t+f(\zeta_x)]\,(t,x)\geq 0$. For simplicity we assume that $(t,x)=(0,0)$ and $[h-\zeta](0,0)=0$. Moreover we assume that the minimum is strict. Indeed, if we choose $\varepsilon>0$ and consider $\zeta^\varepsilon(t,x):=\zeta(t,x)+\varepsilon(t^2+x^2)$, then $[h-\zeta^\varepsilon]$ has a strict minimum at $(0,0)$ and $[\zeta^\varepsilon_t+f(\zeta^\varepsilon_x)]\,(0,0)=[\zeta_t+f(\zeta_x)]\,(0,0)$.

For any $\delta>0$ consider

$$\Omega_\delta \;:=\; \text{connected component of } \{(t,x) : [h-\zeta](t,x)<\delta\} \text{ containing } (0,0)\,.$$

Since h is continuous and the origin is a strict minimum, Ω_δ is an open set and $\operatorname{diam}(\Omega_\delta)\downarrow 0$ as $\delta\downarrow 0$. We introduce the notation

$$\langle g\rangle_\delta \;:=\; \fint_{\Omega_\delta} g(t,x)\,dt\,dx \;=\; \frac{1}{|\Omega_\delta|}\int_{\Omega_\delta} g(t,x)\,dt\,dx\,.$$

By definition of h,

$$\left\langle\begin{pmatrix}\zeta_t\\ \zeta_x\end{pmatrix}\right\rangle_\delta = -\left\langle\begin{pmatrix}(h-\zeta)_t\\ (h-\zeta)_x\end{pmatrix}\right\rangle_\delta + \left\langle\begin{pmatrix}-f(u)\\ u\end{pmatrix}\right\rangle_\delta. \tag{19}$$

For δ sufficiently small we have $\Omega_\delta\subset\subset B_1$. Thus

$$\langle(h-\zeta)_t\rangle_\delta \;=\; \fint_{\Omega_\delta}(h-\zeta)_t \;=\; \frac{1}{|\Omega_\delta|}\int_{\Omega_\delta}(h-\zeta)_t \;=\; \frac{1}{|\Omega_\delta|}\int_{B_1}(\min\{h-\zeta-\delta,0\})_t\,. \tag{20}$$

Since the function $\min\{h-\zeta-\delta,0\}$ is continuous and identically zero on a neighborhood of ∂B_1, the right hand side of (20) vanishes. The same argument applies to $\langle(h-\zeta)_x\rangle_\delta$. Hence, from (19) we get

$$\left\langle\begin{pmatrix}\zeta_t\\ \zeta_x\end{pmatrix}\right\rangle_\delta = \left\langle\begin{pmatrix}-f(u)\\ u\end{pmatrix}\right\rangle_\delta. \tag{21}$$

On the other hand we have

$$\left\langle\begin{pmatrix}\zeta_t\\ \zeta_x\end{pmatrix}\cdot\begin{pmatrix}\eta(u)\\ q(u)\end{pmatrix}\right\rangle_\delta = \left\langle\begin{pmatrix}-f(u)\\ u\end{pmatrix}\cdot\begin{pmatrix}\eta(u)\\ q(u)\end{pmatrix}\right\rangle_\delta - \left\langle\begin{pmatrix}(h-\zeta)_t\\ (h-\zeta)_x\end{pmatrix}\cdot\begin{pmatrix}\eta(u)\\ q(u)\end{pmatrix}\right\rangle_\delta. \tag{22}$$

Note that

$$\begin{aligned}
-\left\langle\begin{pmatrix}(h-\zeta)_t\\ (h-\zeta)_x\end{pmatrix}\cdot\begin{pmatrix}\eta(u)\\ q(u)\end{pmatrix}\right\rangle_\delta \;&=\; -\frac{1}{|\Omega_\delta|}\int_{B_1}\begin{pmatrix}(\min\{h-\zeta-\delta,0\})_t\\ (\min\{h-\zeta-\delta,0\})_x\end{pmatrix}\cdot\begin{pmatrix}\eta(u)\\ q(u)\end{pmatrix}\\
&=\; \frac{1}{|\Omega_\delta|}\int_{B_1}\min\{h-\zeta-\delta,0\}\,\big([\eta(u)]_t+[q(u)]_x\big)\\
&\overset{(8)}{\geq}\; 0\,.
\end{aligned} \tag{23}$$

Thus we obtain from (22)

$$\left\langle\begin{pmatrix}\zeta_t\\ \zeta_x\end{pmatrix}\cdot\begin{pmatrix}\eta(u)\\ q(u)\end{pmatrix}\right\rangle_\delta \geq \left\langle\begin{pmatrix}-f(u)\\ u\end{pmatrix}\cdot\begin{pmatrix}\eta(u)\\ q(u)\end{pmatrix}\right\rangle_\delta.$$

Therefore, with $C = C(\|u\|_\infty)$,

$$\begin{aligned}
&\left\langle \begin{pmatrix} -f(u) \\ u \end{pmatrix} \cdot \begin{pmatrix} \eta(u) \\ q(u) \end{pmatrix} \right\rangle_\delta - \left\langle \begin{pmatrix} -f(u) \\ u \end{pmatrix} \right\rangle_\delta \cdot \left\langle \begin{pmatrix} \eta(u) \\ q(u) \end{pmatrix} \right\rangle_\delta && (24)\\
&\overset{(23)}{\leq} \left\langle \begin{pmatrix} \zeta_t \\ \zeta_x \end{pmatrix} \cdot \begin{pmatrix} \eta(u) \\ q(u) \end{pmatrix} \right\rangle_\delta - \left\langle \begin{pmatrix} -f(u) \\ u \end{pmatrix} \right\rangle_\delta \cdot \left\langle \begin{pmatrix} \eta(u) \\ q(u) \end{pmatrix} \right\rangle_\delta \\
&\overset{(21)}{=} \left\langle \begin{pmatrix} \zeta_t \\ \zeta_x \end{pmatrix} \cdot \begin{pmatrix} \eta(u) \\ q(u) \end{pmatrix} \right\rangle_\delta - \left\langle \begin{pmatrix} \zeta_t \\ \zeta_x \end{pmatrix} \right\rangle_\delta \cdot \left\langle \begin{pmatrix} \eta(u) \\ q(u) \end{pmatrix} \right\rangle_\delta \\
&\leq C \sup_{\Omega_\delta} \left| \begin{pmatrix} \zeta_t \\ \zeta_x \end{pmatrix} - \left\langle \begin{pmatrix} \zeta_t \\ \zeta_x \end{pmatrix} \right\rangle_\delta \right| . && (25)
\end{aligned}$$

We can now apply Proposition 3.2 and obtain

$$\begin{aligned}
&\frac{c^2}{3} \left\langle (u - \langle u \rangle_\delta)^4 \right\rangle_\delta \\
&\leq \left\langle \begin{pmatrix} -f(u) \\ u \end{pmatrix} \cdot \begin{pmatrix} \eta(u) \\ q(u) \end{pmatrix} \right\rangle_\delta - \left\langle \begin{pmatrix} -f(u) \\ u \end{pmatrix} \right\rangle_\delta \cdot \left\langle \begin{pmatrix} \eta(u) \\ q(u) \end{pmatrix} \right\rangle_\delta \\
&\leq C \sup_{\Omega_\delta} \left| \begin{pmatrix} \zeta_t \\ \zeta_x \end{pmatrix} - \left\langle \begin{pmatrix} \zeta_t \\ \zeta_x \end{pmatrix} \right\rangle_\delta \right| .
\end{aligned}$$

Since ζ is smooth, we deduce

$$\lim_{\delta \downarrow 0} \left\langle (u - \langle u \rangle_\delta)^4 \right\rangle_\delta \;=\; 0 . \tag{26}$$

Since f is Lipschitz on $\big[-\|u\|_\infty, \|u\|_\infty\big]$, we have

$$\left| \langle f(u) \rangle_\delta - f\big(\langle u \rangle_\delta\big) \right| \;\leq\; C \langle |u - \langle u \rangle_\delta| \rangle_\delta \;\leq\; C \left\langle (u - \langle u \rangle_\delta)^4 \right\rangle_\delta^{1/4},$$

so that (26) yields

$$\lim_{\delta \downarrow 0} \left| \langle f(u) \rangle_\delta - f\big(\langle u \rangle_\delta\big) \right| \;=\; 0 .$$

In view of (21) this translates into

$$\lim_{\delta \downarrow 0} \left| -\langle \zeta_t \rangle_\delta - f\big(\langle \zeta_x \rangle_\delta\big) \right| \;=\; 0 .$$

Since ζ is smooth, this yields as desired

$$-\zeta_t(0,0) - f\big(\zeta_x(0,0)\big) \;=\; 0 .$$

5. Proof of Theorem 2.4

5.1. **Continuity.** Let h be as in Theorem 2.4. Then h is continuous.

Remark 5.1. The following statement holds: If $h \in L^1_{loc}(\mathbf{R}^2)$, $h_t \in L^2_{loc}(\mathbf{R}^2)$ and $h_x \in L^4_{loc}(\mathbf{R}^2)$, then

$$\sup_{(t_0,x_0),(t_1,x_1) \in K} \frac{|h(t_1,x_1) - h(t_0,x_0)|}{|t_1 - t_0|^{1/5} + |x_1 - x_0|^{1/3}} < \infty \qquad \text{for every bounded } K \subset\subset \mathbf{R}^2. \tag{27}$$

However, in what follows we give a simpler proof of the continuity of h, based on the additional informations provided by (9) and (10).

Testing with a cut-off function η in x we obtain from (10) that $\int \frac{1}{2}\eta u^2\,dx$ is locally bounded in t, that is,

$$u \in L^\infty_{loc}\big(\mathbf{R}_t, L^2_{loc}(\mathbf{R}_x)\big)\,.$$

On one hand, because of $h_x = u$, this yields Hölder continuity in x, uniformly in t, i.e.

$$h \in L^\infty_{loc}\big(\mathbf{R}_t, C^{0,\frac{1}{2}}_{loc}(\mathbf{R}_x)\big)\,. \tag{28}$$

On the other hand, because of $h_t = -\frac{1}{2}u^2$, this gives Lipschitz continuity in t w.r.t. the L^1–norm in x

$$h \in C^{0,1}_{loc}\big(\mathbf{R}_t, L^1_{loc}(\mathbf{R}_x)\big)\,. \tag{29}$$

We now argue "by interpolation" that (28) & (29) imply Hölder continuity in t, uniformly in x, i.e.

$$h \in L^\infty_{loc}\big(\mathbf{R}_x, C^{0,\frac{1}{3}}_{loc}(\mathbf{R}_t)\big)\,. \tag{30}$$

Indeed, let $\xi \in C^\infty_c(\mathbf{R}_x)$ be non-negative with supp $\xi \subset (-1,1)$ and $\int_{\mathbf{R}} \xi\,dx = 1$. Set $\xi_\varepsilon(x) := \frac{1}{\varepsilon}\xi(\frac{x}{\varepsilon})$ and let $\star$ denote the convolution in the x–variable. We have

$$\begin{aligned}
&\big|h(t_1,x) - h(t_2,x)\big| \\
&\quad\le\quad \big|h(t_1,x) - \big(h\star\xi_\varepsilon\big)(t_1,x)\big| \;+\; \big|\big(h\star\xi_\varepsilon\big)(t_1,x) - \big(h\star\xi_\varepsilon\big)(t_2,x)\big| \\
&\qquad\quad + \big|\big(h\star\xi_\varepsilon\big)(t_2,x) - h(t_2,x)\big| \\
&\quad\le\quad \varepsilon^{1/2}\sup_{y,|x-y|\le\varepsilon}\frac{\big|h(t_1,x)-h(t_1,y)\big|}{|x-y|^{1/2}} \;+\; \sup_{\mathbf{R}}|\xi|\cdot\frac{1}{\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon}\big|h(t_1,y)-h(t_2,y)\big|\,dy \\
&\qquad\quad + \varepsilon^{1/2}\sup_{y,|x-y|\le\varepsilon}\frac{\big|h(t_2,x)-h(t_2,y)\big|}{|x-y|^{1/2}} \\
&\quad\overset{(28),(29)}{\le}\quad C\left(\varepsilon^{1/2} + \frac{|t_1-t_2|}{\varepsilon}\right).
\end{aligned}$$

Choosing $\varepsilon = |t_1 - t_2|^{2/3}$ yields (30).

5.2. **Viscosity subsolution.** Since h is continuous, the fact that h is a viscosity subsolution follows from the argument given in Subsection 4.1 with $f(u) = u^2/2$.

5.3. **Viscosity supersolution.** The goal is proving the following

Proposition 5.2. *Let ζ be a smooth function with $\zeta(0,0) = 0$ and such that $h-\zeta$ has a minimum in $(0,0)$. Then $[\zeta_t(0,0) + \frac{1}{2}\zeta_x^2(0,0)] = 0$.*

To simplify the notation we will write $g(\varepsilon,\delta) \lesssim h(\varepsilon,\delta)$ whenever there exist constants $C_1, C_2 > 0$ such that $g(\varepsilon,\delta) \le C_1 h(\varepsilon,\delta)$ for $|\delta|, |\varepsilon| \le C_2$.

Proof. Without loss of generality we may assume $h(0,0) = \zeta(0,0) = 0$. For $0 < \varepsilon \le 1$ we set $\zeta_\varepsilon := \zeta - \varepsilon|(t,x)|$. Then $h - \zeta_\varepsilon$ has a strict minimum at 0. Given $\delta > 0$, we introduce the notation

$$\Omega_{\varepsilon,\delta} \;:=\; \big\{(t,x) : [h-\zeta_\varepsilon](t,x) < \delta\big\}\,.$$

Since $h - \zeta_\varepsilon \ge [h-\zeta](t,x) + \varepsilon|(t,x)| \ge [h-\zeta](0,0) + \varepsilon|(t,x)| = \varepsilon|(t,x)|$, we have

$$\Omega_{\varepsilon,\delta} \;\subset\; \big\{(t,x) : \varepsilon|(t,x)| < \delta\big\} \;=\; B_{\delta/\varepsilon}\big((0,0)\big)\,. \tag{31}$$

We introduce the notation

$$\langle u \rangle_{\varepsilon,\delta} \;:=\; \frac{1}{|\Omega_{\varepsilon,\delta}|}\int_{\Omega_{\varepsilon,\delta}} u(t,x)\,dt\,dx\,.$$

First Step We start by observing:

$$\left\langle \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix} \right\rangle_{\varepsilon,\delta} \;=\; \left\langle \begin{pmatrix} -\frac{1}{2}u^2 \\ u \end{pmatrix} \right\rangle_{\varepsilon,\delta}. \tag{32}$$

Indeed

$$\left\langle \begin{pmatrix} -\frac{1}{2}u^2 \\ u \end{pmatrix} \right\rangle_{\varepsilon,\delta} - \left\langle \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix} \right\rangle_{\varepsilon,\delta} \;=\; \left\langle \begin{pmatrix} (h-\zeta_\varepsilon)_t \\ (h-\zeta_\varepsilon)_x \end{pmatrix} \right\rangle_{\varepsilon,\delta}.$$

Note that

$$\langle h - \zeta_{\varepsilon t} \rangle_{\varepsilon,\delta} \;=\; \frac{1}{|\Omega_{\varepsilon,\delta}|}\int_{\mathbf{R}^2} (\min\{h-\zeta_\varepsilon-\delta,0\})_t \;=\; 0\,,$$

and an analogous computation gives $\langle (h-\zeta_\varepsilon)_x \rangle_{\varepsilon,\delta} = 0$. This establishes (32). An easy consequence of (32) is

$$\langle u^2 \rangle_{\varepsilon,\delta} \;=\; \langle \zeta_{\varepsilon t} \rangle_{\varepsilon,\delta} \lesssim 1\,. \tag{33}$$

Jensen inequality gives $\langle |u| \rangle_{\varepsilon,\delta} \lesssim 1$. Thus we get

$$\left\langle \left| \begin{pmatrix} h_t \\ h_x \end{pmatrix} \right| \right\rangle_{\varepsilon,\delta} \lesssim 1\,.$$

This translates into

$$\left\langle \left| \begin{pmatrix} (h-\zeta_\varepsilon)_t \\ (h-\zeta_\varepsilon)_x \end{pmatrix} \right| \right\rangle_{\varepsilon,\delta} \;\le\; \left\langle \left| \begin{pmatrix} h_t \\ h_x \end{pmatrix} \right| \right\rangle_{\varepsilon,\delta} + \left\langle \left| \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix} \right| \right\rangle_{\varepsilon,\delta} \lesssim 1 \tag{34}$$

Second Step We now prove that

$$\delta^2 \lesssim |\Omega_{\varepsilon,\delta}|\,. \tag{35}$$

Indeed, (34) can be reformulated as

$$\int_{\mathbf{R}^2} \left| \begin{pmatrix} (\min\{h-\zeta_\varepsilon-\delta,0\})_t \\ (\min\{h-\zeta_\varepsilon-\delta,0\})_x \end{pmatrix} \right| \;=\; \int_{\Omega_{\varepsilon,\delta}} \left| \begin{pmatrix} (h-\zeta_\varepsilon)_t \\ (h-\zeta_\varepsilon)_x \end{pmatrix} \right| \lesssim |\Omega_{\varepsilon,\delta}|\,. \tag{36}$$

By (36) and Sobolev inequality,

$$\left[\int_{\mathbf{R}^2} \big(\min\{h-\zeta_\varepsilon-\delta,0\}\big)^2\right]^{1/2} \;\lesssim |\Omega_{\varepsilon,\delta}|\,. \tag{37}$$

Hölder inequality yields

$$-\int_{\mathbf{R}^2} \min\{h-\zeta_\varepsilon-\delta,0\} \lesssim |\Omega_{\varepsilon,\delta}|^{1/2} \left[\int_{\mathbf{R}^2} \big(\min\{h-\zeta_\varepsilon-\delta,0\}\big)^2\right]^{1/2} \lesssim |\Omega_{\varepsilon,\delta}|^{3/2}\,. \tag{38}$$

Note that

$$I(\delta) \;:=\; -\int_{\mathbf{R}^2} \min\{h-\zeta_\varepsilon-\delta,0\} \;=\; \int_0^\delta |\Omega_{\varepsilon,s}|\,ds\,.$$

Thus we obtain the differential inequality

$$I(\delta) \overset{(38)}{\lesssim} |\Omega_{\varepsilon,\delta}|^{3/2} = \left[\frac{d}{d\delta} I(\delta)\right]^{3/2} .$$

Since $I(\delta) > 0$ for $\delta > 0$ we get $1 \lesssim \frac{d}{d\delta}\left[\big(I(\delta)\big)^{1/3}\right]$, which yields $\delta^3 \lesssim I(\delta)$. Note that $|\Omega_{\varepsilon,\delta}|$ is a non–decreasing function of δ. Hence,

$$\delta|\Omega_{\varepsilon,\delta}| \ \geq \ \int_0^\delta |\Omega_{\varepsilon,s}|\, ds \ \gtrsim \ \delta^3 ,$$

which establishes (35).

Third Step We now argue that

$$\left\langle \begin{pmatrix} -u^2/2 \\ u \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle_{\varepsilon,\delta} - \left\langle \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle_{\varepsilon,\delta} \lesssim \frac{\mu\big(B_{\delta/\varepsilon}(0,0)\big)}{\delta} . \tag{39}$$

Indeed, we have

$$\begin{aligned}
&\left\langle \begin{pmatrix} -u^2/2 \\ u \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle_{\varepsilon,\delta} - \left\langle \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle_{\varepsilon,\delta} \\
&= \left\langle \begin{pmatrix} (h-\zeta_\varepsilon)_t \\ (h-\zeta_\varepsilon)_x \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle_{\varepsilon,\delta} \\
&= \frac{1}{|\Omega_{\varepsilon,\delta}|} \int_{\Omega_{\varepsilon,\delta}} \begin{pmatrix} (h-\zeta)_t \\ (h-\zeta)_x \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \\
&= \frac{1}{|\Omega_{\varepsilon,\delta}|} \int_{\mathbf{R}^2} \begin{pmatrix} \big(\min\{h-\zeta_\varepsilon-\delta,0\}\big)_t \\ \big(\min\{h-\zeta_\varepsilon-\delta,0\}\big)_x \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \\
&= \frac{1}{|\Omega_{\varepsilon,\delta}|} \int_{\mathbf{R}^2} \big(-\min\{h-\zeta_\varepsilon-\delta,0\}\big) \left(\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3}\right)_x \right) \\
&\leq \frac{\delta}{|\Omega_{\varepsilon,\delta}|} \mu(\Omega_{\varepsilon,\delta}) \ \leq \ \frac{\delta}{|\Omega_{\varepsilon,\delta}|} \mu\big(B_{\delta/\varepsilon}\big((0,0)\big)\big) \overset{(35)}{\lesssim} \frac{\mu\big(B_{\delta/\varepsilon}\big((0,0)\big)\big)}{\delta} .
\end{aligned}$$

This establishes (39).

Fourth Step We now prove

$$\left\langle u^4 \right\rangle_{\varepsilon,\delta} \lesssim \varepsilon^{-1} \quad \text{and} \quad \left\langle |u|^3 \right\rangle_{\varepsilon,\delta} \lesssim \varepsilon^{-1/2} . \tag{40}$$

Indeed

$$
\begin{aligned}
\left\langle u^4\right\rangle_{\varepsilon,\delta} &\lesssim \left\langle \begin{pmatrix} -u^2/2 \\ u \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle_{\varepsilon,\delta} \\
&\overset{(39)}{\lesssim} \left\langle \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle_{\varepsilon,\delta} + \frac{\mu\big(B_{\delta/\varepsilon}\big((0,0)\big)\big)}{\delta} \\
&\lesssim \sup_{\Omega_{\varepsilon,\delta}} \left| \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix} \right| \left\langle \left| \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right| \right\rangle_{\varepsilon,\delta} + \frac{1}{\varepsilon}\left[\frac{\varepsilon}{\delta}\mu\big(B_{\delta/\varepsilon}\big((0,0)\big)\big)\right] \\
&\lesssim \left\langle u^2\right\rangle_{\varepsilon,\delta} + \left\langle |u|^3\right\rangle_{\varepsilon,\delta} + \varepsilon^{-1} \overset{(33)}{\lesssim} 1 + \left[\left\langle u^2\right\rangle_{\varepsilon,\delta}\right]^{1/2}\left[\left\langle u^4\right\rangle_{\varepsilon,\delta}\right]^{1/2} + \varepsilon^{-1}\,.
\end{aligned}
$$

By Young's inequality, this yields $\langle u^4\rangle_{\varepsilon,\delta} \lesssim \varepsilon^{-1}$. Hölder inequality gives

$$
\left\langle |u|^3\right\rangle_{\varepsilon,\delta} \le \left[\left\langle u^4\right\rangle_{\varepsilon,\delta}\right]^{1/2}\left[\left\langle u^2\right\rangle_{\varepsilon,\delta}\right]^{1/2} \lesssim \varepsilon^{-1/2}\,.
$$

Fifth Step We now argue that

$$
\left| \left\langle \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle_{\varepsilon,\delta} - \left\langle \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix} \right\rangle_{\varepsilon,\delta} \cdot \left\langle \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle_{\varepsilon,\delta} \right| \lesssim \varepsilon^{1/2} + \frac{\delta}{\varepsilon^{3/2}}\,. \tag{41}
$$

Indeed

$$
\begin{aligned}
&\left| \left\langle \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle_{\varepsilon,\delta} - \left\langle \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix} \right\rangle_{\varepsilon,\delta} \cdot \left\langle \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle_{\varepsilon,\delta} \right| \\
&\le \sup_{\Omega_{\varepsilon,\delta}} \left| \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix} - \left\langle \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix} \right\rangle_{\varepsilon,\delta} \right| \left\langle \left| \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right| \right\rangle_{\varepsilon,\delta} \\
&\le \left[\operatorname*{osc}_{\Omega_{\varepsilon,\delta}} \begin{pmatrix} \zeta_{\varepsilon t} \\ \zeta_{\varepsilon x} \end{pmatrix}\right] \left(\left\langle u^2\right\rangle_{\varepsilon,\delta} + \left\langle |u|^3\right\rangle_{\varepsilon,\delta}\right) \\
&\le \left[\operatorname*{osc}_{\Omega_{\varepsilon,\delta}} \begin{pmatrix} \zeta_{t} \\ \zeta_{x} \end{pmatrix} + 2\varepsilon\right] \left(\left\langle u^2\right\rangle_{\varepsilon,\delta} + \left\langle |u|^3\right\rangle_{\varepsilon,\delta}\right) \\
&\overset{(31),(33),(40)}{\lesssim} \left(\frac{\delta}{\varepsilon} + \varepsilon\right)\left(1 + \varepsilon^{-1/2}\right) \lesssim \frac{\delta}{\varepsilon^{3/2}} + \varepsilon^{1/2}\,.
\end{aligned}
$$

Sixth Step Combining (32), (39) and (41) we get

$$
\begin{aligned}
&\left\langle \begin{pmatrix} -u^2/2 \\ u \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle_{\varepsilon,\delta} - \left\langle \begin{pmatrix} -u^2/2 \\ u \end{pmatrix} \right\rangle_{\varepsilon,\delta} \cdot \left\langle \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle_{\varepsilon,\delta} \\
&\qquad \lesssim \frac{\delta}{\varepsilon^{3/2}} + \varepsilon^{1/2} + \frac{\mu\big(B_{\delta/\varepsilon}\big((0,0)\big)\big)}{\delta}\,.
\end{aligned} \tag{42}
$$

According to Proposition 3.2 (b) we obtain

$$
\left\langle \left(u - \langle u\rangle_{\varepsilon,\delta}\right)^4 \right\rangle_{\varepsilon,\delta} \lesssim \frac{\delta}{\varepsilon^{3/2}} + \varepsilon^{1/2} + \frac{\mu\big(B_{\delta/\varepsilon}\big((0,0)\big)\big)}{\delta}\,. \tag{43}
$$

On the other hand we have

$$
\begin{aligned}
\left|\zeta_t(0,0)+\frac{1}{2}\zeta_x^2(0,0)\right| &\overset{(31)}{\lesssim} \left|\langle\zeta_t\rangle_{\varepsilon,\delta}+\frac{1}{2}\left[\langle\zeta_x\rangle_{\varepsilon,\delta}\right]^2\right|+\frac{\delta}{\varepsilon}\\
&\lesssim \left|\langle\zeta_{\varepsilon t}\rangle_{\varepsilon,\delta}+\frac{1}{2}\left[\langle\zeta_{\varepsilon x}\rangle_{\varepsilon,\delta}\right]^2\right|+\varepsilon+\frac{\delta}{\varepsilon}\\
&\overset{(32)}{=} \left|-\left\langle\frac{u^2}{2}\right\rangle_{\varepsilon,\delta}+\frac{1}{2}\left[\langle u\rangle_{\varepsilon,\delta}\right]^2\right|+\varepsilon+\frac{\delta}{\varepsilon}\\
&= \frac{1}{2}\left\langle\left(u-\langle u\rangle_{\varepsilon,\delta}\right)^2\right\rangle_{\varepsilon,\delta}+\varepsilon+\frac{\delta}{\varepsilon}\\
&\lesssim \left[\left\langle\left(u-\langle u\rangle_{\varepsilon,\delta}\right)^4\right\rangle_{\varepsilon,\delta}\right]^{1/2}+\varepsilon+\frac{\delta}{\varepsilon}\\
&\overset{(43)}{\lesssim} \left(\frac{\delta}{\varepsilon^{3/2}}+\varepsilon^{1/2}+\frac{\mu\big(B_{\delta/\varepsilon}((0,0))\big)}{\delta}\right)^{1/2}+\varepsilon+\frac{\delta}{\varepsilon}\,.
\end{aligned}
$$

Letting first δ and then ε go to 0, we get $\zeta_t(0,0)+\frac{1}{2}\zeta_x^2(0,0)=0$. □

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