DELTA-SHOCKS, THE RANKINE–HUGONIOT CONDITIONS, AND SINGULAR SUPERPOSITION OF DISTRIBUTIONS

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Abstract. The problem of defining *δ-shock wave type solutions* of hyperbolic systems of conservation laws in connection with the constructing *singular superpositions* (*products*) of distributions is studied. We illustrate this problem by constructing δ-shock wave type solutions for two systems. One of them,

$$u_t + \quad f(u) - v \quad_x = 0, \qquad v_t + \quad g(u) \quad_x = 0,$$

is a generalization of the well-known Keyfitz–Kranzer system, where $f(u)$ and $g(u)$ are polynomials of degree n and $n+1$, respectively, n is an even integer. The other one is the system

$$u_t + \quad f(u) \quad_x = 0, \qquad v_t + \quad vg(u) \quad_x = 0,$$

where $f(u)$, $g(u)$ are smooth functions. As far as we know, exact δ-shock wave type solutions for the first system have never been constructed.

1. Introduction

1.1. **Singular solutions to systems of conservation laws.** Let us consider the hyperbolic system of conservation laws

$$L_1[u,v] = u_t + \big(F(u,v)\big)_x = 0, \quad L_2[u,v] = v_t + \big(G(u,v)\big)_x = 0, \tag{1.1}$$

where $F(u,v)$, $G(u,v)$ are smooth functions, *linear* with respect to v; $u = u(x,t), v = v(x,t) \in \mathbb{R}$; $x \in \mathbb{R}$. As is well known, even in the case of smooth (and, certainly, in the case of discontinuous) initial data (u^0, v^0), this system may have discontinuous solutions. In this case, it is said that a pair $(u,v) \in L^\infty\big(\mathbb{R} \times (0,\infty); \mathbb{R}^2\big)$ is a *generalized solution* of the Cauchy problem (1.1) with the initial data (u^0, v^0) if the integral identities

$$\begin{aligned}
\int_0^\infty \int \Big(u\varphi_t + F(u,v)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty \int \Big(v\varphi_t + G(u,v)\varphi_x\Big)\,dx\,dt + \int v^0(x)\varphi(x,0)\,dx &= 0
\end{aligned} \tag{1.2}$$

hold for all compactly supported test functions $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\,\infty))$, where $\int \,\cdot\, dx$ denotes an improper integral $\int_{-\infty}^{\infty} \,\cdot\, dx$.

It is well known [1], [10], [11], [12], [13], [14], [25] that there are "nonclassical" situations when the Riemann problem does not possess a weak L^∞-solution except for some particular initial data. In contrast to the standard results of existence of weak solutions to strictly hyperbolic systems, here the *linear* component of the solution v may contain Dirac measures and must be sought in the space of measures,

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while the first component u has bounded variation. In order to solve the Cauchy problem in this nonclassical situation, it is necessary to introduce new singularities called δ-shocks, which are solutions of the hyperbolic system (1.1), such that the *linear* component of the solution can have the form $v(x,t) = V(x,t) + e(x,t)\delta(\Gamma)$, Γ is a connected graph in the upper half-plane $\{(x,t) : x \in \mathbb{R}, t \ge 0\}$, $V \in L^{\infty}\big(\mathbb{R} \times (0,\infty);\mathbb{R}\big)$, $e \in C^1(\Gamma)$.

Several approaches to constructing *δ-shock type solutions* are known. An apparent difficulty in defining such solutions arises due to the fact that, to introduce a definition of the δ-shock type solution, we need to define a *singular superposition* of distributions (for example, the *product of the Heaviside function and the delta function*). We also need to define *in which sense* a distributional solution satisfies nonlinear systems.

In particular, it is well known, that for some cases of system (1.1) the Cauchy problem with the initial data

$$u^0(x) = u_0 + u_1 H(-x), \quad v^0(x) = v_0 + v_1 H(-x), \tag{1.3}$$

where u_0, u_1, v_0, v_1 are constants and $H(\xi)$ is the Heaviside function, may admit a *δ-shock wave type solution*, i.e., a generalized solution of the form

$$\begin{aligned} u(x,t) &= u_0 + u_1 H(-x+ct), \\ v(x,t) &= v_0 + v_1 H(-x+ct) + e(t)\delta(-x+ct), \end{aligned} \tag{1.4}$$

where $e(t)$ is a smooth function such that $e(0) = 0$ and $\delta(\xi)$ is the Dirac delta function.

For example, in [11], in order to construct a δ-shock wave type solution of the system

$$L_{21}[u] = u_t + \big(f(u)\big)_x = 0, \quad L_{22}[u,v] = v_t + \big(g(u)v\big)_x = 0, \tag{1.5}$$

(here $F(u,v) = f(u)$, $G(u,v) = vg(u)$) this system is reduced to a system of Hamilton–Jacobi equations, and then the Lax formula is used. In [10], a δ-shock wave type solution of system (1.5) is constructed as self-similar viscosity limits. In [14], to construct a δ-shock wave type solution of system (1.5) for the case $g(u) = f'(u)$, the problem of *multiplication of distributions* is solved by using the definition of Volpert's averaged superposition [27]. In [20], a general framework for *nonconservative product*

$$g(u)\frac{du}{dx} \tag{1.6}$$

was introduced, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded Borel function and $u : (a,b) \to \mathbb{R}^n$ is a discontinuous function of bounded variation. In the framework of the approach [20] the Cauchy problems for nonlinear hyperbolic systems in nonconservative form can be considered [14], [15], [16]. Note that in [15], [16], for nonconservative systems the notion of generalized solution *does depend on the specific family of paths*, which *can not be derived from the hyperbolic system only*.

In [26], for the system

$$u_t + (u^2)_x = 0, \qquad v_t + (uv)_x = 0, \tag{1.7}$$

(here $F(u,v) = u^2$, $G(u,v) = vu$) with the initial data (1.3), the δ-shock wave type solution is defined as a *measure-valued solution*.

In [13] for the system

$$u_t + (u^2 - v)_x = 0, \qquad v_t + \Big(\frac{1}{3}u^3 - u\Big)_x = 0 \tag{1.8}$$

(here $F(u,v) = u^2 - v$, $G(u,v) = \frac{1}{3}u^3 - u$) with the initial data (1.3) the δ-shock wave type *approximate solution* was studied. But the notion of the *exact δ-shock* *solution* has *not been defined*. In order to construct *approximate solutions*, the Colombeau theory approach, as well as the Dafermos–DiPerna regularization (under assumption that Dafermos profiles exist), and the box approximations are used. In [21] the existence of Dafermos profiles for singular shocks is proved. In [22], a class of problems for which the lowest-order asymptotic approximations to Dafermos profiles can be constructed is identified. System (1.8) is an example of a system satisfying general hypotheses of paper [22].

In [3], [4]– [9], [23], [24] a new approach to solving the problem of the propagation and interaction of singular fronts was developed. This approach was called the *weak asymptotics method*. The *key role* in this method is played by the definition of a *weak asymptotic solution* of the Cauchy problem, which admits passing to the limit in the *weak sense* as $\varepsilon \to 0$, where ε is the regularization parameter. Using V. P. Maslov's idea, this method permits to derive the Rankine–Hugoniot conditions directly from the differential equations *considered in the weak sense*. V. P. Maslov's *algebras of singularities* are essential in our method [18], [19], [2]. By using the *weak asymptotics method* in above mentioned papers, the *dynamics of propagation and interaction* of different nonlinear waves (infinitely narrow δ-solitons, shocks, δ-shocks) of nonlinear equations and hyperbolic systems of conservation laws is studied. In the framework of the *weak asymptotics method* [7]– [9] new Definition 2.1, of a δ-shocks type solution for (1.1) was introduced. This definition is a *natural generalization* of the *usual* system of integral identities (1.2).

1.2. **Main results.** In Sec. 2 we introduce the definition of a *δ-shock wave type solution* for system (1.1), as well as the definition of a *weak asymptotic solution*, which is one of the most important notions in the *weak asymptotics method*. In this section we also derive δ-shock Rankine–Hugoniot conditions. In order to construct a *weak asymptotic solution* of our problems, some weak asymptotics are constructed in Sec. 6.

In Sec. 3 we study the problem of propagation of a δ-shock in the system

$$\begin{array}{rcccl} L_{11}[u,v] & = & u_t + \big(f(u) - v\big)_x & = & 0, \\ L_{12}[u,v] & = & v_t + \big(g(u)\big)_x & = & 0, \end{array} \tag{1.9}$$

where

$$f(u) = \sum_{k=0}^{n} A_k u^k, \quad A_n \neq 0, \qquad g(u) = \sum_{k=0}^{n+1} B_k u^k, \quad B_{n+1} \neq 0, \tag{1.10}$$

are polynomials, n is an even number. The well known Keyfitz–Kranzer system (1.8) is a particular case of system (1.9). Thus we solve the Cauchy problem for system (1.9) with the *δ-shock front initial data*

$$\begin{array}{rcl} u^0(x) & = & u_0^0(x) + u_1^0(x)H(-x), \\ v^0(x) & = & v_0^0(x) + v_1^0(x)H(-x) + e^0\delta(-x), \end{array} \tag{1.11}$$

where $u_k^0(\mathrm{x})$, $v_k^0(x)$, $k = 0, 1$ are given smooth functions, e^0 is a given constant.

In Sec. 4, the problem of propagation of the δ-shock in system (1.5), solved in [6]– [9] is considered.

Remark 1.1. The Keyfitz–Kranzer system (1.8) and system (1.9) differ from system (1.5) and have a *specific "strange"* property. Although *δ-shock wave type solutions* of the Cauchy problems (1.9), (1.11) and (1.5), (1.11) satisfy the same integral identity (2.1), in systems (1.9), (1.8) have *no balance* of singularities. If (u, v) is a δ-shock type solution (1.4) of system (1.8) then u contains the Heaviside function H, and v contains the Heaviside function H and δ-function (see (1.4). Thus, $u^2 - v$ contains the distributions H, δ, and $\frac{1}{3}u^3 - u$ contains the distribution H. It is clear that the term $(u^2-v)_x$ contains H, δ, δ', while the term u_t contains *only* H and δ. Analogously, the term v_t contains H, δ, δ', but the term $(u^3/3-u)_x$ contains *only* H, δ. Seemingly, it is impossible to obtain δ-shock type solutions for systems (1.8) and (1.9). Nevertheless, in Sec. 3, we prove that *there are exact solutions of this type*. First, δ-shock wave type solutions for *specific* systems (1.8), (1.9) were constructed in [23] for piecewise constant initial data.

The problem of defining *δ-shock wave type solutions* for the Cauchy problems (1.9), (1.11) and (1.5), (1.11) in connection with the construction of *singular superpositions* (*products*) of distributions is discussed in Sec. 5. We stress that the *"right" singular superpositions of distributions* (5.6)–(5.9) can be obtained *only in the context of constructing weak asymptotic solutions* to these Cauchy problems.

It remains to note that, since in the "specific" systems (1.9) and (1.8) there are no terms of the type of (1.6) (see (5.6), (5.7)), it is *impossible* to construct a *δ-shock wave type solution* for them by using the *nonconservative product* [15], [16], [20].

1.3. **The scheme of the weak asymptotics method.** According to our method, we shall seek a *δ-shock wave type solution* of the Cauchy problems (1.9), (1.11) and (1.5), (1.11) in the form

$$\begin{aligned} u(x,t) &= u_0(x,t)+u_1(x,t)H(-x+\phi(t)), \\ v(x,t) &= v_0(x,t)+v_1(x,t)H(-x+\phi(t))+e(t)\delta(-x+\phi(t)), \end{aligned} \tag{1.12}$$

where $u_0(x,t)$, $u_1(x,t)$, $v_0(x,t)$, $v_1(x,t)$, $e(t)$, $\phi(t)$ are desired functions. This singular ansatz *preserves* the structure of the initial data (1.11). Within the framework of the *weak asymptotics method*, we find a *δ-shock wave type solution* (1.12) as a weak limit

$$u(x,t)=\lim_{\varepsilon\to+0}u(x,t,\varepsilon), \qquad v(x,t)=\lim_{\varepsilon\to+0}v(x,t,\varepsilon), \tag{1.13}$$

of the *weak asymptotic solution* $(u(x,t,\varepsilon),v(x,t,\varepsilon))$ to this Cauchy problem.

We will construct a *weak asymptotic solution* of the Cauchy problem as the sum of the singular ansatz regularized *with respect to singularities* $H(-x+\phi(t))$ and $\delta(-x+\phi(t))$, and *corrections*:

$$\begin{aligned} u(x,t,\varepsilon) &= \widetilde{u}(x,t,\varepsilon)+R_u(x,t,\varepsilon), \\ v(x,t,\varepsilon) &= \widetilde{v}(x,t,\varepsilon)+R_v(x,t,\varepsilon), \end{aligned}$$

where a pair of functions $\big(\widetilde{u}(x,t,\varepsilon),\widetilde{v}(x,t,\varepsilon)\big)$ is a *regularization* of the singular ansatz (1.12), and the *corrections* $R_u(x,t,\varepsilon)$, $R_v(x,t,\varepsilon)$ are the desired functions, which must admit the estimates:

$$R_j(x,t,\varepsilon)=o_{\mathcal{D}'}(1), \quad \frac{\partial R_j(x,t,\varepsilon)}{\partial t}=o_{\mathcal{D}'}(1), \quad \varepsilon\to+0, \qquad j=u,v. \tag{1.14}$$

Let us note that *choosing the corrections* is an *essential* part of the "right" construction of the *weak asymptotic solution* [6]– [9], [23], [24] (see Remarks 3.1, 4.1, and Sec. 5).

We shall construct a regularization $f(x,\varepsilon)$ of the distribution $f(x)\in\mathcal{D}'(\mathbb{R})$ as

$$f(x,\varepsilon)=f(x)*\frac{1}{\varepsilon}\omega\left(\frac{x}{\varepsilon}\right), \quad \varepsilon>0, \tag{1.15}$$

where $*$ is a convolution, and a mollifier $\omega(\eta)$ has the following properties: (a) $\omega(\eta)\in C^\infty(\mathbb{R})$, (b) $\omega(\eta)$ has a compact support or decreases sufficiently rapidly as $|\eta|\to\infty$, (c) $\int\omega(\eta)\,d\eta=1$, (d) $\omega(\eta)\ge 0$, (e) $\omega(-\eta)=\omega(\eta)$. We have $\lim\limits_{\varepsilon\to+0}\big\langle f(\xi,\varepsilon),\ \phi(\xi)\big\rangle=\big\langle f,\ \phi\big\rangle$ for all $\phi\in\mathcal{D}(\mathbb{R})$.

Thus, we will seek a *weak asymptotic solution* in the form

$$\begin{aligned} u(x,t,\varepsilon) &= u_0(x,t) + u_1(x,t)H_u(-x+\phi(t),\varepsilon) \\ &\qquad\qquad\qquad\qquad + R_u(x,t,\varepsilon), \\ v(x,t,\varepsilon) &= v_0(x,t) + v_1(x,t)H_v\big(-x+\phi(t),\varepsilon\big) \\ &\qquad\qquad + e(t)\delta\big(-x+\phi(t),\varepsilon\big) + R_v(x,t,\varepsilon), \end{aligned} \tag{1.16}$$

where according to (1.15),

$$\delta(x,\varepsilon) = \frac{1}{\varepsilon}\omega_\delta\big(x/\varepsilon\big), \tag{1.17}$$

is a regularization of the δ-function, and

$$H_j(x,\varepsilon) = \omega_{0j}\Big(\frac{x}{\varepsilon}\Big) = \int_{-\infty}^{x/\varepsilon} \omega_j(\eta)\,d\eta, \quad j = u, v \tag{1.18}$$

are regularizations of the Heaviside function $H(x)$. Here the mollifiers $\omega_u(\eta)$, $\omega_v(\eta)$, $\omega_\delta(\eta)$ have properties (a)–(e). It is clear that $\omega_{0j}(\eta) \in C^\infty(\mathbb{R})$, $\lim_{\eta\to+\infty}\omega_{0j}(\eta) = 1$, $\lim_{\eta\to-\infty}\omega_{0j}(\eta) = 0$, $j = u, v$.

Let $\lambda_1(u,v)$, $\lambda_2(u,v)$ be the eigenvalues of the characteristic matrix of system (1.1). As in [10], [13], [26], we use the "overcompression" condition

$$\begin{aligned} \lambda_1(u_+,v_+) \le \quad \dot\phi(t) \quad \le \lambda_1(u_-,v_-), \\ \lambda_2(u_+,v_+) \le \quad \dot\phi(t) \quad \le \lambda_2(u_-,v_-), \end{aligned} \tag{1.19}$$

as the admissibility condition for the δ-shocks, where $\dot\phi(t)$ is the velocity of motion of the δ-shock front, and $u_- = u_0 + u_1$, $v_- = v_0 + v_1$ and $u_+ = u_0$, $v_+ = v_0$ are the respective left- and right-hand values of u, v on the discontinuity curve. It means that all characteristics on both sides of the discontinuity are in-coming.

2. δ-Shock wave type solutions

2.1. **Generalized solutions.** Suppose that $\Gamma = \{\gamma_i : i \in I\}$ is a connected graph in the upper half-plane $\{(x,t) : x \in \mathbb{R},\ t \in [0,\infty)\} \in \mathbb{R}^2$ containing smooth arcs γ_i, $i \in I$, and I is a finite set. By I_0 we denote a subset of I such that an arc γ_k for $k \in I_0$ starts from the points of the x-axis; $\Gamma_0 = \{x_k^0 : k \in I_0\}$ is the set of initial points of arcs γ_k, $k \in I_0$.

Let $(u^0(x), v^0(x))$ be *δ-shock wave type initial data*, i.e.,

$$v^0(x) = V^0(x) + e^0\delta(\Gamma_0),$$

where $u^0, V^0 \in L^\infty\big(\mathbb{R};\mathbb{R}\big)$, and $e^0\delta(\Gamma_0) \stackrel{def}{=} \sum_{k\in I_0} e_k^0\delta(x - x_k^0)$, e_k^0 are constants, $k \in I_0$.

Let us introduce the definition of a *δ-shock wave type solution* for system (1.1).

Definition 2.1. ([7]– [9]) A pair of distributions $\big(u(x,t), v(x,t)\big)$ and graph Γ, where $v(x,t)$ is represented in the form of the sum

$$v(x,t) = V(x,t) + e(x,t)\delta(\Gamma),$$

$u, V \in L^\infty\big(\mathbb{R}\times(0,\,\infty);\mathbb{R}\big)$, $e(x,t)\delta(\Gamma) \stackrel{def}{=} \sum_{i\in I} e_i(x,t)\delta(\gamma_i)$, $e_i(x,t) \in C^1(\Gamma)$, $i \in I$, is called a *generalized δ-shock wave type solution* of system (1.1) with the initial data

$(u^0(x), v^0(x))$ if the integral identities

$$\begin{array}{rcl} \displaystyle\int_0^\infty \int \Big(u\varphi_t + F(u,V)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &=& 0, \\ \displaystyle\int_0^\infty \int \Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl && \\ \displaystyle + \int V^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} e_k^0\varphi(x_k^0,0) &=& 0, \end{array} \tag{2.1}$$

hold for all test functions $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\,\infty))$, where $\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}$ is the tangential derivative on the graph Γ, $\int_{\gamma_i}\cdot\,dl$ is a line integral over the arc γ_i.

2.2. **The Rankine–Hugoniot conditions.**

Theorem 2.1. *Let us assume that $\Omega\subset\mathbb{R}\times(0,\,\infty)$ is some region cut by a smooth curve Γ into a left- and right-hand parts $\Omega_\mp$, $(u(x,t),v(x,t))$ and Γ is a generalized δ-shock wave type solution of system* (1.1) *and $(u(x,t),v(x,t))$ is smooth in $\Omega_\pm$. Then the Rankine–Hugoniot conditions for δ-shocks*

$$\begin{array}{rcl} \big[F(u,v)\big]_\Gamma\nu_1 + \big[u\big]_\Gamma\nu_2 &=& 0, \\ \big[G(u,v)\big]_\Gamma\nu_1 + \big[v\big]_\Gamma\nu_2 &=& \frac{\partial e(x,t)|_\Gamma}{\partial \mathbf{l}}, \end{array} \tag{2.2}$$

hold along Γ, where $\mathbf{n}=(\nu_1,\nu_2)$ is the unit normal to the curve Γ pointing from Ω_- into Ω_+, $\mathbf{l}=(-\nu_2,\nu_1)$,

$$\big[h(u,v)\big]\Big|_\Gamma = \Big(h(u_-,v_-) - h(u_+,v_+)\Big)\Big|_\Gamma$$

is a jump in function $h(u(x,t),v(x,t))$ across the discontinuity curve Γ, $(u_\mp,v_\mp)$ are respective left- and right-hand values of (u,v) on the discontinuity curve.

If $\Gamma=\{(x,t): x=\phi(t)\}$, $\Omega_\pm=\{(x,t):\pm(x-\phi(t))>0\}$ then relations (2.2) *can be rewritten as*

$$\begin{array}{rcl} \dot\phi(t) &=& \frac{[F(u,v)]}{[u]}\Big|_{x=\phi(t)}, \\ \dot e(t) &=& \Big([G(u,v)] - [v]\frac{[F(u,v)]}{[u]}\Big)\Big|_{x=\phi(t)}, \end{array} \tag{2.3}$$

where $e(t)\stackrel{def}{=} e(x,t)\big|_{x=\phi(t)}$, and $\dot{(\cdot)}=\frac{d}{dt}(\cdot)$.

Proof. Selecting the test function $\varphi(x,t)$ with compact support in $\Omega_\pm$, we deduce from (2.1) that (1.1) hold in $\Omega_\pm$, respectively. Now choosing a test function $\varphi(x,t)$ with support in Ω, we deduce from the second identity (2.1) that

$$0 = \int_0^\infty \int \Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt$$

$$= \int\int_{\Omega_-} \Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt + \int\int_{\Omega_+} \Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt.$$

Next, integrating by parts, we obtain

$$\int\int_{\Omega_\pm} \Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt$$

$$= -\int\int_{\Omega_\pm} \Big(V_t + \big(G(u,V)\big)_x\Big)\varphi\,dx\,dt \mp \int_\Gamma \Big(\nu_2 v_\pm + \nu_1 G(u_\pm,v_\pm)\Big)\varphi\,dl$$

$$= \mp \int_\Gamma \Big(\nu_2 v_\pm + \nu_1 G(u_\pm,v_\pm)\Big)\varphi\,dl,$$

owing to (1.1). Adding the last relations, we have

$$\int_0^\infty \int \Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt = \int_\Gamma \Big(\big[G(u,v)\big]\nu_1 + \big[v\big]\nu_2\Big)\varphi(x,t)\,dl \tag{2.4}$$

for all $\varphi(x,t) \in \mathcal{D}(\Omega)$.

Now integrating by parts we can easily see that

$$\int_\Gamma e(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\,dl = -\int_\Gamma \frac{\partial e(x,t)}{\partial \mathbf{l}}\varphi(x,t)\,dl, \tag{2.5}$$

where $\frac{\partial}{\partial \mathbf{l}}e(x,t)|_\Gamma = \frac{\partial}{\partial t}e(x,t)\big|_\Gamma \nu_1 - \frac{\partial}{\partial x}e(x,t)\big|_\Gamma \nu_2$.

Adding (2.4) and (2.5), we deduce

$$\int_\Gamma \bigg(\big[G(u,v)\big]\nu_1 + \big[v\big]\nu_2 - \frac{\partial e(x,t)}{\partial \mathbf{l}}\bigg)\varphi(x,t)\,dl = 0$$

for all $\varphi(x,t) \in \mathcal{D}(\Omega)$. Thus the second relation (2.2) holds.

We obtain the proof of the first relation (2.2) using formula (2.4).

If $\Gamma = \{(x,t) : x = \phi(t)\}$ then $\mathbf{n} = (\nu_1,\nu_2) = \frac{\big(1,-\dot\phi(t)\big)}{\sqrt{1+(\dot\phi(t))^2}}$, $\mathbf{l} = \frac{\big(\dot\phi(t),1\big)}{\sqrt{1+(\dot\phi(t))^2}}$, and

$$\frac{\partial \varphi(x,t)\big|_\Gamma}{\partial \mathbf{l}} = \frac{1}{\sqrt{1+(\dot\phi(t))^2}}\frac{d\varphi(\phi(t),t)}{dt}. \tag{2.6}$$

In view of (2.6), relations (2.2) imply (2.3).

The first equation (2.2) (or (2.3)) is the *standard* Rankine–Hugoniot condition. The left-hand side of the second equation (2.2) (or (2.3)) is called the *Rankine–Hugoniot deficit*.

The system of *δ-shocks* integral identities (2.1) is a *natural generalization* of the system of integral identities (1.2). The integral identities (2.1) differ from (1.2) by an additional term

$$\int_\Gamma e(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\,dl = \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\,dl$$

in the second identity. This term appears due to the *Rankine–Hugoniot deficit*.

2.3. **Weak asymptotic solutions.** Denote by $O_{\mathcal{D}'}(\varepsilon^\alpha)$ the collection of distributions $f(x,t,\varepsilon) \in \mathcal{D}'(\mathbb{R}_x)$ such that

$$\langle f(x,t,\varepsilon),\ \psi(x)\rangle = O(\varepsilon^\alpha),$$

for any test function $\psi(x) \in \mathcal{D}(\mathbb{R}_x)$. Moreover, $\langle f(x,t,\varepsilon),\ \psi(x)\rangle$ is a continuous function in t, where the estimate $O(\varepsilon^\alpha)$ is understood in the standard sense and is uniform with respect to t. The relation $o_{\mathcal{D}'}(\varepsilon^\alpha)$ is understood in a corresponding way.

Definition 2.2. ([6]– [9]) A pair of functions $\big(u(x,t,\varepsilon), v(x,t,\varepsilon)\big)$ smooth as $\varepsilon > 0$ is called a *weak asymptotic solution* of system (1.1) with the initial data $(u^0(x), v^0(x))$ if

$$\begin{aligned} L_1[u(x,t,\varepsilon), v(x,t,\varepsilon)] &= o_{\mathcal{D}'}(1),\\ L_2[u(x,t,\varepsilon), v(x,t,\varepsilon)] &= o_{\mathcal{D}'}(1),\\ u(x,0,\varepsilon) &= u^0(x) + o_{\mathcal{D}'}(1),\\ v(x,0,\varepsilon) &= v^0(x) + o_{\mathcal{D}'}(1), \quad \varepsilon \to +0, \end{aligned} \tag{2.7}$$

where the first two estimates are uniform in t.

Constructing the *weak asymptotic solution* and multiplying the first two relations (2.7) by a test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0, \infty))$, integrating these relations by parts and then passing to the limit as $\varepsilon \to +0$, we will see that the pair of distributions (1.13) satisfy the integral identities (2.1).

3. Propagation of δ-shocks in system (1.9)

3.1. **Weak asymptotic solution.** Consider the Cauchy problem (1.9), (1.11). In this case the graph Γ contains only one arc. Suppose this arc has the form $\Gamma = \{(x,t) : x = \phi(t)\}$, and hence $e(x,t)\big|_{\Gamma} = e(t)$. The first step of our approach is to find a *weak asymptotic solution* of the Cauchy problem (1.9), (1.11).

The eigenvalues of the characteristic matrix of system (1.9) are

$$\lambda_{1,2}(u) = \frac{1}{2}\Big(f'(u) \pm \sqrt{\big(f'(u)\big)^2 - 4g'(u)}\Big), \qquad \big(f'(u)\big)^2 \geq 4g'(u).$$

We assume that the "overcompression" condition (1.19) is satisfied.

We will seek a *δ-shock wave type solution* in the form (1.12) and a *weak asymptotic solution* in the form (1.16). Since the *generalized δ-shock wave type solution* is defined as a weak limit (1.13) of (1.16), in view of the estimates (1.14), the corrections $R_u(x,t,\varepsilon)$, $R_v(x,t,\varepsilon)$ *do not make a contribution* to the generalized solution of the problem. However, according to (3.9), (3.10), these terms *make a contribution* to the weak asymptotics of the superposition $f\big(u(x,t,\varepsilon)\big) - v(x,t,\varepsilon)$ and $g\big(u(x,t,\varepsilon)\big)$, and hence play an *essential* role in the construction of the generalized solution to the problem. Without introducing these terms, we cannot solve the Cauchy problem with arbitrary initial data and cannot construct the "right" singular superpositions (see Remarks 3.1).

Here we choose the *corrections* in the special form

$$\begin{aligned} R_u(x,t,\varepsilon) &= P(t)\tfrac{1}{\varepsilon^{1/n}}\Omega_P\Big(\tfrac{-x+\phi(t)}{\varepsilon}\Big) \\ &\qquad\qquad +Q(t)\tfrac{1}{\varepsilon^{1/(n+1)}}\Omega_Q\Big(\tfrac{-x+\phi(t)}{\varepsilon}\Big), \\ R_v(x,t,\varepsilon) &= 0, \end{aligned} \tag{3.1}$$

where $P(t)$, $Q(t)$ are the desired functions, $\frac{1}{\varepsilon}\Omega_P^n\big(x/\varepsilon\big)$, $\frac{1}{\varepsilon}\Omega_Q^{n+1}\big(x/\varepsilon\big)$ are regularizations (1.17) of the delta function, mollifiers $\Omega_P(\eta)$, $\Omega_Q(\eta)$ have properties (a)–(c). Consequently, estimates (1.14) hold.

In addition to (3.1), we can choose mollifiers $\Omega_P(\eta)$, $\Omega_Q(\eta)$ such that

$$\begin{aligned} \int \Omega_P^k(\eta)\Omega_Q^{n+1-k}(\eta)\,d\eta = 0, \qquad & k = 1, 2, \dots n+1, \\ \int \Omega_Q^{n+1}(\eta)\,d\eta \neq 0, \qquad & \int \Omega_P^n(\eta)\,d\eta \neq 0. \end{aligned} \tag{3.2}$$

In particular, for system (1.8) $f(u) = u^2$, $g(u) = \frac{1}{3}u^3 - u$ and relations (3.2) have the form $\int \Omega_P^3(\eta)\,d\eta = 0$, $\int \Omega_P^2(\eta)\Omega_Q(\eta)\,d\eta = 0$, $\int \Omega_P(\eta)\Omega_Q^2(\eta)\,d\eta = 0$.

Theorem 3.1. *Let*

$$\lambda_+(u_0^0(0)) \leq \frac{[f(u^0)] - [v^0]}{[u^0]}\bigg|_{x=0} \leq \lambda_-(u_0^0(0) + u_1^0(0)), \tag{3.3}$$

then there exists $T>0$ such that, for $t\in[0,T)$, the Cauchy problem (1.9), (1.11) *has a weak asymptotic solution* (1.16), (3.1), (3.2) *if and only if*

$$\begin{array}{rcl} L_{11}[u_+,v_+] &=& 0, \quad x>\phi(t), \\ L_{11}[u_-,v_-] &=& 0, \quad x<\phi(t), \\ L_{12}[u_+,v_+] &=& 0, \quad x>\phi(t), \\ L_{12}[u_-,v_-] &=& 0, \quad x<\phi(t), \\ \dot\phi(t) &=& \frac{[f(u)]-[v]}{[u]}\Big|_{x=\phi(t)}, \\ \dot e(t) &=& \Big([g(u)]-[v]\frac{[f(u)]-[v]}{[u]}\Big)\Big|_{x=\phi(t)}, \end{array} \tag{3.4}$$

$$\begin{array}{rcl} P(t) &=& \Big(\frac{e(t)}{aA_n}\Big)^{1/n}, \\ Q(t) &=& \Big\{\frac{e(t)}{cB_{n+1}}\Big(\frac{[f(u)]-[v]}{[u]}-\frac{1}{A_n}\Big(B_n+ \\ && (n+1)B_{n+1}\big(u_0+\frac{b}{a}u_1\big)\Big|_{x=\phi(t)}\Big)\Big)\Big\}^{1/(n+1)}, \end{array} \tag{3.5}$$

where $u_+=u_0$, $v_+=v_0$, $u_-=u_0+u_1$, $v_-=v_0+v_1$,

$$a=\int\Omega_P^n(\eta)\,d\eta>0, \quad b=\int\omega_{0u}(\eta)\Omega_P^n(\eta)\,d\eta, \quad c=\int\Omega_Q^{n+1}(\eta)\,d\eta\neq0. \tag{3.6}$$

The initial data for system (3.4), (3.5) *are defined from* (1.11), *and*

$$\begin{array}{rcl} e(0) &=& e^0, \\ P(0) &=& \Big(\frac{e^0}{aA_n}\Big)^{1/n}, \\ Q(0) &=& \Big\{\frac{e^0}{cB_{n+1}}\Big(\frac{[f(u)]-[v]}{[u]}-\frac{1}{A_n}\Big(B_n \\ && +(n+1)\big(u_0+\frac{b}{a}u_1\big)B_{n+1}\Big)\Big)\Big\}^{1/(n+1)}\Big|_{x=0}. \end{array}$$

Proof. With the help of (3.2), (3.6) and relations (6.1) from Lemma 6.1, we find the following weak asymptotics

$$\begin{array}{rcl} R^k(x,t,\varepsilon) &=& o_{\mathcal{D}'}(1), \quad k\le n-1, \\ R^n(x,t,\varepsilon) &=& aP^n(t)\delta(-x+\phi(t))+o_{\mathcal{D}'}(1), \\ R^{n+1}(x,t,\varepsilon) &=& cQ^{n+1}(t)\delta(-x+\phi(t))+o_{\mathcal{D}'}(1), \\ H(-x+\phi(t),\varepsilon)R^n(x,t,\varepsilon) &=& bP^n(t)\delta(-x+\phi(t))+o_{\mathcal{D}'}(1), \end{array} \tag{3.7}$$

where a, b, c are defined by (3.6).

Using relations (6.1) from Lemma 6.1, one can calculate

$$\begin{array}{rcl} \big(u(x,t,\varepsilon)\big)^k &=& u_0^k+\big((u_0+u_1)^k-u_0^k\big)H(-x+\phi(t)) \\ && +o_{\mathcal{D}'}(1), \quad k\le n-1, \\ \big(u(x,t,\varepsilon)\big)^n &=& u_0^n+\big((u_0+u_1)^n-u_0^n\big)H(-x+\phi(t)) \\ && +R^n(x,t,\varepsilon)+o_{\mathcal{D}'}(1), \\ \big(u(x,t,\varepsilon)\big)^{n+1} &=& u_0^{n+1}+\big((u_0+u_1)^{n+1}-u_0^{n+1}\big)H(-x+\phi(t)) \\ && +(n+1)\big(u_0+u_1H(-x+\phi(t),\varepsilon)\big) \\ && \times R^n(x,t,\varepsilon)+R^{n+1}(x,t,\varepsilon)+o_{\mathcal{D}'}(1). \end{array} \tag{3.8}$$

Taking into account relations (3.7), (3.8), we obtain

$$\begin{array}{l} f\big(u(x,t,\varepsilon)\big)=f(u_0)+\Big(f(u_0+u_1)-f(u_0)\Big)H(-x+\phi(t)) \\ \qquad +aA_nP^n(t)\delta(-x+\phi(t))+o_{\mathcal{D}'}(1), \\ g\big(u(x,t,\varepsilon)\big)=g(u_0)+\Big(g(u_0+u_1)-g(u_0)\Big)H(-x+\phi(t)) \end{array} \tag{3.9}$$

$$+\Big\{aB_nP^n(t)+(n+1)\big(au_0+bu_1\big)B_{n+1}P^n(t)$$
$$+cB_{n+1}Q^{n+1}(t)\Big\}\delta(-x+\phi(t))+o_{\mathcal{D}'}(1),\quad \varepsilon\to+0. \tag{3.10}$$

Substituting the smooth ansatz (1.16) and relations (3.9), (3.10) into the left-hand side of system (1.9), we obtain, up to $o_{\mathcal{D}'}(1)$, the following relations

$$\begin{aligned}
&L_{11}[u(x,t,\varepsilon),v(x,t,\varepsilon)]\\
&\quad= L_{11}[u_0,v_0]+\Big\{\frac{\partial u_1}{\partial t}+\frac{\partial}{\partial x}\big[f(u)-v\big]\Big\}H(-x+\phi(t))\\
&\qquad+\Big\{[u]\dot\phi(t)-\big[f(u)-v\big]\Big\}\delta(-x+\phi(t))\\
&\qquad+\Big\{e(t)-aA_nP^n(t)\Big\}\delta'(-x+\phi(t))+o_{\mathcal{D}'}(1),
\end{aligned}\tag{3.11}$$

$$\begin{aligned}
&L_{12}[u(x,t,\varepsilon),v(x,t,\varepsilon)]\\
&\quad= L_{22}[u_0,v_0]+\Big\{\frac{\partial v_1}{\partial t}+\frac{\partial}{\partial x}\Big[g(u)\Big]\Big\}H(-x+\phi(t))\\
&\quad=\Big\{[v]\dot\phi(t)+\dot e(t)-\Big[g(u)\Big]\Big\}\delta(-x+\phi(t))\\
&\quad+\Big\{e(t)\dot\phi(t)-aB_nP^n(t)-(n+1)\big(au_0+bu_1\big)B_{n+1}P^n(t)\\
&\qquad-cB_{n+1}Q^{n+1}(t)\Big\}\delta'(-x+\phi(t))+o_{\mathcal{D}'}(1),\quad \varepsilon\to+0.
\end{aligned}\tag{3.12}$$

Here we take into account estimates (1.14).

Setting the left-hand side of (3.11), (3.12) equal to zero, we obtain the necessary and sufficient conditions for the first two equalities (2.7), i.e., systems (3.4), (3.5).

Consider the Cauchy problem

$$\begin{array}{lcl}
L_{11}[u,V] &=& 0,\quad u(x,0)=u^0(x),\\
L_{12}[u,V] &=& 0,\quad V(x,0)=V^0(x)=v_0^0(x)+v_1^0(x)H(-x),
\end{array}\tag{3.13}$$

assuming that condition (3.3) holds. The last condition means that $(u^0(x),V^0(x))$ is entropy initial data. According to [17, Ch.4.2.], we extend a pair of functions

$$\begin{array}{rl}
\big(u_+^0(x)=u_0^0(x),\ V_+^0(x)=v_0^0(x)\big), & \qquad x\le 0,\\
\big(u_-^0(x)=u_0^0(x)+u_1^0(x),\ V_-^0(x)=v_0^0(x)+v_1^0(x)\big), & \qquad x\ge 0,
\end{array}$$

in a bounded C^1 fashion and continue to denote the extended pair of functions by $\big(u_\pm^0(x),V_\pm^0(x)\big)$. By $\big(u_\pm(x,t),V_\pm(x,t)\big)$ we denote the C^1 solutions of the problems

$$\begin{array}{lclclcl}
L_{11}[u,V] &=& 0, & & u_\pm(x,0) &=& u_\pm^0(x),\\
L_{12}[u,V] &=& 0, & & V_\pm(x,0) &=& V_\pm^0(x),
\end{array}$$

which, according to [17, Ch.2.1.], exist for small enough time interval $[0,\ T_1]$. The pair $\big(u_\pm(x,t),V_\pm(x,t)\big)$ determines a two-sheeted covering of the plane (x,t). Next, we define the function $x=\phi(t)$ as a solution of the problem

$$\dot\phi(t)=\frac{f(u_-(x,t))-f(u_+(x,t))-V_-(x,t))+V_+(x,t)}{u_-(x,t)-u_+(x,t)}\bigg|_{x=\phi(t)},$$

$\phi(0)=0$. It is clear that there exists a unique function $\phi(t)$ for sufficiently short times $[0,\ T_2]$. Setting $T=\min(T_1,T_2)$, we define the shock solution by

$$(u(x,t),V(x,t))=\left\{\begin{array}{lcl}
(u_+(x,t),V_+(x,t)), & x &>& \phi(t),\\
(u_-(x,t),V_-(x,t)), & x &<& \phi(t).
\end{array}\right.$$

Thus the first five equations of system (3.4) define a unique solution of the Cauchy problem (3.13) for $t\in[0,\ T)$. Solving this problem, we obtain $u(x,t)$, $V(x,t)$, $\phi(t)$.

Then, substituting these functions into (3.4), (3.5), we obtain $e(t)$, $v(x,t) = V(x,t) + e(t)\delta(-x+\phi(t))$, and $P(t)$, $Q(t)$. It is clear that mollifiers $\Omega_P(\eta)$, $\Omega_Q(\eta)$ can be chosen to satisfy relations (3.2).

3.2. **δ-Shock wave type solution.** At the second step, using the *weak asymptotic solution* constructed by Theorem 3.1, we obtain a generalized solution of the Cauchy problem (1.9), (1.11).

Theorem 3.2. *There exists $T > 0$ given by Theorem* 3.1 *such that the Cauchy problem* (1.9), (1.11), (3.3) *for $t \in [0, T)$ has a unique generalized solution* (1.12), *which satisfies the integral identities* (2.1)*:*

$$
\begin{aligned}
\int_0^T \int \Big(u\varphi_t + \big(f(u) - V\big)\varphi_x\Big)\,dx\,dt \qquad\qquad & \\
+ \int u^0(x)\varphi(x,0)\,dx &= 0, \\
\int_0^T \int \Big(V\varphi_t + g(u)\varphi_x\Big)\,dx\,dt + \int V^0(x)\varphi(x,0)\,dx & \\
+ \int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl + e^0\varphi(0,0) &= 0,
\end{aligned}
\tag{3.14}
$$

where $\Gamma = \{(x,t) : x = \phi(t),\ t \in [0,\ T)\}$,

$$
\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl = \int_0^T e(t)\frac{d\varphi(\phi(t),t)}{dt}\,dt,
$$

$V(x,t) = v_0(x,t) + v_1(x,t)H(-x+\phi(t))$, $\frac{d\varphi(\phi(t),t)}{dt} = \varphi_t(\phi(t),t) + \dot\phi(t)\varphi_x(\phi(t),t)$ *(see* (2.6)*), and functions $u_k(x,t)$, $v_k(x,t)$, $\phi(t)$, $e(t)$ are defined by system* (3.4)*.*

Proof. By Theorem 3.1 we have the following estimates:

$$
L_{11}[u(x,t,\varepsilon)] = o_{\mathcal{D}'}(\varepsilon), \quad L_{12}[u(x,t,\varepsilon), v(x,t,\varepsilon)] = o_{\mathcal{D}'}(\varepsilon).
$$

Let us apply the left-hand and right-hand sides of these relations to an arbitrary test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,T))$. Since for $\varepsilon > 0$ the functions $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ are smooth, then integrating by parts, we obtain

$$
\begin{aligned}
&\int_0^T \int \Big(u(x,t,\varepsilon)\varphi_t(x,t) + \Big(f(u(x,t,\varepsilon)) - v(x,t,\varepsilon)\Big)\varphi_x(x,t)\Big)\,dxdt \\
&\qquad\qquad + \int u(x,0,\varepsilon)\varphi(x,0)\,dx = o(1), \\
&\int_0^T \int \Big(v(x,t,\varepsilon)\varphi_t(x,t) + g(u(x,t,\varepsilon))\varphi_x(x,t)\Big)\,dxdt \\
&\qquad\qquad + \int v(x,0,\varepsilon)\varphi(x,0)\,dx = o(1), \quad \varepsilon \to +0.
\end{aligned}
$$

Passing to the limit as $\varepsilon \to +0$, and taking into account (1.16), (3.1), (3.9), (3.10), (3.5), and the fact that

$$
\lim_{\varepsilon\to+0}\int_0^T \int_{-\infty}^{\infty} e(t)\delta\big(-x+\phi(t),\varepsilon\big)\varphi(x,t)\,dxdt = \int_0^T e(t)\varphi(\phi(t),t)\,dt,
$$

$$
\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} e(0)\delta\big(-x,\varepsilon\big)\varphi(x,0)\,dx = e(0)\varphi(0,0),
$$

we obtain the integral identities (3.14). According to Theorem 3.1, system (3.4) has a unique solution.

The fifth and sixth equations of systems (3.4) are the Rankine–Hugoniot conditions of δ-shocks, and the right-hand side of the sixth equation is the *Rankine–Hugoniot deficit.*

Corollary 3.1. *For* $t \in [0, \infty)$, *the Cauchy problem* (1.9), (1.11), (3.3) (u_k^0, v_k^0, $k = 1, 2$ *are constants*) *has a unique generalized solution* (1.12), *where*

$$\begin{aligned} \phi(t) &= \tfrac{[f(u)]-[v]}{[u]}t, \\ e(t) &= e^0 + \Big([g(u)] - [v]\tfrac{[f(u)]-[v]}{[u]}\Big)t. \end{aligned}$$

By Theorem 3.2 and Corollary 3.1 we can obtain a δ-shock type solution of the Cauchy problem (1.8), (1.11).

Remark 3.1. To find a generalized solution of the Cauchy problem (1.9), (1.11) and (1.8), (1.11) we construct a weak asymptotic solution of problem (1.16), where the functions $u_k(x,t)$, $v_k(x,t)$, $\phi(t)$, $e(t)$, $k = 0, 1$ are determined by relations (3.4) and the functions $\omega_{0u}(\eta)$, $\Omega_P(\eta)$, $\Omega_Q(\eta)$, $P(t)$, $Q(t)$ are determined by relations (3.2), (3.5), (3.6).

In view of estimate (1.14) (see also formulas (5.6), (5.7) below), the generalized solution (1.12) of the Cauchy problem *does not depend* on correction functions $P(t)$, $Q(t)$. However, according to (3.5), without introducing the terms

$$P(t)\frac{1}{\varepsilon^{1/n}}\Omega_P\Big(\frac{-x+\phi(t)}{\varepsilon}\Big), \qquad Q(t)\frac{1}{\varepsilon^{1/(n+1)}}\Omega_Q\Big(\frac{-x+\phi(t)}{\varepsilon}\Big),$$

we cannot solve the Cauchy problem which admits δ-shocks. If we introduce only the first term, we cannot solve the Cauchy problem with an *arbitrary initial data* (1.11), but *only* with initial values determined by the relation

$$\frac{[f(u)]-[v]}{[u]} = \frac{1}{A_n}\Big(B_n + (n+1)\Big(u_0 + \frac{b}{a}u_1\Big)B_{n+1}\Big), \tag{3.15}$$

where the constants a, b are defined by (3.6). This is related to the fact that system (3.4), (3.15) is overdetermined.

Without introducing the *corrections* we cannot also construct the *"right" singular superpositions* (5.6), (5.7) in Sec. 5.

4. Propagation of δ-shocks in system (1.5)

Let us consider the Cauchy problem (1.5), (1.11), where $u_1^0(0) > 0$. The eigenvalues of the characteristic matrix of system (1.5) are $\lambda_1(u) = f'(u)$, $\lambda_2(u) = g(u)$. We shall assume that

$$f''(u) > 0, \quad g'(u) > 0, \quad f'(u) \le g(u), \tag{4.1}$$

i.e., the "overcompression" condition (1.19) is satisfied.

We will seek a *δ-shock wave type solution* in the form (1.12), a *weak asymptotic solution* in the form (1.16), and choose corrections in the form

$$R_u(x,t,\varepsilon) = 0, \qquad R_v(x,t,\varepsilon) = R(t)\frac{1}{\varepsilon}\Omega''\Big(\frac{-x+\phi(t)}{\varepsilon}\Big), \tag{4.2}$$

where $R(t)$ is a continuous function, $\varepsilon^{-3}\Omega''\big(x/\varepsilon\big)$ is a regularization of the distribution $\delta''(x)$, $\Omega(\eta)$ has the properties (a)–(c) (see Sec. 1). It is clear that estimates (1.14) hold.

In [6]– [9] the following theorems were proved.

Theorem 4.1. *There exists $T > 0$ such that, for $t \in [0, T)$, the Cauchy problem* (1.5), (1.11), (4.1) *has a weak asymptotic solution* (1.16), (4.2) *if and only if*

$$\begin{array}{rcl}
L_{21}[u_0] & = & 0, \quad x > \phi(t), \\
L_{21}[u_0+u_1] & = & 0, \quad x < \phi(t), \\
L_{22}[u_0, v_0] & = & 0, \quad x > \phi(t), \\
L_{22}[u_0+u_1, v_0+v_1] & = & 0, \quad x < \phi(t), \\
\dot{\phi}(t) & = & \frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}, \\
\dot{e}(t) & = & \Big([vg(u)] - [v]\frac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)},
\end{array} \tag{4.3}$$

$$R(t) = \frac{e(t)}{c(t)}\bigg(\frac{[f(u)]}{[u]}\bigg|_{x=\phi(t)} - a(t)\bigg), \tag{4.4}$$

where

$$\begin{array}{rcl}
a(t) & = & \int g\big(u_-(x,t)\omega_{0u}(\eta) + u_+(x,t)(1-\omega_{0u}(\eta))\big)\big|_{x=\phi(t)}\omega_\delta(\eta)\,d\eta, \\
c(t) & = & \int g\big(u_-(x,t)\omega_{0u}(\eta) + u_+(x,t)(1-\omega_{0u}(\eta))\big)\big|_{x=\phi(t)}\Omega''(\eta)\,d\eta \neq 0,
\end{array} \tag{4.5}$$

$u_- = u_0 + u_1$, $v_- = v_0 + v_1$, $u_+ = u_0$, $v_+ = v_0$. *The initial data for system* (4.3), (4.4) *are defined from* (1.11), *and*

$$\phi(0) = 0, \quad R(0) = \frac{e^0}{c(0)}\bigg(\frac{[f(u^0)]}{[u^0]}\bigg|_{x=0} - a(0)\bigg).$$

In [6]– [9], to prove Theorem 4.1 we use the weak asymptotics $v(x,t,\varepsilon)g\big(u(x,t,\varepsilon)\big)$, $f\big(u(x,t,\varepsilon)\big)$ given by Lemma 6.2.

Theorem 4.2. *Assume that conditions* (4.1) *are satisfied. Then, for $t \in [0, T)$, where $T > 0$ is given by Theorem* 4.1, *the Cauchy problem* (1.5), (1.11), *has a unique generalized solution* (1.12), *which satisfies the integral identities* (2.1)*:*

$$\begin{array}{rcl}
\int_0^T \int \Big(u\varphi_t + f(u)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx & = & 0, \\
\int_0^T \int \Big(\varphi_t + g(u)\varphi_x\Big)V\,dx\,dt + \int V^0(x)\varphi(x,0)\,dx & & \\
\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl + e^0\varphi(0,0) & = & 0,
\end{array} \tag{4.6}$$

for all $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0, T))$, where $\Gamma = \{(x,t) : x = \phi(t), \quad t \in [0,\ T)\}$,

$$\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl = \int_0^T e(t)\Big(\varphi_t(\phi(t),t) + \dot{\phi}(t)\varphi_x(\phi(t),t)\Big)\,dt,$$

$V(x,t) = v_0 + v_1 H(-x + \phi(t))$. *Here functions $u_k(x,t)$, $v_k(x,t)$, $k = 0, 1$, $\phi(t)$, $e(t)$ are defined by system* (4.3) *with the initial data defined from* (1.11), $\phi(0) = 0$.

Corollary 4.1. *For $t \in [0, \infty)$, the Cauchy problem* (1.5), (1.11) (u_k^0, v_k^0, $k = 1, 2$ *are constants*) *has a unique generalized solution* (1.12), *where*

$$\begin{array}{rcl}
\phi(t) & = & \frac{[f(u)]}{[u]}t, \\
e(t) & = & e^0 + \Big([g(u)v] - \frac{[f(u)]}{[u]}[v]\Big)t.
\end{array}$$

Remark 4.1. According to (4.4), (4.5), without introducing the *corrections* (4.2) we can *only* solve the Cauchy problem with initial data determined by the relation

$$\frac{\big[f(u(x,t))\big]}{\big[u(x,t)\big]}\bigg|_{x=\phi(t)} = \int g\big(u_0(\phi(t),t) + u_1(\phi(t),t)\omega_{0u}(\eta)\big)\omega_\delta(\eta)\,d\eta.$$

In this case we cannot construct the *"right" singular superpositions* (5.8), (5.9) defined in Sec. 5.

5. Singular superpositions (products) of distributions

5.1. **Singular superpositions.** It seems natural to introduce the product of the Heaviside function and delta function as the weak limit of the product of their regularizations. Then, according to the second relation (6.1), we have

$$\overbrace{H(x)\delta(x)} \stackrel{def}{=} \lim_{\varepsilon\to+0} H(x,\varepsilon)\delta(x,\varepsilon) = B_1\delta(x), \tag{5.1}$$

where $B_1 = \int \omega_0(\eta)\omega_\delta(\eta)\,d\eta$. The product (5.1) defined in this way *depends* on the mollifiers ω, ω_δ, i.e., on the regularizations of distributions $H(x)$, $\delta(x)$.

In a similar way, we can introduce the *singular superpositions* $f\big(u(x,t)\big) - v(x,t)$, $g\big(u(x,t)\big)$, where distributions $u(x,t)$, $v(x,t)$ are given by (1.12) and polynomials $f(u)$, $g(u)$ are given by (1.10). Using regularizations of distributions (1.12) $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ given by (1.16), (3.1), (3.2) and weak asymptotics (3.9), (3.10), we define *singular superpositions* by the following definition:

$$\begin{aligned}\overbrace{f\big(u(x,t)\big) - v(x,t)} &\stackrel{def}{=} \lim_{\varepsilon\to+0}\Big(f\big(u(x,t,\varepsilon)\big) - v(x,t,\varepsilon)\Big) = f(u_0) - v_0 \\ &+ \big[f(u) - v\big]H(-x+\phi(t)) + \Big\{aA_nP^n(t) - e(t)\Big\}\delta(-x+\phi(t)),\end{aligned} \tag{5.2}$$

$$\begin{aligned}\overbrace{g\big(u(x,t)\big)} &\stackrel{def}{=} \lim_{\varepsilon\to+0} g\big(u(x,t,\varepsilon)\big) = g(u_0) + \big[g(u)\big]H(-x+\phi(t)) \\ &+\Big\{aB_nP^n(t) + (n+1)\big(au_0+bu_1\big)B_{n+1}P^n(t) + cB_{n+1}Q^{n+1}(t)\Big\}\delta(-x+\phi(t)).\end{aligned} \tag{5.3}$$

where the correction functions $P(t)$, $Q(t)$ are given by (3.5), and a, b, c by (3.6), and the limits are understood in the weak sense.

Let $f(u)$, $g(u)$ be smooth functions. In the same way, using regularizations of distributions (1.12) $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ given by (1.16), (4.2) and weak asymptotics given by Lemma 6.2, we define the *singular superpositions*:

$$\overbrace{f\big(u(x,t)\big)} \stackrel{def}{=} \lim_{\varepsilon\to+0} f\big(u(x,t,\varepsilon)\big) = f(u_0) + \big[f(u)\big]H(-x+\phi(t)), \tag{5.4}$$

$$\begin{aligned}\overbrace{v(x,t)g\big(u(x,t)\big)} &\stackrel{def}{=} \lim_{\varepsilon\to+0} v(x,t,\varepsilon)g\big(u(x,t,\varepsilon)\big) = v_0g(u_0) \\ &+ \big[g(u)v\big]H(-x+\phi(t)) + \Big\{e(t)a(t) + R(t)c(t)\Big\}\delta(-x+\phi(t)),\end{aligned} \tag{5.5}$$

where $a(t)$, $c(t)$ are defined by (4.5).

It is easy to see that the *singular superpositions* (5.2)–(5.5) *depend* on the regularizations of the Heaviside function, delta function and the correction functions $P(t)$, $Q(t)$, $R(t)$. This fact means that the above introduced *singular superpositions* are *not unique*.

5.2. **"Right" singular superpositions.** However, in the context constructing of *weak asymptotic solutions* of the Cauchy problems we can define explicit formulas for the *"right" singular superpositions*.

Namely, substituting $P(t)$, $Q(t)$ given by (3.5) into expressions (5.2), (5.3), we obtain *"right" unique singular superpositions*:

$$\begin{aligned}f\big(u(x,t)\big) - v(x,t) &\stackrel{def}{=} \lim_{\varepsilon\to+0}\Big(f\big(u(x,t,\varepsilon)\big) - v(x,t,\varepsilon)\Big) \\ &= f(u_0) - v_0 + \big[f(u) - v\big]H(-x+\phi(t)),\end{aligned} \tag{5.6}$$

$$g\big(u(x,t)\big) \stackrel{def}{=} \lim_{\varepsilon\to+0}\Big(g\big(u(x,t,\varepsilon)\big)\Big)$$
$$= g(u_0) + \big[g(u)\big]H(-x+\phi(t)) + e(t)\frac{\big[f(u)\big]}{[u]}\delta(-x+\phi(t)). \tag{5.7}$$

Substituting $R(t)$ given by (4.4) into expressions (5.4), (5.5), we obtain *"right" unique singular superpositions*:

$$f\big(u(x,t)\big) \stackrel{def}{=} \lim_{\varepsilon\to+0} f\big(u(x,t,\varepsilon)\big) = f(u_0) + \big[f(u)\big]H(-x+\phi(t)), \tag{5.8}$$

$$v(x,t)g\big(u(x,t)\big) \stackrel{def}{=} \lim_{\varepsilon\to+0} v(x,t,\varepsilon)g\big(u(x,t,\varepsilon)\big) = v_0 g(u_0)$$
$$+\big[vg(u)\big]H(-x+\phi(t)) + e(t)\frac{[f(u)]}{[u]}\delta(-x+\phi(t)). \tag{5.9}$$

In (5.6)–(5.9) the distributions $u(x,t)$, $v(x,t)$ are defined by (1.12).

In contrast to (5.2)–(5.5), where $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ are regularizations of distributions (1.12), in (5.6), (5.7), and (5.8), (5.9), $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ give the *weak asymptotic solution* of the Cauchy problem (1.9), (1.11), and (1.5), (1.11), respectively.

It is clear that the *unique "right" singular superpositions* (5.6)–(5.9) *are independent* of the regularizations of the Heaviside function, delta function and the correction functions and can be obtained *only by the construction of a weak asymptotic solution* of the Cauchy problem.

In fact, by (5.9) we *define* the *unique "right" product* of the Heaviside function and the delta function in the context of the Cauchy problem (1.5), (1.11). Setting $R(t)=0$ and comparing formulas (5.5) and (5.9), we readily see that to construct *unique "right" product* we must choose the mollifiers ω_u, ω_δ in (4.5) such that

$$a(t) = \int g\big(u_-(x,t)\omega_{0u}(\eta) + u_+(x,t)(1-\omega_{0u}(\eta))\big)\big|_{x=\phi(t)}\omega_\delta(\eta)\,d\eta$$
$$= \frac{[f(u)]}{[u]}\bigg|_{x=\phi(t)}. \tag{5.10}$$

In particular, for system (1.7) (here $f(u)=u^2$, $g(u)=u$) the *unique "right" product* of the Heaviside function and the delta function is defined as

$$e(t)\delta(-x+\phi(t))u(x,t)$$
$$= e(t)\delta(-x+\phi(t))\begin{cases} u_-(x,t), & x<\phi(t),\\ u_+(x,t), & x>\phi(t),\end{cases}$$
$$= \big(u_-(x,t)+u_+(x,t)\big)e(t)\delta(-x+\phi(t)).$$

Here according to (5.10), the mollifiers are such that

$$\int \omega_{0u}(\eta)\omega_\delta(\eta)\,d\eta = \frac{u_-(x,t)}{u_-(x,t)-u_+(x,t)}\bigg|_{x=\phi(t)},$$

and according to (1.19), (4.1), $u_+(\phi(t),t)\le 0$.

As was already mentioned above, systems (1.9) and (1.8) have a *specific "strange"* property and, in contrast to system (1.5), formulas (5.6), (5.7) *do not define (!)* the *product of the Heaviside function and the δ-function*. Moreover, although (according to (1.12)), $u(x,t)$ *does not depend (!)* on the term $e(t)\delta(-x+\phi(t))$, the right-hand side of the *"right" singular superposition* (5.7) *does depend (!)* on this term. Thus one can say that the term $e(t)\delta(-x+\phi(t))$ "appears in (5.7) from nothing". Analogously, the left-hand side in (5.6) *depends* on $e(t)\delta(-x+\phi(t))$, but the right-hand side in (5.6) *does not depend* on this term.

Thus a *"right" singular superposition* is determined only in the context of solving the Cauchy problem. If we knew the *"right" singular superpositions* (5.6), (5.7) and (5.8), (5.9) in advance then Theorem 3.2 and Theorem 4.2 could be proved explicitly by substituting these superpositions into (1.9), (3.14) and (1.5), (4.6), respectively.

6. Some weak asymptotic expansions

In order to find a *weak asymptotic solution* of the Cauchy problems (1.9), (1.11) and (1.5), (1.11), we need weak asymptotics calculated in the following lemmas.

Lemma 6.1. *Let* $\delta(x,\varepsilon)=\frac{1}{\varepsilon}\omega_\delta\Big(\frac{x}{\varepsilon}\Big)$, $\frac{1}{\varepsilon}\Omega\Big(\frac{x}{\varepsilon}\Big)$ *be regularizations* (1.17) *of the delta function, and* $H(\xi,\varepsilon)=\omega_0\Big(\frac{\xi}{\varepsilon}\Big)=\int_{-\infty}^{\frac{x}{\varepsilon}}\omega(\eta)\,d\eta$, *be regularization* (1.18) *of the Heaviside function* $H(x)$, $x\in\mathbb{R}$. *Then*

$$\begin{aligned}\big(H(\xi,\varepsilon)\big)^r &= H(\xi)+O_{\mathcal{D}'}(\varepsilon),\\ \Big(H(x,\varepsilon)\Big)^r\delta(x,\varepsilon) &= B_r\delta(x)+O_{\mathcal{D}'}(\varepsilon),\\ \delta(x,\varepsilon)\Big(\omega\Big(\frac{x}{\varepsilon}\Big)\Big)^r &= A_r\delta(x)+O_{\mathcal{D}'}(\varepsilon),\quad \varepsilon\to+0,\end{aligned}\tag{6.1}$$

where $B_r=\int\omega_0^r(\eta)\omega_\delta(\eta)\,d\eta$, $A_r=\int\omega_\delta(\eta)\Omega^r(\eta)\,d\eta$, $r=1,2,\dots$.

Proof. From (1.18), we obviously have the first relation in (6.1). Making the change of variables $x=\varepsilon\eta$, we obtain

$$\Big\langle\frac{1}{\varepsilon}\omega_\delta\big(\frac{x}{\varepsilon}\big)\Big(\omega_0\big(\frac{x}{\varepsilon}\big)\Big)^r,\psi(x)\Big\rangle$$
$$=\int\omega_0^r(\eta)\omega_\delta(\eta)\psi(\varepsilon\eta)\,d\eta=B_r\psi(0)+O(\varepsilon),\quad \varepsilon\to+0,$$

for all $\psi(x)\in\mathcal{D}(\mathbb{R})$, i.e., the second relation is proved. Since $\omega_\delta(\eta)\Omega^r(\eta)$ decreases sufficiently rapidly as $|\eta|\to\infty$, then following the same reasoning, we prove the third relation:

$$\Big\langle\frac{1}{\varepsilon}\omega_\delta\big(\frac{x}{\varepsilon}\big)\Big(\Omega\big(\frac{x}{\varepsilon}\big)\Big)^r,\psi(x)\Big\rangle=\int\omega_\delta(\eta)\Omega^r(\eta)\psi(\varepsilon\eta)\,d\eta$$
$$=A_r\psi(0)+O(\varepsilon),\quad \varepsilon\to+0,\quad \forall\ \psi(x)\in\mathcal{D}(\mathbb{R}),\quad r=1,2,\dots.$$

Lemma 6.2. ([5, Corollary 1.1.], [6]– [8]) *If* $f(u)$, $g(u)$ *are smooth functions, and* $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ *are defined by* (1.16), (4.2) *then*

$$f\big(u(x,t,\varepsilon)\big)=f(u_0)+\big[f(u)\big]H(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon),\quad \varepsilon\to+0,$$
$$\begin{aligned}v(x,t,\varepsilon)g\big(u(x,t,\varepsilon)\big)&=g(u_0)v_0+\big[g(u)v\big]H(-x+\phi(t))\\ &+\Big\{e(t)a(t)+R(t)c(t)\Big\}\delta(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon),\quad \varepsilon\to+0,\end{aligned}$$

where $a(t)$, $c(t)$ *are defined by* (4.5).

References

[1] F. Bouchut, *On zero pressure gas dynamics*, Advances in Math. for Appl. Sci., World Scientific, **22**, (1994), 171–190.

[2] V. G. Danilov, V. P. Maslov, V. M. Shelkovich, *Algebra of singularities of singular solutions to first-order quasilinear strictly hyperbolic systems*, Theor. Math. Phys., **114**, no. 1, (1998), 1–42.

[3] V. G. Danilov, G. A. Omel′yanov, V. M. Shelkovich, *Weak Asymptotics Method and Interaction of Nonlinear Waves*, in Mikhail Karasev (ed.), "Asymptotic Methods for Wave and Quantum Problems", Amer. Math. Soc. Transl., Ser. 2, **208**, 2003, 33–165.

[4] V. G. Danilov, V. M. Shelkovich, *Propagation and interaction of nonlinear waves to quasilinear equations*, Hyperbolic problems: Theory, Numerics, Applications (Eighth International Conference in Magdeburg, February/March 2000, v.I). International Series of Numerical Mathematics, v. 140, Birkhäuser Verlag Basel/Switzerland, 2001, 267–276.

[5] V. G. Danilov and V. M. Shelkovich, *Propagation and interaction of shock waves of quasilinear equation*, Nonlinear Studies, **8**, no. 1, (2001), 135–169.

[6] V. G. Danilov, V. M. Shelkovich, *Propagation and interaction of delta-shock waves of a hyperbolic system of conservation laws*, In Hou, Thomas Y.; Tadmor, Eitan (Eds.), Hyperbolic Problems: Theory, Numerics, Applications. Proceedings of the Ninth International Conference on Hyperbolic Problems held in CalTech, Pasadena, March 25-29, 2002, Springer Verlag, 2003, 483–492.

[7] V. G. Danilov, V. M. Shelkovich, *Delta-shock wave type solution of hyperbolic systems of conservation laws*, pp.27, Preprint 2003-052 at the url:
http://www.math.ntnu.no/conservation/2003/052.html

[8] V. G. Danilov, V. M. Shelkovich, *Dynamics of propagation and interaction of delta-shock waves in conservation law systems*, pp.40, Preprint 2003-068 at the url:
http://www.math.ntnu.no/conservation/2003/068.html (to be published in Journal of Differential Equations)

[9] V. G. Danilov, V. M. Shelkovich, *Propagation and interaction of δ-shock waves to hyperbolic systems of conservation laws*, Dokl. Ross. Akad. Nauk, **394**, no. 1, (2004), 10–14. English transl. in Russian Doklady Mathematics., **69**, no. 1, (2004).

[10] G. Ercole, *Delta-shock waves as self-similar viscosity limits*, Quart. Appl. Math., **LVIII**, no. 1, (2000), 177–199.

[11] Feiming Huang, *Existence and uniqueness of discontinuous solutions for a class nonstrictly hyperbolic systems*, In Chen, Gui-Qiang (ed.) et al. Advances in nonlinear partial differential equations and related areas. Proceeding of conf. dedicated to prof. Xiaqi Ding, China, 1997, 187–208.

[12] B. L. Keyfitz, *Conservation laws, delta-shocks and singular shocks*, In M. Grosser, G. Hormann and M. Oberguggenberger (Eds.), Nonlinear Theory of Generalized Functions, Chapman & Hall/CRC, 1999, 99–112.

[13] B. Lee Keyfitz and H. C. Kranzer, *Spaces of weighted measures for conservation laws with singular shock solutions*, Journal of Differential Equations, **118**, (1995), 420–451.

[14] Philippe Le Floch, *An existence and uniqueness result for two nonstrictly hyperbolic systems*, Nonlinear evolution equations that change type, Springer-Verlag, 1990, 126–138.

[15] Philippe Le Floch, *Entropy weak solutions to nonlinear hyperbolic systems under nonconservative form*, Commun. in partial differential equations, **13**, no. 6, (1988), 669–727.

[16] Philippe Le Floch, Tai-Ping Liu, *Existense theory for nonlinear hyperbolic systems in nonconservative form*, Forum Math., **5**, (1993), no. 3, 261–280.

[17] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Springer-Verlag New York, Berlin, Heidelberg, Tokyo, 1984.

[18] V. P. Maslov, *Three algebras corresponding to nonsmooth solutions of systems of quasilinear hyperbolic equations*, Uspekhi Mat. Nauk, **35**, (1980), no. 2, 252–253. (Russian).

[19] V. P. Maslov, *Non-standard characteristics in asymptotical problems*, In: Proceeding of the International Congress of Mathematicians, August 16-24, 1983, Warszawa, vol. I, Amsterdam–New York–Oxford: North-Holland, 1984, 139–185.

[20] G. Dal Maso, P. G. Le Floch, and F. Murat, *Definition and weak stability of nonconservative products*, J. Math. Pures Appl., **74**, (1995), 483–548.

[21] S. Schecter, *Existence of Dafermos profiles for singular shocks*, Preprint 2003-051 at the url:
http://www.math.ntnu.no/conservation/2003/051.html

[22] M. Sever, *Viscous structure of singular shocks*, Nonlinearity, **15**, (2002), 705–725.

[23] V. M. Shelkovich, *Delta-shock waves of a class of hyperbolic systems of conservation laws*, in A. Abramian, S. Vakulenko, V. Volpert (Eds.), "Patterns and Waves", St. Petersburg, 2003, 155–168.

[24] V. M. Shelkovich, *A specific hyperbolic system of conservation laws admitting delta-shock wave type solutions*, pp.29, Preprint 2003-059 at the url:
http://www.math.ntnu.no/conservation/2003/059.html

[25] Wancheg Shen, Tong Zhang, *The Riemann problem for the transportaion equations in gas dynamics*, Memoirs of the Amer. Math. Soc., **137**, no. 654, (1999), 1–77.

[26] Dechun Tan, Tong Zhang and Yuxi Zheng, *Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws*, Journal of Differential Equations, **112**, (1994), 1–32.

[27] A. I. Volpert, *The space BV and quasilinear equations*, Math. USSR Sb., **2**, (1967), 225–267.

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