# WELL–POSEDNESS FOR A CLASS OF HYPERBOLIC SYSTEMS OF CONSERVATION LAWS IN SEVERAL SPACE DIMENSIONS

LUIGI AMBROSIO, FRANÇOIS BOUCHUT, CAMILLO DE LELLIS

Abstract. In this paper we consider a system of conservation laws in several space dimensions whose nonlinearity is due only to the modulus of the solution. This system, first considered by Keyfitz and Kranzer in one space dimension, has been recently studied by many authors. In particular, using standard methods from DiPerna–Lions theory, we improve the results obtained by the first and third author, showing existence, uniqueness and stability results in the class of functions whose modulus satisfies, in the entropy sense, a suitable scalar conservation law. In the last part of the paper we consider a conjecture on renormalizable solutions and show that this conjecture implies another one recently made by Bressan in connection with the system of Keyfitz and Kranzer.



## 1. Introduction

In this note we consider the Cauchy problem for the system of conservation laws

$$\begin{cases} \partial_t u_i + \sum_{\alpha=1}^{n} \partial_{x_\alpha}(f_\alpha(|u|)u_i) \;=\; 0 \\ \\ u_i(0,\cdot) \;=\; \overline{u}_i(\cdot) \end{cases} \tag{1}$$

where $u = (u_1, \ldots, u_k) : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^k$. The system (1) was first considered by Keyfitz and Kranzer in one space dimension in [9] and then studied by several authors (see [6], [8], and [11] for the literature on it).

As a partial answer to a conjecture of Bressan (see Section 3.3 of [4]), in [2] it was shown the existence of weak solutions to (1) when $|\overline{u}| \in L^\infty \cap BV(\mathbf{R}^n)$ and $|\overline{u}| \geq c > 0$ for some $c$. Following the suggestion of the last section of [4] these solutions were constructed with the following method (an higher dimensional analog of what applied earlier in one space dimension; see for example Section 8.2 of [11]). We first find the modulus $\rho := |u|$ by solving, in the sense of Kruzhkov, the conservation law

$$\begin{cases} \partial_t \rho + \sum_{\alpha=1}^{n} \partial_{x_\alpha}\big(f_\alpha(\rho)\rho\big) \;=\; 0 \\ \\ \rho(0,\cdot) \;=\; \overline{\rho}(\cdot)\,. \end{cases} \tag{2}$$

Then we construct an approximation scheme for the ODE $\dot{x}(t) = f(\rho(t, x(t)))$, which formally gives, via the method of characteristics, the angular part $\theta = u/\rho$ of the solution. In this construction we used the results of [1], where the first author extended the DiPerna–Lions theory to $BV$ vector fields satisfying natural $L^\infty$ bounds, as in [7], on the distributional divergence.

In this paper we show how the results of [1] on transport equations with BV coefficients and standard arguments from DiPerna–Lions theory yield a straightforward proof of the existence of such solutions, without passing through approximations of the ODE $\dot{x}(t) = f(\rho(t, x(t)))$. This proof gives also uniqueness and stability under perturbation of the initial data and allows to remove the assumption $|\overline{u}| \geq c > 0$. See Theorem 2.6 for the precise statements.

In the last section we formulate a conjecture (see Conjecture 4.3) which is closely related to the one of Section 3.3 of [4]. Indeed, using arguments from DiPerna–Lions theory, we show that a positive answer to Conjecture 4.3 would give a positive answer to that of Bressan (see Proposition 4.4).

## 2. Preliminaries and statement of Theorem 2.6

Before stating the main theorem, we recall the notion of entropy solution of a scalar conservation law and the classical theorem of Kruzhkov, which provides existence, stability and uniqueness of entropy solutions to the Cauchy problem for scalar laws.

**Definition 2.1.** *Let $g \in W^{1,\infty}_{loc}(\mathbf{R}, \mathbf{R}^n)$. A pair $(\eta, q)$ of functions $\eta \in W^{1,\infty}_{loc}(\mathbf{R}, \mathbf{R})$, $q \in W^{1,\infty}_{loc}(\mathbf{R}, \mathbf{R}^n)$ is called an* entropy–entropy flux pair *relative to $g$ if*

$$q' \;=\; \eta' g' \qquad \mathcal{L}^1\text{–almost everywhere on } \mathbf{R}. \tag{3}$$

*If, in addition, $\eta$ is a convex function, then we say that $(\eta, q)$ is a* convex entropy–entropy flux *pair. A weak solution $\rho \in L^\infty(\mathbf{R}^+_t \times \mathbf{R}^n_x)$ of*

$$\begin{cases} \partial_t \rho + \mathrm{div}_x[g(\rho)] \;=\; 0 \\ \\ \rho(0, \cdot) \;=\; \overline{\rho}(\cdot) \end{cases} \tag{4}$$

*is called an entropy solution if $\partial_t[\eta(\rho)] + \mathrm{div}_x[q(\rho)] \leq 0$ in the sense of distributions for every convex entropy–entropy flux pair $(\eta, q)$.*

In what follows, we say that $\rho \in L^\infty(\mathbf{R}^+ \times \mathbf{R}^n)$ *has a strong trace $\overline{\rho}$ at* 0 if for every bounded $\Omega \subset \mathbf{R}^n$ we have

$$\lim_{T \downarrow 0} \frac{1}{T} \int_{[0,T] \times \Omega} |\rho(t, x) - \overline{\rho}(x)| \, dx \;=\; 0\,.$$

**Theorem 2.2** ([10] Kruzhkov)**.** *Let $g \in W^{1,\infty}_{loc}(\mathbf{R}, \mathbf{R}^n)$ and $\overline{\rho} \in L^\infty$. Then there exists a* unique *entropy solution $\rho$ of (4) with a strong trace at 0. If in addition $\overline{\rho} \in BV_{loc}(\mathbf{R}^n)$, then, for every open set $A \subset\subset \mathbf{R}^n$ and for every $T \in ]0, \infty[$, there exists an open set $A' \subset\subset \mathbf{R}^n$ (whose diameter depends only on $A$, $T$, $g$ and $\|\overline{\rho}\|_\infty$) such that*

$$\|\rho\|_{BV(]0,T[\times A)} \;\leq\; \|\overline{\rho}\|_{BV(A')}\,. \tag{5}$$

**Remark 2.3.** In many cases the requirement that $\rho$ has strong trace at 0 is not necessary. Indeed, when $g$ is sufficiently regular and satisfies suitable assumptions of genuine nonlinearity, Vasseur proved in [12] that *any* entropy solution has a strong trace at 0.

**Definition 2.4.** *A weak solution $u$ of (1) is called a* renormalized entropy solution *if $|u|$ is an entropy solution of the scalar law (2) with a strong trace at 0.*

The suggestion of using the terminology "renormalized entropy solutions" has been taken from [8]. This terminology is more appropriate than the one of "entropy solutions" used in [2], because the usual notion of *entropy* (or *admissible*) solution of a hyperbolic system of conservation laws does not coincide with the one of renormalized entropy solutions. Let us recall the usual notion of entropy solution for systems (cp. Section 4.3 of [6]):

**Definition 2.5.** *Let $F_\alpha : \mathbf{R}^k \to \mathbf{R}^k$, $\alpha = 1, \ldots, n$, be Lipschitz and consider the system*

$$\partial_t u + \sum_{\alpha=1}^{n} \partial_{x_\alpha} [F_\alpha(u)] \;=\; 0 \qquad u : \Omega \subset \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^k . \tag{6}$$

*A pair $(\eta, q)$ of functions $\eta \in W^{1,\infty}_{loc}(\mathbf{R}^k, \mathbf{R})$, $q \in W^{1,\infty}_{loc}(\mathbf{R}^k, \mathbf{R}^n)$ is called a* convex entropy–entropy flux pair *for the system (6) if $\eta$ is convex and if for every open set $\Omega$ and for every smooth solution $u$ of (6) we have $\partial_t[\eta(u)] + \mathrm{div}_x[q(u)] = 0$. A weak solution $u$ of (6) is called an* entropy *solution if for for every convex entropy–entropy flux pair we have $\partial_t[\eta(u)] + \mathrm{div}_x[q(u)] \leq 0$ in the sense of distributions.*

Indeed it can be shown that, already in one space dimension, there exist entropy solutions of (1) which are not renormalized entropy solutions, see for example section 3.1 of [4]. On the other hand one can show (at least when $f \in C^2$) that every $C^2$ entropy $\eta$ for (1) is of the form $h(|v|) + |v| H(v/|v|)$ for $|v| > 0$ (see for example Lemma 1.1 of [8]). Thus, it follows from Corollary 3.4 of [1] (see Lemma 2.8 below) that if $u$ is a renormalized entropy solution, then $u$ satisfies $\partial_t[\eta(u)] + \mathrm{div}_x[q(u)] \leq 0$ for every convex entropy–entropy flux pair which is $C^2$ on $\mathbf{R}^k \setminus \{0\}$.

**Theorem 2.6.** *Let $f \in W^{1,\infty}_{loc}(\mathbf{R}, \mathbf{R}^k)$ and $|\overline{u}| \in L^\infty \cap BV_{loc}$. Then there exists a unique renormalized entropy solution $u$ of (1). If $\overline{u}^j$ is a sequence of initial data such that*

(a) *$|\overline{u}^j| \leq C$ for some constant $C$,*
(b) *for every bounded open set $\Omega$, there is a constant $C(\Omega)$ such that $\big\| |\overline{u}^j| \big\|_{BV(\Omega)} \leq C(\Omega)$,*
(c) *$\overline{u}^j \to \overline{u}$ strongly in $L^1_{loc}$,*

*then the corresponding renormalized entropy solutions $u^j$ converge strongly in $L^1_{loc}$ to $u$.*

The proof of the theorem follows from the theory of renormalized solutions to the transport equation

$$\partial_t(\rho w) + \mathrm{div}_x(g w) \;=\; 0, \qquad w : \Omega \subset \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^k \tag{7}$$

and from the results of [1].

**Definition 2.7** ([7] DiPerna–Lions renormalized solutions)**.** *Let $\Omega \subset \mathbf{R} \times \mathbf{R}^n$ be open and assume that $(\rho, g) \in L^\infty(\Omega, \mathbf{R} \times \mathbf{R}^n)$ satisfy $\partial_t \rho + \mathrm{div}_x g = 0$ in the sense of distributions. A $w \in L^\infty(\Omega, \mathbf{R}^k)$ is called a* renormalized solution *of (7) if for every $h \in C^1(\mathbf{R}^k)$ we have*

$$\partial_t(\rho h(w)) + \mathrm{div}_x(g h(w)) \;=\; 0 \qquad in\ \mathcal{D}'(\Omega).$$

Corollary 3.4 and Remark 3.5(3) of [1] give

**Lemma 2.8.** *If $(\rho, g)$ in Definition 2.7 are in $BV_{loc}$, then every bounded weak solution $w : \Omega \to \mathbf{R}^k$ of (7) is a renormalized solution.*

**Remark 2.9** (Initial conditions)**.** If $(z,m)\in L^\infty(\mathbf{R}\times\mathbf{R}^n,\mathbf{R}\times\mathbf{R}^n)$ satisfy $\partial_t z+\mathrm{div}_x m=0$ in $]0,T[\times\mathbf{R}^n$, then by simply testing the equation against appropriate smooth functions we get that for every $\varphi\in C^\infty_c(\mathbf{R}^n)$ the map $t\to\int_{\mathbf{R}^n}\varphi(x)z(t,x)\,dx$ coincides almost everywhere with a uniformly continuous function. Hence, by a density argument we conclude that, after discarding a set of $t$'s of measure 0,

$$t\to z(t,\cdot)\ \text{has a weak limit in}\ L^1_{loc}\ \text{as}\ t\downarrow 0. \tag{8}$$

This gives that $z$ has a "weak" trace $\overline{z}$ at 0. Assume $\rho$ and $g$ are as in Definition 2.7 and that $w$ solves $\partial_t(\rho w)+\mathrm{div}_x(gw)=0$ in $]0,T[\times\mathbf{R}^n$. Apply (8) to $z=\rho$ and $m=g$ and denote by $\rho(0,\cdot)$ the trace of $\rho$ at 0. Then note that we can apply (8) to $z=\rho w$ and $m=gw$. Thus, it is natural to understand a bounded weak solution $w$ of

$$\begin{cases}\partial_t(\rho w)+\mathrm{div}_x(gw) &= 0\\[2ex] \rho w(0,\cdot) &= \rho(0,\cdot)\overline{w}(\cdot)\ ,\end{cases}$$

as a function which solves the first equation in the sense of distribution and such that $\rho w$ has $\rho(0,\cdot)\overline{w}(\cdot)$ as weak trace at 0. It can be easily checked that this is equivalent to the usual notion of solution in the sense of distributions, i.e. for every $\varphi\in C^\infty_c(\mathbf{R}\times\mathbf{R}^n)$ we have

$$\int_{\mathbf{R}^+\times\mathbf{R}^n} w(t,y)\left[\rho(t,y)\partial_t\varphi(t,y)+g(t,y)\cdot\nabla_x\varphi(t,y)\right]dy\,dt \;=\; \int_{\mathbf{R}^n}\rho(0,x)\overline{w}(x)\varphi(0,x)\,dx\,.$$

It turns out that this weak trace property is not sufficient for some of our arguments as this notion of trace is, in general, not stable under left composition. In all the cases treated here we could overcome this issue by using Theorem 3.3 of [3], where the author proved strong $L^1$ continuity in $t$ for solutions of transport equations with BD coefficients. However we prefer to give a more elementary argument which uses suitable extensions of $\rho$, $g$, and $u$ to negative times.

Proposition 2.10 below is a direct consequence of DiPerna–Lions theory of renormalized solutions. Its proof is based on a classical inequality for transport equations with finite speed of propagation. This inequality is stated independently in Lemma 2.11 and for the reader's convenience we report a proof of it.

**Proposition 2.10.** *Let $\rho,g\in L^\infty\cap BV_{loc}(\mathbf{R}^+\times\mathbf{R}^n)$ be such that $\partial_t\rho+\mathrm{div}_x g=0$, $\rho(0,\cdot)\in BV_{loc}$, and $|g|\le c\rho$ for some constant $c$. Assume $u^1$, $u^2$ are bounded weak solutions of*

$$\begin{cases}\partial_t(\rho u^i)+\mathrm{div}_x(gu^i) &= 0\\[2ex] \rho u^i(0,\cdot) &= \rho(0,\cdot)\overline{u}^i(\cdot)\ .\end{cases} \tag{9}$$

*If $\rho(0,\cdot)\overline{u}^1=\rho(0,\cdot)\overline{u}^2(x)$, then $\rho u^1=\rho u^2$.*

**Lemma 2.11.** *Let $z\in L^\infty([0,T]\times\mathbf{R}^n)$ and $m\in L^\infty([0,T]\times\mathbf{R}^n,\mathbf{R}^n)$ be such that*

- $\partial_t z+\mathrm{div}_x m\le 0$*;*
- $t\to z(t,\cdot)$ *is weakly continuous in* $L^1_{loc}$*;*
- $|m|\le Cz$*.*

*Then, for every $\tau \in [0,T]$, $x_0 \in \mathbf{R}^n$ and $R>0$, we have*

$$\int_{B_R(x_0)} z(\tau,x)\,dx \;\le\; \int_{B_{R+C\tau}(x_0)} z(0,x)\,dx\,. \tag{10}$$

*Proof.* Without loss of generality we assume $x_0=0$. Let $\chi_\varepsilon \in C^\infty\big(\mathbf{R}^+\big)$ be such that

$$\chi_\varepsilon = 1 \text{ on } [0,1]\,, \qquad \chi_\varepsilon = 0 \text{ on } [1+\varepsilon,+\infty[\,, \quad \text{and} \quad \chi'_\varepsilon \le 0\,.$$

Define the test function $\varphi(t,x) := \chi_\varepsilon\left(\frac{|x|}{R+C(\tau-t)}\right)$. Note that $\varphi$ is nonnegative and belongs to $C^\infty([0,\tau]\times\mathbf{R}^n)$. Since $t\to z(t,\cdot)$ is weakly continuous in $L^1_{loc}$, we can test $\partial_t z + \mathrm{div}_x m = 0$ with $\varphi(t,x)\mathbf{1}_{[0,\tau]}(t)$. Indeed let $\mu$ be the measure $\partial_t z + \mathrm{div}_x m$ and let $0<\tau_1<\tau_2<\tau$. Consider a standard family of nonnegative mollifiers $\xi^\delta \in C^\infty(\mathbf{R})$ and set $\zeta^\delta := \mathbf{1}_{[\tau_1,\tau_2]} * \xi^\delta$. Testing $\partial_t z + \mathrm{div}_x m = \mu$ with $\varphi(t,x)\zeta^\delta(t)$ we get

$$\int z(s,y)\varphi(s,y)\left[\xi^\delta(\tau_2-s)-\xi^\delta(\tau_1-s)\right]ds\,dy \;=\; \int \zeta^\delta\left[z\,\partial_t\varphi + m\cdot\nabla_x\varphi\right] + \int \zeta^\delta\varphi d\mu\,. \tag{11}$$

Note that $\int \zeta^\delta d\mu \le 0$. Moreover, by the weak continuity of $t\to z(t,\cdot)$, the integrals $\int z(s,y)\varphi(s,y)\xi^\delta(\tau_i-s)\,ds\,dy$ converge to $\int \varphi(\tau_i,x)z(\tau_i,x)\,dx$ as $\delta\downarrow 0$. Hence, in the limit we get

$$\int_{[\tau_1,\tau_2]\times\mathbf{R}^n}\left[z\,\partial_t\varphi + m\cdot\nabla_x\varphi\right] \;\ge\; \int_{\mathbf{R}^n}\varphi(\tau_2,x)z(\tau_2,x)\,dx - \int_{\mathbf{R}^n}\varphi(\tau_1,x)z(\tau_1,x)\,dx\,.$$

Then, letting $\tau_2\uparrow\tau$ and $\tau_1\downarrow 0$ we get

$$\int_{[0,\tau]\times\mathbf{R}^n}\left[z\,\partial_t\varphi + m\cdot\nabla_x\varphi\right] \;\ge\; \int_{\mathbf{R}^n}\varphi(\tau,x)z(\tau,x)\,dx - \int_{\mathbf{R}^n}\varphi(0,x)z(0,x)\,dx\,. \tag{12}$$

We compute $z(s,y)\partial_t\varphi(s,y) + m(s,y)\cdot\nabla_x\varphi(s,y)$ as

$$\chi'_\varepsilon\left(\frac{|y|}{R+C(\tau-s)}\right)\left[\frac{C|y|z(s,y)}{(R+C(\tau-s))^2} + \frac{y\cdot m(s,x)}{|y|(R+C(\tau-s))}\right]\,. \tag{13}$$

Letting $\alpha := |y|/((R+C(\tau-s))$, the expression in (13) becomes

$$\frac{\chi'_\varepsilon(\alpha)}{R+C(\tau-s)}\left[Cz\alpha + m\cdot\frac{y}{|y|}\right]\,.$$

For $\alpha\le 1$ we have $\chi'_\varepsilon(\alpha)=0$, whereas for $\alpha\ge 1$ we have $\chi'_\varepsilon(\alpha)\le 0$ and $Cz\alpha\ge|m|$. Thus we conclude that the integrand of the left hand side of (12) is nonpositive. Hence

$$\int_{\mathbf{R}^n}\chi_\varepsilon\left(\frac{|x|}{R}\right)z(\tau,y)\,dx \;\le\; \int_{\mathbf{R}^n}\chi_\varepsilon\left(\frac{|x|}{R+C\tau}\right)dy\,.$$

Letting $\varepsilon\downarrow 0$ we get (10). □

*Proof of Proposition 2.10.* By linearity, it is sufficient to prove that, if $u$ is a bounded weak solution of (9) and $\overline{u}(\cdot)\rho(0,\cdot)\equiv 0$, then $\rho u = 0$. We extend $(\rho,g)$ to the whole $\mathbf{R}^+\times\mathbf{R}^n$ by setting

$$a(t,x) \;:=\; \begin{cases}\rho(0,x) & \text{for } t<0\\ \rho(t,x) & \text{for } t\ge 0\end{cases} \qquad b(t,x) \;:=\; \begin{cases}0 & \text{for } t<0\\ g(t,x) & \text{for } t\ge 0\,.\end{cases}$$

Clearly $a$, $b \in BV_{loc}(\mathbf{R}\times\mathbf{R}^n)$ and $\partial_t a+\mathrm{div}_x b = 0$. Now extend $u$ to negative times by setting $u(t,x) = \overline{u}(x)$. Then we have $\partial_t(au) + \mathrm{div}_x(bu) = 0$. Applying Lemma 2.8, if we denote by $v$ the square of $|u|$, we have

$$\begin{cases} \partial_t(av) + \mathrm{div}_x(bv) \;=\; 0 \qquad \text{on } \mathbf{R}\times\mathbf{R}^n \\ \\ v \;= 0 \qquad \text{on } \{t<0\}. \end{cases} \tag{14}$$

Thanks to Remark 2.9, the map $t \to a(t,\cdot)v(t,\cdot)$ is weakly continuous in $L^1_{loc}$, and hence $av(0,\cdot) = 0$. We can apply Lemma 2.11 with $z = \rho v$ and $m = bv$. We conclude that $z$ is identically 0. This implies that $\rho u = 0$ on $\{t > 0\}$, which is the desired conclusion. □

## 3. Proof of Theorem 2.6

3.1. **Existence.** Fix $\overline{u} \in L^\infty(\mathbf{R}^n)$ with $\overline{\rho} := |\overline{u}| \in BV_{loc}$. Set

$$\overline{\theta}(x) \;:=\; \begin{cases} \overline{u}(x)/|\overline{\rho}|(x) & \text{when } |\overline{\rho}|(x) > 0 \\ (1,0,\ldots,0) & \text{otherwise.} \end{cases}$$

Let $\rho$ be the entropy solution of the scalar law (2) with initial data $\overline{\rho}$ as strong trace. It follows from Theorem 2.2 that $\rho \in BV_{loc}$. Consider the Cauchy problem for the transport equation

$$\begin{cases} \partial_t(\rho\theta) + \mathrm{div}_x(f(\rho)\rho\theta) \;=\; 0 \\ \\ \rho\theta(0,\cdot) \;=\; \rho(0,\cdot)\overline{\theta}(\cdot)\,. \end{cases} \tag{15}$$

Let us approximate $\rho, f(\rho)\rho$ with a sequence of uniformly bounded and smooth functions $\rho^j, g^j$ such that:

$$\rho^j \to \rho,\ \rho^j(0,\cdot) \to \rho(0,\cdot), \text{ and } g^j \to f(\rho)\rho \text{ strongly in } L^1_{loc}; \tag{16}$$

$$\partial_t\rho^j + \mathrm{div}_x g^j = 0; \tag{17}$$

$$\rho^j \;\geq\; j^{-1}\,. \tag{18}$$

Let us solve the Cauchy problem

$$\begin{cases} \partial_t(\rho^j\theta^j) + \mathrm{div}_x(g^j\theta^j) \;=\; 0 \\ \\ \rho^j\theta^j(0,\cdot) \;=\; \rho(0,\cdot)\overline{\theta}(\cdot)\,. \end{cases} \tag{19}$$

This can be done with the method of characteristics. Indeed set $f^j := g_j/\rho_j$ and solve the ODE

$$\begin{cases} \dfrac{d}{dt}\Phi^j(t,x) \;=\; f^j(t,\Phi^j(t,x)) \\ \\ \Phi^j(0,x) \;=\; x\,. \end{cases}$$

Since $f^j$ is smooth there exists a unique smooth $\Phi^j$ and for each $t$ the map $\Phi^j(t,\cdot) : \mathbf{R}^n \to \mathbf{R}^n$ is invertible. Denoting by $\Psi^j(t,\cdot)$ its inverse, $\theta^j(t,x)$ is given by $\overline{\theta}(\Psi^j(t,x))$.

Clearly we get $\|\theta^j\|_\infty = 1$. Thus, up to subsequences, we can assume that $\theta^j$ converges to a bounded $\theta$ weakly* in $L^\infty$. Hence, by (16) $\rho^j\theta^j \to \rho\theta$ and $g^j\theta^j \to f(\rho)\rho\theta$ in the sense of distributions, from which we conclude that $\theta$ is a bounded weak solution of (15).

As in Proposition 2.10, we extend the vector field $(\rho, f(\rho)\rho)$ to the whole $\mathbf{R}\times\mathbf{R}^n$ by setting

$$a(t,x) := \begin{cases} \rho(0,x) & \text{for } t<0 \\ \rho(t,x) & \text{for } t\geq 0 \end{cases} \qquad b(t,x) := \begin{cases} 0 & \text{for } t<0 \\ \rho(t,x)f(\rho(t,x)) & \text{for } t\geq 0\,. \end{cases}$$

Then we extend $\theta$ to negative times by setting $\theta(t,x)=\overline{\theta}(x)$ for $t<0$. Thus $\partial_t(a\theta)+\mathrm{div}_x(b\theta)=0$ on the whole $\mathbf{R}^+\times\mathbf{R}^n$. Lemma 2.8 implies that $\theta$ is a renormalized solution. Thus $|\theta|^2$ satisfies $\partial_t(a|\theta|^2)+\mathrm{div}_x(b|\theta|^2)=0$ in the sense of distributions. Note that $|\theta(t,x)|^2$ is identically 1 on $\{t<0\}$. Thus, $|\theta|^2$ is a weak solution of

$$\begin{cases} \partial_t(\rho|\theta|^2)+\mathrm{div}_x(f(\rho)\rho|\theta|^2) \;=\; 0 \\ \\ \rho|\theta|^2(0,\cdot) \;=\; \rho(0,\cdot)\,. \end{cases} \tag{20}$$

Since the function identically equal to 1 is a weak solution of the same Cauchy problem, Proposition 2.10 implies that $\rho|\theta|^2=\rho$ on $\mathbf{R}\times\mathbf{R}^n$.

3.2. **Uniqueness.** If $u^1$ and $u^2$ are two renormalized entropy solutions of (1), then $|u^1|=|u^2|$ because of Theorem 2.2. Let $\rho:=|u^1|=|u^2|$ and define

$$\theta^i(t,x) \;:=\; \begin{cases} u^i(t,x)/\rho(t,x) & \text{when } \rho(t,x)\neq 0 \\ 0 & \text{when } \rho(t,x)=0, \end{cases}$$

$$\overline{\theta}(x) \;=\; \begin{cases} \overline{u}(x)/|\overline{u}|(x) & \text{when } |\overline{u}|(x)\neq 0 \\ 0 & \text{when } |\overline{u}|(x)=0. \end{cases}$$

Thus $\theta^i$ solve the transport problem

$$\begin{cases} \partial_t(\rho\theta^i)+\mathrm{div}_x(\rho f(\rho)\theta^i) \;=\; 0 \\ \\ \rho\theta^i(0,\cdot) \;=\; \rho(0,\cdot)\overline{\theta}(\cdot)\;. \end{cases} \tag{21}$$

Hence, thanks to Proposition 2.10, $u^1=\rho\theta^1=\rho\theta^2=u^2$.

3.3. **Stability.** Let $\overline{u}, u, \overline{u}^j, u^j$ be as in the statement of the Theorem. Recall that $\rho^j:=|u^j|$ is the entropy solution of the scalar law

$$\begin{cases} \partial_t\rho^j+\mathrm{div}_x(f(\rho^j)\rho^j) \;=\; 0 \\ \\ \rho^j(0,\cdot) \;=\; |\overline{u}^j|(\cdot)\,, \end{cases}$$

Hence Theorem 2.2 and condition (b) imply that $\|\rho^j\|_{BV(]0,T[\times\Omega)}\leq C(T,\Omega)<\infty$ for every $T>0$ and for every bounded open set $\Omega\subset\mathbf{R}^n$. Hence the sequence $|\rho^j|$ is strongly precompact in $L^1_{loc}$. Since $\rho$ is the unique entropy solution of

$$\begin{cases} \partial_t\rho+\mathrm{div}_x(f(\rho)\rho) \;=\; 0 \\ \\ \rho(0,\cdot) \;=\; |\overline{u}|(\cdot)\,, \end{cases}$$

we conclude that $\rho^j\to\rho$ strongly in $L^1_{loc}$.

Set $\theta^j:=u^j/\rho_j$ where $\rho_j\neq 0$ and $\theta^j:=0$ everywhere else. Define $\overline{\theta}^j$ analogously. Take a subsequence $j(l)$ such that the $\theta^{j(l)}$'s and the $\overline{\theta}^{j(l)}$'s converge weakly$^*$ in $L^\infty$ to bounded functions $\theta^\infty$ and $\overline{\theta}^\infty$. Recall that $\rho^{j(l)} \to \rho$ strongly in $L^1_{loc}(\mathbf{R}^+ \times \mathbf{R}^n)$ and that $\rho^{j(l)}(0,\cdot) = |\overline{u}^{j(l)}(\cdot)| \to |\overline{u}(\cdot)| = \rho(0,\cdot)$ strongly in $L^1_{loc}(\mathbf{R}^n)$. Hence, we can pass to the limit in the problems

$$\begin{cases} \partial_t(\rho^{j(l)}\theta^{j(l)}) + \mathrm{div}_x(f(\rho^{j(l)})\rho^{j(l)}\theta^{j(l)}) \;=\; 0 \\ \\ \rho^{j(l)}\theta^{j(l)}(0,\cdot) \;=\; \rho^{j(l)}(0,\cdot)\overline{\theta}^{j(l)}(\cdot)\,, \end{cases}$$

and conclude that $\theta^\infty$ solves

$$\begin{cases} \partial_t(\rho\theta^\infty) + \mathrm{div}_x(f(\rho)\rho\theta^\infty) \;=\; 0 \\ \\ \rho\theta^\infty(0,\cdot) \;=\rho(0,\cdot)\overline{\theta}^\infty(\cdot)\,. \end{cases}$$

Set $\theta := u/|u|$ where $\rho \neq 0$ and $\theta := 0$ everywhere else, and define $\overline{\theta}$ in an analogous way. Then $\theta$ solves

$$\begin{cases} \partial_t(\rho\theta) + \mathrm{div}_x(f(\rho)\rho\theta) \;=\; 0 \\ \\ \rho\theta(0,\cdot) \;=\; \rho(0,\cdot)\overline{\theta}(\cdot)\,, \end{cases}$$

Thanks to assumption (c), we have $\rho(0,\cdot)\overline{\theta}^\infty(\cdot) = \rho(0,\cdot)\overline{\theta}(\cdot)$. Hence we can apply Proposition 2.10 to conclude that $\rho(t,x)\theta^\infty(t,x) = \rho(t,x)\theta(t,x)$ for almost every $(t,x)$. Hence, $u^{j(l)}$ converges weakly$^*$ to $u$. Since $|u^{j(l)}| \to |u|$ strongly in $L^1_{loc}$, we conclude that $u^{j(l)} \to u$ strongly in $L^1_{loc}$.

The argument above shows that from every subsequence $\{u^{j(l)}\} \subset \{u^j\}$ we can extract a further subsequence which converges strongly in $L^1_{loc}$ to $u$. This implies that the whole sequence $\{u^j\}$ converges to $u$.

## 4. Bressan's conjecture

In this section we show that a suitable renormalization property is closely related to a conjecture recently made by Bressan on the compactness of Lagrangian flows. We also indicate some cases when the compactness is known.

**Conjecture 4.1.** *[Bressan [4]] Let $f^j : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^n$ be a sequence of smooth maps and define $\Phi^j : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^n$ by*

$$\begin{cases} \dfrac{d}{dt}\Phi^j(t,x) \;=\; f^j(t,\Phi^j(t,x)) \\ \\ \Phi^j(0,x) \;=\; x\,. \end{cases}$$

*Denote by $J^j(t,x)$ the Jacobian determinants of $\nabla_x\Phi^j(t,x)$. Assume that*

$$\|f^j\|_\infty + \|\nabla_{t,x}f^j\|_{L^1} \;\le\; C \qquad \text{and} \qquad C^{-1} \le J^j \le C\,,$$

*for some constant $C$. Then $\Phi^j$ is strongly precompact in $L^1_{loc}$.*

**Remark 4.2.** Let $\Psi^j(t,\cdot)$ be the inverse of $\Phi^j(t,\cdot)$ and let $\rho^j(t,x) := 1/J^j(t,\Psi^j(t,x))$. Note that $\partial_t\rho^j + \mathrm{div}_x(f^j\rho^j) = 0$ and that, thanks to our assumptions, $C \ge \rho^j \ge C^{-1}$. We can assume that, up to subsequences, $\rho^j$ converges to a bounded function $\rho$ weakly$^*$ in $L^\infty$. Moreover, again up to subsequences, we can assume that $f^j$ converges strongly in $L^1$ to a BV map $f$. This gives that $\partial_t\rho + \mathrm{div}_x(\rho f) = 0$.

In view of DiPerna–Lions theory, the following seems closely related to Conjecture 4.1:

**Conjecture 4.3.** *Let $f \in L^\infty \cap BV_{loc}(\mathbf{R}\times\mathbf{R}^n)$ and let $\rho \in L^\infty(\mathbf{R}\times\mathbf{R}^n)$ be such that*

$$\rho \geq c > 0 \qquad \textit{and} \qquad \partial_t\rho + \mathrm{div}_x(\rho f) \;=\; 0\,.$$

*Then every bounded weak solution $w$ of $\partial_t(\rho w) + \mathrm{div}_x(\rho f w) \;=\; 0$ is a renormalized solution.*

Indeed we will prove

**Proposition 4.4.** *Conjecture 4.3 implies Conjecture 4.1.*

**Remark 4.5** (Absolutely continuous divergence)**.** Note that the results of [1] give a positive answer to Conjecture 4.3 when $\mathrm{div}_x f$ is absolutely continuous. Indeed fix $w : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}^k$ as in Conjecture 4.3 and consider the vector–valued map $v := (\rho, \rho w)$. Note that $\partial_t v + \mathrm{div}_x(vf) = 0$. In view of point (3) of Remark 3.5 of [1], $v$ is a renormalized solution, which in this case means that

$$\partial_t(H(v)) + \mathrm{div}_x(fH(v)) \;=\; \left(\sum_{j=1}^{k+1} v_j \frac{\partial H}{\partial y_j}(v) - H(v)\right)\mathrm{div}_x f \qquad \text{for all } H \in C^1(\mathbf{R}^{k+1}). \tag{22}$$

Let $\zeta : \mathbf{R}^{k+1} \to \mathbf{R}^k$ be a smooth map such that $\zeta(y) = (y_2/y_1, \ldots, y_k/y_1)$ on $y_1 \geq C^{-1}$. For every $h \in C^1(\mathbf{R}^k)$ consider the map $H(y) := y_1 h(\zeta(y))$ and not that

$$H(v) = \rho h(w) \quad \text{and} \quad \sum_{j=1}^{k+1} v_j \frac{\partial H}{\partial y_j}(v) - H(v) \;=\; 0\,.$$

Thus from (22) we get $\partial_t(\rho h(w)) + \mathrm{div}_x(\rho h(w) f) = 0$.

Concerning Conjecture 4.1, we know from Theorem 6.5 of [1] that it holds provided the $f^j$'s converge to an $f$ whose divergence is absolutely continuous and with negative part in $L^\infty$. The proof in [1] actually uses only the lower bound on $J^j$ and not the upper bound.

Note that Proposition 4.4 yields a proof of Conjecture 4.1 under the only assumption that the divergence of $f$ is absolutely continuous.

Before coming to the proof of Proposition 4.4 we need two lemmas. Up to subsequences, we assume that $\rho^j \to \rho$ weakly$^*$ in $L^\infty$ and that $f^j \to f$ strongly in $L^1_{loc}$. Thus $\rho$ satisfies the transport equation

$$\partial_t\rho + \mathrm{div}_x(\rho f) = 0\,,$$

and, by Remark 2.9, the map $t \to \rho(t,\cdot)$ is weakly continuous in $L^1_{loc}$. Arguing as in Remark 2.9, it is easy to see that for every test function $\varphi \in C^\infty_c(\mathbf{R}^n)$ the functions $t \to \int \varphi(x)\rho^j(t,x)\,dx$ are equicontinuous and thus they converge uniformly to the map $t \to \int \varphi(x)\rho(t,x)dx$. Since $\rho^j(0,x) = 1$, we conclude that

$$\rho(t,\cdot) \;\to\; 1 \quad \text{weakly}^* \text{ in } L^\infty \text{ for } t \downarrow 0. \tag{23}$$

In view of this, $w \in L^\infty$ is a weak solution of

$$\left\{\begin{array}{l} \partial_t(\rho w) + \mathrm{div}_x(\rho f w) \;=\; 0 \\[2ex] \rho w(0,\cdot) \;=\; \rho(0,\cdot)\overline{w}(\cdot) \end{array}\right. \tag{24}$$

if for every $\varphi \in C_c^\infty(\mathbf{R}\times\mathbf{R}^n)$ we have

$$\int_{\mathbf{R}^+\times\mathbf{R}^n} \rho(t,y)w(t,y)\left[\partial_t\varphi(t,y)+f(t,y)\cdot\nabla_x\varphi(t,y)\right]dy\,dt \;=\; \int_{\mathbf{R}^n}\rho(0,x)\overline{w}(x)\varphi(0,x)\,dx\,.$$

**Lemma 4.6.** *Let $\rho^j$ and $f^j$ be as in Remark 4.2 and assume that $\rho^j \to \rho$ weakly* in $L^\infty$ and $f^j \to f$ strongly in $L^1_{loc}$. Assume that Conjecture 4.3 holds. If $w:\mathbf{R}^+\times\mathbf{R}^n\to\mathbf{R}^k$ is a bounded weak solution of (24), then for every $h\in C^1(\mathbf{R}^k)$ the function $h(w)$ is a weak solution of*

$$\left\{\begin{array}{l} \partial_t(\rho h(w))+\mathrm{div}_x(\rho f h(w)) \;=\; 0 \\ \\ \rho h(w(0,\cdot)) \;=\; \rho(0,\cdot)h(\overline{w}(\cdot)) \end{array}\right. \tag{25}$$

**Lemma 4.7.** *Let $\rho^j$ and $f^j$ be as in Remark 4.2 and assume that $\rho^j\to\rho$ weakly* in $L^\infty$ and $f^j\to f$ strongly in $L^1_{loc}$. If $\overline{w}\in L^\infty$ and Conjecture 4.3 holds, then there exists a unique weak solution of (24). Moreover, if $w^j$ are the solutions of*

$$\left\{\begin{array}{l} \partial_t(\rho^j w^j)+\mathrm{div}_x(\rho^j f^j w^j) \;=\; 0 \\ \\ \rho w^j(0,\cdot) \;=\; \rho(0,\cdot)\overline{w}(\cdot)\,, \end{array}\right. \tag{26}$$

*then $w^j\to w$ strongly in $L^1_{loc}$.*

*Proof of Lemma 4.6.* Conjecture 4.3 implies that

$$\partial_t(\rho h(w))+\mathrm{div}_x(\rho f h(w)) \;=\; 0$$

in the sense of distributions. Thus, in order to prove the lemma we have to ensure that $h(w)$ satisfies the right boundary conditions.

As in the proof of Lemma 2.10, we extend the field $\rho$ and $f$ to maps $a$ and $b$ defined on the whole $\mathbf{R}\times\mathbf{R}^n$ by setting

$$a(t,x) \;:=\; \left\{\begin{array}{ll} 1 & \text{for } t<0 \\ \rho(t,x) & \text{for } t\geq 0 \end{array}\right. \qquad b(t,x) \;:=\; \left\{\begin{array}{ll} 0 & \text{for } t<0 \\ f(t,x) & \text{for } t\geq 0\,. \end{array}\right.$$

Clearly $b\in BV_{loc}(\mathbf{R}\times\mathbf{R}^n)$. Moreover, thanks to (23), it is clear that $\partial_t a+\mathrm{div}_x(ab)=0$. Extend $w$ to negative times by setting

$$w(t,x) \;:=\; \overline{w}(x) \qquad \text{for } t<0.$$

Then $\partial_t(aw)+\mathrm{div}_x(baw)=0$ on the whole $\mathbf{R}\times\mathbf{R}^n$. Applying Conjecture 4.3 to the field $(a,b)$ we get that $\partial_t(ah(w))+\mathrm{div}_x(abh(w))=0$ on the whole $\mathbf{R}\times\mathbf{R}^n$. Since $h(w(t,x))=h(\overline{w}(x))$ for $t<0$, clearly $h(w)$ is a weak solution of (26). □

*Proof of Lemma 4.7.* For the uniqueness we repeat the same proof of Lemma 2.10. First of all, by linearity it is sufficient to prove the lemma when $\overline{w}=0$. Then we set $v:=|w|^2$ and by Lemma 4.6 we get that $v$ is a weak solution of

$$\left\{\begin{array}{l} \partial_t(\rho v)+\mathrm{div}_x(\rho f v) \;=\; 0 \\ \\ \rho v(0,\cdot) \;=\; 0\,. \end{array}\right. \tag{27}$$

Thanks to Remark 2.9, the map $t \to \rho(t,\cdot)v(t,\cdot)$ is weakly continuous in $L^1_{loc}$. Thus we can apply Lemma 2.11 to $z = \rho v$ and $h = \rho f v$. We conclude that $\rho v = 0$ and, since $\rho > 0$, we get $v = 0$.

We now pass to the second part of the lemma. Let us fix $w^j$ as in the statement. By possibly extracting a subsequence, assume that $\rho^j w^j$ converge, weakly* in $L^\infty$, to a function $v$. Set $w := v/\rho$. Since $f^j \to f$ strongly in $L^1_{loc}$, it follows easily that $w$ is a weak solution of (24). Since $\rho^j$ and $f^j$ are smooth, for every $h \in C^1$ we have

$$\begin{cases} \partial_t(\rho^j h(w^j)) + \mathrm{div}_x(\rho f h(w)) \;=\; 0 \\ \\ \rho h(w(0,\cdot)) \;=\; \rho(0,\cdot)\overline{w}(\cdot)\,. \end{cases} \tag{28}$$

As above, we can assume that, up to subsequences, $\rho^j h(w^j)$ converge to a function $z$, weakly* in $L^\infty$. Setting $\tilde{z} = z/\rho$, we find that $\tilde{z}$ solves

$$\begin{cases} \partial_t(\rho\tilde{z}) + \mathrm{div}_x(\rho f \tilde{z}) \;=\; 0 \\ \\ \rho\tilde{z}(0,\cdot) \;=\; \rho(0,\cdot)h(\overline{w}(\cdot))\,. \end{cases} \tag{29}$$

Thanks to Lemma 4.6, $h(w^j)$ solves the same transport problem. Thus, by the first part of the lemma, we have that $\tilde{z} = h(w)$. Hence for every such $h$, $\rho^j h(w^j)$ converges (weakly* in $L^\infty$) to $\rho h(w)$.

Fix a bounded set $\Omega \subset \mathbf{R}^+ \times \mathbf{R}^n$. Then

$$\int_\Omega \rho^j |w^j - w|^2 \;=\; \int_{\mathbf{R}^+\times\mathbf{R}^n} \mathbf{1}_\Omega \rho^j |w^j|^2 + \int_{\mathbf{R}^+\times\mathbf{R}^n} \mathbf{1}_\Omega \rho^j |w|^2 - 2\int_{\mathbf{R}^+\times\mathbf{R}^n} \mathbf{1}_\Omega \rho^j w^j \cdot w\,.$$

Since $\rho^j \overset{*}{\rightharpoonup} \rho$, $\rho^j w^j \overset{*}{\rightharpoonup} \rho w$, and $\rho^j|w^j|^2 \overset{*}{\rightharpoonup} \rho|w|^2$, we get that $\int_\Omega \rho^j|w^j - w|^2 \;\downarrow\; 0$. Since $\rho^j \geq C^{-1}$, we conclude that $\int_\Omega |w^j - w|^2 \downarrow 0$. □

*Proof of Proposition 4.4.* Let $f^j$ and $\Phi^j$ be as in Conjecture 4.1 and define $\Psi^j$ as in Remark 4.2. Without loosing our generality we assume that $f^j \to f$ strongly in $L^1_{loc}$. Fix $T > 0$ and consider the ODE

$$\begin{cases} \dfrac{d}{dt}\Lambda^j(t,x) \;=\; f^j(t,\Lambda(t,x)) \\ \\ \Lambda^j(T,x) \;=\; x\,. \end{cases}$$

Note that $\Lambda^j(t,\cdot) = \Phi^j(t,\Psi^j(T,\cdot))$. Thus, if we denote by $\tilde{J}(t,\cdot)$ the Jacobian of $\Lambda^j(t,\cdot)$, we get that $0 \leq C^{-2} \leq \tilde{J}(t,\cdot) \leq C^2$. Denote by $\Gamma^j(t,\cdot)$ the inverse of $\Lambda^j(t,\cdot)$ and set $\tilde{\rho}^j(t,x) := 1/\tilde{J}(t,\Gamma^j(t,x))$. Moreover, for every $\overline{w} \in L^\infty(\mathbf{R}^n,\mathbf{R}^n)$ define the function $w^j(t,x) := \overline{w}(\Gamma^j(t,x))$. Clearly we have

$$\begin{cases} \partial_t(\tilde{\rho}^j w^j) + \mathrm{div}_x(\tilde{\rho}^j f^j w^j) \;=\; 0 \\ \\ \rho^j w^j(x,T) \;=\; \overline{w}(x)\,. \end{cases}$$

We claim that the $\tilde{\rho}^j$'s have a unique weak* limit. Indeed, assume that $\tilde{\rho}$ and $\hat{\rho}$ are weak* limits of two convergent subsequences of $\tilde{\rho}^j$'s. Then $\partial_t\tilde{\rho}^j + \mathrm{div}_z(f\tilde{\rho}) = 0$ and $\partial_t\hat{\rho}^j + \mathrm{div}_z(f\hat{\rho}) =$

0. Moreover, thanks to the discussion preceding (23), both $\tilde{\rho}$ and $\hat{\rho}$ have weak trace equal to 1 at $t = T$. Thus, if we set $v := \hat{\rho}/\tilde{\rho}$ we have

$$\begin{cases} \partial_t(\tilde{\rho}v) + \mathrm{div}_x(\tilde{\rho}vf) \ = \ 0 \\ \\ \tilde{\rho}v(0,\cdot) \ = \ 1\,. \end{cases}$$

Since the 1 is a weak solution of the same Cauchy problem, by Lemma 4.7 we have that $w = 1$, and hence $\tilde{\rho} = \hat{\rho}$.

Note that there exists a constant $C$ such that $|\Gamma^j(t,x) - x| \le C(T-t)$ for every $t$, $x$ and $j$. Fix $r > 0$ and choose $R > 0$ so large that $R - CT > r$. Let $\overline{w}$ be the vector valued map $x \to x\,\mathbf{1}_{B_R(0)}(x)$. Thus, for every $t < T$ and every $|x| < r$, $w^j(t,x)$ is equal to the vector $\Gamma^j(t,x)$. Thanks to Lemma 4.7, $w^j$ converges strongly in $L^1_{loc}$ to a unique $w$. Hence, by the arbitrariness of $r$ we conclude that $\Gamma^j$ converges to a unique $\Gamma$ strongly in $L^1_{loc}$.

For each $x$, $\Gamma^j(\cdot,x)$ is a Lipschitz curve, with Lipschitz constant uniformly bounded. Thus we infer that, for a.e. $x$, $\Gamma^j(\cdot,x)$ converges uniformly to the curve $\Gamma(\cdot,x)$ on $[0,T]$. Hence, we conclude that, after possibly changing $\Gamma$ on a set of measure 0, for every $t \ge 0$ the maps $\Gamma^j(t,\cdot)$ converge to $\Gamma(t,\cdot)$ in $L^1_{loc}(\mathbf{R}^n)$.

Since $\Gamma^j(0,\cdot) = \Phi^j(T,\cdot)$ we conclude that for every $T$ there exists a $\Phi(T,\cdot)$ such that $\Phi^j(T,\cdot)$ converges to $\Phi(T,\cdot)$ in $L^1_{loc}(\mathbf{R}^n)$. Since $\Phi^j$ is locally uniformly bounded, we conclude that $\Phi^j$ converges to $\Phi$ strongly in $L^1_{loc}(\mathbf{R}^+ \times \mathbf{R}^n)$. $\square$

## References


[1] L.Ambrosio: *Transport equation and Cauchy problem for BV vector fields.* Preprint, 2003 (available at http://cvgmt.sns.it).

[2] L.Ambrosio, C.De Lellis: *Existence of solutions for a class of hyperbolic systems of conservation laws in several space dimensions.* To appear in IMRN.

[3] F.Bouchut: *Renormalized solutions to the Vlasov equation with coefficients of bounded variation.* Arch. Ration. Mech. Anal. **157** (2001), 75–90.

[4] A.Bressan: *An ill posed Cauchy problem for a hyperbolic system in two space dimensions.* Preprint, 2003.

[5] A.Bressan: *A lemma and a conjecture on the cost of rearrangements.* Preprint, 2003.

[6] C.Dafermos: *Hyperbolic conservation laws in continuum physics.* Springer-Verlag, Berlin, 1999.

[7] R.DiPerna, P.L.Lions: *Ordinary differential equations, transport theory and Sobolev spaces.* Invent. Math., **98** (1989), 511–517.

[8] H.Frid: *Asymptotic stability of non–planar riemann solutions for a special class of multi–d systems of conservation laws.* Preprint, 2003.

[9] B.L.Keyfitz, H.C.Kranzer: *A system of nonstrictly hyperbolic conservation laws arising in elasticity theory.* Arch. Rational Mech. Anal. **72** (1980), 219–241.

[10] S.Kruzhkov: *First-order quasilinear equations with several space variables.* Math. USSR Sbornik, **10** (1970), 217–273.

[11] D.Serre: *System of conservation laws I, II.* Cambridge University Press, 1999.

[12] A.Vasseur: *Strong traces for solutions of multidimensional scalar conservation laws.* Arch. Ration. Mech. Anal., **160** (2001), 181–193.



Luigi Ambrosio, Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy

*E-mail address*: l.ambrosio@sns.it

François Bouchut, Département de mathématiques et applications, ENS, 45, rue d'Ulm, F-75230 Paris cedex 05, France
*E-mail address*: Francois.Bouchut@ens.fr

Camillo De Lellis, Max–Planck Institute for Mathematics in the Sciences, Inselstr. 22, D-04103 Leipzig, Germany
*E-mail address*: delellis@mis.mpg.de