BLOW–UP OF THE BV NORM IN THE MULTIDIMENSIONAL KEYFITZ AND KRANZER SYSTEM

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ABSTRACT. We consider the Cauchy problem for the system $\partial_t u_i + \mathrm{div}_z(g(|u|)u_i) = 0$ $i \in \{1, \dots, k\}$, in m space dimensions and with $g \in C^3$. When $k \geq 2$ and $m = 2$ we show a wide choice of g's for which the BV norm of admissible solutions can blow up, even when the initial data have arbitrarily small oscillation, arbitrarily small total variation, and are bounded away from the origin. When $m \geq 3$ we show that this occurs whenever g is not constant, i.e. unless the system reduces to k decoupled transport equations with constant coefficients.

1. INTRODUCTION

Let us consider the system of conservation laws

$$\partial_t U + \mathrm{div}_z[F(U)] \;=\; 0 \qquad U : \Omega \subset \mathbf{R}_t \times \mathbf{R}^m_z \to \mathbf{R}^k_u\,, \tag{1}$$

where $F : \mathbf{R}^k \to \mathbf{R}^m \times \mathbf{R}^k$ is a C^3 function. In what follows we will use the notation $F = (F_1, \dots, F_m)$, where each F_i is a map from $\mathbf{R}^k$ to $\mathbf{R}^k$ and we restrict our attention to *symmetric* systems, i.e. systems for which $DF_i|_V$ is a symmetric matrix for every i and for every $V \in \mathbf{R}^k$. It is known, by a result of Rauch based on a previous remark of Brenner for linear hyperbolic systems (see [6] and [14]), that BV–estimates (and L^p estimates for $p \neq 2$) fail for all the systems (1) which do not satisfy the commutator conditions

$$DF_i|_V \cdot DF_j|_V \;=\; DF_j|_V \cdot DF_i|_V \qquad \text{for every } V \in \mathbf{R}^k. \tag{2}$$

When $n = 2$, it was proved in [9] that (2) is also sufficient to get L^p estimates for every $p \leq 2$ and, under additional conditions, also for $p = \infty$.

A particular class of systems that satisfy the commutators condition (2) is given by

$$\begin{cases} \partial_t u + \mathrm{div}_z[g(|u|)u] \;=\; 0 \\ \\ u(0,\cdot) \;=\; u_0\,, \end{cases} \qquad u : \mathbf{R}^+_t \times \mathbf{R}^m_z \to \mathbf{R}^k\,. \tag{3}$$

In [2], using the extension of the DiPerna–Lions theory to BV fluxes (recently done in [1]), the authors have proved that, if $|u_0| \in BV_{loc}(\mathbf{R}^m)$ and $g \in C^1$, then there exists a unique weak solution of (3) such that $\rho = |u|$ is an entropy solution of the scalar law

$$\begin{cases} \partial_t \rho + \mathrm{div}_z[g(\rho)\rho] \;=\; 0 \\ \\ \rho(0,\cdot) \;=\; |u_0| \end{cases} \tag{4}$$

We say that such a u is a *renormalized entropy solution*.

Taking into account the BV theory for one–dimensional hyperbolic systems of conservation laws, it is natural to ask the question whether BV estimates hold for renormalized entropy solutions of (3), at least when the initial data are uniformly close to a constant and have small BV norm. Moreover, since one might wonder whether the notion of renormalized entropy solution is really a suitable one, one can raise the question whether BV estimates are possible for some other notion of admissible solutions. In this paper we prove in Theorem 2.3 that for a wide choice of g's there are initial data u_0 (with arbitrarily small BV norm and uniformly close to a constant $c \neq 0$) such that the BV norm of admissible solutions u of (3) blows up instantaneously, no matter what criterion of admissibility is chosen among the ones proposed in the literature. When m in (3) is strictly bigger than 2, such initial data can be constructed for every g which is not constant, i.e. for every system of the form (3) which does not reduce to k decoupled transport equations with constant coefficients.

2. Statement of the result and preliminaries

2.1. **Statement of the result.** In order to state Theorem 2.3 we first need some definitions.

Definition 2.1. *A pair of functions $\eta \in W^{1,\infty}(\mathbf{R}^k, \mathbf{R})$, $q \in W^{1,\infty}(\mathbf{R}^k, \mathbf{R}^m)$ is called an* entropy–entropy flux pair *for* (1) *if $Dq = D\eta \cdot DF$.*

General hyperbolic systems of conservation laws may not have any entropy–entropy flux pair besides the ones where η is an affine function. However, the systems (3) always have an abundance of entropies.

Definition 2.2. *Let $U_0 \in L^\infty$. Then U is called an* entropy solution *of*

$$\begin{cases} \partial_t U + \mathrm{div}_z[G(U)] \;=\; 0 \\ \\ U(0,\cdot) \;=\; U_0\,, \end{cases} \tag{5}$$

if, for every convex entropy-entropy flux pair η, q and for every smooth test function $\psi \geq 0$,

$$\int_{t>0}\int_{\mathbf{R}^m} [\partial_t\psi(t,z)\,\eta(U(t,z)) + \nabla_z\psi(t,z)\cdot q(U(t,z))]\,dt\,dz + \int_{\mathbf{R}^m} \psi(0,z)\eta(U_0(z))\,dz \;\geq\; 0\,. \tag{6}$$

The (nonpositive) entropy production *measure $\partial_t[\eta(U)] + \mathrm{div}_z[q(U)]$ will be denoted by μ_η.*

A lemma of [12] shows that every entropy η for the system (3) is of the form $\xi(|u|) + |u|G(u/|u|)$, where ξ must be an entropy for the scalar law $\partial_t\rho + \mathrm{div}_z(g(\rho)\rho) = 0$. Clearly, if η is convex, then ξ must be also convex. From the proof of [2], it follows that for all entropies of the form $|u|G(u/|u|)$, the entropy production vanishes. Hence, since $|u|$ is an entropy solution of the corresponding scalar law, we conclude that *renormalized entropy* solutions of (3) are indeed *entropy* solutions. However, there are some entropy solutions which are not necessarily renormalized. See for instance the example in [7] (or top of page 188 of [8]).

The chain rule of Vol'pert for BV functions implies that, if U is a BV entropy solution of (5), then $\mu_\eta(\Omega \setminus J_U) = 0$, i.e. μ_η "lives" on the jump set J_U (which in this case is called the set of shocks of the solution U). We refer to the next subsection for the relevant definitions. For BV weak solutions U of (1), other criteria of admissibility have been studied in the literature. We remark that all of them are based on imposing some conditions on the set of shocks and these conditions turn out to be stronger than the entropy ones. Hence the notion of entropy solution is the weakest among the ones proposed in the literature. We are now in the position to state the main remark of this paper

Theorem 2.3. *Let $k \geq 2$, $m \geq 3$, $g \in C^3_{loc}$ and let $c \in \mathbf{R}^k \setminus \{0\}$ such that $g'(|c|) \neq 0$. Then there exists a sequence of of initial data $u_0^n : \mathbf{R}^m \to \mathbf{R}^k$ such that*

- $\|u_0^n - c\|_{BV(\mathbf{R}^m)} + \|u_0^n - c\|_\infty \to 0$ *for $n \uparrow \infty$;*
- *If u^n is* any *bounded entropy solution of* (3) *with initial data u_0^n, then for any $T > 0$ there exists $R > 0$ such that $\|u^n\|_{BV(]0,T[\times B_R)} = \infty$.*

When $m = 2$ the same statement holds if in addition we assume that $g''(|c|)$ is parallel to $g'(|c|)$ (or vanishes).

2.2. **BV functions and Vol'pert chain rule.** For definitions, statements, and proofs of the claims contained in this subsection, we refer to the book [5].

When $u : \mathbf{R}^l \to \mathbf{R}^k$ is a BV function, we define S_u as the set of points x where u is not approximately continuous and J_u as the set of points x where u "jumps", i.e. where it has approximate right and left limits with respect some fixed direction $\nu(x) \in \mathbf{S}^{l-1}$. The BV Structure Theorem states that J_u is a codimension 1 rectifiable set and that $\mathcal{H}^{l-1}(S_u \setminus J_u) = 0$, where $\mathcal{H}^{l-1}$ denotes the $l-1$–dimensional Hausdorff measure. Moreover $\nu(x)$ can be chosen so that $\nu : J_u \to \mathbf{S}^{l-1}$ is a Borel measurable function and coincides with the approximate normal to J_u. With this choice of $\nu(x)$ we denote by $u^+(x)$ and $u^-(x)$ respectively the right and left approximate limit of u at x. For any $x \notin S_u$ we denote by $\tilde{u}(x)$ the approximate limit of u at x. The BV Structure Theorem implies also that the matrix–valued measure Du can be written as

$$Du \;=\; \mu \;+\; \big[(u^+ - u^-) \otimes \nu\big]\, \mathcal{H}^{l-1} \llcorner J_u \,, \tag{7}$$

where

- $\mathcal{H}^{l-1} \llcorner J_u$ denotes the Borel measure given by

$$\mathcal{H}^{l-1} \llcorner J_u\,(A) \;=\; \mathcal{H}^{l-1}(A \cap J_u) \qquad \text{for any Borel set } A;$$

- $\mu(A) = 0$ for any Borel set A of Hausdorff dimension $l - 1$.

We denote μ by $D_d u$ and we call it *diffused part of the measure Du* (in the notation of [5] this measure is just the sum of the absolutely continuous part $D_a u$ and of the Cantor part $D_c u$). We can now state the chain rule of Vol'pert for BV functions.

Theorem 2.4. *Let $u \in BV(\mathbf{R}^l, \mathbf{R}^k)$ and $\eta \in C^1(\mathbf{R}^k, \mathbf{R}^h)$. Then $\eta \circ u \in BV$ and*

$$D[\eta \circ u] \;=\; [D\eta \circ \tilde{u}] \cdot D_d u \;+\; \big\{[\eta(u^+) - \eta(u^-)] \otimes \nu\big\}\, \mathcal{H}^{l-1} \llcorner J_u \,. \tag{8}$$

Remark 2.5. In [3] the authors proved a suitable extension of Theorem 2.4 to $\eta \in W^{1,\infty}$. In what follows we will sometimes consider the measure $D[\eta \circ u]$ for η which are merely Lipschitz. However we will not need the general result of [3]. Indeed, in some cases we will be able to use some "ad hoc" considerations, whereas in all the other cases considered, Proposition 2.6 below will be sufficient for our purposes.

Proposition 2.6. *Let $u \in BV(\mathbf{R}^l, \mathbf{R}^k)$ and $\eta \in W^{1,\infty}(\mathbf{R}^k, \mathbf{R}^h)$. Then $\eta \circ u \in BV$ and*

$$D[\eta \circ u] \;=\; \mu \;+\; \big\{[\eta(u^+) - \eta(u^-)] \otimes \nu\big\}\, \mathcal{H}^{l-1} \llcorner J_u \,. \tag{9}$$

where $\mu(G) = 0$ for every $G \subset J_u$.

2.3. **Riemann Problem for scalar laws.** Let us consider the Cauchy problem

$$\left\{\begin{array}{rcl} \partial_t\rho + \mathrm{div}_x[h(\rho)] &=& 0 \\ \\ \rho(0,\cdot) &=& \rho^{in} \end{array}\right. \qquad \rho : \mathbf{R}^+_t \times \mathbf{R}^m_x \;\to\; \mathbf{R}\,, \tag{10}$$

where $h : \mathbf{R} \to \mathbf{R}^m$ is of class C^3. Fix $\beta, \gamma, \alpha \in \mathbf{R}$, set $\varepsilon := \max\{|\alpha - \beta|, |\alpha - \gamma|\}$, and define

$$\rho^{in}(x_1, \ldots, x_m) \;=\; \left\{\begin{array}{ll} \beta & \text{for } x_m < 0 \\ \gamma & \text{for } x_m > 0 \end{array}\right.$$

and consider the entropy solution ρ of (10). It is easy to see that ρ depends only on t and x_m. For each $T > 0$ define:

$$\begin{array}{rcl} \xi &:=& \max\big\{x_m | \rho(T, \cdot, x_m) = \beta\big\} \\ \eta &:=& \min\big\{x_m | \rho(T, \cdot, x_m) = \gamma\big\}\,. \end{array}$$

Then the following lemma holds

Lemma 2.7. *If we denote by h'_m and h''_m the m–th components of the vector–valued functions h' and h'', then there exist constants C and δ (depending only on h) such that*

$$\max\big\{|\xi - Th'_m(\alpha)|, |\eta - Th'_m(\alpha)|\big\} \;\le\; h''_m(\alpha)\varepsilon + C\varepsilon^2 \qquad \textit{for } \varepsilon \le \delta\,. \tag{11}$$

3. Blow–up of the BV norm for renormalized entropy solutions

In this section we prove the following proposition

Proposition 3.1. *Let $k \ge 2$, $m \ge 3$, and $g \in C^3_{loc}$. Then, for every $c \in \mathbf{R}^k$ such that $g'(|c|) \neq 0$, there exists a sequence of initial data $u^n_0 : \mathbf{R}^m \to \mathbf{R}^k$ such that*

- $\|u^n_0 - c\|_{BV(\mathbf{R}^m)} + \|u^n_0 - c\|_\infty \to 0$ *for $n \uparrow \infty$;*
- *If u^n denotes the unique renormalized entropy solution of* (3) *with $u^n(0,\cdot) = u^n_0$, then $u^n(t,\cdot) \notin BV_{loc}(\mathbf{R}^m)$ for every n and for every $t \in]0,1[$.*

When $m = 2$ the same statement holds if in addition $g''(|c|) \parallel g'(|c|)$ or $g''(|c|) = 0$.

Before giving the proof of Proposition 3.1 we consider the special case of system (3) when $g = (f, 0, \ldots, 0)$, that is

$$\left\{\begin{array}{rcl} \partial_t u + \partial_{x_1}[f(|u|)u] &=& 0 \\ \\ u(0,\cdot) &=& u_0\,. \end{array}\right. \tag{12}$$

The following is a corollary of Proposition 3.1

Proposition 3.2. *Let $k \ge 2$, $m \ge 2$ and $c \in \mathbf{R}^m$ be such that $f'(|c|) \neq 0$. Then there exists a sequence of initial data $u^n_0 : \mathbf{R}^m \to \mathbf{R}^k$ as in Proposition 3.1.*

Roughly speaking, the proof of Proposition 3.1 is based on the following remark: When $m = 3$ we can choose initial data, close to a constant, in such a way that the behavior of the renormalized entropy solutions of (3) is close to the behavior of solutions of (12). This seems to be no longer true for $m = 2$, unless $g''(|c|)$ is parallel to $g'(|c|)$ (or $g''(|c|) = 0$). Due to this remark, we choose to give a quick self–contained proof of Proposition 3.2.

3.1. **Proof of Proposition 3.2.** For the sake of simplicity we prove the proposition when $k = m = 2$, $f'(|c|) = 1$, and $f(|c|) = 0$. Only minor adjustments are needed to handle the general case.

Let $\{m_k\}$ be a sequence of positive even numbers such that

$$\sum_k m_k 2^{-k} \;<\; \infty\,. \tag{13}$$

Let $\delta > 0$ be so small that f is injective on $[|c| - 2\delta, |c| + 2\delta]$ and $[f(|c|) - \delta, f(|c|) + \delta]$ is contained in $f([|c| - 2\delta, |c| + 2\delta])$. Then, for k sufficiently large, we define r_k as the unique number in $[-2\delta, 2\delta]$ such that $f(|c| + r_k) = 1 + 2^{-k}$. Notice that for k sufficiently large we have $r_k \leq 2^{-k+1}$. Set $\alpha = c/|c|$ and for every k choose an $\alpha_k \in \mathbf{S}^1$ such that $|\alpha_k - \alpha| = k^{-2}$.

Let I_k be the interval $[2^{-k}, 2^{-k+1}[$ and subdivide it in m_k equal subintervals

$$I_k^j \;:=\; \left[2^{-k} + \frac{(j-1)2^{-k}}{m_k}\,,\, 2^{-k} + \frac{j2^{-k}}{m_k}\right[\qquad j \in \{1, \ldots, m_k\}\,.$$

Next define the functions $\psi_k : \mathbf{R}^2 \to \mathbf{S}^1$ as

$$\psi_k(x,y) \;:=\; \begin{cases} \alpha_k & \text{if } y \in I_k \text{ and } \left[x2^{-k}\right] \text{ is odd} \\ \alpha & \text{otherwise} \end{cases}$$

and the functions $\chi_k : \mathbf{R}^2 \to \mathbf{R}$ as

$$\chi_k(x,y) \;:=\; \begin{cases} r_k & \text{if } y \in I_k^j \text{ for } j \text{ even and } x \in [-M, M] \\ r_{k+1} & \text{if } y \in I_k^j \text{ for } j \text{ odd and } x \in [-M, M] \\ 0 & \text{otherwise}\,, \end{cases}$$

where M is a positive real number to be chosen later. Finally we define

$$\rho_0^n \;:=\; |c| + \sum_{k=n}^{\infty} \chi_n\,;$$

$$\theta_0^n(x,y) \;:=\; \begin{cases} \psi_k(x,y) & \text{if } y \in I_k \text{ for some } k \geq n \text{ and } x \in [-M, M] \\ \alpha & \text{otherwise;} \end{cases}$$

$$u_0^n \;:=\; \rho_0^n \theta_0^n\,.$$

Clearly $\|u_0^n - c\|_\infty \leq (r_n + |c|)|\alpha_n - \alpha| + r_n$. Hence, as $n \uparrow \infty$ we have $\|u_0^n - c\|_\infty \to 0$. Moreover notice that $u_0^n - c$ is supported on $[-M, M] \times [0, 1]$. In order to show that

$$\|u_0^n - c\|_{BV(\mathbf{R}^2)} \;\to\; 0$$

it is sufficient to show

$$\|\rho_0^n - |c|\|_{BV([-2M,2M]^2)} \;\to\; 0 \tag{14}$$

$$\|\theta_0^n - \alpha\|_{BV([-2M,2M]^2)} \;\to\; 0 \tag{15}$$

Note that

$$\|\rho_0^n - |c|\|_{BV([-2M,2M]^2)} \;\leq\; 2M \sum_{k \geq n} m_k r_k + 6M r_n \;\leq\; 4M \sum_{k \geq n} m_k 2^{-k} + 6M r_n$$

and since $\sum_{k\geq n} 2^{-k}m_k$ is summable, we get (14). Moreover,

$$\|\theta^n-\alpha\|_{BV([-2M,2M]^2)} \;\leq\; 2M\sum_{k\geq n}2^{-k}k^{-2}2^k+2M\sum_{k\geq n}\left[k^{-2}+(k+1)^{-2}\right]+6Mn^{-2}$$

and the summability of $\sum k^{-2}$ gives (15).

Now we let u^n be the unique renormalized solution of (12). Recall that $\rho^n := |u^n|$ is the unique entropy solution of (4) with initial data ρ^n_0, and note that, for any y. Since $\rho^n_0(y,\cdot)$ is constant on $[-M,M]$, by the finite speed of propagation of scalar laws, we can choose M large enough so that

$$\rho^n(t,y,x) \;=\; \rho^n_0(y,x) \qquad \text{for } t\in[0,1] \text{ and } x\in[-2,2]\;.$$

Note that M can be chosen independently of n.

Now, let $\theta^n(t,y,x) := u^n/|u^n|(t,y,x)$. Clearly, since $|u^n|(t,\cdot)\in BV\cap L^\infty$ for every t and it is bounded away from zero, it is sufficient to show that $\theta^n(t,\cdot)\not\in BV([-1,1]^2)$ for any $t>0$.

Fix any $(t,y,x)\in[0,1]^3$ and note that:

- If $\rho^n_0(y,0)=|c|$, then $\theta^n(t,y,x)=\theta^n_0(y,x)$;
- If $\rho^n_0(y,x)=|c|+r_k$, then $\theta^n(t,y,x)=\theta^n_0(y,x-2^{-k})$.

Hence, for $j\in\{1,\sigma_k-1\}$, $k\geq n$, and $l\in\{-2^k,\ldots,2^k-1\}$, the function $\theta^n(t,\cdot)$ jumps on the segments

$$S_{j,k,l} \;:=\; \left\{y \;=\; 2^{-k}+\frac{j2^{-k}}{\sigma_k}\quad x\in[l2^{-k},(l+1)2^{-k}]\right\}\;.$$

The total amount of this jump is given by

$$J_k \;:=\; \int_{S_{j,k,l}}\left|\left(\theta^n\right)^+(t,x,y)-\left(\theta^n\right)^-(t,x,y)\right|d\mathcal{H}^1(x) \;=\; t2^{-k}k^{-2}\,.$$

Thus

$$\|\theta^n(t,\cdot)\|_{BV([-1,1]^2)} \;\geq\; 2\sum_{k\geq n}2^k(m_k-1)J_k \;=\; 2t\sum_{k\geq n}(m_k-1)k^{-2}\,. \tag{16}$$

Recall that the bound (13) is the only condition required on the sequence of even numbers $\{m_k\}$. If we set $m_k=2k^2$, then (13) is clearly satisfied, whereas (16) is infinite. □

3.2. **Proof of Proposition 3.1.** For the sake of simplicity we assume $k=2$. Moreover, since the construction is the same, we only present the proof when $m\geq 3$ and, without loosing our generality, we assume $m=3$. We denote by h the function given by $h(\rho)=\rho g(\rho)$ and by α the positive real number $|c|$. Clearly there exists a unit vector $\omega\in\mathbf{R}^3$ such that

$$\omega\cdot g(\alpha) \;=\; \omega\cdot h'(\alpha) \tag{17}$$
$$\omega\cdot g'(\alpha) \;=\; 0 \tag{18}$$
$$\omega\cdot h''(\alpha) \;=\; 0\,. \tag{19}$$

Indeed, since $h'(\alpha)=g(\alpha)+\alpha g'(\alpha)$, (17) reduces to (18). Thus, the conditions above reduce to find a unit vector $\omega\in\mathbf{R}^3$ which is perpendicular to both the vectors $g'(\alpha)$ and $h''(\alpha)$. We fix an orthonormal system of coordinates in $\mathbf{R}^3$ in such a way that $\omega=(0,0,1)$.

First Step Construction of the modulus

Let $\{\sigma_k\}$ be a sequence of vanishing positive real numbers such that $\sum \sigma_k < \infty$ and let $I_k \subset \mathbf{R}$ be the intervals

$$I_0 \ := \ [0, \sigma_1[\qquad\qquad I_k \ := \ \left[\sum_{i \le k-1} \sigma_i, \sum_{i \le k} \sigma_i\right] .$$

Let m_k be a strictly increasing sequence of even integers and divide every I_k in m_k equal subintervals I_k^j for $j \in \{1, \dots m_k\}$. Finally, let $\{a_k\}$ be a vanishing sequence of real numbers and set

$$\rho^{in}(x_1, x_2, x_3) \ := \ \begin{cases} \alpha + a_k & \text{if } x_3 \in I_k^j \text{ for some even } j \\ \alpha & \text{otherwise.} \end{cases}$$

Then, let ρ be the entropy solution of the Cauchy problem

$$\begin{cases} \partial_t \rho + \mathrm{div}_x [h(\rho)] \ = \ 0 \\ \\ \rho(0, \cdot) \ = \ \rho^{in} \end{cases} \tag{20}$$

Clearly ρ is a function of t and x_3 only. Moreover, recalling that $h_3''(\alpha) = 0$, we can apply Lemma 2.7 in order to get the following property.

(T) For every $C_1 > 0$, there exists a $C_2 > 0$ such that if

$$\frac{\sigma_k}{m_k} \ \ge \ C_2 a_k^2 \, , \tag{21}$$

then every I_k^j contains a subinterval J_k^j such that
- The length of J_k^j is bigger than $C_1 a_k^2$;
- For every $(t, \xi_1, \xi_2, \xi_3) \in [0,1] \times \mathbf{R}^2 \times J_k^j$ we have

$$\rho\big(t, \xi_1, \xi_2, \xi_3 + t h_3'(\alpha)\big) \ = \ \rho\big(0, \xi_1, \xi_2, \xi_3\big) \, . \tag{22}$$

For each couple j, k we let $S_{k,j}$ be the stripe

$$S_{k,j} \ := \ \left\{(t, x_1, x_2, x_3) \middle| \, 0 \le t \le 1 \quad \text{and} \quad (x_3 - t h_3'(\alpha)) \in J_k^j \right\}$$

Step 2 The flux generated by ρ.

Denote by $B_R \subset \mathbf{R}^3$ the ball of radius R and centered at the origin. It is easy to check that there exists a constant C_3 such that:

$$\|\rho^{in}\|_{BV(B_R)} \ \le \ C_3 R^3 + C_3 R^2 \left(\sum_k (m_k + 1) |a_k| \right) . \tag{23}$$

Hence, to insure that $\rho^{in} \in BV_{loc}$ it is sufficient to assume

$$\sum_k (m_k + 1) |a_k| \ < \ \infty \, . \tag{24}$$

Assuming that this condition is fulfilled, from the classical result of Kruzkov we get the existence of a constant M such that $\|\rho(t, \cdot)\|_{BV(B_R)} \le \|\rho^{in}\|_{BV(B_{R+Mt})}$. Thus we can apply [4] (see also [2]) to get the existence of a unique regular Lagrangian flow Φ for the ODE

$$\begin{cases} \frac{d}{dt} \Phi(s, \cdot) \ = \ g(\rho(s, \Phi(s, \cdot))) \\ \\ \Phi(0, x) \ = \ x \, . \end{cases}$$

Fix any stripe S_k^j as defined in the first step. Clearly, for a.e. x, every connected component of the intersection of the trajectory curve $\gamma_x := \{\Phi(t,x)|t \in \mathbf{R}\}$ with the stripe S_k^j is a straight segment. If j is even, then this segment is parallel to $(1, g(\alpha))$, otherwise it is parallel to $(1, g(\alpha + a_k))$. Thus, if j is even and $(t,x) \in S_{k,j}$, then the portion of trajectory

$$T_{t,x} := \big\{\Phi(s,\xi) \quad \text{for } \xi \text{ such that } \Phi(t,\xi) = x \text{ and for } s \in [0,t]\big\}$$

is a straight segment contained in $S_{k,j}$.

Let us now turn to the case where j is odd. Note that

$$g(\alpha + a_k) = g(\alpha) + g'(\alpha)a_k + O(a_k^2)\,. \tag{25}$$

Thanks to the properties of $\omega = (0,0,1)$, we have that the segments of the form

$$\big\{(t, \xi + t(g(\alpha) + a_k g'(\alpha)))\big|\, |t| \leq 1 \quad \text{and} \quad (0,\xi) \in S_{k,j}\big\} \tag{26}$$

are subsets of $S_{k,j}$. Now, fix any $C_0 > 0$ and recall (T) of Step 1. From (25) and (26) it follows that, for C_1 in (T) sufficiently large, there exists a subinterval $K_{k,j}$ such that:

- The length of $K_{k,j}$ is bigger than $C_0 a_k^2$;
- If $t \in [0,1]$ and $x_3 - tg'(\alpha) \in K_{k,j}$, then the set
$$T_{t,x} = \big\{\Phi(s,\xi)\big|\, s \in [0,t] \quad \text{and} \quad \Phi(t,\xi) = x\big\}$$
is a straight segment contained in $S_{k,j}$.

We fix $C_0 = 1$ and choose C_1 (and hence C_2) in such a way to ensure the existence of the segments $K_{k,j}$.

Third Step Construction of the angular part.

We recall that $g_3'(|c|) = g'(|c|) \cdot \omega = 0$ and that $g_3'(|c|) \neq 0$. Since the construction of the previous step is independent of the choice of the coordinates x_1 and x_2, we can choose them so that $g'(|c|) = (0, C_4, 0)$, with $C_4 > 0$. Choose the a_k's in such a way that

$$g_2(\alpha + a_k) - g_2(\alpha) = 2^{-k}\,.$$

Then, clearly, there exists a constant C_5 such that

$$\frac{2^{-k}}{C_5} \leq a_k \leq C_5 2^{-k}\,. \tag{27}$$

Let $\eta_k \in \mathbf{S}^1$ be such that $\left|\theta_k - \frac{c}{|c|}\right| = k^{-2}$. Then define

$$\theta^{in}(x_1,x_2,x_3) := \begin{cases} \eta_k & \text{if } x_3 \in I_k \text{ and } [2^k x_2] \text{ is even} \\ c/|c| & \text{otherwise}\,, \end{cases}$$

where by $[2^k x_2]$ we mean the integer part of $2^k x_2$. Set $u^{in} := \rho^{in}\theta^{in}$. Let u be the renormalized entropy solution of

$$\begin{cases} \partial_t u + \mathrm{div}_z[g(|u|)u] = 0 \\ \\ u(0,\cdot) = u^{in}\,. \end{cases} \tag{28}$$

We denote by θ the angular part $u/|u|$. According to [2] and [4], θ is obtained by the formula

$$\theta(t, \Phi(t,\xi)) = \theta^{in}(t,\xi)\,.$$

Now, define the map Φ_t^{-1} as the inverse of the map

$$\Phi_t : \mathbf{R}^3 \ni x \ \to \ \Phi(t,x) \in \mathbf{R}^3 \,.$$

Thanks to [2] and [4], Φ_t^{-1} is well defined for every t (up negligible subsets of $\mathbf{R}^3$). Thus $\theta(t,x) = \theta^{in}(\Phi_t^{-1}(x))$.

Fourth step Choice of parameters.

We will prove that, for an appropriate choice of the various parameters, $u^{in} \in BV_{loc}$, whereas $u(T,\cdot)$ is not in BV_{loc} for any $T \in]0,1]$. Recall that $\rho^{in} = |u^{in}|$ and $\rho(T,\cdot) = |u|$ are both in BV_{loc} and that $0 \le C_6^{-1} \le \rho \le C_6$ for some positive constant C_6. Thus our claim reduces to choosing the parameters σ_k and m_k in such a way that $\theta^{in} \in BV_{loc}$ and $\theta(T,\cdot) \notin BV_{loc}$ for every $t \in]0,1]$. Note that, for some constant C_7,

$$\|\theta^{in}\|_{BV(B_R)} \ \le \ C_7 R^3 + C_7 R^2 \left(\sum_k \frac{2^k}{k^2}\sigma_k + \sum_k k^{-2} \right) . \tag{29}$$

Hence, choosing $\sigma_k = 2^{-k}$ we conclude that $\theta^{in} \in BV(B_R)$ for every $R > 0$.

Now, we choose $m_k = 2k^2$, and since from (27) we have $a_k \le C_5 2^{-k}$, we clearly fulfill the condition (24), which is the only one we required on the sequence $\{m_k\}$. Thus we get

$$\frac{\sigma_k}{m_k} \ = \ k^2 2^{-k+1} \,.$$

Since from (27) we have $a_k^2 \le C_5 2^{-2k}$, clearly (21) is fulfilled for any constant C_2, provided k is large enough. Thus, we get the existence of a constant C_8 such that the segments $K_{k,j}$ of the second step exist for any $k \ge 8$.

Fix $T \in]0,1]$ and $k \ge C_8$. Recalling that $\theta(t,x) = \theta^{in}(\Phi_t^{-1}(x))$ and taking into account the properties of Φ proved in the second step, we conclude what follows

- If $j \in [1, m_k]$ is even and $\xi_{k,j}$ belongs to the segment $J_{k,j}$, then
$$\theta(T, x_1, x_2, \xi_{k,j} + T g_3(\alpha)) \ = \ \begin{cases} \eta_k & \text{if } \left[2^k(x_2 - T g_2(\alpha))\right] \text{ is even} \\ c/|c| & \text{otherwise.} \end{cases}$$
- If $j \in [1, m_k]$ is odd and $\xi_{k,j}$ belongs to the segment $K_{k,j}$, then
$$\theta(T, x_1, x_2, \xi_{k,j} + T g_3(\alpha)) \ = \ \begin{cases} \eta_k & \text{if } \left[2^k(x_2 - T g_2(\alpha + a_k))\right] \text{ is even} \\ c/|c| & \text{otherwise.} \end{cases}$$

Recall that $g_2(\alpha + a_k) - g_2(\alpha) = 2^{-k}$. Thus, for any $j \in [1, m_k - 1]$, we have

$$\begin{aligned} A_{k,j} \ &:= \ \int_{[0,1]^2} |\theta(T, x_1, x_2, \xi_{k,j} + T g_3(\alpha)) - \theta(T, x_1, x_2, \xi_{k,j+1} + T g_3(\alpha))| \, dx_1 \, dx_2 \\ &= \ T \left| \eta_k - \frac{c}{|c|} \right| \ = \ T k^{-2} \,. \end{aligned}$$

Thus,

$$\sum_{k \ge C_8} \ \sum_{1 \le j \le m_k - 1} A_{k,j} \ = \ \sum_{k \ge C_8} \frac{m_k - 1}{k^2} \ = \ \sum_{k \ge C_8} \frac{2k^2 - 1}{k^2} \ = \ \infty \,. \tag{30}$$

Note that if $\theta(T,\cdot)$ were locally in BV, then $\partial_{x_3}\theta(T,\cdot)$ would be a Radon measure. Denote by μ the total variation measure of $\partial_{x_3}\theta(T,\cdot)$ and by $\mathcal{S}_{k,j}$ the stripes

$$\mathcal{S}_{k,j} \;:=\; \left\{(x_1,x_2,x_3)\big|\,(x_1,x_2)\in[0,1]^2 \quad\text{and}\quad (x_3-Tg_3(\alpha))\in[\xi_{k,j},\xi_{k,j+1}]\right\}.$$

Then $A_{k,j}\le\mu(\mathcal{S}_{k,j})$. The $\mathcal{S}_{k,j}$ are pairwise disjoint and for R' sufficiently large, they are all contained in the ball $B_{R'}$. Thus, we would get

$$\sum_{k\ge C_8}\;\sum_{1\le\le m_k-1} A_{k,j} \;\le\; \mu(B_{R'}) \;<\; \infty\,,$$

which contradicts (30). Hence, we conclude that $\theta(T,\cdot)$ is not in $BV(B_{R'})$ for any $T\in]0,1]$.

Fifth Step Truncation of the construction and conclusion.
Next, define $\tilde{u}^n_0:\mathbf{R}^3\to\mathbf{R}^2$ as

$$\tilde{u}^n_0(x_1,x_2,x_3) \;:=\; \begin{cases} u^{in}(x_1,x_2,x_3) & \text{if } x_3\in I_k \text{ for some } k\ge n\\ c & \text{otherwise.}\end{cases}$$

Clearly $\|\tilde{u}^n_0-c\|_\infty+\|\tilde{u}^n_0-c\|_{BV(U)}\to 0$ for every bounded open set $U\subset\mathbf{R}^3$. Moreover, if we denote by $\tilde{u}^n$ the renormalized entropy solution of

$$\begin{cases} \partial_t u+\mathrm{div}_z[g(|u|)u] \;=\; 0\\ \\ u(0,\cdot) \;=\; \tilde{u}^n_0\,, \end{cases} \tag{31}$$

then $\tilde{u}^n(T,\cdot)\notin BV(B_{R'})$ for any $T\in]0,1]$. Finally, let $M>0$ and define

$$u^n_0(x_1,x_2,x_3) \;:=\; \begin{cases} \tilde{u}^n_0(x_1,x_2,x_3) & \text{if } x_1^2+x_2^2+x_3^2\le M\\ c & \text{otherwise}\,.\end{cases}$$

Let u^n be the renormalized entropy solution of

$$\begin{cases} \partial_t u+\mathrm{div}_z[g(|u|)u] \;=\; 0\\ \\ u(0,\cdot) \;=\; u^n_0\,. \end{cases} \tag{32}$$

If M is chosen sufficiently large, then $u^n=\tilde{u}^n$ on $[0,1]\times B_{R'}$ for any n. Hence the sequence $\{u^n_0\}$ has all the properties required by the statement of the proposition. □

4. Proof of Theorem 2.3

Theorem 2.3 is a consequence of Proposition 3.1 and the following

Proposition 4.1. *Let $u_0\in L^\infty(\mathbf{R}^m,\mathbf{R}^k)$ and let C be the closure of the convex hull of its essential image. Assume that*

(a) *Either $0\notin C$ or it is an extremal point of C;*
(b) *u is a bounded entropy solution of (3);*
(c) *$u\in BV(]0,T[\times U)$ for some $T>0$ and for every bounded open $U\subset\mathbf{R}^m$.*

Then u is a renormalized entropy solution on $]0,T[\times\mathbf{R}^m$.

Proof. Define $\rho := |u|$ and $\rho_0 := |u_0|$. The goal is to show that ρ is an entropy solution of the scalar law

$$\begin{cases} \partial_t \rho + \mathrm{div}_z[g(\rho)\rho] \;=\; 0 \\ \\ \rho(0,\cdot) \;=\; |u_0| \end{cases} \tag{33}$$

Actually it is sufficient to show that ρ is a *weak* solution of (33). Indeed, note that for every $\eta : \mathbf{R}^+ \to \mathbf{R}$ which is convex and increasing, $\eta(|u|)$ is a convex entropy for the system (3) (the entropy flux given is of the form $q(|u|)$ for q such that $q' = \eta' g'$). Thus we have

$$\int_{t>0}\int_{\mathbf{R}^m} [\partial_t \psi(t,z)\,\eta(\rho(t,z)) + \nabla_z \psi(t,z)\cdot q(\rho(t,z))]\,dt\,dz + \int_{\mathbf{R}^m} \psi(0,z)\eta(\rho_0(z))\,dz \;\geq\; 0\,, \tag{34}$$

for every nonnegative smooth test function ψ. Moreover, if ρ is a weak solution of (33), then for every linear function $L : \mathbf{R} \to \mathbf{R}$ we get

$$\int_{t>0}\int_{\mathbf{R}^m} [\partial_t \psi(t,z)\,L(\rho(t,z)) + \nabla_z \psi(t,z)\cdot Q(\rho(t,z))]\,dt\,dz + \int_{\mathbf{R}^m} \psi(0,z)L(\rho_0(z))\,dz \;=\; 0\,, \tag{35}$$

where $Q : \mathbf{R} \to \mathbf{R}^m$ is given by $Q = (L(g_1), \ldots, L(g_m))$. Given any convex function ξ we can write it as $L+\eta$, where L is an appropriate linear function and η is increasing on the half–line $\mathbf{R}^+$. Thus, summing (34) and (35), we conclude that ρ satisfies the entropy inequality for ξ, and hence that ρ is an entropy solution of (33).

Moreover, since $u \in BV(]0,T[\times U)$ for every bounded open set U, we have that

$$\lim_{t\downarrow 0} \frac{1}{t}\int_0^t \rho(\cdot, s)\,ds$$

converges strongly to ρ_0 in $L^1_{loc}(\mathbf{R}^m)$. Hence to show that ρ is a weak solution of (33) it is sufficient to show that

$$\text{the Radon measure} \qquad \mu \;=\; \partial_t\rho + \mathrm{div}(\rho g(\rho)) \qquad \text{vanishes on }]0,T[\times\mathbf{R}^m. \tag{36}$$

First Step

We wish to apply Theorem 2.4 to conclude that μ is supported on the jump set (or shock set) J_u. However this is not possible since the function $|u|$ is not C^1 in the origin (compare with Remark 2.5). We approximate this function uniformly with smooth C^1 convex functions of the form $\eta_n(|u|)$. Clearly, also these functions are entropies for the system of Keyfitz and Kranzer and their entropy fluxes are of the form $q_n(|u|)$ for some functions $q_n(t)$ which converge uniformly to $tg(t)$. Let $\nu : J_u \to \mathbf{R}^m$ be a Borel vector field such that $(1,\nu)/\sqrt{1+|\nu|^2}$ is normal to J_u $\mathcal{H}^m$–a.e.. Then, the chain rule of Vol'pert gives that

$$\partial_t[\eta_n(\rho)] + \mathrm{div}_x[q_n(\rho)] \;=\; (1+|\nu|^2)^{-1/2}\big[(\eta(|u^+|)-\eta(|u^-|)) + (q_n(|u^+|)-q_n(|u^-|))\cdot\nu\big]\,\mathcal{H}^m \llcorner J_u\,.$$

Passing to the limit in n we get:

$$\mu \;=\; (1+|\nu|^2)^{-1/2}\big[(|u^+|-|u^-|) + (|u^+|g(|u^+|) - |u^-|g(|u^-|))\cdot\nu\big]\,\mathcal{H}^m \llcorner J_u\,. \tag{37}$$

Thus, we must prove that

$$\big(1+g(|u^+|)\cdot\nu\big)|u^+| \;=\; \big(1+g(|u^-|)\cdot\nu\big)|u^-| \qquad \mathcal{H}^m\text{–a.e. on } J_u. \tag{38}$$

In what follows, for the sake of simplicity, we will drop the "$\mathcal{H}^m$–a.e.".

Since u is a weak solution of (3), using Proposition 2.6 we get

$$\big(g(|u^+|)\cdot\nu+1\big)u^+ \;=\; \big(g(|u^-|)\cdot\nu+1\big)u^-\,, \tag{39}$$

and hence

$$\big|g(|u^+|)\cdot\nu+1\big||u^+| \;=\; \big|g(|u^-|)\cdot\nu+1\big||u^-|\,. \tag{40}$$

If $|u^+|$ (or $|u^-|$) vanishes, (38) follows trivially. Hence, after setting $\rho^\pm := |u^\pm|$ we restrict our attention to the subset of J_u given by $G := \{\rho^+ \neq 0 \neq \rho^-\}$. On this set we define $\theta^\pm := u^\pm/\rho^\pm$ and we note that (39) becomes

$$\alpha\theta^+ \;:=\; \big[\big(g(\rho^+)\cdot\nu+1\big)\rho^+\big]\theta^+ \;=\; \big[\big(g(\rho^-)\cdot\nu+1\big)\rho^-\big]\theta^- \;=:\; \beta\theta^-\,, \tag{41}$$

whereas (38) reduces to $\alpha=\beta$. If $\theta^+=\theta^-$, then (38) follows easily. Thus we restrict our attention on $G'\subset G$ given by $G' := \{\theta^+\neq\theta^-\}$.

Recall that for any Lipschitz function $h:\mathbf{S}^{m-1}\to\mathbf{R}$, the pair

$$\left(|u|h\left(\frac{u}{|u|}\right), |u|g(|u|)h\left(\frac{u}{|u|}\right)\right)$$

is an entropy–entropy flux pair. In this case, applying Proposition 2.6, we conclude, whenever the function $u\to|u|g(u/|u|)$ is convex, the measure

$$\Big\{h(\theta^+)\big[\rho^+\big(1+g(\rho^+)\cdot\nu\big)\big]-h(\theta^-)\big[\rho^-\big(1+g(\rho^-)\cdot\nu\big)\big]\Big\}\,\mathcal{H}^m \llcorner G' \tag{42}$$

is nonpositive, that is $\alpha h(\theta^+)-\beta h(\theta^-)\leq 0$ on G'.

In the next step we will prove that, if D is the closure of the convex hull of the essential image of $u|_{]0,T[\times\mathbf{R}^m}$, then either $0\notin D$, or 0 is an extremal point of D. Hence we conclude that 0 does not belong to the segment joining $\theta^- = u^-/|u^-|$ with $\theta^+ = u^+/|u^+|$. This condition implies that for any pair of real numbers (γ^+,γ^-) there exists a convex function of the form $|u|h(u/|u|)$ such that $h(\theta^-)=\gamma^-$ and $h(\theta^+)=\gamma^+$. From this it follows that $\alpha\gamma^+-\beta\gamma^-\leq 0$ for any choice of $\gamma^\pm$. Hence $\alpha=\beta=0$.

Second Step

In order to complete the proof it remains to show that, if D denotes the closure of the convex hull of the essential image of $u|_{]0,T[\times\mathbf{R}^m}$, then either the origin is not contained in D, or it is an extremal point of D. Recalling (a), this property is true for the closure of the convex hull of u_0. Choose $\xi_1,\ldots,\xi_k$ unit vectors of $\mathbf{R}^k$ such that

$$C \;\subset\; \big\{x|\,x\cdot\xi_i\leq 0 \quad \text{for every } i\big\}$$

and 0 is an extremal point of $\big\{x|\,x\cdot\xi_i\leq 0\,\text{for every } i\big\}$. We will show that the essential image of u is contained in $\{x|\,x\cdot\xi_i=0\}$ for every i.

Fix i and denote by $\eta:\mathbf{R}^k\to\mathbf{R}$, $q:\mathbf{R}^k\to\mathbf{R}^m$ the functions

$$\eta(U) \;:=\; \begin{cases} 0 & \text{if } \xi_i\cdot U\leq 0\\ \xi_i\cdot U & \text{otherwise.}\end{cases} \qquad\qquad q(U) \;:=\; g(|U|)\eta(U)\,.$$

Note that (η,q) is a convex entropy–entropy flux pair. Clearly $\eta(u_0)=0$ and a standard argument borrowed from the theory of Kruzkov for scalar laws implies that $\eta(u)=0$. We include this argument for the convenience of the reader.

Since $\eta(u_0) = 0$ the boundary term in the entropy inequality (6) disappears. Using Fubini–Tonelli Theorem, we extend this inequality to

$$\int_0^\tau \int_{\mathbf{R}^m} \eta(u)\partial_t\psi + q(u)\cdot\nabla_z\psi(s,x)\,ds\,dx - \int_{\mathbf{R}^m} \eta(u(\tau,x))\psi(\tau,x)\,dx \geq 0\,, \tag{43}$$

valid for a.e. τ and every test function $\psi \in C^\infty(\mathbf{R}\times\mathbf{R}^m)$. Since u is bounded, there is a constant C such that $|g(|U|)| \leq C$, and hence $C\eta(u) \geq |q(u)|$. It is not difficult to show that for every $R > 0$ there exists a nonnegative test function ψ such that

$$\begin{array}{rcll} -\partial_t\psi & \geq & C|\nabla_z\psi| & \text{on } [0,\tau]\times\mathbf{R}^m \\ \psi(\tau,x) & = & 1 & \text{for every } x\in B_R. \end{array}$$

For such a test function the first term in the left hand side of (43) is nonpositive. Thus we get

$$\int_{B_R} \eta(u(\tau,x))\,dx \;\leq\; \int_{\mathbf{R}^m} \eta(u(\tau,x))\psi(\tau,x)\,dx \;\leq\; 0\,.$$

Since this inequality holds for every $R > 0$ and for a.e. $\tau \in]0,T[$, we get that $\eta(u) \leq 0$. This completes the proof. □

5. Final remarks

5.1. **Bounds in different spaces.** Varying the parameters involved in the proof of Proposition 3.2, one can construct initial data u_0^n with renormalized solutions u^n which have the following properties:

- The sequence $\{u_0^n\}$ satisfies requirement (a) above;
- The restriction of u^n to $\{t\}\times[0,1]^m$ is piecewise constant on the elements U_k^n of a countable partition by open sets;
- $J_{u^n} := \big(\{t\}\times[0,1]^m\big)\setminus\bigcup U_k^n$ is the union of countably many Lipschitz hypersurfaces;
- If we denote by $(u^n)^+$ and $(u_n)^-$ the right and left traces of u^n on J_{u^n}, then

$$\int_{J_{u^n}} \big|(u^n)^+ - (u^n)^-\big|^j\,d\mathcal{H}^{m-1} \;=\; \infty \qquad \text{for every positive } j. \tag{44}$$

Similar examples are possible in the more general case, but the j for which (44) holds will depend on the number of space variables and on the regularity of g. Building on them, it would be interesting to understand if also $W^{\alpha,p}$ norms (for $\alpha < 1$) blow up in finite time. However, this requires more than (44), in particular it requires more subtle computations involving the dislocation of the jump set, see for instance [11].

5.2. **Uniqueness of entropy solutions.** In view of the proof of Proposition 4.1 the following seems likely:

Conjecture 5.1. *Let u be a bounded entropy solution of* (3) *and denote by C the closure of the convex hull of its essential image. If $0 \notin C$ or if it is an extremal point of C, then u is a renormalized entropy solution.*

When the map $\rho \to h(\rho) = g(\rho)\rho$ satisfies suitable assumptions of genuine nonlinearity, one might hope to follow an approach similar to the one of Proposition 4.1. Indeed, fix an entropy solution u of (3) and set $\rho := |u|$. Then

$$\partial_t[\eta(\rho)] + \mathrm{div}_x[q(\rho)] \quad \text{is a nonpositive measure}$$

for every η which is convex and increasing and every q such that $q' = \eta' h'$. Clearly, every C^2 function $\xi : [0, \|\rho\|_\infty] \to \mathbf{R}$ can be written as the difference of two increasing convex functions. Thus we conclude that

$$\partial_t[\xi(\rho)] + \mathrm{div}_x[q(\rho)] \quad \text{is a } \textit{Radon} \text{ measure}$$

for every $\xi \in C^2$ and every q such that $q' = \xi' h'$. Thus, we are in the position of applying Theorem 2.4 of [10] to conclude that

Theorem 5.2. *There exists a codimension* 1 *rectifiable set* $J_\rho \subset \mathbf{R}^+ \times \mathbf{R}^m$ *such that*

- ρ *has left and right approximate limits* $\mathcal{H}^m$*–a.e. on* J_ρ*;*
- ρ *has vanishing mean oscillation on* $\mathbf{R}^+ \times \mathbf{R}^m \setminus J_\rho$*.*

Hence, as a first step one might hope to prove that the measure $\mu = \partial_t \rho + \mathrm{div}_x[h(\rho)]$ is concentrated on the set J_ρ. As a second step some generalization of the proof of Proposition 4.1 could give that $\mu \llcorner J_\rho$ vanishes, and hence that $\mu = 0$.

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