A NOTE ON ADMISSIBLE SOLUTIONS OF 1D SCALAR CONSERVATION LAWS AND 2D HAMILTON–JACOBI EQUATIONS

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Abstract. Let $\Omega \subset \mathbb{R}^2$ be an open set and $f \in C^2(\mathbb{R})$ with $f'' > 0$. In this note we prove that entropy solutions of $D_t u + D_x f(u) = 0$ belong to $SBV_{loc}(\Omega)$. As a corollary we prove the same property for gradients of viscosity solutions of planar Hamilton–Jacobi PDEs with uniformly convex hamiltonians.

1. Introduction

In this paper we consider entropy solutions of the scalar conservation law

$$D_t u + D_x[f(u)] \; = \; 0 \qquad \text{in } \Omega \tag{1.1}$$

and viscosity solutions of the planar Hamilton–Jacobi PDE

$$H(\nabla v) \; = \; 0 \qquad \text{in } \Omega, \tag{1.2}$$

where H and f are C^2 and locally uniformly convex. In these cases it is known that u and ∇v belong to $BV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$, i.e. that the distributions Du and $D\nabla u$ are vector (resp. matrix) valued Radon measures. The rough picture that one has in mind when describing such solutions is the one of piecewise C^1 functions with discontinuities of jump type. The space of BV functions enjoys good functional analytic properties, but the behaviour of a generic BV function can be indeed very far from the picture above.

Following [3], given $w \in BV(\mathbb{R}^m, \mathbb{R}^k)$ we decompose Dw into three mutually singular measures: $Dw = D^a w + D^c w + D^j w$. $D^a w$ is the part of the measure which is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^m$. $D^j w$ is called *jump* part and it is concentrated on the rectifiable $m-1$ dimensional set J where the function u has jump discontinuities (in an appropriate measure–theoretic sense: see Section 2). $D^c w$ is called the *Cantor part*, it is singular with respect to $\mathscr{L}^m$ and it satisfies $D^c w(E) = 0$ for every Borel set E with $\mathscr{H}^{m-1}(E) < \infty$. When $m = 1$, $D^j w$ consists of a countable sum of weighted Dirac masses, whereas $D^c w$ is the non–atomic singular part of the measure. A typical example of $D^c w$ is the derivative of the Cantor–Vitali ternary function (see for instance Example 1.67 of [3]).

In [5] the authors introduced the space of special functions of bounded variations, denoted by SBV, which consists of the functions $w \in BV$ such that $D^c w = 0$. This space played an important role in the last years, in connection with problems coming from the theory of image segmentation and with variational problems in fracture mechanics (see [3] and the references quoted therein for a detailed presentation of this subject).

It is natural to ask whether entropy solutions of (1.1) and gradients of viscosity solutions of (1.2) are locally SBV and, as far as we know, this question has never been addressed in the literature. Our interest is in part motivated by some measure–theoretic questions arisen in [2].

In the following remark we single out a canonical representative in the equivalence class of u for which more precise informations, of pointwise tipe, are available.

Remark 1.1. *Let $u \in L^\infty(\Omega)$ be a weak solution of* (1.1) *and assume $]t_1, t_2[\times J \subset \Omega$ for some open set $J \subset \mathbb{R}$. Using the equation one can prove that for every $\tau \in]t_1, t_2[$ the functions $f_\varepsilon(x) = \int_\tau^{\tau+\varepsilon} u(x,t)dt$ have a unique limit f in $L^\infty(J)$ weak* for $\varepsilon \downarrow 0$ (see for instance Theorem 4.1.1 of* [4]*). Therefore from now on we fix the convention that $u(\tau, \cdot) = f(\cdot)$.*

The following is the main result of this note.

Theorem 1.2. *Let $u \in L^\infty(\Omega)$ be an entropy solution of* (1.1) *with $f \in C^2(\mathbb{R})$ locally uniformly convex. Then there exists $S \subset \mathbb{R}$ at most countable such that $\forall \tau \in \mathbb{R} \setminus S$ the following holds:*

$$u(\tau, \cdot) \in SBV_{loc}(\Omega_\tau) \qquad \text{with } \Omega_\tau := \{x \in \mathbb{R} : \ (\tau, x) \in \Omega\}. \tag{1.3}$$

From this theorem, using the slicing theory of BV functions, we obtain:

Corollary 1.3. *Let $f \in C^2(\mathbb{R})$ be locally uniformly convex and let $u \in L^\infty(\Omega)$ be an entropy solution of* (1.1)*. Then $u \in SBV_{loc}(\Omega)$.*

Eventually, via a local change of coordinates we apply the previous result to the Hamilton–Jacobi case:

Corollary 1.4. *Let $H \in C^2(\mathbb{R}^2)$ be locally uniformly convex and let $u \in W^{1,\infty}(\Omega)$ be a* viscosity *solution of $H(\nabla u) = 0$. Then $\nabla u \in SBV_{loc}(\Omega)$.*

As we show in Remark 3.3, Theorem 1.2 is optimal. Also the regularity results obtained in the two corollaries seem to be optimal, in view of the fact that shocks do occur and that the gradients of viscosity solutions of Hamilton-Jacobi PDEs can jump along hypersurfaces. Our result applies in particular to the distance function $\mathrm{dist}(x, K)$, which solves the eikonal equation $|\nabla u|^2 - 1 = 0$ in the viscosity sense in $\Omega = \mathbb{R}^2 \setminus K$. In this connection, we mention the paper [8], where the authors establish among other things the SBV regularity in any space dimension, but under some regularity assumptions on K.

It would be interesting to extend these results to

(a) BV admissible solutions of genuinely nonlinear systems of conservation laws in 1 space dimension;
(b) Viscosity solutions of uniformly convex Hamilton–Jacobi PDEs in higher dimensions.

The proof of Theorem 1.2 uses at the very end a variational principle, due to Hopf and Lax. However, it might be that combining part of this proof with the theory of characteristics for systems of conservation laws (as developed in [4]) one could be able to extended Theorem 1.2 at least to the case (a).

2. Preliminaries

2.1. BV **and** SBV **spaces.** In what follows $\mathscr{L}^d$ and $\mathscr{H}^n$ denote respectively the Lebesgue measure on $\mathbb{R}^d$ and the n–th dimensional Hausdorff measure on Euclidean spaces. A set $J \subset \mathbb{R}^d$ is said *countably $\mathscr{H}^n$–rectifiable* (or briefly rectifiable) if there exist countably many n–dimensional Lipschitz graphs Γ_i such that $\mathscr{H}^n(J \setminus \bigcup \Gamma_i) = 0$. Given a Borel measure μ and a Borel set A we denote by $\mu \llcorner A$ the measure given by $\mu \llcorner A(C) = \mu(A \cap C)$.

The *approximate discontinuity set* $S_w \subset \Omega$ of a locally summable function $w : \Omega \subset \mathbb{R}^d \to \mathbb{R}^m$ and the *approximate limit* are defined as follows: $x \notin S_w$ if and only if there exists $z \in \mathbb{R}^m$ satisfying

$$\lim_{r \downarrow 0} r^{-d} \int_{B_r(x)} |w(y) - z| \, dy = 0.$$

The vector z, if it exists, is unique and denoted by $\tilde{w}(x)$, the approximate limit of w at x. It is easy to check that the set S_w is Borel and that $\tilde{w}$ is a Borel function in its domain (see §3.6 of [3] for the details). By Lebesgue differentiation theorem the set S_w is Lebesgue negligible and $\tilde{w} = w$ $\mathscr{L}^d$-a.e. in $\Omega \setminus S_w$.

In a similar way one can define the *approximate jump set* $J_w \subset S_w$, by requiring the existence of $a, b \in \mathbb{R}^m$ with $a \neq b$ and of a unit vector ν such that

$$\lim_{r \downarrow 0} r^{-d} \int_{B_r^+(x,\nu)} |w(y) - a| \, dy = 0, \qquad \lim_{r \downarrow 0} r^{-d} \int_{B_r^-(x,\nu)} |w(y) - b| \, dy = 0,$$

where

$$\begin{cases} B_r^+(x,\nu) := \{y \in B_r(x) : \ \langle y - x, \nu \rangle > 0\}, \\ B_r^-(x,\nu) := \{y \in B_r(x) : \ \langle y - x, \nu \rangle < 0\}. \end{cases} \tag{2.1}$$

The triplet (a, b, ν), if it exists, is unique up to a permutation of a and b and a change of sign of ν. We denote it by $(w^+(x), w^-(x), \nu(x))$, where $w^\pm(x)$ are called *approximate one-sided limits* of w at x. It is easy to check that the set J_w is Borel and that $w^\pm$ and ν can be chosen to be Borel functions in their domain (see again §3.6 of [3] for details).

The following structure theorem, essentially due to Federer and Vol'pert, holds (see for instance Theorem 3.77 and Proposition 3.92 of [3]):

Theorem 2.1. *Let $w \in BV(\Omega)$. Then $\mathscr{H}^{d-1}(S_w \setminus J_w) = 0$ and J_w is a countably $\mathscr{H}^{d-1}$–rectifiable set. If we denote by $D^a w$ the absolutely continuous part of Dw and by $D^s w$ the singular part, then $D^s w$ can be written as $D^j w + D^c w$, where*

$$D^j w \ = \ (w^+ - w^-) \otimes \nu_{J_w} \mathscr{H}^{d-1} \llcorner J_w \,, \tag{2.2}$$

$$D^c w(E) \ = \ 0 \qquad \text{for any Borel set } E \text{ with } \mathscr{H}^{d-1}(E) < \infty. \tag{2.3}$$

When $\Omega \subset \mathbb{R}$ we have the following refinement (see for instance Theorem 3.28 of [3]):

Proposition 2.2. *Let $w \in BV(\Omega)$ and let $\Omega \subset \mathbb{R}$. Then $S_w = J_w$, $\tilde{w}$ is continuous on $\Omega \setminus J_w$ and $\tilde{w}$ has* classical *left and right limits (which coincide with $w^\pm(x)$) at any $x \in J_w$.*

Therefore

$$D^j w = \sum_{x \in J_w} (w^+(x) - w^-(x))\delta_x .$$

2.2. **Hopf–Lax formula and characteristics.** Let $f \in C^2$ be locally uniformly convex, $u_0 \in L^1(\mathbb{R})$ and let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ be the entropy solution of the Cauchy problem

$$\begin{cases} D_t u + D_x[f(u)] = 0 \\ u(0,\cdot) = u_0 . \end{cases} \tag{2.4}$$

Then u can be computed by using a variational principle, the so-called Hopf–Lax formula. In particular we have the following well-known theorem.

Theorem 2.3 (Hopf–Lax formula)**.** *Let $u_0 \in L^1(\mathbb{R})$, let $f : \mathbb{R} \to \mathbb{R}$ be C^2 and locally uniformly convex and set*

$$v_0(y) := \int_{-\infty}^{y} u_0(s)\, ds \qquad y \in \mathbb{R}.$$

Let

$$v(t,x) := \min \left\{ t f^* \left(\frac{x-y}{t} \right) + v_0(y) : \ y \in \mathbb{R} \right\} . \tag{2.5}$$

Then the following statements hold:

(i) *For any $t > 0$ there exists a countable set S_t such that the minimum is attained at a unique point $y(t,x)$ for any $x \notin S_t$.*

(ii) *The map $x \mapsto y(t,x)$ is nondecreasing in its domain, its jump set is S_t and $v(t,\cdot)$ is differentiable at any $x \notin S_t$, with*

$$f'(v_x(t,x)) = \frac{x - y(t,x)}{t} . \tag{2.6}$$

In particular $v_x(t,\cdot)$ is continuous on $\mathbb{R} \setminus S_t$.

(iii) *There exists a constant C such that*

$$v_x(t, x+y) \le v_x(t,x) + \frac{C}{t} y \qquad \text{whenever } y \ge 0 \text{ and } x,\, x+y \notin S_t. \tag{2.7}$$

This is called Oleinik E–condition.

(iv) *v is a Lipschitz map and $u := v_x$ is the unique entropy solution of* (1.1) *with the initial condition $u(0,\cdot) = u_0$.*

(v) *If $t_n \to t > 0$, then $v_x(t_n,\cdot) \to v_x(t,\cdot)$ in L^1_{loc}.*

Proof. For a proof of point (i), of the fact that $x \mapsto y(t,x)$ is nondecreasing, and of the fact that S_t is the set of discontinuities of $y(t,\cdot)$ we refer for instance to Theorem 1 of §3.4.2 of [6]. For (iii) and (iv) we refer to Theorem 2 of §3.4.2, to the first lemma of §3.4.3 and to Theorem 3 of the same section of [6].

(ii) It remains to prove that $v(t,\cdot)$ is diffrentiable on $\mathbb{R}\setminus S_t$ and that (2.6) hold. Since v is Lipschitz, $v(t,\cdot)$ is differentiable almost everywhere. In Theorem 1 of §3.4.2 of [6] it is shown that (2.6) holds for a.e. x. Since $f'\in C^1$ and $f''>0$, if we define

$$w(x) \;:=\; f'^{-1}\left(\frac{x-y(t,x)}{t}\right)$$

we conclude that the discontinuity set of w is precisely S_t and that $w(z)=v_x(t,z)$ for $\mathscr{L}^1$–a.e. x. Fix a point $z\notin S_t$ and note that

$$|v(\zeta)-v(z)-(\zeta-z)w(z)| \;\leq\; \left|\int_z^\zeta \big(w(\xi)-w(z)\big)\,d\xi\right| \;=\; o(|\zeta-z|).$$

We conclude that $v(t,\cdot)$ is differentiable at z and that its derivative is equal to $w(z)$.

(v) Note that v_x is locally uniformly bounded, and thus it suffices to prove that $v_x(t_n,\cdot)\to v_x(t,\cdot)$ pointwise almost everywhere. Fix $x\notin S_t$ and let y^* be a cluster point of the sequence $\{y(t_n,x)\}$. The variational principle yields that y^* is a minimizer for the right hand side of (2.5). Since $y(t,x)$ is the unique minimizer of this function, we conclude that $y^*=y(t,x)$. Therefore $y(t_n,x)\to y(t,x)$ for every $x\notin S_t$. From (2.6) we get the same convergence for $v_x(t_n,\cdot)$. This concludes the proof. □

We can use the Hopf–Lax variational principle to define backward characteristics emanating from points (t,x) with $x\in\mathbb{R}\setminus S_t$. We refer to Chapters X and XI of [4] for a different and more general approach to the theory of characteristics, based on differential inclusions.

Definition 2.4 (Characteristics)**.** *Let $x\notin S_t$. The segment joining (t,x) with $(0,y(t,x))$ will be called (backward minimal)* characteristic *emanating from (t,x). These segments, when parametrized with constant speed on the interval $[0,t]$, are minimizers of the variational problem related to the Hopf–Lax formula*

$$\min\left\{\int_0^t f^*(\dot\gamma(s))\,ds+v_0(\gamma(0)):\;\gamma\in C^1\left([0,t];\mathbb{R}\right),\;\gamma(t)=x\right\}.$$

Indeed, the strict convexity of f^* forces the minimizers to be straight lines and forces a constant speed parameterization.

The monotonicity of $y(t,\cdot)$ immediately implies that characteristics emanating from points $x,\,y\notin S_t$ with $x\neq y$ do not intersect in the open upper half plane. It turns out that the minimality of characteristics easily implies that two different characteristics starting even at different times are either one contained in the other or do not intersect (see Figure 1).

Proposition 2.5 (No-crossing of characteristics)**.** *Let $t>0$ and $x_0\notin S_t$. Let also $s\in]0,t]$ and $x_0'\notin S_s$. Then the characteristic emanating from (t,x) and the one emanating from (s,x_0) do not intersect in the upper half plane $\{\tau>0\}$, unless the first contains the second.*

Proof. By the previous remarks we can assume with no loss of generality that $s \in]0,t[$. Assume by contradiction that there is an intersection at (s_*, x_*) with $s_* \in]0,s]$. Let

$$\gamma(\tau) := \begin{cases} y'_0 + \dfrac{\tau}{s_*}(x_* - y'_0) & \text{if } \tau \in [0, s_*]; \\ x''_0 + \dfrac{\tau - s_*}{t - s_*}(x_0 - x_*) & \text{if } \tau \in [s_*, t]\,. \end{cases}$$

The definition of v gives

$$v(t,x_0) \geq \int_0^t f^*(\dot\gamma)\,d\tau + v_0(y'_0) = s_* f^*\left(\frac{x_* - y'_0}{s_*}\right) + (t - s_*) f^*\left(\frac{x_0 - x_*}{t - s_*}\right) + v_0(y'_0)\,,$$

with a strict inequality if $(x_0 - x_*)/(t - s_*)$ and $(x_* - y'_0)/s_*$ are not equal. On the other hand, the minimality of the segment joining (s_*, x_*) to $(0, y'_0)$ gives

$$s_* f^*\left(\frac{x_* - y'_0}{s_*}\right) + v_0(y'_0) = v(s_*, x_*)$$

and the so-called dynamic programming principle (see for instance [6]) gives

$$v(t, x_0) = (t - s_*) f^*\left(\frac{x_0 - x_*}{t - s_*}\right) + v(s_*, x_*)\,.$$

As a consequence equality must hold and the two segments are parallel. $\square$

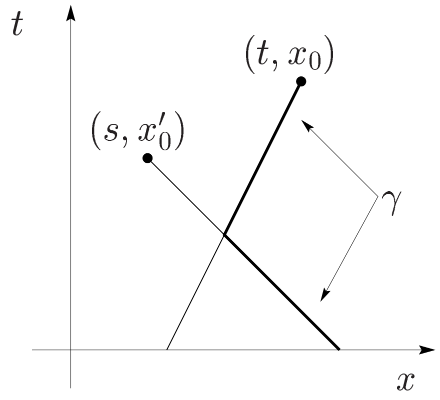

FIGURE 1. The "crossing" of two characteristics would give a minimizer γ (in the Hopf–Lax variational principle) which is not a straight line.

3. Proof of Theorem 1.2

Definition 3.1 (Characteristic cones). *The backward characteristic cone $C_{x,\tau}$ emanating from $x \in S_\tau$ is defined as the open triangle having*

$$(\tau, x)\,, \qquad (y^-(\tau, x), 0)\,, \qquad (y^+(\tau, x), 0)$$

as vertices. Notice that due to the no crossing of characteristics two cones $C_{\tau,x}$ and $C_{t,y}$ are either one contained in the other or disjoint. We define also

$$C_\tau := \bigcup_{x\in S_\tau} C_{\tau,x}\,. \tag{3.1}$$

We remark that the two "diagonal" segments which define the characteristic cone coincide with the minimal and maximal backward characteristics as defined in [4].

3.1. **Proof of Theorem 1.2. Step 1** Preliminary remarks.

Let us fix $(\tau,\xi)\in\Omega$ and r such that $B_r(\tau,\xi)\subset\Omega$. Thanks to the finite speed of propagation, there exists a positive ρ such that the values of u in the ball $B_\rho(\tau,\xi)$ depend only on the values of u on the segment $\{t=\tau-2\rho\}\cap B_r(\tau,\xi)$. Thus, if we denote by w the entropy solution of the Cauchy problem

$$\begin{cases} D_t w + D_x[f(w)] \;=\; 0 & \text{for } t>\tau-2\rho\\[2ex] w(\tau-2\rho,x) \;=\; u(\tau-2\rho,x)\mathbf{1}_{B_r(\tau,\xi)}(\tau-2\rho,x) & \text{for every } x\in\mathbb{R}, \end{cases}$$

we get that $w=u$ on $B_\rho(\tau,\xi)$. Moreover, note that $w(t,\cdot)\in BV$ for every $t>\tau-2\rho$. Thus it suffices to prove the theorem under the additional assumptions that $\Omega=\{t>0\}$ and that $u(0,\cdot)$ is a bounded compactly supported BV function.

Under this assumption we know that $u=v_x$, where v is given by the Hopf–Lax formula (2.5). Moreover, from Theorem 2.3(v) and Remark 1.1, for every $t>0$ we have $u(t,\cdot)=v_x(t,\cdot)$. Since $u(0,\cdot)$ is compactly supported we know that for every constant T there exist constants R and c_1 such that the support of $u(t,\cdot)$ is contained in $\{|x|\le R\}$ and the total variation of $Du(t,\cdot)$ is bounded by a constant c_1 for $t\in[0,T]$.

For each t we denote by μ_t the Cantor part of the measure $D_x u(t,\cdot)$ and by ν_t the jump part. Using this notation, (1.3) is equivalent to prove that

$$\mu_t = 0 \text{ except for an at most countable set of } t\text{'s}. \tag{3.2}$$

Oleinik's estimate (2.7) implies that the singular measures μ_t and ν_t are both nonpositive and that the left and right limits $u^\pm(t,x)$ of $u(t,\cdot)$ are well defined. Recall also that the semi-monotonicity of $u(t,\cdot)$ gives

$$u^+(t,x)-u^-(t,y) = Du(t,\cdot)([x,y]) \qquad \text{whenever } x<y. \tag{3.3}$$

Step 2 Definition of a functional $F(t)$.

Let $y(t,\cdot)$ be the nondecreasing map in Theorem 2.3, defined out of S_t. We define the open intervals

$$I_{t,x} :=]y^-(t,x),y^+(t,x)[\,, \qquad\qquad I_t := \bigcup_{x\in S_t} I_{t,x}.$$

From (2.6) it follows immediately that

$$\mathscr{L}^1(I_{t,x}) \;\le\; -c_2\nu_t(\{x\})\,, \tag{3.4}$$

for some constant c_2 depending only on $\|u\|_\infty$ and on $\|f''\|_{L^\infty([-\|u\|_\infty,\|u\|_\infty])}$.

We set

$$F(t) \ := \ \mathscr{L}^1(I_t) \ = \ \sum_{x\in S_t}\mathscr{L}^1(I_{t,x})\,,$$

where the second equality follows from the no-crossing property of characteristics. From (3.4) we conclude

$$F(t) \ \leq \ -c_2\nu_t(\mathbb{R}) \ \leq \ c_2|Du(t,\cdot)|(\mathbb{R}) \ \leq \ c_1c_2 \qquad \forall t\in[0,T]\,. \tag{3.5}$$

Let us prove now that $I_s\subset I_t$ whenever $s\leq t$. Indeed, if $x\in S_s$ the no-crossing property of characteristics gives that $I_{s,x}$ has an empty intersection with the image of the nondecreasing map $y(t,\cdot)$, defined on $\mathbb{R}\setminus S_t$. Therefore $I_{s,x}$ must be contained in one of the piecewise disjoint jump intervals $I_{t,y}$, $y\in S_t$. Hence, taking into account (3.5), we obtain that

$$F \text{ is a nondecreasing bounded function in } [0,T]. \tag{3.6}$$

As usual we denote by $F(t^+)$ the right limit of F at t. Next we will prove that for any integer k we have

$$\tau_0\geq T/k>0 \text{ and } \mu_{\tau_0}(\mathbb{R})\leq -1/k \qquad \Longrightarrow \qquad F(\tau_0^+) \ \geq \ F(\tau_0)+c_3\,, \tag{3.7}$$

where c_3 is a strictly positive constant which depends on $\|u\|_\infty$, T, k and f. Clearly (3.6) and (3.7) imply that all sets

$$\big\{\tau\in[T/k,T[\ \big|\ \mu_\tau(\mathbb{R})\leq -1/k\big\}$$

are finite. Thus the claim of the theorem is reduced to prove (3.7).

Step 3 Proof of (3.7). Recalling the definition of C_τ given in (3.1), we need the following

Lemma 3.2. *Let $\tau_0>0$. Then, for μ_{τ_0}-a.e. x there exists $\eta>0$ such that*

$$\{\tau_0\}\times]x-\eta,x+\eta[\subset C_\tau.$$

We first show how to conclude (3.7) from the lemma. We fix $\tau>\tau_0\geq T/k$ and, to simplify the notation, we use μ and ν in place of μ_{τ_0} and ν_{τ_0}, and denote by σ the full distributional derivative of $u(\tau_0,\cdot)$. Denote by E the set of x's for which Lemma 3.2 applies and such that

$$\lim_{\eta\downarrow 0}\frac{\eta+|\sigma-\mu|([x-\eta,x+\eta])}{-\mu([x-\eta,x+\eta])} \ = \ 0. \tag{3.8}$$

Besicovitch differentiation theorem gives that $\mu(\mathbb{R}\setminus E)=0$ and (3.3) gives

$$\lim_{\eta\downarrow 0}\frac{u^-(\tau_0,x-\eta)-u^+(\tau_0,x+\eta)}{-\mu([x-\eta,x+\eta])} \ = \ 1 \qquad \forall x\in E\,. \tag{3.9}$$

For every $x\in E$ and for every $\eta>0$ such that $x\pm\eta\notin S_{\tau_0}$ we denote by $J_{x,\eta}$ the interval $]y(\tau_0,x-\eta),y(\tau_0,x+\eta)[$, i.e. (see (2.6))

$$J_{x,\eta}=\big]x-\eta-\tau_0f'(u(\tau_0,x-\eta)),x+\eta-\tau_0f'(u(\tau_0,x+\eta))\big[\,.$$

From (3.9) and the fact that μ is a nonpositive measure, it follows that for η sufficiently small we have $f'(u(\tau_0, x+\eta)) \le f'(u(\tau_0, x-\eta))$. Hence we can write

$$\begin{aligned}\mathscr{L}^1(J_{x,\eta}) &= 2\eta + \tau_0\Big[f'(u(\tau_0, x+\eta)) - f'(u(\tau_0, x-\eta))\Big] \\ &\ge 2\eta + \frac{T}{k}\min_{|t|\le\|u\|_\infty} f''(t)\Big[(u(\tau_0, x+\eta) - u(\tau_0, x-\eta)\Big].\end{aligned}$$

Hence, from (3.9) we conclude

$$\mathscr{L}^1(J_{x,\eta}) \ \ge\ -c_4\mu([x-\eta, x+\eta])\,, \tag{3.10}$$

for η sufficiently small, where c_4 is a positive constant depending only on T, k and f.

Due to the no-crossing property of characteristics (see Figure 2) we have that $J_{x,\eta}$ can only intersect the intervals $I_{\tau_0,y}$ emanating from a point y in $[x-\eta, x+\eta]$, so that recalling (3.4) we obtain

$$\mathscr{L}^1(J_{x,\eta}\cap I_{\tau_0}) \ =\ \sum_{y\in S_{\tau_0}\cap[x-\eta,x+\eta]} \mathscr{L}^1(I_{\tau_0,y}) \ \le\ -c_2\nu([x-\eta, x+\eta])\,.$$

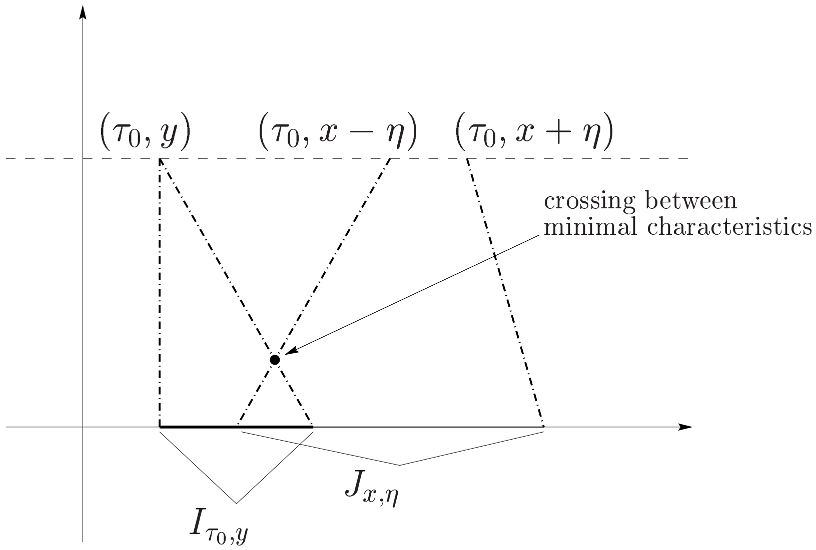

FIGURE 2. $y \notin [x-\eta, x+\eta]$ and $J_{x,\eta}\cap I_{\tau_0,y} \ne \emptyset$ would violate the no-crossing property.

From (3.8) and (3.10) it follows that for any $x \in E$ we have

$$\mathscr{L}^1(J_{x,\eta}\setminus I_{\tau_0}) \ \ge\ -\frac{c_4}{2}\mu([x-\eta, x+\eta]) \tag{3.11}$$

provided η is small enough. Using Besicovitch covering lemma, we can cover μ–a.e. E with pairwise disjoint intervals $K_j = [x_j - \eta_i, x_j + \eta_j]$ such that (3.11) and the conclusion of Lemma 3.2 both hold for $x = x_j$ and $\eta = \eta_j$. Note that the intervals J_{x_j,η_j} are pairwise disjoint as well (again due to the no-crossing property of characteristics) and that, thanks to Lemma 3.2, they belong to I_τ. Hence, recalling that $-\mu(\mathbb{R}) \geq k$, we get

$$\begin{aligned} F(\tau) - F(\tau_0) &\geq \sum_j \mathscr{L}^1(J_{x_j,\eta_j} \setminus I_{\tau_0}) \geq -\sum_j \frac{c_4}{2}\mu([x_j-\eta_j, x_j+\eta_j]) \\ &\geq -\frac{c_4}{2}\mu(E) = -\frac{c_4}{2}\mu(\mathbb{R}) \geq \frac{c_4}{2k} =: c_3\,. \end{aligned}$$

This gives the claim (3.7), and reduces the theorem to Lemma 3.2.

Proof of Lemma 3.2. We will prove that the conclusion of the lemma holds for any x which satisfies the following conditions:

$$x \notin S_{\tau_0} \quad \text{and} \quad \lim_{\eta\downarrow 0} \frac{u(\tau_0,x)-u(\tau_0,x-\eta)}{\eta} = -\infty. \tag{3.12}$$

By Besicovitch differentiation theorem on intervals, the measure μ_{τ_0} is concentrated on E. Hence, in what follows, we fix $\tau > \tau_0$ and x such that (3.12) holds and our goal is to prove that for η small enough $\{\tau_0\}\times]x-\eta, x+\eta[\subset C_\tau$.

Let us define $w(\tau,\xi) = u(t+\tau_0, x+\xi)$. Clearly $D_t w + D_x[f(w)] = 0$. Hence it is sufficient to prove the following statement:

Assume w is a bounded solution of $D_t w + D_x[f(w)] = 0$ on $\mathbb{R}^+ \times \mathbb{R}$, such that $w(0,\cdot)$ is a compactly supported BV function. Assume that the following two conditions hold:

$$0 \notin S_{w(0,\cdot)} \quad \textit{and} \quad \lim_{\eta\downarrow 0} \frac{w(0,0)-w(0,-\eta)}{\eta} = -\infty\,. \tag{3.13}$$

Then $0 \in I_\tau$ for any $\tau > 0$.

We argue by contradiction. If the claimed statement is not true, then $0 \notin I_\tau$ for some τ and therefore for any $n \in N$ we can find $x_n \notin S_\tau$ such that $z_n = y(\tau, x_n) \in]-\frac{1}{n}, \frac{1}{n}[$. Recall that z_n is the unique minimum of the function

$$\xi \mapsto L_n(\xi) := \tau g\left(\frac{x_n-\xi}{\tau}\right) + \int_{-\infty}^{\xi} w(0,s)\,ds\,,$$

with $g := f^*$. Since the slopes $(x_n - z_n)/\tau$ are uniformly bounded, $|x_n|$ is uniformly bounded as well, and hence we can assume that a subsequence of $\{x_n\}$, not relabeled, converges to $x \in \mathbb{R}$. Then 0 is a minimizer (not necessarily unique) of the function

$$\xi \mapsto L(\xi) := \tau g\left(\frac{x-\xi}{\tau}\right) + \int_{-\infty}^{\xi} w(0,s)\,ds\,.$$

Since by (3.13) $w(0,\cdot)$ is continuous at 0, we have that L is differentiable at 0, and since 0 is a minimizer we have

$$0 = L'(0) = -g'\left(\frac{x}{\tau}\right) + w(0,0)\,. \tag{3.14}$$

We will show that if $\eta > 0$ is sufficiently small we have $L(-\eta) < L(0)$, contradicting the minimality of 0. Recall that g is C^2. So for some constant D we have

$$\left|\tau g\left(\frac{x}{\tau}\right) - \tau g\left(\frac{x+\eta}{\tau}\right) + \eta g'\left(\frac{x}{\tau}\right)\right| \;\leq\; D\eta^2\,.$$

Hence we can write

$$\tau g\left(\frac{x}{\tau}\right) - \tau g\left(\frac{x+\eta}{\tau}\right) \;\geq\; -\eta g'\left(\frac{x}{\tau}\right) - D\eta^2\,. \tag{3.15}$$

In order to estimate $L(0) - L(-\eta)$ it remains to bound

$$\int_{-\infty}^{0} w(0,\zeta)\,d\zeta - \int_{-\infty}^{-\eta} w(0,\zeta)\,d\zeta \;=\; \int_{-\eta}^{0} w(0,\zeta)\,d\zeta$$

$$=\;\; \eta w(0,0) + \int_{-\eta}^{0} (w(0,\zeta) - w(0,0))\,d\zeta\,. \tag{3.16}$$

Let us fix now a large constant E. Notice that (3.13) gives

$$\int_{-\eta}^{0} w(0,\zeta) - w(0,0)\,d\zeta \geq E\int_{-\eta}^{0} -\zeta\,d\zeta = \frac{E}{2}\eta^2 \tag{3.17}$$

for $\eta > 0$ small enough. From (3.15) and (3.16) we get

$$L(0) - L(-\eta) \;\geq\; \eta\left[w(0,0) - g'\left(\frac{x}{\tau}\right)\right] + \left(\frac{E}{2} - D\right)\eta^2 \tag{3.18}$$

for $\eta > 0$ small enough. Recalling (3.14) we finally get

$$L(0) - L(-\eta) \;\geq\; \left(\frac{E}{2} - D\right)\eta^2\,. \tag{3.19}$$

Note that D is a fixed constant, whereas E can be chosen arbitrarily large, provided η is sufficiently small. Hence, this means that for η sufficiently small $L(0) > L(-\eta)$. □

Proof of Corollary 1.3. The slicing theory of BV functions shows that the Cantor part of the 2-dimensional measure $D_x u$ is the integral with respect to t of the Cantor parts of $Du(t,\cdot)$ (see Theorem 3.108 of [3] for a precise statement). Therefore Theorem 1.2 implies that the measure $D_x u$ has no Cantor part. Using the chain rule of Vol'pert (see Theorem 3.96 of [3]) and equation (1.1), we get that $D_t u$ has no Cantor part as well. Thus, we finally infer that $u \in SBV_{loc}(\Omega)$. □

Remark 3.3. *It is not difficult to show that Theorem 1.2 is optimal. Indeed, let $v : \mathbb{R} \to [0,1]$ be any continuous non–increasing function which does not belong to $SBV_{loc}(\mathbb{R})$. For any $x \in \mathbb{R}$ let r_x be the straight line which passes through $(1,x)$ and has slope $(1, v(x))$. Since v is non–increasing, for any pair $\{r_x, r_y\}_{x\neq y}$ we have $r_x \cap r_y \cap \{t \leq 1\} = \emptyset$. Therefore, there exists a unique function $\tilde{u} \in W^{1,\infty}_{loc}(]-\infty,1[\times\mathbb{R})$ which is constantly equal to $v(x)$ on every r_x. From the classical method of characteristics it follows that $\tilde{u}$ is a solution of $D_t\tilde{u} + D_x(\tilde{u}^2/2) = 0$.*

Set $u_0(x) := \tilde{u}(0,x)$ and let u be the entropy solution of

$$\left\{\begin{array}{rcl} D_t u + D_x\left(\frac{u^2}{2}\right) & = & 0 \\ \\ u(0,\cdot) & = & u_0\,. \end{array}\right. \tag{3.20}$$

Since $\tilde{u}$ is locally Lipschitz, $\tilde{u}$ is an entropy solution of (3.20) in $\{t<1\}$. Therefore we conclude that $\tilde{u}=u$ on $]0,1[\times\mathbb{R}$ and that $u(1,\cdot)=v\notin SBV_{loc}(\mathbb{R})$. By the finite speed of propagation, if we choose M large enough and we define $\overline{u}_0 := u_0\mathbf{1}_{[-M,M]}$, the corresponding entropy solution $\overline{u}$ has $\overline{u}(1,\cdot)\notin SBV_{loc}$.

Arguing in a similar way, for every $m>0$ we can find $u_0^m\in BV(\mathbb{R})$ such that

- *$\|u_0^m\|_{BV}\le m$ and the support of u_0^m is contained in $[-m,m]$;*
- *If u^m is the entropy solution of (3.20) with initial data u_0^m, then there exists $\tau\in]0,m[$ such that $u^m(\tau,\cdot)\notin SBV([-m,m])$.*

Let $C>2$ and $\{m_i\}$ be a decreasing sequence of positive numbers such that $\sum m_i<\infty$. Set

$$\sigma_j := C\sum_{i\le j} m_i \qquad\qquad \hat{u}_0(x) \;:=\; \sum_j u_0^{m_j}(x-\sigma_j)$$

and let $\hat{u}$ be the corresponding entropy solution of (3.20). By the finite speed of propagation, there exists a $C>0$ such that $\hat{u}(t,x)=u^{m_j}(t,x)$ for every $(t,x)\in]0,m_j[\times]\sigma_j-m_j,\sigma_j+m_j[$ and for every j. Therefore we conclude that $\{t\in]0,1[:\hat{u}(t,\cdot)\notin SBV_{loc}\}$ is infinite.

4. Proof of Corollary 1.4

We first recall the definition of semiconcave functions.

Definition 4.1. *Let $\alpha:\Omega\subset\mathbb{R}^2\to\mathbb{R}$. We say that α is semiconcave if there exists $c\in\mathbb{R}$ such that $\alpha(x)-c|x|^2$ is a concave function.*

The proof is based on Theorem 1.2 and on the following lemma concerning differentiability points of semiconcave functions.

Lemma 4.2. *Let $\alpha:\Omega\subset\mathbb{R}^2\to\mathbb{R}$ be a semiconcave function and let Σ be the set of its differentiability points. Then*

(i) *$\Omega\setminus\Sigma$ is countably $\mathscr{H}^1$-rectifiable;*
(ii) *$\nabla\alpha$ is continuous on Σ.*

Proof. Without loosing our generality we assume that α is concave and Ω is convex. Statement (i) is well-known in any Euclidean space, and a simple proof of it is given in [1], with references to more precise results. Statement (ii) can be obtained noticing that at differentiability points the subdifferential

$$\partial\alpha(x) := \left\{p\in\mathbb{R}^2:\ \alpha(y)\le\alpha(x)+\langle p,y-x\rangle\ \ \forall y\in\mathbb{R}^2\right\}$$

contains only $\nabla\alpha(x)$. On the other hand, the graph of the subdifferential $\{(x,p):\ p\in\partial\alpha(x)\}$ is clearly closed in $\Omega\times\mathbb{R}^2$ and this immediately leads to the stated continuity property. □

Proof of Theorem 1.4. Let $\tilde{\Omega} \subset\subset \Omega$ and recall that, since H is strictly convex, u is semiconcave in $\tilde{\Omega}$ (see for instance [7]). Without loosing our generality we can assume that 0 is a regular value of H, otherwise $\{H = 0\}$ consists of one point and the statement is trivial.

Denoting by Σ the set of points where u is differentiable, we will prove that there exists an open set $A \supset \tilde{\Omega} \cap \Sigma$ such that $\nabla u \in SBV(A)$. This implies

$$|D^c \nabla u|(\tilde{\Omega}) = |D^c \nabla u|(\tilde{\Omega} \setminus A) \leq |D^c \nabla u|(\tilde{\Omega} \setminus \Sigma)$$

and since, by Lemma 4.2(i), the set $\tilde{\Omega} \setminus \Sigma$ is countably $\mathscr{H}^{d-1}$-rectifiable, we obtain from (2.3) that $|D^c \nabla u| = 0$ in $\tilde{\Omega}$, i.e. $\nabla u \in SBV(\tilde{\Omega}; \mathbb{R}^2)$. In order to obtain an open set A with the stated properties it suffices to show that for any $x \in \tilde{\Omega} \cap \Sigma$ there exists $r > 0$ such that $\nabla u \in SBV(B_r(x); \mathbb{R}^2)$.

Fix $x \in \tilde{\Omega} \cap \Sigma$; from Lemma 4.2(ii) it follows that

$$\lim_{r \downarrow 0} \|\nabla u - v\|_{L^\infty(B_r(x))} \;=\; 0 \quad \text{with} \quad v := \nabla u(x). \tag{4.1}$$

Since 0 is a regular value of H we can fix a system of coordinates (z, y) on $\mathbb{R}^2$ such that

$$\frac{\partial H}{\partial z}(v) \;>\; 0 \qquad \frac{\partial H}{\partial y}(v) \;=\; 0\,.$$

We let ρ be sufficiently small, so that there exists $h \in C^2(\mathbb{R})$ such that

$$\{H = 0\} \cap B_\rho(v) \;=\; \{(-h(y), y)\} \cap B_\rho(v)\,.$$

Then

$$h'(y) \;=\; \frac{\frac{\partial H}{\partial y}(-h(y), y)}{\frac{\partial H}{\partial z}(-h(y), y)}\,, \qquad h'(0) = 0\,, \qquad h''(0) \;=\; \frac{\frac{\partial^2 H}{\partial y^2}(v)}{\frac{\partial H}{\partial z}(v)} \;>\; 0\,.$$

So we can assume in addition that h is strictly convex on $[-\rho, \rho]$.

We use (4.1) to find $r > 0$ such that the (essential) range of $\nabla u|_{B_r(x)}$ is contained in $B_\rho(v)$. Therefore

$$\partial_z u + h(\partial_y u) \;=\; 0 \quad \mathscr{L}^2\text{-a.e. in } B_r(x). \tag{4.2}$$

Hence, if we define $w := \partial_y u$ we get:

$$D_z w + D_y[h(w)] \;=\; D_y \partial_z u + D_y[h(\partial_y u)] \;=\; D_y\big\{\partial_z u + h(\partial_y u)\big\} \;=\; 0$$

in the sense of distributions. Moreover, from semiconcavity of u we get that there exists $C > 0$ such that $D_y w = D_{yy} u \;\leq\; C\mathscr{L}^2$. This means that w satisfies Oleinik's E-condition, and hence is an entropy solution of $D_z w + D_y[h(w)] = 0$. From Corollary 1.3 we conclude that $\partial_y u = w \in SBV(B_r(x))$. Applying Vol'pert's chain rule (see Theorem 3.96 of [3]), from (4.2) we conclude that $\partial_z u = -h(\partial_y u) \in SBV(B_r(x))$. □

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