STRUCTURAL PROPERTIES OF STRESS RELAXATION AND CONVERGENCE FROM VISCOELASTICITY TO POLYCONVEX ELASTODYNAMICS

CORRADO LATTANZIO AND ATHANASIOS E. TZAVARAS

ABSTRACT. We consider a model of stress relaxation approximating the equations of elastodynamics. Necessary and sufficient conditions are derived for the model to be equipped with a global free energy and to have positive entropy production, and the resulting class allows for both convex and non-convex equilibrium potentials. For convex equilibrium potentials, we prove a strong dissipation estimate and two relative energy estimates: for the relative entropy of the relaxation process and for the modulated relative energy. Both give convergence results to smooth solutions. For polyconvex equilibrium potentials, an augmenting of the system of polyconvex elastodynamics and the null-Lagrangian structure suggest an appropriate notion of relative energy. We prove convergence of viscosity approximations to polyconvex elastodynamics in the regime the limit solution remains smooth. A modulated relative energy is also obtained for the polyconvex case which shows stability of relaxation approximations.

CONTENTS

1. Introduction

A continuous medium with nonlinear elastic response is described by the system

$$\partial_t^2 y_i = \partial_\alpha T_{i\alpha}(\nabla_x y), \tag{1.1}$$

where $y : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$, $d = 2, 3$, describes the motion and T is the Piola–Kirchoff stress tensor. For hyperelastic materials T is generated by a stored energy function

$$T_{i\alpha}(F) = \frac{\partial W(F)}{\partial F_{i\alpha}}\,, \tag{s}$$

an assumption which is motivated by considerations of thermodynamics. The system of elastodynamics (1.1) is often recast as a system of conservation laws,

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha T_{i\alpha}(F)\,, \end{aligned} \tag{1.2}$$

for the velocity $v_i = \partial_t y_i$ and the deformation gradient $F_{i\alpha} = \partial_\alpha y_i$. The equivalence of (1.1) and (1.2) holds for solutions (v, F) with F a gradient, $F = \nabla y$, a property equivalent to the set of differential constraints

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0\,. \tag{1.3}$$

The constraints (1.3) are an involution [12]: if they are satisfied at $t = 0$ then $(1.2)_1$ propagates (1.3) to hold for all times.

The first objective of this article is to study the mechanical and mathematical ramifications of the approximation of (1.2) by a theory of stress relaxation,

$$
\begin{aligned}
\partial_t F_{i\alpha} &= \partial_\alpha v_i \\
\partial_t v_i &= \partial_\alpha S_{i\alpha} \\
\partial_t (S_{i\alpha} - f_{i\alpha}(F)) &= -\frac{1}{\epsilon}(S_{i\alpha} - T_{i\alpha}(F)).
\end{aligned}
\tag{1.4}
$$

This model may be visualized in the context of viscoelasticity with memory

$$
S = f(F) + \int_{-\infty}^{t} \frac{1}{\epsilon} e^{-\frac{1}{\epsilon}(t-\tau)} h(F(\cdot,\tau))\, d\tau
$$

with the equilibrium stress $T(F)$ decomposed into an elastic and viscoelastic contribution, $T(F) = f(F) + h(F)$, $f = \frac{\partial W_I}{\partial F}$ and $T = \frac{\partial W}{\partial F}$, and a kernel exhibiting a single relaxation time $\frac{1}{\epsilon}$. The approximation (1.4) is consistent with the second law of thermodynamics, provided the potential of the instantaneous elastic response W_I dominates the potential of the equilibrium response W. On the other hand, consistency with thermodynamics does not require convexity of W and, accordingly, we study the relaxation limit $\epsilon \to 0$ in two distinctive cases: (i) when W is convex, (ii) when W is polyconvex.

The second objective is to consider the viscosity approximation of (1.2)

$$
\begin{aligned}
\partial_t F_{i\alpha} &= \partial_\alpha v_i \\
\partial_t v_i &= \partial_\alpha T_{i\alpha}(F) + \epsilon \partial_\alpha \partial_\alpha v_i
\end{aligned}
\tag{1.5}
$$

for polyconvex equilibrium potentials, $W(F) = g(F, \operatorname{cof} F, \det F)$ with g convex, and to prove convergence of (1.5) to (1.2) so long as the solutions of (1.2) remain smooth. The notion of relative energy ([11], [13, Thm 5.2.1]) is an efficient tool for proving convergence when W is convex. Using the null-Lagrangian structure and an extension of polyconvex elastodynamics to a symmetric hyperbolic system [15], we show how to devise the appropriate notion of relative energy and prove convergence of (1.5) for W polyconvex.

A premiss of this work is to devise and compare various structural identities for relaxation systems, using (1.4) as a case study, and to pinpoint the notions of relative energy and *modulated relative energy* as efficient tools for proving convergence of relaxation approximations in the smooth regime. This is an alternative to the usual approach based on analysis of the linearized collision operator (*e.g.* [6], [23]).

We outline next the main results: The first task is to determine under which conditions the model (1.4) is endowed with the analog of the

H-theorem in kinetic theory of gases. This will yield the structure that in the theory of relaxation systems [7] is called an "entropy function". To this end, the model is embedded within a theory of stress-relaxation with thermal effects. This model is interpreted within the framework of thermo-mechanical theories with internal state variables [9], and consistency with thermodynamics is equivalent to the existence of a free energy function Ψ that exhibits entropy production. Conditions for the existence of the free energy function are well understood for processes taking values in a neighborhood of the equilibrium "Maxwellian" manifold [21, 24], but the issue becomes complex when the free energy and consistency with the second law is requested on the entire state space. In Section 2 we address this issue for the stress relaxation theory (2.12) and show that, for an instantaneous elastic response derived from a potential

$$f(F,\theta) = \frac{\partial W_I(F,\theta)}{\partial F} \, ,$$

a necessary and sufficient condition for the existence of a global free energy function is the function h describing the viscoelastic stresses to be a dissipative map,

$$\big(h(F_2,\theta) - h(F_1,\theta)\big) \cdot (F_2 - F_1) \le 0 \, , \quad \forall F_1, F_2 \, , \, \forall \theta \, ,$$

with inverse h^{-1} derived from a potential (see Proposition 2.2 and [22] for 1-d stress relaxation).

In Section 3, we consider isothermal stress-relaxation processes. Isothermal processes are determined by (1.4). Under conditions of consistency with the second law of thermodynamics,

$$\begin{gathered} T(F) = \frac{\partial W(F)}{\partial F} = f(F) + h(F) \\ f(F) = \frac{\partial W_I(F)}{\partial F} \, , \quad h(F) = -\frac{\partial W_v(F)}{\partial F} \, , \; W_v = W_I - W \text{ convex,} \end{gathered} \tag{H}$$

the mechanical energy of the stress-relaxation theory

$$\mathcal{E} = \frac{1}{2}|v|^2 + \Psi(F, S - f(F))$$

dissipates according to the H-theorem

$$\begin{aligned}\partial_t\left(\frac{1}{2}|v|^2+\Psi(F,S-f(F))\right)&-\partial_\alpha(v_i S_{i\alpha})\\&+\frac{1}{\epsilon}(F_{i\alpha}-h^{-1}_{i\alpha}(S-f(F)))(S_{i\alpha}-T_{i\alpha}(F))=0\,.\end{aligned}$$

The last term stands for the dissipation of viscoelastic stresses and is positive under (H).

Consistency with the second law of thermodynamics does not require convexity of the equilibrium stored energy W, but rather that the instantaneous elastic potential dominates the equilibrium potential, that is $W_I - W$ convex. (This is noted for the one-dimensional isothermal model in [16], and for relaxation theories in gas dynamics in [21].) The convexity of $W_I - W$ can be motivated by the Chapman-Enskog expansion of kinetic theory (see Section 3.1). Accordingly, we study the behavior of (1.4) as $\epsilon \to 0$ in two distinctive cases: (i) W convex and (ii) W polyconvex.

When the equilibrium potential W is convex the nonequilibrium free energy $\Psi = \Psi(F,A)$ is also convex (Proposition 3.1). In order to compare a smooth solution $(v,F,S-f(F))$ of the relaxation system (1.4) to a smooth solution $(\hat v,\hat F)$ of the equilibrium system (1.2), we consider the relative energy between the relaxing and the "equilibrium" solution,

$$\begin{aligned}\mathcal{E}_r:=\frac{1}{2}|v-\hat v|^2&+\Psi(F,S-f(F))-\Psi(\hat F,h(\hat F))\\&-\frac{\partial\Psi}{\partial F}(\hat F,h(\hat F))\cdot(F-\hat F)-\frac{\partial\Psi}{\partial A}(\hat F,h(\hat F))\cdot(S-f(F)-h(\hat F))\,.\end{aligned}$$

For Ψ uniformly convex, $\mathcal{E}_r$ is equivalent to the L^2-norm of the solution difference and satisfies the identity

$$\begin{aligned}\partial_t\mathcal{E}_r&-\partial_\alpha\Big((v_i-\hat v_i)(S_{i\alpha}-T_{i\alpha}(\hat F))\Big)\\&+\frac{1}{\epsilon}(F_{i\alpha}-h^{-1}_{i\alpha}(S-f(F)))(S_{i\alpha}-T_{i\alpha}(F))\\&\quad=(\partial_\alpha\hat v_i)\Big(S_{i\alpha}-T_{i\alpha}(\hat F)-\frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}(\hat F)(F_{j\beta}-\hat F_{j\beta})\Big)\,.\end{aligned}$$

This relative energy identity serves for proving stability and convergence of smooth solutions of the nonlinear viscoelastic model (1.4) towards smooth solutions of the elasticity equations (1.2) for convex equilibrium potentials (see Theorem 3.3 in Section 3.2).

In Section 4 we consider (1.4) with *linear* instantaneous elastic response, $f(F) = EF$, E constant. The resulting model is equivalent to regularization of (1.2) by a wave operator

$$\begin{aligned} \partial_t F_{i\alpha} - \partial_\alpha v_i &= 0 \\ \partial_t v_i - \partial_\alpha T_{i\alpha}(F) &= \epsilon E \partial_\alpha \partial_\alpha v_i - \epsilon \partial_t^2 v_i \,. \end{aligned} \tag{1.6}$$

The mechanical energy of (1.2) reads

$$\mathcal{E}_\infty = \frac{1}{2}|v|^2 + W(u)\,,$$

and it is easily seen that it does not dissipate along the relaxation dynamics. ($\mathcal{E}_\infty$ should not be confused with $\mathcal{E}$, it corresponds to the limit values of $\mathcal{E}$ along equilibria of the relaxation dynamics.) However, one may define the *modulated energy*

$$\mathcal{E}_m = \frac{1}{2}|v + \epsilon v_t|^2 + W(F) + \epsilon \partial_\alpha v_i T_{i\alpha}(F) + \epsilon^2 \frac{E}{2} \sum_{\alpha=1}^{3} |\partial_\alpha v|^2$$

amounting to a correction of $\mathcal{E}_\infty$ by higher order contributions of acoustic waves. Under the subcharacteristic condition

$$E > \nabla_F^2 W \tag{sc}$$

the modulated energy is positive definite and dissipates according to (4.5). The latter provides a stronger dissipation estimate than the H-theorem and generalizes in multi-d the analysis pursued in [22]. We also define the *modulated relative energy* between two smooth solutions (v, F) of (1.6) and $(\hat{v}, \hat{F})$ of (1.2) by

$$\begin{aligned} \mathcal{E}_{md} :=& \frac{1}{2}|v - \widehat{v} + \epsilon \partial_t (v - \widehat{v})|^2 + W(F) - W(\widehat{F}) - \frac{\partial W}{\partial F_{i\alpha}}(\widehat{F})\,(F_{i\alpha} - \widehat{F}_{i\alpha}) \\ &+ \frac{1}{2}\epsilon^2 E \sum_{\alpha=1}^{3} |\partial_\alpha (v - \widehat{v})|^2 + \epsilon \partial_\alpha (v_i - \widehat{v}_i)(T_{i\alpha}(F) - T_{i\alpha}(\widehat{F}))\,. \end{aligned}$$

Under conditions of uniform convexity and (sc), the quantity $\mathcal{E}_{md}$ satisfies the identity (4.11) in Lemma 4.3, and gives rise to a second stability and convergence framework for smooth solutions (see Theorem 4.4). The reader is referred to [22, 5, 1] for similar identities in other contexts.

Convexity of the elastic stored energy $W(F)$ is, in general, incompatible with the requirement of material frame indifference for the equations (1.2) (*e.g.* [8]), and various weaker notions have been proposed as replacements of convexity in the theory of elastostatics (see [3] for an updated survey). One such notion is that of polyconvexity [2], namely that W is written as a convex function of its minors,

$$W(F) = g(\Phi(F)) \,, \quad \Phi(F) := (F, \operatorname{cof} F, \det F) \,, \tag{pc}$$

with $g = g(F, Z, w) = g(\Xi)$ a convex function of $\Xi \in \mathbb{R}^{19}$. Due to some recently discovered [17] kinematic identities on the null-Lagrangians $\Phi(F)$,

$$\partial_t \Phi^A(F) = \partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \Big) \,,$$

the system of polyconvex elastodynamics has the striking property that, for F satisfying (1.3) it can be embedded into the enlarged system [15]

$$\begin{aligned} \partial_t \Xi^A &= \partial_\alpha \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} v_i \right) \\ \partial_t v_i &= \partial_\alpha \left(\frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \right) . \end{aligned} \tag{1.7}$$

The precise sense of the embedding is that for data (v_0, Ξ_0), with $\Xi_0 = \Phi(F_0)$ and $F_0 = \nabla y(\cdot, 0)$, the resulting solution (v, Ξ) has the properties $\Xi = \Phi(F)$, with $F = \nabla y$, and (v, F) solves (1.2). Moreover, system (1.7), (1.3) admits the entropy pair

$$\partial_t \Big(\frac{1}{2} |v|^2 + g(\Xi) \Big) - \partial_\alpha \Big(\sum_{i,A} v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \Big) = 0$$

and is thus symmetrizable. Similarly, solutions (v, F) of the viscosity approximation (1.5), (1.3) can be embedded to the enlarged system (5.10), while solutions of relaxation approximations by linear wave operator (1.6), (1.3) are embedded to the system (6.2). Besides the embedding to a hyperbolic system with a convex entropy, the extension of polyconvex elastodynamics has more appealing properties: (i) it is connected with a convex minimization algorithm with the kinematic constraints inducing constrained minimization problems [15], and (ii) it is respected by the natural approximations of viscosity or relaxation by wave operator.

In Section 5.2 we exploit the extended systems in order to prove a relative energy identity: If (v, F) is a solution of the viscosity approximation (1.5) and $(\widehat{v}, \widehat{F})$ a smooth solution of (1.2) then we show the relative energy calculation (5.11). The relative energy

$$H_r = \frac{1}{2}|v - \widehat{v}|^2 + g(\Phi(F)) - g(\Phi(\widehat{F})) - \frac{\partial g}{\partial \Xi^A}(\Phi(\widehat{F}))(\Phi(F)^A - \Phi(\widehat{F})^A)$$

and the associated flux

$$Q_r^\alpha = \left(\frac{\partial g}{\partial \Xi^A}(\Phi(F)) - \frac{\partial g}{\partial \Xi^A}(\Phi(\widehat{F})) \right) (v_i - \widehat{v}_i) \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}}$$

are the basis for proving convergence of zero-viscosity limits for polyconvex energies (Theorem 5.3) at least as long as the limit solution remains smooth. The relative energy computation (Lemma 5.2) uses in an essential way the null-Lagrangian structure (5.5) of $\Phi(F)$ and the extensions (1.7) and (5.10).

Other examples of systems that can be augmented to a symmetrizable system are the Born-Infeld system [4] and certain nonlinear models in electromagnetism [20]. C.M. Dafermos informed us that a connection of involutions in relative energy calculations is observed in [20], and that relative energy calculations can be performed without prior knowledge of the extended system under the framework of contingent conservation laws [14].

As already noted, the relaxation model can be a thermodynamically admissible without the equilibrium stored energy being necessarily convex (see [16], [21] and Section 2). In Section 6 we show that the relaxation model (1.4) provides a stable approximation for (1.2) for polyconvex stored energies (at least for smooth solutions). The notions of modulated energy and modulated relative energy are extended for polyconvex equilibrium potentials and we show that as long as a subcharacteristic condition is fulfilled the relaxation approximation is stable (Theorem 6.2) and converges to polyconvex elastodynamics in the smooth regime (Theorem 6.4). Finally, in the appendix we compare the dissipation structure provided by the H-theorem with the dissipation structure provided by the modulated energy. The two structures differ away from the equilibrium manifold, their Chapman-Enskog expansions agree near the equilibrium manifold up to order $O(\epsilon)$, and differ already at the order $O(\epsilon^2)$.

2. Free energy function

In this section we determine necessary and sufficient conditions for a model of viscoelastic stress-relaxation to be equipped with a global free energy. To this end, the relaxation model will be embedded in a framework of thermomechanical theories with internal state variables. The free energy provides a global entropy dissipation structure valid on the whole state-space and yields an entropy function for the stress relaxation process.

2.1. **Thermomechanical theories with internal variables.**

Thermomechanical theories with internal state variables are applicable in contexts where some physical quantities, which from a phenomenological standpoint determine the composition of the material, undergo relaxation towards equilibrium. They are used in diverse modeling contexts, including relaxation of internal energy, viscoelasticity, or models in plasticity. The consistency of internal variables' theories with the Clausius-Duhem inequality [9] provides conditions on constitutive relations so that the theory exhibits entropy dissipation. The resulting entropy dissipation inequality yields for these phenomenological theories the analog of the H–theorem.

In the context of internal variable theories the thermomechanical process is described by $(\chi(x,t), \theta(x,t), A(x,t))$, where

$$\chi : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3\,, \quad \theta : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}\,, \quad A : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^N$$

stand respectively for the mechanical motion, the temperature and the (possibly vector or tensor) field of internal state variables. As usual we introduce the velocity $v = \partial_t \chi$ and the deformation gradient $F = \nabla_x \chi$ connected through the kinematic compatibility relation

$$\partial_t F = \nabla v\,.$$

The internal variables A are assumed to evolve according to a differential law

$$\partial_t A = D(F, \theta, A) \tag{2.1}$$

generated by a "dissipative" vector field D, which is such that as $t \to \infty$ the dynamics of A stabilizes to an equilibrium response $A_\infty := h(F, \theta)$.

Naturally, $h(F,\theta)$ is an equilibrium for the dynamics in (2.1)

$$D(F,\theta,h(F,\theta)) = 0\,.$$

The thermomechanical process satisfies the balance laws of mass momentum and energy, which in Lagrangian coordinates take the form

$$\begin{aligned}\partial_t \rho_0 &= 0\\ \rho_0 \partial_t v &= \operatorname{Div} S + \rho_0 b\\ \rho_0 \partial_t \left(e + \frac{1}{2}|v|^2 \right) &= \operatorname{Div}(v\cdot S + Q) + \rho_0 v \cdot b + \rho_0 r\,.\end{aligned}$$

The balance of angular momentum reduces to the relation $SF^T = FS^T$ which is usually viewed as a constraint on constitutive relations. As usual, ρ_0 is the referential density, S is the Piola–Kirchoff stress tensor, b the body force per unit mass, e the internal energy, Q the (referential) heat flux vector and r the radiating heat density per unit mass. Also, η will denote the specific entropy and $\psi := e - \theta\eta$ the Helmholtz free energy. For homogeneous materials ρ_0 is constant and the balance of mass is automatically satisfied.

It is a premiss of continuum thermomechanics that the process should comply with the second law of thermodynamics, in the form of the Clausius–Duhem inequality

$$\rho_0 \partial_t \eta \geq \operatorname{Div} \frac{Q}{\theta} + \rho_0 \frac{r}{\theta}\,. \tag{2.2}$$

For smooth processes (2.2) is viewed as constraining the format of the constitutive relations and through a standard procedure (see [9] or [22]) restricts the form of constitutive functions.

In the case of theories with internal variables the outline is as follows: one starts with general functional dependences relating the dependent variables ψ, S, η and Q to the prime variables F, θ, A and $g := \nabla\theta$, the temperature gradient. The fields b, r are viewed as externally supplied and the requirement of compatibility with the Clausius-Duhem inequality is re-expressed in the format: any smooth thermomechanical process (χ,θ,A) consistent with (2.1) must satisfy the energy dissipation inequality

$$\rho_0 \partial_t \psi + \rho_0 \eta \partial_t \theta - \operatorname{tr} S \partial_t F^T - \frac{1}{\theta} Q \cdot \nabla\theta \leq 0\,.$$

Using appropriate test processes it turns out the constitutive functions are of the form

$$
\begin{aligned}
\psi &= \Psi(F,\theta,A) \\
S &= \rho_0 \frac{\partial \Psi}{\partial F}(F,\theta,A) \\
\eta &= -\frac{\partial \Psi}{\partial \theta}(F,\theta,A) \\
Q &= Q(F,\theta,A,g)
\end{aligned}
\tag{2.3}
$$

and should comply with the inequality

$$
-\rho_0 \frac{\partial \Psi}{\partial A}(F,\theta,A)\cdot D(F,\theta,A) + \frac{1}{\theta} Q(F,\theta,A,g)\cdot g \geq 0\,, \quad \forall F\,,\theta\,,A\,. \tag{2.4}
$$

To proceed further one needs to identify the free energy Ψ by solving a system of a differential equation and a differential inequality. Typically, this system can be integrated in a neighborhood of the equilibrium manifold, and this issue is systematically analyzed in [21]. The identification of the free energy becomes a complex problem when we request that the free energy is defined on the whole state space. This is achieved in various specific models of relaxation of internal energy [10, 22, 24] or for certain (mostly one-dimensional) models of stress relaxation in viscoelasticity [16, 22] or in relaxation for gas dynamics [18]. In the following section we derive necessary and sufficient conditions for a theory of stress-relaxation with thermal effects to admit a globally defined free energy.

2.2. **A theory of stress relaxation with thermal effects.** We next consider a theory with internal variables where A is a tensor describing viscoelastic stresses, and S, A satisfy

$$
\begin{aligned}
S &= f(F,\theta) + A \\
\partial_t A &= -\frac{1}{\epsilon}\big(A - h(F,\theta)\big)\,.
\end{aligned}
\tag{2.5}
$$

In addition, the heat flux is postulated to be Fourier heat conduction

$$
Q(F,\theta,g,A) = k(F,\theta,A) g
$$

with conductivity $k \geq 0$.

In this theory, an additive decomposition of the stress tensor S is postulated into an instantaneous elastic part $f(F,\theta)$ and a viscoelastic part A

undergoing stress relaxation with relaxation time $\frac{1}{\epsilon}$, $\epsilon > 0$. The long-time response of the material is determined by the equilibrium stress $T(F,\theta)$ which is accordingly decomposed,

$$T(F,\theta) = f(F,\theta) + h(F,\theta)\,,$$

into the elastic and the viscoelastic stress contribution. Alternatively, the constitutive theory for the stress (2.5) can also be expressed in the form,

$$S = f(F,\theta) + \int_{-\infty}^{t} \frac{1}{\epsilon} e^{-\frac{1}{\epsilon}(t-\tau)} h(F(\cdot,\tau),\theta(\cdot,\tau))\,d\tau\,,$$

of a theory with fading memory with kernel $k = \frac{1}{\epsilon}e^{-\frac{1}{\epsilon}s}$ comprising of only one relaxation time.

In view of (2.3), (2.4), the theory of stress relaxation is compatible with the Clausius-Duhem inequality, if there is a globally defined free energy solving the problem:

$$\rho_0 \frac{\partial \Psi(F,\theta,A)}{\partial F} = f(F,\theta) + A \tag{2.6}$$

$$\rho_0 \frac{\partial \Psi(F,\theta,A)}{\partial A} \cdot (A - h(F,\theta)) \geq 0\,. \tag{2.7}$$

(Given two tensors F, G we use the notation $F \cdot G = F_{i\alpha}G_{i\alpha} = \mathrm{tr}(FG^T)$ with summation convention over repeated indices.)

Henceforth, we assume that $f(0,\theta) = h(0,\theta) = 0$ and that the instantaneous elastic response derives from a potential

$$f(F,\theta) = \rho_0 \frac{\partial W_I(F,\theta)}{\partial F}\,. \tag{h$_1$}$$

Integration of (2.6) yields

$$\rho_0 \Psi(F,\theta,A) = \rho_0 W_I(F,\theta) + A \cdot F + G(A,\theta)\,, \tag{2.8}$$

and to fulfill (2.7) the integrating factor $G(A,\theta)$ has to be selected so that $j(A,\theta) = -\nabla_A G(A,\theta)$ satisfies

$$(F - j(A,\theta)) \cdot (A - h(F,\theta)) \geq 0\,, \quad \forall F,\theta,A\,. \tag{2.9}$$

Note that θ is a parameter in the above inequality. We establish necessary and sufficient conditions for the solution of (2.9). The solvability is related to the theory of monotone maps (*e.g.* [19]).

Lemma 2.1. *Let h, j be continuous maps. The maps h, j satisfy*

$$(F - j(A)) \cdot (A - h(F)) \geq 0\,, \quad \forall A\,, F\,, \tag{2.10}$$

if and only if h is invertible, j is invertible, $h = j^{-1}$ and h, j are both dissipative.

Proof. Fix $\hat{A} = h(\hat{F})$ and consider $A = \hat{A} + te_{i\alpha}$ where $t \in \mathbb{R}$ and $e_{i\alpha} = e_i \otimes E_\alpha$ the standard basis in the space of tensors. Then,

$$(\hat{F} - j(\hat{A} + te_{i\alpha})) \cdot (\hat{A} + te_{i\alpha} - h(\hat{F})) \geq 0\,, \quad \forall t \in \mathbb{R}\,.$$

We conclude $(\hat{F} - j(\hat{A})) \cdot e_{i\alpha} = 0$ and thus $\hat{F} = j(\hat{A})$. In a similar fashion, if $\hat{F} = j(\hat{A})$ then we apply (2.10) to the test function $F = \hat{F} + te_{i\alpha}$, $t \in \mathbb{R}$ and conclude $\hat{A} = h(\hat{F})$.

In summary, $\hat{A} = h(\hat{F})$ if and only if $\hat{F} = j(\hat{A})$, and thus h, j are invertible with $j = h^{-1}$. Applying once more (2.10) between the pairs F_2 and $F_1 = j(A_1)$ we deduce that

$$(h(F_2) - h(F_1)) \cdot (F_2 - F_1) \leq 0\,, \quad \forall F_1, F_2\,,$$

that is h is a dissipative map. The same statement shows that j is a dissipative map as well. The converse is obvious. □

Lemma 2.1 provides a characterization of stress relaxation models that are (globally) compatible with the Clausius-Duhem inequality.

Proposition 2.2. *Let f, h be smooth maps satisfying $f(0,\theta) = h(0,\theta) = 0$ and hypothesis (h_1). There exists a global free energy $\Psi(F,\theta,A)$ satisfying (2.6) and (2.7) if and only if $h(\cdot,\theta)$ is a dissipative map with its inverse $j = h^{-1}$ a gradient. If h in addition satisfies that $\nabla_F h$ is invertible then h is the negative gradient of a convex function $W_v(F,\theta)$,*

$$h(F,\theta) = -\rho_0 \frac{\partial W_v(F,\theta)}{\partial F}\,. \tag{h$_2$}$$

The free energy is defined by (2.8) with $G(\cdot,\theta)$ a convex function such that $\nabla_A G = -h^{-1}$.

Proof. Suppose there is a free energy Ψ solution of (2.6), (2.7). Then Ψ has the form (2.8) and $j = -\nabla_A G$ satisfies (2.9). Lemma 2.1 then implies that $h = j^{-1}$ is a dissipative map. The converse also follows from the same lemma.

The maps h and h^{-1} are connected by $h \circ h^{-1} = id$. If $\nabla_F h$ is invertible then

$$\nabla_A (h^{-1}) = (\nabla_F h)^{-1} \ .$$

Since h^{-1} is the gradient of a concave function then $\nabla_A(h^{-1})$ is symmetric. Therefore $\nabla_F h$ is also symmetric and there is a convex function $W_v(F, \theta)$ generating h via (h_2). □

Remark 2.3. It is instructive to summarize the results in terms of stored energy functions. When f is generated by a stored energy via (h_1) and h is generated by a stored energy via (h_2), the equilibrium stress is

$$T(F;\theta) = f(F;\theta) + h(F;\theta) = \rho_0 \frac{\partial W(F,\theta)}{\partial F} \,, \quad \text{with } W = W_I - W_v \,.$$

The compatibility of the model with the Clausius-Duhem inequality requires that $W_v = W_I - W$ is convex,

$$-\nabla_F h(F;\theta) = \rho_0 \nabla_F^2 W_v(F,\theta) = \rho_0 \nabla_F^2 (W_I(F,\theta) - W(F,\theta)) > 0 \,. \tag{2.11}$$

In summary, we have considered a model of viscoelastic stress relaxation (with thermal effects)

$$\begin{cases} \partial_t F_{i\alpha} = \partial_\alpha v_i \\ \rho_0 \partial_t v_i = \partial_\alpha S_{i\alpha} + \rho_0 b_i \\ \rho_0 \partial_t \left(e + \frac{1}{2}|v|^2 \right) = \partial_\alpha (v_i S_{i\alpha} + k \partial_\alpha \theta) + \rho_0 v_i b_i + \rho_0 r \\ \partial_t (S_{i\alpha} - f_{i\alpha}(F,\theta)) = -\frac{1}{\epsilon}(S_{i\alpha} - T_{i\alpha}(F,\theta)) \end{cases} \tag{2.12}$$

and derived in Proposition 2.2 conditions for global compatibility with the Clausius-Duhem inequality. Under such conditions the model is equipped with the entropy dissipation identity

$$\begin{aligned} &\rho_0 \partial_t \eta - \partial_\alpha \left(k \frac{1}{\theta} \partial_\alpha \theta \right) \\ &\qquad = \frac{1}{\epsilon} \rho_0 \frac{1}{\theta} \frac{\partial \Psi}{\partial A_{i\alpha}} \left(A_{i\alpha} - h_{i\alpha}(F,\theta) \right) + k \frac{|\nabla \theta|^2}{\theta^2} + \rho_0 \frac{r}{\theta} \,. \end{aligned} \tag{2.13}$$

Under the framework of Proposition 2.2 the first term on the right of (2.13) captures the dissipation due to the viscoelastic stresses and is positive.

3. Isothermal stress relaxation and the H-theorem

Next, the case of isothermal stress relaxation is considered. From a mechanics viewpoint, isothermal conditions are achieved by regulating the radiation heat supply r so as to keep the temperature constant, $\theta = \theta_0$, and accordingly the heat flux $Q = 0$. The energy equation is automatically satisfied and the equations determining the isothermal process account solely for mechanical effects:

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha S_{i\alpha} \\ \partial_t (S_{i\alpha} - f_{i\alpha}(F)) &= -\frac{1}{\epsilon}(S_{i\alpha} - T_{i\alpha}(F)) \end{aligned} \tag{3.1}$$

where for simplicity we took $\rho_0 = 1$ and $b = 0$.

3.1. **H-theorem.** Henceforth, we work under the framework

$$\begin{aligned} T(F) = \frac{\partial W(F)}{\partial F} &= f(F) + h(F)\,, \\ \text{where} \quad f(F) &= \frac{\partial W_I(F)}{\partial F}\,, \quad h(F) = -\frac{\partial W_v(F)}{\partial F}\,, \\ \text{and } W_v &= W_I - W \text{ is convex.} \end{aligned} \tag{a}$$

The implications of (a) are outlined in Proposition 2.2 and Remark 2.3. The isothermal model is compatible with the Clausius-Duhem inequality, and the entropy dissipation inequality (2.13) together with the energy balance law $(2.12)_3$ give (by eliminating r) an equation for the dissipation of the mechanical energy. This reads:

$$\begin{aligned} \partial_t \left(\frac{1}{2}|v|^2 + \Psi(F, S - f(F)) \right) &- \partial_\alpha (v_i S_{i\alpha}) \\ + \frac{1}{\epsilon}(F_{i\alpha} - h^{-1}_{i\alpha}(S - f(F)))&(S_{i\alpha} - T_{i\alpha}(F)) = 0\,. \end{aligned} \tag{3.2}$$

Note that, under (a), the last term expresses the dissipation arising from the viscoelastic stresses and is positive.

Equation (3.2) provides for the relaxation model (3.1) an analog of the H-theorem, and yields what is called in the literature of relaxation systems an "entropy" function [7]. It is worth emphasizing that the existence of the free-energy functional $\Psi(F, A)$ does not require any convexity assumption on the equilibrium potential $W(F)$, but rather that the equilibrium potential is *dominated* by the instantaneous potential (see (2.11)).

Condition (a) should be compared to the classical *subcharacteristic condition* which is typically needed to control hyperbolic–hyperbolic relaxation limits (*e.g.* [7]). Another perspective to (a) is provided by the classical Chapman–Enskog expansion for the relaxation limit of (3.1). Indeed, if we set

$$S_{i\alpha} = S^0_{i\alpha} + \epsilon S^1_{i\alpha} + O(\epsilon^2)$$

in (3.1), the Chapman–Enskog procedure leads to the expression for $S_{i\alpha}$:

$$S_{i\alpha} = T_{i\alpha}(F) - \epsilon \left(\frac{\partial^2 W(F)}{\partial F_{i\alpha} \partial F_{j\beta}} - \frac{\partial^2 W_I(F)}{\partial F_{i\alpha} \partial F_{j\beta}} \right) \partial_\beta v_j + O(\epsilon^2)\,.$$

Thus, the first order correction of system (3.1) is given by

$$\begin{cases} \partial_t F_{i\alpha} = \partial_\alpha v_i \\ \partial_t v_i = \partial_\alpha T_{i\alpha}(F) + \epsilon \partial_\alpha \left(\left(\frac{\partial^2 W_I(F)}{\partial F_{i\alpha} \partial F_{j\beta}} - \frac{\partial^2 W(F)}{\partial F_{i\alpha} \partial F_{j\beta}} \right) \partial_\beta v_j \right), \end{cases}$$

which is a dissipative approximation if and only if (a) holds.

Next, conditions for convexity of the free energy function are derived (*cf.* [16], [22] for 1-d cases).

Proposition 3.1. *(i) If a function $\Psi(F, A)$ of the form*

$$\Psi(F, A) = W_I(F) + A_{i\alpha} F_{i\alpha} + G(A) \tag{3.3}$$

is convex $\forall (F, A)$, then the functions $W_I(F)$ and $G(A)$ are uniformly convex.

(ii) Let $h(F)$ be globally invertible. Assume that, for some constants δ_1, $\delta_2 > 0$, we have

$$\nabla_F f(F) = \nabla^2_F W_I(F) \geq \delta_1 I > 0 \tag{3.4}$$

and

$$\delta_2 I \geq -\nabla_F h(F) = \nabla^2_F (W_I - W)(F) > 0\,. \tag{3.5}$$

If $\delta_1 \geq \delta_2$ then the function $\Psi(F, A)$ defined in (3.3), with $G(A)$ given by $\nabla_A G(A) = -h^{-1}(A)$, is convex in (F, A). If $\delta_1 > \delta_2$ then $\Psi(F, A)$ is uniformly convex.

Remark 3.2. The conditions in Proposition 3.1 involve the convexity of W_I and (in an indirect way) the convexity of W. Indeed, observe that $\delta_1 >$

δ_2 implies that $\nabla_F^2 W_I > -\nabla_F h = \nabla_F^2(W_I - W)$, that is the equilibrium potential W is convex, $\nabla_F^2 W > 0$.

Proof of Proposition 3.1. We use the notation $H_\Psi(F,A)[(H,K),(H,K)]$ for the quadratic form associated with the Hessian of Ψ at the point (F,A) evaluated along (H,K); similarly $H_{W_I}(F)[H,H]$ is the quadratic form associated with Hessian of W_I at F along H. We then have

$$H_\Psi(F,A)[(H,K),(H,K)] = H_{W_I}(F)[H,H] + 2H \cdot K + H_G(A)[K,K]\,.$$

To show (i) we proceed by contradiction. Assume there exist sequences F_n and H_n with $|H_n| = 1$ such that

$$H_{W_I}(F_n)[H_n,H_n] \to 0\,, \qquad\qquad \text{as } n \to +\infty\,,$$

Then, for a fixed $\hat{A}$,

$$\begin{aligned} &H_\Psi(F_n,\hat{A})[(\lambda H_n,H_n),(\lambda H_n,H_n)] \\ &\quad = \lambda^2 H_{W_I}(F_n)[H_n,H_n] + 2\lambda|H_n|^2 + H_G(\hat{A})[H_n,H_n] \\ &\quad \le \lambda^2 H_{W_I}(F_n)[H_n,H_n] + (2\lambda + M)|H_n|^2\,, \end{aligned} \tag{3.6}$$

where $M > 0$ is a fixed constant depending on $\hat{A}$, and by choosing $2\lambda < -M$ and n sufficiently large in (3.6), we have $H_\Psi(F_n,\hat{A})[(\lambda H_n,H_n),(\lambda H_n,H_n)] < 0$, which is in contradiction with the convexity of $\Psi(F,A)$. Clearly the same argument applies to the function $G(A)$.

Let now (3.4), (3.5) hold with $\frac{\delta_1}{\delta_2} \ge 1$. Then

$$\nabla_A^2 G(A) \ge \frac{1}{\delta_2} I$$

and thus, for all $\delta > 0$,

$$\begin{aligned} H_\Psi(F,A)[(H,K),(H,K)] &\ge \delta_1|H|^2 + 2H \cdot K + \frac{1}{\delta_2}|K|^2 \\ &\ge \left(\delta_1 - \frac{1}{\delta}\right)|H|^2 + \left(\frac{1}{\delta_2} - \delta\right)|K|^2\,. \end{aligned}$$

If $\frac{\delta_1}{\delta_2} > 1$, we select δ such that $\frac{1}{\delta_2} > \delta > \frac{1}{\delta_1}$ and deduce uniform convexity. In the limit case $\frac{\delta_1}{\delta_2} = 1$ we only deduce convexity. $\square$

Under (3.4) and (3.5) the function Ψ is uniformly convex, and we may assume without loss of generality $\Psi(0,0) = \nabla_{(F,A)}\Psi(0,0) = 0$. Then (3.2) provides an L^2 estimate and control of the distance from the equilibrium manifold, namely

$$\begin{aligned}\int_{\mathbb{R}^3} \big(|v|^2 + |F|^2 + |S - f(F)|^2\big)\, dx + \frac{1}{\epsilon}\int_0^{+\infty}\int_{\mathbb{R}^3} |S - T(F)|^2 dxdt \\ \le C\int_{\mathbb{R}^3} \big(|v|^2 + |F|^2 + |S - f(F)|^2\big)\Big|_{t=0} dx\,. \end{aligned} \tag{3.7}$$

3.2. **Relative energy.** Let (v, F, S) be a smooth solution of (3.1) and $(\hat{v}, \hat{F})$ be a smooth solution of the elasticity system

$$\begin{aligned} \partial_t \hat{F}_{i\alpha} &= \partial_\alpha \hat{v}_i \\ \partial_t \hat{v}_i &= \partial_\alpha T_{i\alpha}(\hat{F})\,. \end{aligned} \tag{3.8}$$

The notion of relative energy [11] is used in order to establish stability of classical solutions in systems of conservation laws or convergence of viscosity approximations.

We show here that guided by the appropriate thermodynamics framework, this idea can be adapted in the context of relaxation approximations. For the relaxation system (3.1), define the relative energy $\mathcal{E}_r(v, F, A; \hat{v}, \hat{F}, h(\hat{F}))$ generated by the mechanical energy (of the isothermal relaxation model) relative to an equilibrium,

$$\begin{aligned}\mathcal{E}_r := \frac{1}{2}|v - \hat{v}|^2 + \Psi(F, S - f(F)) - \Psi(\hat{F}, h(\hat{F})) \\ - \frac{\partial \Psi}{\partial F}(\hat{F}, h(\hat{F}))\cdot(F - \hat{F}) - \frac{\partial \Psi}{\partial A}(\hat{F}, h(\hat{F}))\cdot(S - f(F) - h(\hat{F}))\,.\end{aligned}$$

From the thermodynamic relations in Section 2 we have

$$\begin{aligned}\frac{\partial \Psi}{\partial F}(F, h(F)) &= \frac{\partial W_I}{\partial F} + h(F) = \frac{\partial W}{\partial F}\,, \\ \frac{\partial \Psi}{\partial A}(F, h(F)) &= F + \nabla_A G(h(F)) = 0\end{aligned}$$

and (by selecting an appropriate normalization) $\Psi(F, h(F)) = W(F)$. The relative entropy then reads

$$\mathcal{E}_r = \frac{1}{2}|v - \hat{v}|^2 + \Psi(F, S - f(F)) - W(\hat{F}) - \frac{\partial W}{\partial F}(\hat{F})\cdot(F - \hat{F})\,, \tag{3.9}$$

while the associated relative fluxes turn out of the form

$$\mathcal{F}_r^\alpha = (v_i - \hat{v}_i)(S_{i\alpha} - T_{i\alpha}(\hat{F}))\,.$$

The relative energy computation is performed as follows: observe that (v, F, S) satisfies (3.2) and that $(\hat{v}, \hat{F})$ being smooth satisfies the energy identity

$$\partial_t \frac{1}{2}\Big(|\hat{v}|^2 + W(\hat{F})\Big) - \partial_\alpha\Big(\hat{v}_i T_{i\alpha}(\hat{F})\Big) = 0\,. \tag{3.10}$$

From

$$\begin{aligned} \partial_t(F_{i\alpha} - \hat{F}_{i\alpha}) &= \partial_\alpha(v_i - \hat{v}_i) \\ \partial_t(v_i - \hat{v}_i) &= \partial_\alpha(S_{i\alpha} - T_{i\alpha}(\hat{F})) \end{aligned}$$

and (3.8) we derive the identity

$$\begin{aligned} &\partial_t\Big(\frac{\partial W}{\partial F_{i\alpha}}(\hat{F})(F_{i\alpha} - \hat{F}_{i\alpha}) + \hat{v}_i(v_i - \hat{v}_i)\Big) \\ &\qquad - \partial_\alpha\Big(T_{i\alpha}(\hat{F})(v_i - \hat{v}_i) + \hat{v}_i(S_{i\alpha} - T_{i\alpha}(\hat{F}))\Big) \\ &\quad = \partial_t\Big(\frac{\partial W}{\partial F_{i\alpha}}(\hat{F})\Big)(F_{i\alpha} - \hat{F}_{i\alpha}) + (\partial_t\hat{v}_i)(v_i - \hat{v}_i) \\ &\qquad - (\partial_\alpha T_{i\alpha}(\hat{F}))(v_i - \hat{v}_i) - (\partial_\alpha\hat{v}_i)(S_{i\alpha} - T_{i\alpha}(\hat{F})) \\ &\quad = -(\partial_\alpha\hat{v}_i)\Big(S_{i\alpha} - T_{i\alpha}(\hat{F}) - \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}(\hat{F})(F_{j\beta} - \hat{F}_{j\beta})\Big)\,. \end{aligned} \tag{3.11}$$

Then combining (3.2), (3.10) and (3.11) we deduce

$$\begin{aligned} &\partial_t\mathcal{E}_r - \partial_\alpha\Big((v_i - \hat{v}_i)(S_{i\alpha} - T_{i\alpha}(\hat{F}))\Big) \\ &\quad + \frac{1}{\epsilon}(F_{i\alpha} - h^{-1}_{i\alpha}(S - f(F)))(S_{i\alpha} - T_{i\alpha}(F)) \\ &\quad = (\partial_\alpha\hat{v}_i)\Big(S_{i\alpha} - T_{i\alpha}(\hat{F}) - \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}(\hat{F})(F_{j\beta} - \hat{F}_{j\beta})\Big)\,. \end{aligned} \tag{3.12}$$

The relative energy identity can be used to obtain stability and convergence of the relaxation system (3.1) as long as the solution of (1.2) remains smooth.

Theorem 3.3. *Assume that W_I, W satisfy for some $\gamma_I > \gamma_v > 0$ and $M > 0$ hypotheses* (a),

$$\nabla_F^2 W_I(F) \geq \gamma_I I > \gamma_v I \geq \nabla_F^2(W_I - W)(F) > 0\,, \tag{b}$$

$$|\nabla_F^2 W_I(F)| \leq M\,. \quad |\nabla^3 W(F)| \leq M\,, \quad \forall F\,. \tag{c}$$

Let $(v^\epsilon, F^\epsilon, S^\epsilon)$ be smooth solutions of (3.1) and $(\widehat{v}, \widehat{F})$ be a smooth solution of (1.2) defined on $\mathbb{R}^3 \times [0,T]$ and emanating from smooth data $(v_0^\epsilon, F_0^\epsilon, S_0^\epsilon)$ and $(\widehat{v}_0, \widehat{F}_0)$. Then the relative energy $\mathcal{E}_r$ defined in (3.9) satisfies (3.12), and for $R > 0$ there exist constants s and $C = C(R, T, \gamma_I, \gamma_v, M, \nabla\widehat{v}) > 0$

independent of ϵ such that

$$\int_{|x|<R} \mathcal{E}_r(x,t)dx \le C\left(\int_{|x|<R+st} \mathcal{E}_r(x,0)dx + \epsilon\right).$$

In particular, if the data satisfy

$$\int_{|x|<R+sT} \mathcal{E}_r(x,0)dx \longrightarrow 0\,, \quad \text{as } \epsilon \downarrow 0\,,$$

then

$$\sup_{t\in[0,T]} \int_{|x|<R} \left(|v^\epsilon - \widehat{v}|^2 + |F^\epsilon - \widehat{F}|^2 + |A^\epsilon - h(\widehat{F})|^2\right) dx \longrightarrow 0\,.$$

Proof. Fix $R > 0$, $t \in [0,T)$ and consider the cone

$$\mathcal{C}_t = \{(x,\tau) : 0 < \tau < t,\ \ |x| < R + s(t-\tau)\}$$

where s is a constant to be selected. The aim is to monitor the quantity

$$\varphi(\tau) = \int_{|x|<R+s(t-\tau)} \mathcal{E}_r(x,\tau)dx\,, \quad 0 \le \tau \le t\,.$$

Proposition 3.1 and (b) imply that $\Psi(F,A)$ is uniformly convex and thus for some $c = c(\gamma_I, \gamma_v) > 0$

$$\mathcal{E}_r \ge c\big(|v-\hat{v}|^2 + |F-\hat{F}|^2 + |A-h(\hat{F})|^2\big)\,.$$

Proposition 2.2, together with (a) and (b), imply $\nabla_A G = -h^{-1}$,

$$\nabla_A^2 G(A) = \big(-\nabla_F h\big)^{-1} = \big(\nabla_F^2 (W_I - W)\big)^{-1} \ge \frac{1}{\gamma_v} I$$

and

$$\begin{aligned} D &:= (F - h^{-1}(S - f(F)))\cdot(S - T(F)) \\ &= (\nabla_A G(A) - \nabla_A G(h(F)))\cdot(A - h(F)) \\ &\ge \frac{1}{\gamma_v}|A - h(F)|^2 \\ &= \frac{1}{\gamma_v}|S - T(F)|^2\,. \end{aligned} \tag{3.13}$$

Observe next that by (a) and (c)

$$\begin{aligned} \sum_\alpha |\mathcal{F}_r^\alpha|^2 &= \sum_\alpha \Big|\sum_i (v_i - \hat{v}_i)(S_{i\alpha} - T_{i\alpha}(\hat{F}))\Big|^2 \\ &\le |v-\widehat{v}|^2 |S - T(\widehat{F})|^2 \\ &\le |v-\widehat{v}|^2 (|A - h(\widehat{F})| + |f(F) - f(\widehat{F})|)^2 \\ &\le C\big(|v-\hat{v}|^2 + |F-\hat{F}|^2 + |A-h(\hat{F})|^2\big)^2 \end{aligned}$$

and thus we can select s so that

$$\Big(\sum_\alpha |\mathcal{F}_r^\alpha|^2 \Big)^{1/2} \le s \mathcal{E}_r \,. \tag{3.14}$$

Consider now the identity (3.12),

$$\begin{aligned} \partial_t \mathcal{E}_r - \partial_\alpha \mathcal{F}_r^\alpha + \frac{1}{\epsilon} D = Q \,, \\ Q = (\partial_\alpha \hat{v}_i) \Big(S_{i\alpha} - T_{i\alpha}(\hat{F}) - \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}}(\hat{F})(F_{j\beta} - \hat{F}_{j\beta}) \Big) \,, \end{aligned}$$

in the weak form

$$\begin{aligned} - \iint \Big(\mathcal{E}_r \partial_t \phi - \mathcal{F}_r^\alpha \partial_\alpha \phi - \frac{1}{\epsilon} \phi D \Big) dx d\tau \\ - \int \mathcal{E}_r(x,0) \phi(x,0) dx = \iint \phi Q dx d\tau \end{aligned} \tag{3.15}$$

with ϕ Lipschitz continuous with compact support in $\mathbb{R}^d \times [0,T)$. Following [13, Thm 5.2.1], with R, $t \in [0,T)$, s fixed as precised above, and $\delta > 0$ such that $t + \delta < T$, we select the test function $\phi(x,\tau) = \theta(\tau)\psi(x,\tau)$ where

$$\theta(\tau) = \begin{cases} 1 & 0 \le \tau < t \\ 1 - \frac{1}{\delta}(\tau - t) & t \le \tau \le t + \delta \\ 0 & t + \delta \le \tau \,, \end{cases}$$

$$\psi(x,\tau) = \begin{cases} 1 & \tau > 0 \,, \ |x| - R - s(t-\tau) < 0 \\ 1 - \frac{1}{\delta}\big(|x| - R - s(t-\tau)\big) & \tau > 0 \,, \ 0 < |x| - s(t-\tau) - R < \delta \\ 0 & \tau > 0 \,, \ \delta < |x| - R - s(t-\tau) \,. \end{cases}$$

Then (3.15) gives

$$\begin{aligned} & \frac{1}{\delta} \int_t^{t+\delta} \int_{|x|<R} \mathcal{E}_r dx d\tau + \frac{1}{\delta} \int_0^t \int_{0<|x|-R-s(t-\tau)<\delta} (s\mathcal{E}_r - \sum_\alpha \frac{x_\alpha}{|x|} \mathcal{F}_r^\alpha) dx d\tau \\ & \qquad + \frac{1}{\epsilon} \int_0^t \int_{|x|<R+s(t-\tau)} D dx d\tau + O(\delta) \\ & = \int_{|x|<R+st} \mathcal{E}_r(x,0) dx + \int_0^t \int_{|x|<R+s(t-\tau)} Q dx d\tau \,. \end{aligned} \tag{3.16}$$

By (3.14) the second term in (3.16) is positive. Letting $\delta \to 0$ and using (3.13), we have

$$\begin{aligned} \int_{|x|<R} \mathcal{E}_r(x,t) dx + \frac{c}{\epsilon} \iint_{\mathcal{C}_t} |S - T(F)|^2 dx d\tau \\ \le \int_{|x|<R+st} \mathcal{E}_r(x,0) dx + \iint_{\mathcal{C}_t} |Q| dx d\tau \,. \end{aligned} \tag{3.17}$$

In order to handle the right hand side of (3.17) we use the bounds

$$\int_{\mathcal{C}_t} \big|\partial_\alpha \hat{v}_i \big(T_{i\alpha}(F) - T_{i\alpha}(\widehat{F}) - \frac{\partial T_{i\alpha}(\widehat{F})}{\partial F_{j\beta}}(F_{j\beta} - \widehat{F}_{j\beta})\big)\big| dx \le C \int_{\mathcal{C}_t} |F - \widehat{F}|^2 dx \,,$$
$$\int_{\mathcal{C}_t} |\partial_\alpha \hat{v}_i (S_{i\alpha} - T_{i\alpha}(F))| dx \le \frac{c}{\epsilon} \int_{\mathcal{C}_t} |S - T(F)|^2 dx + C\epsilon \,,$$

where C is a positive constant depending on the $\big(L^\infty \cap L^2\big)(\mathcal{C}_t)$-norm of $\nabla \hat{v}$ on the cone $\mathcal{C}_t = \{0 < \tau < t, |x| < |x| < R + s(t-\tau)\}$, the bound M and T. We obtain

$$\varphi(t) \le \varphi(0) + C\left(\epsilon + \int_0^t \varphi(\tau)\right)$$

and conclude via the Gronwall lemma. □

4. Strong dissipation and modulated relative energy for convex equilibrium potentials

In this section we analyze the case of a viscoelastic material with linear instantaneous response and uniformly convex equilibrium potential. Consider the system

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha S_{i\alpha} \\ \partial_t (S_{i\alpha} - EF) &= -\frac{1}{\epsilon}(S_{i\alpha} - T_{i\alpha}(F)) \,, \end{aligned} \tag{4.1}$$

under the hypotheses

$$T_{i\alpha}(F) = \frac{\partial W(F)}{\partial F_{i\alpha}}$$

where the stored energy satisfies

$$\gamma I \le \nabla_F^2 W(F) \le \Gamma I \tag{4.2}$$

for some constants γ, Γ, with $\Gamma < E$, and $\forall F$. We shall assume, without loss of generality, $W(0) = \nabla_F W(0) = 0$, and note that the instantaneous potential $\frac{1}{2}E|F|^2$ dominates the equilibrium potential $W(F)$ and thus a global free energy exists for this model.

The system (4.1) can be rewritten as a regularization of the equilibrium system by a wave operator,

$$\begin{aligned} \partial_t F_{i\alpha} - \partial_\alpha v_i &= 0 \\ \partial_t v_i - \partial_\alpha T_{i\alpha}(F) &= \epsilon E \partial_\alpha \partial_\alpha v_i - \epsilon \partial_t^2 v_i \,. \end{aligned} \tag{4.3}$$

This reformulation of (4.1) allows to uncover the dissipative nature of the relaxation process. It is possible to correct the mechanical energy, by incorporating higher order corrections associated with acoustic waves, so that the resulting *modulated energy*

$$\begin{aligned}\mathcal{E}_m = \frac{1}{2}|v|^2 + \epsilon v_i \partial_t v_i + \frac{1}{2}\epsilon^2 \lambda |\partial_t v|^2 \\ + W(F) + \frac{1}{2}\epsilon^2 \lambda E \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 + \epsilon \lambda \partial_\alpha v_i T_{i\alpha}(F)\end{aligned} \tag{4.4}$$

is positive definite and dissipates. This idea is introduced in [22] for the one-dimensional variant of (4.1) and was effective in connection with a relaxation approximation of the Euler equations [5]. It provides an estimate that captures the dissipative nature of the relaxation process, under the condition

$$E > \nabla_F^2 W(F)\,.$$

We call it "strong dissipation estimate" to contrast it to the weaker dissipation captured by the H-theorem in (3.2).

Lemma 4.1. *Any smooth solution of (4.3) verifies*

$$\begin{aligned}\partial_t \mathcal{E}_m - \partial_\alpha \big(v_i T_{i\alpha}(F) + \epsilon E v_i \partial_\alpha v_i + \epsilon^2 \lambda E \partial_t v_i \partial_\alpha v_i + \epsilon \lambda \partial_t v_i T_{i\alpha}(F) \big) \\ + \epsilon \left(E \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 - \lambda \partial_\alpha v_i \frac{\partial^2 W(F)}{\partial F_{i\alpha} F_{j\beta}} \partial_\beta v_j \right) + \epsilon(\lambda - 1)|\partial_t v|^2 = 0\,,\end{aligned} \tag{4.5}$$

where λ is an arbitrary constant.

Proof. We start with the usual energy of the equilibrium system. We multiply $(4.3)_1$ by $T_{i\alpha}(F)$ and $(4.3)_2$ by v_i and integrate by parts to get

$$\begin{aligned}\partial_t \left[\frac{1}{2}|v|^2 + \epsilon v_i \partial_t v_i + W(F) \right] - \partial_\alpha [v_i T_{i\alpha}(F) + \epsilon E v_i \partial_\alpha v_i] \\ + \epsilon E \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 - \epsilon |\partial_t v|^2 = 0\,.\end{aligned} \tag{4.6}$$

We next add corrections to (4.6) in order to obtain a coercive and dissipative energy, under appropriate conditions on the constant E. To this end, multiply $(4.3)_2$ by $\epsilon \lambda \partial_t v_i$ to obtain

$$\epsilon \lambda |\partial_t v|^2 = \epsilon \lambda \partial_t v_i \partial_\alpha T_{i\alpha}(F) - \partial_t \left[\frac{1}{2}\epsilon^2 \lambda |\partial_t v|^2 \right]$$

$$+ \partial_\alpha[\epsilon^2 \lambda E \partial_t v_i \partial_\alpha v_i] - \partial_t \left[\frac{1}{2} \epsilon^2 \lambda E \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 \right] . \tag{4.7}$$

Next, using the identity

$$\begin{aligned}
\partial_t v_i \partial_\alpha T_{i\alpha}(F) &= \partial_\alpha v_i \partial_t T_{i\alpha}(F) - \partial_t[\partial_\alpha v_i T_{i\alpha}(F)] + \partial_\alpha[\partial_t v_i T_{i\alpha}(F)] \\
&= \partial_\alpha v_i \frac{\partial^2 W(F)}{\partial F_{i\alpha} \partial F_{j\beta}} \partial_\beta v_j - \partial_t[\partial_\alpha v_i T_{i\alpha}(F)] + \partial_\alpha[\partial_t v_i T_{i\alpha}(F)]
\end{aligned}$$

we rearrange terms in (4.7) to get

$$\begin{aligned}
&\partial_t \left[\frac{1}{2} \epsilon^2 \lambda |\partial_t v|^2 + \frac{1}{2} \epsilon^2 \lambda E \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 + \epsilon \lambda \partial_\alpha v_i T_{i\alpha}(F) \right] \\
&\quad - \partial_\alpha[\epsilon^2 \lambda E \partial_t v_i \partial_\alpha v_i + \epsilon \lambda \partial_t v_i T_{i\alpha}(F)] \\
&\quad + \epsilon \lambda |\partial_t v|^2 - \epsilon \lambda \partial_\alpha v_i \frac{\partial^2 W(F)}{\partial F_{i\alpha} \partial F_{j\beta}} \partial_\beta v_j = 0 .
\end{aligned}$$

Finally, adding the above equality to (4.6) we get (4.5). □

If $E > 0$ is sufficiently large the quantity $\mathcal{E}_m$ in (4.4) is positive definite and decays along smooth solutions of (4.3). More precisely, we assume an enforced version of the subcharacteristic condition (2.11):

$$E > \Gamma , \qquad E > \frac{\Gamma^2}{\gamma} , \tag{4.8}$$

where γ and Γ are defined in (4.2), and prove a stability estimate in the L^2-norm

$$\psi(t) := \int_{\mathbb{R}^3} \Big(|v(x,t)|^2 + |F(x,t)|^2 + \epsilon^2 \Big(|\partial_t v(x,t)|^2 + \sum_{\alpha=1}^{3} |\partial_\alpha v(x,t)|^2 \Big) \Big) dx .$$

Theorem 4.2. *Assume that (4.2) and (4.8) hold and let $(v(x,t), F(x,t))$ be a smooth solution of (4.3), with data (v_0, F_0), which decay sufficiently fast to zero as $|x| \to +\infty$. Then*

$$O(1)\psi(t) \le \int_{\mathbb{R}^3} \mathcal{E}_m(x,t) dx \le \int_{\mathbb{R}^3} \mathcal{E}_m(x,0) dx \le O(1)\psi(0) .$$

Proof. Thanks to (4.8), there exists a constant $\lambda > 1$ such that

$$E > \lambda \Gamma , \qquad E > \lambda \frac{\Gamma^2}{\gamma} ,$$

and let us consider relation (4.5) for this fixed λ. We first prove that the energy $\mathcal{E}_m$ defined in (4.4) is coercive. To this end, we have

$$\epsilon v_i \partial_t v_i \geq -\frac{1}{2\delta_1}|v|^2 - \frac{1}{2}\epsilon^2 \delta_1 |\partial_t v|^2 \,,$$

for a fixed constant $1 < \delta_1 < \lambda$ and

$$\begin{aligned}
\epsilon\lambda \partial_\alpha v_i T_{i\alpha}(F) &\geq -\frac{1}{2\delta_2}\lambda\epsilon^2 E \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 - \frac{\delta_2 \lambda}{2E} \sum_{i,\alpha=1}^{3} |T_{i\alpha}(F)|^2 \\
&= -\frac{1}{2\delta_2}\lambda\epsilon^2 E \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 - \frac{\delta_2 \lambda}{2E} \left| \frac{\partial^2 W(\widetilde{F})}{\partial F_{i\alpha} \partial F_{j\beta}} F_{j\beta} \right|^2 \\
&\geq -\frac{1}{2\delta_2}\lambda\epsilon^2 E \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 - \frac{\delta_2 \lambda}{2E} \Gamma^2 |F|^2 \,,
\end{aligned}$$

where δ_2 is a fixed constant such that

$$1 < \delta_2 < \frac{E\gamma}{\lambda \Gamma^2} \,.$$

Hence, since $W(F) \geq \frac{\gamma}{2}|F|^2$, the above estimates imply there exists a positive constant $C = C(E, \Gamma, \gamma)$ such that

$$\mathcal{E}_m \geq \frac{1}{C}\left(|v|^2 + |F|^2 + \epsilon^2 \left(\partial_t |v|^2 + \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 \right) \right) \,.$$

Moreover, since $E > \lambda\Gamma$,

$$\epsilon \left(E \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 - \lambda \partial_\alpha v_i \frac{\partial^2 W(F)}{\partial F_{i\alpha} F_{j\beta}} \partial_\beta v_j \right) > 0$$

and $\epsilon(\lambda - 1)|\partial_t v|^2 > 0$, because $\lambda > 1$. Integrating (4.5) in x and t, we end up with

$$\int_{\mathbb{R}^3} \mathcal{E}_m(x,t)\,dx \leq \int_{\mathbb{R}^3} \mathcal{E}_m(x,0)\,dx \,,$$

which provides the result. □

Let now (v, F) be a smooth solution of (4.3) and $(\widehat{v}, \widehat{F})$ be a smooth solution of the limit system

$$\begin{cases} \partial_t \widehat{F}_{i\alpha} = \partial_\alpha \widehat{v}_i \\ \partial_t \widehat{v}_i = \partial_\alpha T_{i\alpha}(\widehat{F}) \,. \end{cases} \tag{4.9}$$

The objective is to control the L^2-norm of the difference thus proving rigorously the relaxation limit from (4.3) towards smooth solutions of (4.9). To this end, a variant of the relative energy is introduced accounting for higher order corrections introduced by a modulated energy, in the spirit of [5, 1]. We define the *modulated relative energy*

$$
\begin{aligned}
\mathcal{E}_{md} :=& \frac{1}{2}|v-\widehat{v}|^2 + +\epsilon(v_i-\widehat{v}_i)\partial_t(v_i-\widehat{v}_i) + \frac{1}{2}\epsilon^2\lambda|\partial_t(v-\widehat{v})|^2 \\
&+ W(F) - W(\widehat{F}) - \frac{\partial W}{\partial F_{i\alpha}}(\widehat{F})\,(F_{i\alpha}-\widehat{F}_{i\alpha}) \\
&+ \frac{1}{2}\epsilon^2\lambda E\sum_{\alpha=1}^{3}|\partial_\alpha(v-\widehat{v})|^2 + \epsilon\lambda\partial_\alpha(v_i-\widehat{v}_i)(T_{i\alpha}(F)-T_{i\alpha}(\widehat{F}))
\end{aligned}
\tag{4.10}
$$

and the associated flux

$$
\begin{aligned}
\mathcal{F}_{\alpha,md} :=&(v_i-\widehat{v}_i)(T_{i\alpha}(F)-T_{i\alpha}(\widehat{F})) \\
&+ \epsilon E(v_i-\widehat{v}_i)\partial_\alpha(v_i-\widehat{v}_i) + \epsilon^2\lambda E\partial_t(v_i-\widehat{v}_i)\partial_\alpha(v_i-\widehat{v}_i) \\
&+ \epsilon\lambda\partial_t(v_i-\widehat{v}_i)(T_{i\alpha}(F)-T_{i\alpha}(\widehat{F}))
\end{aligned}
$$

and establish the following remarkable identity.

Lemma 4.3. *Let (v,F) and $(\widehat{v},\widehat{F})$ be smooth solutions of (4.3) and (4.9) respectively. Then we have*

$$
\begin{aligned}
&\partial_t\mathcal{E}_{md} - \partial_\alpha\mathcal{F}_{\alpha,md} \\
&\quad + \epsilon\left(E\sum_{\alpha=1}^{3}|\partial_\alpha(v-\widehat{v})|^2 - \lambda\partial_\alpha(v_i-\widehat{v}_i)\frac{\partial^2 W(F)}{\partial F_{i\alpha}\partial F_{j\beta}}\partial_\beta(v_j-\widehat{v}_j)\right) \\
&\quad + \epsilon(\lambda-1)|\partial_t(v-\widehat{v})|^2 \\
&= \partial_\alpha v_i\left(T_{i\alpha}(F)-T_{i\alpha}(\widehat{F}) - \frac{\partial T_{i\alpha}(\widehat{F})}{\partial F_{j\beta}}(F_{j\beta}-\widehat{F}_{j\beta})\right) \\
&\quad + \epsilon(v_i-\widehat{v}_i)(E\partial_\alpha\partial_\alpha\widehat{v}_i - \partial_t^2\widehat{v}_i) + \epsilon^2\lambda\partial_t(v_i-\widehat{v}_i)(E\partial_\alpha\partial_\alpha\widehat{v}_i - \partial_t^2\widehat{v}_i) \\
&\quad + \epsilon\lambda\partial_\alpha(v_i-\widehat{v}_i)\left(\frac{\partial T_{i\alpha}(F)}{\partial F_{j\beta}} - \frac{\partial T_{i\alpha}(\widehat{F})}{\partial F_{j\beta}}\right)\partial_t\widehat{F}_{j\beta},
\end{aligned}
\tag{4.11}
$$

where λ is an arbitrary constant.

Proof. The argument to derive (4.11) combines a relative energy argument with a (higher order) modulated energy correction of the relative energy in the spirit of (4.5). We mention only the main differences relative to the previous case.

The energy estimates for the relaxing system (4.3) and the equilibrium system (4.9) are given by

$$\partial_t \left[\frac{1}{2}|v|^2 + W(F)\right] - \partial_\alpha [v_i T_{i\alpha}(F)] = \epsilon E v_i \partial_\alpha \partial_\alpha v_i - \epsilon v_i \partial_t^2 v_i \,, \tag{4.12}$$

$$\partial_t \left[\frac{1}{2}|\widehat{v}|^2 + W(\widehat{F})\right] - \partial_\alpha [\widehat{v}_i T_{i\alpha}(\widehat{F})] = 0 \,. \tag{4.13}$$

The differences $v - \widehat{v}$ and $F - \widehat{F}$ verify the equations

$$\partial_t (F_{i\alpha} - \widehat{F}_{i\alpha}) = \partial_\alpha (v_i - \widehat{v}_i) \,,$$

$$\begin{aligned} \partial_t (v_i - \widehat{v}_i) &= \partial_\alpha (T_{i\alpha}(F) - T_{i\alpha}(\widehat{F})) + \epsilon E \partial_\alpha \partial_\alpha (v_i - \widehat{v}_i) - \epsilon \partial_t^2 (v_i - \widehat{v}_i) \\ &\quad + \epsilon (E \partial_\alpha \partial_\alpha \widehat{v}_i - \partial_t^2 \widehat{v}_i) \,. \end{aligned} \tag{4.14}$$

Using the symmetry of $\frac{\partial T_{i\alpha}(\widehat{F})}{\partial F_{j\beta}} = \frac{\partial^2 W(\widehat{F})}{\partial F_{i\alpha} \partial F_{j\beta}}$, we obtain

$$\begin{aligned} &\partial_t \Big(T_{i\alpha}(\widehat{F})(F_{i\alpha} - \widehat{F}_{i\alpha}) + \widehat{v}_i (v_i - \widehat{v}_i) \Big) \\ &\quad - \partial_\alpha \Big(T_{i\alpha}(\widehat{F})(v_i - \widehat{v}_i) + \widehat{v}_i (T_{i\alpha}(F) - T_{i\alpha}(\widehat{F})) \Big) \\ &\quad = -\partial_\alpha \widehat{v}_i \left(T_{i\alpha}(F) - T_{i\alpha}(\widehat{F}) - \frac{\partial T_{i\alpha}(\widehat{F})}{\partial F_{j\beta}} (F_{j\beta} - \widehat{F}_{j\beta}) \right) \\ &\quad + \epsilon E \widehat{v}_i \partial_\alpha \partial_\alpha v_i - \epsilon \widehat{v}_i \partial_t^2 v_i \,. \end{aligned} \tag{4.15}$$

Subtracting (4.13) and (4.15) from (4.12) and rearranging the derivatives as in the proof of Lemma 4.1, we end up with

$$\begin{aligned} &\partial_t \left[\frac{1}{2}|v - \widehat{v}|^2 + W(F) - W(\widehat{F}) - T_{i\alpha}(\widehat{F})(F_{i\alpha} - \widehat{F}_{i\alpha}) + \epsilon (v_i - \widehat{v}_i) \partial_t (v_i - \widehat{v}_i) \right] \\ &\quad - \partial_\alpha [v_i T_{i\alpha}(F) - \widehat{v}_i T_{i\alpha}(\widehat{F}) - T_{i\alpha}(\widehat{F})(v_i - \widehat{v}_i) - \widehat{v}_i (T_{i\alpha}(F) - T_{i\alpha}(\widehat{F})) \\ &\quad + \epsilon E (v_i - \widehat{v}_i) \partial_\alpha (v_i - \widehat{v}_i)] \\ &\quad + \epsilon E \sum_{\alpha=1}^{3} |\partial_\alpha (v - \widehat{v})|^2 - \epsilon |\partial_t (v - \widehat{v})|^2 \\ &\quad = \partial_\alpha \widehat{v}_i \left(T_{i\alpha}(F) - T_{i\alpha}(\widehat{F}) - \frac{\partial T_{i\alpha}(\widehat{F})}{\partial F_{j\beta}} (F_{j\beta} - \widehat{F}_{j\beta}) \right) \\ &\quad + \epsilon (v_i - \widehat{v}_i)(E \partial_\alpha \partial_\alpha \widehat{v}_i - \partial_t^2 \widehat{v}_i) \,. \end{aligned} \tag{4.16}$$

The above estimate is corrected by adding the relation

$$\partial_t \left[\frac{1}{2} \epsilon^2 \lambda |\partial_t (v - \widehat{v})|^2 + \frac{1}{2} \epsilon^2 \lambda E \sum_{\alpha=1}^{3} |\partial_\alpha (v - \widehat{v})|^2 \right]$$

$$
\begin{aligned}
&- \partial_\alpha[\epsilon^2 \lambda E \partial_t (v_i - \widehat{v}_i) \partial_\alpha (v_i - \widehat{v}_i)] \\
&+ \epsilon^2 \lambda |\partial_t (v - \widehat{v})|^2 - \epsilon \lambda \partial_t (v_i - \widehat{v}_i) \partial_\alpha (T_{i\alpha}(F) - T_{i\alpha}(\widehat{F})) \\
&= \epsilon^2 \lambda \partial_t (v_i - \widehat{v}_i)(E \partial_\alpha \partial_\alpha \widehat{v}_i - \partial_t^2 \widehat{v}_i)
\end{aligned} \tag{4.17}
$$

obtained by multiplying (4.14) by $\epsilon\lambda\partial_t(v_i - \widehat{v}_i)$ and rearranging derivatives.

The last term in the left of (4.17) can be recast in the form (by interchanging the x and t derivatives)

$$
\begin{aligned}
&- \partial_t (v_i - \widehat{v}_i) \partial_\alpha (T_{i\alpha}(F) - T_{i\alpha}(\widehat{F})) = -\partial_\alpha (v_i - \widehat{v}_i) \partial_t (T_{i\alpha}(F) - T_{i\alpha}(\widehat{F})) \\
&\quad + \partial_t [\partial_\alpha (v_i - \widehat{v}_i)(T_{i\alpha}(F) - T_{i\alpha}(\widehat{F})] - \partial_\alpha [\partial_t (v_i - \widehat{v}_i)(T_{i\alpha}(F) - T_{i\alpha}(\widehat{F})] \\
&\quad = -\partial_\alpha (v_i - \widehat{v}_i) \frac{\partial^2 W(F)}{\partial F_{i\alpha} \partial F_{j\beta}} \partial_\beta (v_j - \widehat{v}_j) \\
&\quad - \partial_\alpha (v_i - \widehat{v}_i) \left(\frac{T_{i\alpha}(F)}{\partial F_{j\beta}} - \frac{T_{i\alpha}(\widehat{F})}{\partial F_{j\beta}} \right) \partial_t \widehat{F}_{j\beta} \\
&\quad + \partial_t [\partial_\alpha (v_i - \widehat{v}_i)(T_{i\alpha}(F) - T_{i\alpha}(\widehat{F})] - \partial_\alpha [\partial_t (v_i - \widehat{v}_i)(T_{i\alpha}(F) - T_{i\alpha}(\widehat{F})] \,.
\end{aligned}
$$

Using that identity, (4.17) and (4.16) yield (4.11). □

The relative energy is estimated by using condition (4.8) and the uniform convexity of the potential $W(F)$. Introduce the L^2-norm of the difference of two solutions

$$
\psi_d(t) := \int_{\mathbb{R}^3} \left(|v - \widehat{v}|^2 + |F - \widehat{F}|^2 + \epsilon^2 \Big(|\partial_t (v - \widehat{v})|^2 + \sum_{\alpha=1}^{3} |\partial_\alpha (v - \widehat{v})|^2 \Big) \right) dx \,.
$$

Theorem 4.4. *Assume (4.2), (4.8) and* $|\nabla_F^3 W(F)| \leq M$ *for some* $M > 0$. *Let* (v^ϵ, F^ϵ) *and* $(\widehat{v}, \widehat{F})$ *be smooth solutions of (4.3) and (4.9) respectively, defined on* $\mathbb{R}^3 \times [0,T]$, *decaying sufficiently fast to zero as* $|x| \to +\infty$ *and emanating from smooth data* $(v_0^\epsilon, F_0^\epsilon)$ *and* $(\widehat{v}_0, \widehat{F}_0)$. *Then, there exists a constant* $C = C(T, E, \gamma, \Gamma, M, \widehat{v}, \widehat{F}) > 0$ *independent of* ϵ *such that*

$$
\psi_d(t) \leq C \left(\psi_d(0) + \epsilon^2 \right) . \tag{4.18}
$$

If moreover $\psi_d^\epsilon(0) \to 0$ *as* $\epsilon \downarrow 0$, *then*

$$
\sup_{t \in [0,T]} \Big(\|v^\epsilon(\cdot,t) - \widehat{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} + \|F^\epsilon(\cdot,t) - \widehat{F}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \Big) \longrightarrow 0 \,,
$$

as $\epsilon \downarrow 0$.

Proof. Let us denote by $\varphi(t)$ the quantity

$$\varphi(t) = \int_{\mathbb{R}^3} \mathcal{E}_{md}(v, F; \widehat{v}, \widehat{F}) dx \,,$$

where $\mathcal{E}_{md}$ is defined in (4.10). Then, following the proof of Theorem 4.2, conditions (4.2) and (4.8) imply

$$\frac{1}{C}\psi_d(t) \le \varphi(t) \le C\psi_d(t)$$

and

$$\varphi(t) \le \varphi(0) + \int_0^t \int_{\mathbb{R}^3} |\mathcal{R}^\epsilon| dx ds \,, \tag{4.19}$$

where $C = C(E, \gamma, \Gamma, t, \widehat{v}, \widehat{F}) > 0$ is a given positive constant and $\mathcal{R}^\epsilon$ stands for the right hand side of (4.11) which we shall estimate term by term. Let C be a positive constant depending on E, γ, Γ, M and on the equilibrium solution $(\widehat{v}, \widehat{F})$ and its derivatives. Then

$$\int_{\mathbb{R}^3} \Big| \partial_\alpha \widehat{v}_i \big(T_{i\alpha}(F) - T_{i\alpha}(\widehat{F}) - \frac{\partial T_{i\alpha}(\widehat{F})}{\partial F_{j\beta}} (F_{j\beta} - \widehat{F}_{j\beta}) \big) \Big| dx \le C \int_{\mathbb{R}^3} |F - \widehat{F}|^2 dx \,,$$

$$\int_{\mathbb{R}^3} \big| \epsilon (v_i - \widehat{v}_i)(E \partial_\alpha \partial_\alpha \widehat{v}_i - \partial_t^2 \widehat{v}_i) \big| \, dx \le C \Big(\epsilon^2 + \int_{\mathbb{R}^3} |v - \widehat{v}|^2 dx \Big) \,,$$

$$\int_{\mathbb{R}^3} \big| \epsilon^2 \lambda \partial_t (v_i - \widehat{v}_i)(E \partial_\alpha \partial_\alpha \widehat{v}_i - \partial_t^2 \widehat{v}_i) \big| \, dx \le C \Big(\epsilon^2 + \int_{\mathbb{R}^3} \epsilon^2 |\partial_t (v - \widehat{v})|^2 dx \Big) \,,$$

$$\int_{\mathbb{R}^3} \Big| \epsilon \lambda \partial_\alpha (v_i - \widehat{v}_i) \Big(\frac{\partial T_{i\alpha}(F)}{\partial F_{j\beta}} - \frac{\partial T_{i\alpha}(\widehat{F})}{\partial F_{j\beta}} \Big) \partial_t \widehat{F}_{j\beta} \Big| dx$$

$$\le C \int_{\mathbb{R}^3} \Big(\epsilon^2 \sum_{\alpha=1}^{3} |\partial_\alpha (v - \widehat{v})|^2 + |F - \widehat{F}|^2 \Big) dx \,.$$

Thus relation (4.19) becomes

$$\varphi(t) \le \varphi(0) + C\epsilon^2 t + C \int_0^t \varphi(s) ds$$

which implies (4.18) in view of the aforementioned coercive nature of $\varphi(t)$ and thanks to the Gronwall lemma. As a consequence, we obtain convergence of the relaxation system as long as the limit solution remains smooth. □

Remark 4.5. 1. The essential hypotheses of Theorem 4.4 are that $W(F)$ is convex and satisfies the condition $E > \nabla^2 W$. The rest of the hypotheses are there to account for the lack of *a-priori* L^∞-bounds, and could be removed if a-priori L^∞ estimates were available.

2. The same stability argument applies to the approximation of elastodynamics by viscoelasticity of the rate type,

$$\begin{cases} \partial_t F_{i\alpha} = \partial_\alpha v_i \\ \partial_t v_i = \partial_\alpha T_{i\alpha}(F) + \epsilon \partial_\alpha \partial_\alpha v_i \, . \end{cases} \tag{4.20}$$

In (4.20), the viscoelastic stress of the rate type is

$$S_{i\alpha} = \frac{\partial W(F)}{\partial F_{i\alpha}} + \epsilon \partial_\alpha v_i \, .$$

This result is a direct consequence of the general result in [13, Thm 5.2.1]. By contrast, for the case of relaxation one needs the corrections resulting in the modulated relative energy, or the approach of relative energy employed in Theorem 3.3.

5. Polyconvex elastodynamics

Consider now the system of elastodynamics

$$\partial_t^2 y_i = \partial_\alpha T_{i\alpha}(\nabla_x y) \tag{5.1}$$

for $y \in \mathbb{R}^3$. Equation (5.1) can be rewritten as a system of conservation laws for the velocity $v_i = \partial_t y_i$ and the deformation gradient $F_{i\alpha} = \partial_\alpha y_i$ as follows

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha T_{i\alpha}(F) \, . \end{aligned} \tag{5.2}$$

The equivalence of the two formulations holds for functions F that are gradients. Note that $F = \nabla y$ if and only if it satisfies

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0 \tag{5.3}$$

and, technically, the system (5.1) is equivalent to (5.2) subject to the differential constraint (5.3). The latter relation is an involution [12]: if it is satisfied for the initial data then $(5.2)_1$ propagates (5.3) to hold for all times. Therefore, for the equivalence of the two formulations, it suffices that (5.3) is satisfied for the initial data.

5.1. The symmetrizable extension of polyconvex elastodynamics.

Consider next the uniformly polyconvex case, when

$$T(F) = \frac{\partial W(F)}{\partial F}$$

and the stored energy $W : \mathrm{Mat}^{3\times 3} \to [0,\infty)$ factorizes as a uniformly convex function of the minors of F:

$$W(F) = g \circ \Phi(F)\,,$$

with $g : \mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R} \to \mathbb{R}$ uniformly convex and

$$\Phi(F) = (F, \operatorname{cof} F, \det F)\,. \tag{5.4}$$

Here the cofactor matrix $\operatorname{cof} F$ and the determinant $\det F$ are

$$\begin{aligned}
(\operatorname{cof} F)_{i\alpha} &= \frac{1}{2}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}F_{k\gamma}\,,\\
\det F &= \frac{1}{6}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{i\alpha}F_{j\beta}F_{k\gamma} = \frac{1}{3}(\operatorname{cof} F)_{i\alpha}F_{i\alpha}\,.
\end{aligned}$$

We review a symmetrizable extension of polyconvex elastodynamics [15], based on certain kinematic identities on $\det F$ and $\operatorname{cof} F$ from [17]. The components of $\Phi^A(F)$ in (5.4), for $A = 1, \dots, 19$, are null Lagrangians and satisfy for any smooth map $y(x,t)$ the identities

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla y)\right) \equiv 0$$

or equivalently

$$\partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right) = 0\,, \quad \forall F \text{ with } \partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0\,. \tag{5.5}$$

The kinematic compatibility equation $(5.2)_1$ implies

$$\begin{aligned}
\partial_t \Phi^A(F) &= \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}}\partial_\alpha v_i\\
&= \partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)v_i\right) - v_i\partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right)\\
&= \partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)v_i\right), \quad \forall F \text{ with } \partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0\,.
\end{aligned}$$

This motivates to embed (5.2) into the system of conservation laws

$$\begin{aligned}
\partial_t v_i &= \partial_\alpha\left(\frac{\partial g}{\partial \Xi^A}(\Xi)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right)\\
\partial_t \Xi^A &= \partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)v_i\right).
\end{aligned} \tag{5.6}$$

Note that $\Xi = (F, Z, w)$ takes values in $\mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R} \simeq \mathbb{R}^{19}$ and is treated as a new dependent variable. (Since the components of F constitute the first nine components of Ξ, equation $(5.2)_1$ is included as the first part of $(5.6)_2$.)

The extension has the following properties:

(i) If $F(\cdot,0)$ is a gradient then $F(\cdot,t)$ remains a gradient $\forall t$.

(ii) If $F(\cdot,0)$ is a gradient and $\Xi(\cdot,0)=\Phi(F(\cdot,0))$ then $F(\cdot,t)$ remains a gradient and $\Xi(\cdot,t)=\Phi(F(\cdot,t))$, $\forall t$. In other words, the system of elastodynamics can be visualized as constrained evolution of (5.6).

(iii) The enlarged system admits a strictly convex entropy
$$\eta(v,\Xi)=\frac{1}{2}|v|^2+g(\Xi)$$
and is thus symmetrizable (along solutions that are gradients).

Remark 5.1. 1. The relations
$$\partial_t\Phi^A(F)=\partial_\alpha\left(\frac{\partial\Phi^A}{\partial F_{i\alpha}}(F)v_i\right)$$
do not form what is usually called entropy-entropy flux pairs in the theory of hyperbolic conservation laws, because they hold under the differential structure (5.3).

2. Property (iii) is again based on the null-Lagrangian structure and η is not an entropy in the usual sense of the theory of conservation laws. To see that, let us rewrite system (5.6) as
$$U_t-\partial_\alpha A^\alpha(U)=0\,, \tag{5.7}$$
where
$$U=\begin{pmatrix} v_i\\ \Xi^A\end{pmatrix}\,, \qquad\qquad A^\alpha(U)=\begin{pmatrix}\frac{\partial g(\Xi)}{\partial\Xi^A}\frac{\partial\Phi^A(F)}{\partial F_{i\alpha}}\\ \frac{\partial\Phi^A(F)}{\partial F_{i\alpha}}v_i\end{pmatrix}\,.$$
Then $(\eta(U),q^\alpha(U))$ is an entropy pair for (5.7) if it satisfies
$$\sum_{k=1}^{22}\frac{\partial\eta(U)}{\partial U_k}\frac{\partial A^\alpha(U)_k}{\partial U_j}=\frac{\partial q^\alpha(U)}{\partial U_j}\,,\quad \text{for any } \alpha=1,2,3 \text{ and } j=1,\dots,22\,.$$
Now, our pair (η,q^α) does not verify the above definition, because
$$\begin{aligned}&\sum_{i=1}^{3}v_i\frac{\partial}{\partial U_j}\left(\frac{\partial g(\Xi)}{\partial\Xi^A}\frac{\partial\Phi^A(F)}{\partial F_{i\alpha}}\right)+\sum_{A=1}^{19}\frac{\partial g(\Xi)}{\partial\Xi^A}\frac{\partial}{\partial U_j}\left(\frac{\partial\Phi^A(F)}{\partial F_{i\alpha}}v_i\right)\\ &\quad=\frac{\partial}{\partial U_j}\left(\sum_{i,A}v_i\frac{\partial g(\Xi)}{\partial\Xi^A}\frac{\partial\Phi^A(F)}{\partial F_{i\alpha}}\right)+\sum_{i,A}v_i\frac{\partial g(\Xi)}{\partial\Xi^A}\frac{\partial}{\partial U_j}\frac{\partial\Phi^A(F)}{\partial F_{i\alpha}}\end{aligned}\tag{5.8}$$

and the last term of the above equality is different from zero when we differentiate the terms cof F or det F of $\Phi(F)$ with respect to $U_j = F_{i\alpha}$. However, η verifies

$$\partial_t \left[\frac{1}{2}|v|^2 + g(\Xi)\right] - \partial_\alpha \left[\sum_{i,A} v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}}\right] = 0\,,$$

along F that are gradients as is easily verified using (5.5). The extra term in (5.8) vanishes, when we multiply this equation by $\partial_\alpha U_j$ and sum with respect to j and α, because

$$\sum_{i,A} v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \partial_\alpha \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} = 0\,,$$

by (5.5). This particular nature of η is crucial in the relative energy estimate of the following section.

5.2. **Relative energy for polyconvex elastodynamics: Viscosity approximation.** Let $(\widehat{v}, \widehat{\Xi}) \in \mathbb{R}^{22}$ be a smooth solution of

$$\begin{cases} \partial_t \widehat{\Xi}^A = \partial_\alpha \left(\frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \widehat{v}_i \right) \\ \partial_t \widehat{v}_i = \partial_\alpha \left(\frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) \end{cases} \tag{5.9}$$

and let (v, Ξ) be a solution of the viscosity approximation of (5.6)

$$\begin{cases} \partial_t \Xi^A = \partial_\alpha \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} v_i \right) \\ \partial_t v_i = \partial_\alpha \left(\frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \right) + \epsilon \partial_\alpha \partial_\alpha v_i\,. \end{cases} \tag{5.10}$$

(System (5.10) bears the same relation to the viscosity approximation (5.17) as (5.9) bears to (5.2).) For both solutions we assume that F satisfies (5.3), or equivalently for the initial data $F(x,0) = \nabla y(x,0)$.

The goal is to obtain a relative energy estimate among the two solutions. To this end, we define the relative entropy

$$\eta_r(v, \Xi; \widehat{v}, \widehat{\Xi}) := \frac{1}{2}|v - \widehat{v}|^2 + g(\Xi) - g(\widehat{\Xi}) - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A}(\Xi^A - \widehat{\Xi}^A)$$

and note that the associated (relative) flux will turn out to be

$$q_r^\alpha(v, \Xi; \widehat{v}, \widehat{\Xi}) := \left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \right)(v_i - \widehat{v}_i) \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}}\,,$$

$\alpha = 1, 2, 3$.

Lemma 5.2. *Let (v, Ξ) and $(\widehat{v}, \widehat{\Xi})$ be smooth solutions of (5.10) and (5.9) respectively. Then*

$$
\begin{aligned}
&\partial_t \eta_r - \partial_\alpha q_r^\alpha + \epsilon \sum_{\alpha=1}^{3} |\partial_\alpha (v - \widehat{v})|^2 \\
&\qquad = Q + \partial_\alpha \big(\epsilon (v_i - \widehat{v}_i) \partial_\alpha (v_i - \widehat{v}_i) \big) + \epsilon (v_i - \widehat{v}_i) \partial_\alpha \partial_\alpha \widehat{v}_i \,,
\end{aligned}
\tag{5.11}
$$

where Q is quadratic of the form

$$
\begin{aligned}
Q := {}& \frac{\partial^2 g(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B} \partial_\alpha \widehat{\Xi}^B \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) (v_i - \widehat{v}_i) \\
&+ \partial_\alpha \widehat{v}_i \left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \right) \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) \\
&+ \partial_\alpha \widehat{v}_i \left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} - \frac{\partial^2 g(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B} (\Xi^B - \widehat{\Xi}^B) \right) \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \,.
\end{aligned}
\tag{5.12}
$$

Proof. Property (5.5) gives for a smooth solution $(\widehat{v}, \widehat{\Xi})$

$$
\partial_t \left[\frac{1}{2} |\widehat{v}|^2 + g(\widehat{\Xi}) \right] - \partial_\alpha \left[\widehat{v}_i \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right]
\tag{5.13}
$$

and for (v, Ξ) solution of (5.10),

$$
\partial_t \left[\frac{1}{2} |v|^2 + g(\Xi) \right] - \partial_\alpha \left[v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \right] = \epsilon v_i \partial_\alpha \partial_\alpha v_i \,.
\tag{5.14}
$$

Subtracting (5.9) from (5.10) we obtain

$$
\partial_t (v_i - \widehat{v}_i) - \partial_\alpha \left(\frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) = \epsilon \partial_\alpha \partial_\alpha v_i \,,
$$

$$
\partial_t (\Xi^A - \widehat{\Xi}^A) - \partial_\alpha \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} v_i - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \widehat{v}_i \right) = 0
$$

which, in turn, implies by (5.9) and the identities (5.5),

$$
\begin{aligned}
&\partial_t \Big[\widehat{v}_i (v_i - \widehat{v}_i) + \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} (\Xi^A - \widehat{\Xi}^A) \Big] \\
&\quad - \partial_\alpha \Bigg[\widehat{v}_i \left(\frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) \\
&\qquad\qquad + \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} v_i - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \widehat{v}_i \right) \Bigg] \\
&= (\partial_t \widehat{v}_i)(v_i - \widehat{v}_i) + \partial_t \left(\frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \right) (\Xi^A - \widehat{\Xi}^A)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \partial_\alpha \widehat{v}_i \left(\frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) \\
&\quad - \partial_\alpha \left(\frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \right) \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} v_i - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \widehat{v}_i \right) + \epsilon \widehat{v}_i \partial_\alpha \partial_\alpha v_i \\
&= -\partial_\alpha \left(\frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \right) \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} v_i - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \widehat{v}_i - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} (v_i - \widehat{v}_i) \right) \\
&\quad - \partial_\alpha \widehat{v}_i \Big(\frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} - \frac{\partial^2 g(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B} \frac{\partial \Phi^B(\widehat{F})}{\partial F_{i\alpha}} (\Xi^A - \widehat{\Xi}^A) \Big) \\
&\quad + \epsilon \widehat{v}_i \partial_\alpha \partial_\alpha v_i \\
&=: \mathcal{I} + \epsilon \widehat{v}_i \partial_\alpha \partial_\alpha v_i \ , .
\end{aligned} \tag{5.15}
$$

Next, we subtract (5.13) and (5.15) from (5.14) and conclude

$$
\begin{aligned}
\partial_t \eta_r - \partial_\alpha \Bigg[v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \widehat{v}_i \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \\
- \widehat{v}_i \left(\frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) \\
- \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} v_i - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \widehat{v}_i \right) \Bigg] \\
= -\mathcal{I} + \epsilon (v_i - \widehat{v}_i) \partial_\alpha \partial_\alpha v_i \, .
\end{aligned} \tag{5.16}
$$

Since

$$
\begin{aligned}
\epsilon (v_i - \widehat{v}_i) \partial_\alpha \partial_\alpha v_i &= \epsilon (v_i - \widehat{v}_i) \partial_\alpha \partial_\alpha (v_i - \widehat{v}_i) + \epsilon (v_i - \widehat{v}_i) \partial_\alpha \partial_\alpha \widehat{v}_i \\
&= \epsilon \partial_\alpha \partial_\alpha \widehat{v}_i (v_i - \widehat{v}_i) + \partial_\alpha [\epsilon (v_i - \widehat{v}_i) \partial_\alpha (v_i - \widehat{v}_i)] - \epsilon \sum_{\alpha=1}^{3} |\partial_\alpha (v - \widehat{v})|^2 \, ,
\end{aligned}
$$

it remains to control the term $\mathcal{I}$. This term is not quadratic in $v - \widehat{v}$ and $\Xi - \widehat{\Xi}$, but may be corrected using a divergence term in order to obtain the quadratic expression in (5.11). Indeed, $\mathcal{I}$ satisfies the chain of identities:

$$
\begin{aligned}
-\mathcal{I} = \partial_\alpha \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) v_i \\
+ \partial_\alpha \widehat{v}_i \frac{\partial g(\Xi)}{\partial \Xi^A} \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) \\
+ \partial_\alpha \widehat{v}_i \left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} - \frac{\partial^2 g(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B} (\Xi^B - \widehat{\Xi}^B) \right) \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\partial^2 g(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B} \partial_\alpha \widehat{\Xi}^B \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) (v_i - \widehat{v}_i) \\
&\quad + \partial_\alpha \widehat{v}_i \left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \right) \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) \\
&\quad + \partial_\alpha \widehat{v}_i \left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} - \frac{\partial^2 g(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B} (\Xi^B - \widehat{\Xi}^B) \right) \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \\
&\quad + \partial_\alpha \left[\widehat{v}_i \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) \right] ,
\end{aligned}
$$

where we used once again the null–Lagrangian property (5.5) for both $\Phi^A(F)$ and $\Phi^A(\widehat{F})$.

Finally, we combine the last term with the flux term of (5.16) to obtain

$$
\begin{aligned}
&\partial_\alpha \Bigg[v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \widehat{v}_i \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \\
&\quad - \widehat{v}_i \left(\frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) \\
&\quad - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} v_i - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \widehat{v}_i \right) \\
&\quad + \widehat{v}_i \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) \Bigg] \\
&\quad = \partial_\alpha \left[\left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \right) (v_i - \widehat{v}_i) \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \right] = \partial_\alpha q_r^\alpha
\end{aligned}
$$

and the proof is complete. $\square$

Due to properties (i) and (ii) of Section 5.1 the relative energy identity (5.11) can be restricted to solutions $(v, \Xi = \Phi(F))$ with $F = \nabla y$, and $(\widehat{v}, \widehat{\Xi} = \Phi(\widehat{F}))$ with $\widehat{F} = \nabla \widehat{y}$. The resulting relative energy and corresponding flux read

$$
\begin{aligned}
H_r &= \eta_r(v, \Phi(F); \widehat{v}, \Phi(\widehat{F})) \\
&= \frac{1}{2}|v - \widehat{v}|^2 + g(\Phi(F)) - g(\Phi(\widehat{F})) - \frac{\partial g}{\partial \Xi^A}(\Phi(\widehat{F}))(\Phi(F)^A - \Phi(\widehat{F})^A) , \\
Q_r^\alpha &= q_r^\alpha(v, \Phi(F); \widehat{v}, \Phi(\widehat{F})) \\
&= \left(\frac{\partial g}{\partial \Xi^A}(\Phi(F)) - \frac{\partial g}{\partial \Xi^A}(\Phi(\widehat{F})) \right) (v_i - \widehat{v}_i) \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}}
\end{aligned}
$$

and will be used to obtain stability estimates in the sequel.

Let (v^ϵ, F^ϵ) satisfy the viscosity approximation

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha T_{i\alpha}(F) + \epsilon \partial_\alpha \partial_\alpha v_i \end{aligned} \tag{5.17}$$

and $(\widehat{v}, \widehat{F})$ be a smooth solution of (5.2). Our next objective is to prove convergence of the viscosity approximation to polyconvex elastodynamics as long as the limit solution is smooth, by controlling the L^2-norm

$$\Psi_d(t) := \int_{\mathbb{R}^d} \Big(|v - \widehat{v}|^2 + |\Phi(F) - \Phi(\widehat{F})|^2 \Big) dx \,, \quad d = 2, 3 \,.$$

To this end, we assume

$$0 < \gamma I \le \nabla^2_\Xi g(\Xi) \le \Gamma I \,, \qquad\qquad |\nabla^3_\Xi g(\Xi)| \le M \,, \tag{5.18}$$

and prove:

Theorem 5.3. *Let $(v^\epsilon, \Phi(F^\epsilon))$, $F^\epsilon = \nabla y^\epsilon$, and $(\widehat{v}, \Phi(\widehat{F}))$, $\widehat{F} = \nabla \widehat{y}$, be smooth solutions of (5.10) and (5.9), defined on $\mathbb{R}^d \times [0,T]$, $d = 2,3$, that decay sufficiently fast as $|x| \to +\infty$ and emanate from data $(v_0^\epsilon, \Phi(F_0^\epsilon))$, $F_0^\epsilon = \nabla y^\epsilon(\cdot, 0)$, and $(\widehat{v}_0, \Phi(\widehat{F}_0))$, $\widehat{F}_0 = \nabla \widehat{y}(\cdot, 0)$. Assume g verifies (5.18). Then, there exists a constant $C = C(T, \gamma, \Gamma, M, \widehat{v}, \widehat{\Xi}) > 0$ such that*

$$\Psi_d(t) \le C \left(\Psi_d(0) + \epsilon^2 \right) \,. \tag{5.19}$$

If moreover the data satisfy $\Psi^\epsilon_d(0) \to 0$ as $\epsilon \downarrow 0$, then

$$\sup_{t \in [0,T]} \Big(\|v^\epsilon(\cdot, t) - \widehat{v}(\cdot, t)\|_{L^2(\mathbb{R}^d)} + \|\Phi(F^\epsilon(\cdot, t)) - \Phi(\widehat{F}(\cdot, t))\|_{L^2(\mathbb{R}^d)} \Big) \to 0 \,,$$

as $\epsilon \downarrow 0$.

Proof. On account of (5.18), there exists a positive constant $C = C(\gamma, \Gamma)$ such that

$$\frac{1}{C} \psi_d(t) \le \varphi(t) := \int_{\mathbb{R}^d} \eta_r(v, \Xi; \widehat{v}, \widehat{\Xi}) dx \le C \psi_d(t) \,,$$

where

$$\psi_d(t) := \int_{\mathbb{R}^d} \Big(|v - \widehat{v}|^2 + |\Xi - \widehat{\Xi}|^2 \Big) dx \,.$$

Integrating (5.11) over $\mathbb{R}^d \times [0,t]$ we obtain

$$\psi_d(t) \le C \left(\psi_d(0) + \int_0^t \int_{\mathbb{R}^d} |\mathcal{R}^\epsilon| \, dx ds \right) \,, \tag{5.20}$$

where $\mathcal{R}^\epsilon = Q + \epsilon(v_i - \widehat{v}_i)\partial_\alpha\partial_\alpha\widehat{v}_i$.

Let now C be a positive constant depending on Γ, M and the functions $\widehat{v}$, $\widehat{\Xi}$ and their derivatives. We have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \left| \frac{\partial^2 g(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B} \partial_\alpha \widehat{\Xi}^B \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) (v_i - \widehat{v}_i) \right| dx \\
&\quad \leq C \int_{\mathbb{R}^d} \left(|v - \widehat{v}|^2 + \Big| \frac{\partial \Phi}{\partial F}(F) - \frac{\partial \Phi}{\partial F}(\widehat{F}) \Big|^2 \right) dx \,, \\
&\int_{\mathbb{R}^d} \left| \partial_\alpha \widehat{v}_i \left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \right) \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) \right| dx \\
&\quad \leq C \int_{\mathbb{R}^d} \left(|\Xi - \widehat{\Xi}|^2 + \Big| \frac{\partial \Phi}{\partial F}(F) - \frac{\partial \Phi}{\partial F}(\widehat{F}) \Big|^2 \right) dx \,, \\
&\int_{\mathbb{R}^d} \left| \partial_\alpha \widehat{v}_i \left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} - \frac{\partial^2 g(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B} (\Xi^B - \widehat{\Xi}^B) \right) \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right| dx \\
&\quad \leq C \int_{\mathbb{R}^d} |\Xi - \widehat{\Xi}|^2 dx \,, \\
&\int_{\mathbb{R}^d} |\epsilon (v_i - \widehat{v}_i) \partial_\alpha \partial_\alpha \widehat{v}_i| \, dx \leq C \left(\epsilon^2 + \int_{\mathbb{R}^d} |v - \widehat{v}|^2 dx \right) .
\end{aligned}
$$

Then (5.20) gives

$$
\begin{aligned}
\psi_d(t) &\leq C\psi_d(0) + C\epsilon^2 t \\
&\qquad + C \int_0^t \Big(|v - \widehat{v}|^2 + |\Xi - \widehat{\Xi}|^2 + \Big| \frac{\partial \Phi}{\partial F}(F) - \frac{\partial \Phi}{\partial F}(\widehat{F}) \Big|^2 \Big) ds \,.
\end{aligned}
\tag{5.21}
$$

From the identities

$$
\frac{\partial \det F}{\partial F_{i\alpha}} = (\operatorname{cof} F)_{i\alpha} \,, \qquad \frac{\partial (\operatorname{cof} F)_{i\alpha}}{\partial F_{j\beta}} = \epsilon_{ijk}\epsilon_{\alpha\beta\gamma} F_{k\gamma} \,,
$$

we have

$$
\Big| \frac{\partial \Phi}{\partial F}(F) - \frac{\partial \Phi}{\partial F}(\widehat{F}) \Big| \leq C |\Phi(F) - \Phi(\widehat{F})| \,.
$$

We now restrict (5.21) to solutions of the form $(v, \Phi(F))$ and $(\widehat{v}, \Phi(\widehat{F}))$ and use the above identities to deduce

$$
\Psi_d(t) \leq C \Big(\Psi_d(0) + \epsilon^2 t + \int_0^t \Psi_d(s) ds \Big) \,,
$$

whence (5.19) follows via the Gronwall lemma. $\square$

6. Strong dissipation and modulated relative energy for the relaxation approximation of polyconvex elastodynamics

Next, consider the extended system

$$
\begin{aligned}
\partial_t \widehat{\Xi}^A &= \partial_\alpha \left(\frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \widehat{v}_i \right) \\
\partial_t \widehat{v}_i &= \partial_\alpha \left(\frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right)
\end{aligned}
\tag{6.1}
$$

and the associated relaxation approximation

$$
\begin{aligned}
\partial_t \Xi^A &= \partial_\alpha \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} v_i \right) \\
\partial_t v_i &= \partial_\alpha S_{i\alpha} \\
\partial_t (S_{i\alpha} - E F_{i\alpha}) &= -\frac{1}{\epsilon} \left(S_{i\alpha} - \widetilde{T}_{i\alpha}(\Xi) \right) .
\end{aligned}
$$

We assume $W(F) = g(\Phi(F))$ is polyconvex, assume (with no loss of generality) $W(0) = \nabla_F W(0) = 0$, and denote by

$$
\widetilde{T}_{i\alpha}(\Xi) = \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} .
$$

As in the previous section, F and $\widehat{F}$ are deformation gradients, or equivalently $F(x,0) = \nabla_x y(x,0)$ and $\widehat{F}(x,0) = \nabla_x \widehat{y}(x,0)$. Also, the relaxation system may be expressed in the form of approximation by wave operator

$$
\begin{aligned}
\partial_t \Xi^A &= \partial_\alpha \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} v_i \right) \\
\partial_t v_i &= \partial_\alpha \widetilde{T}_{i\alpha}(\Xi) + \epsilon E \partial_\alpha \partial_\alpha v_i - \epsilon \partial_t^2 v_i \, .
\end{aligned}
\tag{6.2}
$$

We recall here that (6.1) and (6.2) reduce respectively to

$$
\begin{aligned}
\partial_t \widehat{F}_{i\alpha} &= \partial_\alpha \widehat{v}_i \\
\partial_t \widehat{v}_i &= \partial_\alpha T_{i\alpha}(\widehat{F})
\end{aligned}
\tag{6.3}
$$

and

$$
\begin{aligned}
\partial_t F_{i\alpha} &= \partial_\alpha v_i \\
\partial_t v_i &= \partial_\alpha T_{i\alpha}(F) + \epsilon E \partial_\alpha \partial_\alpha v_i - \epsilon \partial_t^2 v_i
\end{aligned}
\tag{6.4}
$$

for the special solutions $(\widehat{v}, \Phi(\widehat{F}))$ and $(v, \Phi(F))$. Our objective is to study the relaxation limit from (6.4) to polyconvex elastodynamics (6.3), and obtain the structural identities for the modulated energy in Lemma 6.1 and modulated relative energy in Lemma 6.3 in the polyconvex case as well as corresponding stability estimates.

The modulated energy for system (6.4) takes the form

$$\begin{aligned}\mathcal{E}_{pm} &= \frac{1}{2}|v|^2 + \epsilon v_i \partial_t v_i + \frac{1}{2}\epsilon^2 \lambda |\partial_t v|^2 \\ &\quad + g(\Xi) + \frac{1}{2}\epsilon^2 \lambda E \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 + \epsilon \lambda \partial_\alpha v_i \widetilde{T}_{i\alpha}(\Xi)\,.\end{aligned} \tag{6.5}$$

This corrected energy is positive definite and dissipates along the relaxation process and thus gives estimates for the relaxing solutions, provided an appropriate subcharacteristic condition is assumed, namely for E sufficiently big (see (6.11)).

Lemma 6.1. *Any smooth solution of (6.2) verifies*

$$\begin{aligned}&\partial_t \mathcal{E}_{pm} - \partial_\alpha \left(v_i \widetilde{T}_{i\alpha}(\Xi) + \epsilon E v_i \partial_\alpha v_i + \epsilon^2 \lambda E \partial_t v_i \partial_\alpha v_i + \epsilon \lambda \partial_t v_i \widetilde{T}_{i\alpha}(\Xi) \right) \\ &\quad + \epsilon \left(E \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 - \lambda \partial_\alpha v_i \widetilde{W}_{i\alpha}^{j\beta}(\Xi) \partial_\beta v_j \right) + \epsilon(\lambda - 1)|\partial_t v|^2 = 0\,,\end{aligned} \tag{6.6}$$

where

$$\widetilde{W}_{i\alpha}^{j\beta}(\Xi) = \frac{\partial^2 g(\Xi)}{\partial \Xi^A \partial \Xi^B} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \frac{\partial \Phi^B(F)}{\partial F_{j\beta}} + \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial^2 \Phi^A(F)}{\partial F_{i\alpha} \partial F_{j\beta}}$$

and λ is an arbitrary constant.

Proof. Thanks to the null–Lagrangian nature of $\Phi(F)$, the following estimate holds

$$\begin{aligned}&\partial_t \left[\frac{1}{2}|v|^2 + \epsilon v_i \partial_t v_i + g(\Xi) \right] - \partial_\alpha \left[v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} + \epsilon E v_i \partial_\alpha v_i \right] \\ &\quad + \epsilon E \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 - \epsilon |\partial_t v|^2 = 0\,.\end{aligned} \tag{6.7}$$

At this point, we shall correct the above estimate by adding an energy estimate for the wave operator in (6.2). Thus, we proceed as in the proof of Lemma 4.1 to obtain the following relation

$$\begin{aligned}&\partial_t \left[\frac{1}{2}\epsilon^2 \lambda |\partial_t v|^2 + \frac{1}{2}\epsilon^2 \lambda E \sum_{\alpha=1}^{3} |\partial_\alpha v|^2 + \epsilon \lambda \partial_\alpha v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \right] \\ &\quad - \partial_\alpha \left[\epsilon^2 \lambda E \partial_t v_i \partial_\alpha v_i + \epsilon \lambda \partial_t v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \right]\end{aligned}$$

$$+\epsilon\lambda|\partial_t v|^2 - \epsilon\lambda\partial_\alpha v_i \partial_t\left(\frac{\partial g(\Xi)}{\partial \Xi^A}\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}}\right) = 0\,. \tag{6.8}$$

Finally, using once again the null–Lagrangian property of $\Phi(F)$, we have

$$\begin{aligned}&\partial_t\left(\frac{\partial g(\Xi)}{\partial \Xi^A}\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}}\right)\\ &\quad=\left(\frac{\partial^2 g(\Xi)}{\partial \Xi^A\partial\Xi^B}\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}}\frac{\partial \Phi^B(F)}{\partial F_{j\beta}}+\frac{\partial g(\Xi)}{\partial \Xi^A}\frac{\partial^2 \Phi^A(F)}{\partial F_{i\alpha}\partial F_{j\beta}}\right)\partial_\beta v_j\,,\end{aligned}$$

and we sum (6.7) and (6.8) to obtain (6.6). □

To exploit the strong dissipation estimate we assume that g is uniformly convex

$$\nabla^2_{\Xi} g(\Xi) \geq \gamma I, \quad \Xi\in\mathbb{R}^{19}\,, \tag{6.9}$$

and consider a framework of *a priori* uniformly bounded solutions. That is, we assume that the solutions F of (6.4) and $\widehat{F}$ of (6.3) satisfy the uniform bounds

$$|F|,\ |\widehat{F}| \leq M\,. \tag{6.10}$$

Define now $\Gamma > 0$ such that

$$\max_{|F|\leq M} \nabla^2_F W(F) \leq \Gamma I\,.$$

The energy $\mathcal{E}_{pm}$ defined in (6.5) is positive definite and dissipates if the following subcharacteristic condition holds:

$$E > \Gamma\,, \qquad E > \frac{\Gamma^2}{\gamma}\,. \tag{6.11}$$

Under (6.11), we can exploit the structural identity in Lemma 6.1 and obtain a dissipation estimate for the L^2-norm

$$\psi(t) := \int_{\mathbb{R}^3}\Big(|v(x,t)|^2 + |F(x,t)|^2 + \epsilon^2\Big(|\partial_t v(x,t)|^2 + \sum_{\alpha=1}^{3}|\partial_\alpha v(x,t)|^2\Big)\Big)dx\,.$$

Theorem 6.2. *Let $(v, F = \nabla y)$ be a smooth solution of (6.4), with initial datum $(v_0, F_0 = \nabla y(x,0))$, such that condition (6.10) holds and which decays sufficiently fast to zero as $|x| \to +\infty$. Let us assume conditions (6.9) and (6.11) hold. Then*

$$O(1)\psi(t) \leq \int_{\mathbb{R}^3}\mathcal{E}_{pm}(x,t)dx \leq \int_{\mathbb{R}^3}\mathcal{E}_{pm}(x,0)dx \leq O(1)\psi(0)\,.$$

Proof. Set

$$\varphi(t) = \int_{\mathbb{R}^3} E_{pm}(x,t)dx$$

and consider the identity (6.6). For any $\lambda > 1$, by the uniform convexity of g we have for some $c = c(\lambda) > 0$

$$\mathcal{E}_{pm} \geq C(|v|^2 + \epsilon^2|\partial_t v|^2) + \frac{1}{2}\gamma|\Xi|^2 + \frac{1}{2}\epsilon^2\lambda E\sum_{\alpha=1}^{3}|\partial_\alpha v|^2 + \epsilon\lambda\partial_\alpha v_i \widetilde{T}_{i\alpha}(\Xi)\,,$$

for any (v, Ξ).

For a smooth solution of (6.2) that decays sufficiently fast to zero as $|x| \to +\infty$, we obtain upon integrating (6.6) in x and t

$$\begin{aligned}\int_{\mathbb{R}^3}\left(c(|v|^2 + \epsilon^2|\partial_t v|^2) + \frac{1}{2}\gamma|\Xi|^2 + \frac{1}{2}\epsilon^2\lambda E\sum_{\alpha=1}^{3}|\partial_\alpha v|^2 + \epsilon\lambda\partial_\alpha v_i\widetilde{T}_{i\alpha}(\Xi)\right)dx \\ + \epsilon\int_0^t\int_{\mathbb{R}^3}\left(E\sum_{\alpha=1}^{3}|\partial_\alpha v|^2 - \lambda\partial_\alpha v_i\widetilde{W}_{i\alpha}^{j\beta}(\Xi)\partial_\beta v_j\right)dxds \leq \varphi(0)\,. \end{aligned} \tag{6.12}$$

We now employ estimate (6.12) for the special solution $\Xi = \Phi(F)$, with F verifying the aforementioned assumptions. Also $\lambda > 1$ is fixed so that

$$E > \lambda\Gamma\,, \quad E > \lambda\frac{\Gamma^2}{\gamma}\,.$$

Since $|\Phi(F)|^2 \geq |F|^2$ and

$$\widetilde{T}_{i\alpha}(\Phi(F)) = T_{i\alpha}(F) = \frac{\partial W(F)}{\partial F_{i\alpha}}\,, \qquad \widetilde{W}_{i\alpha}^{j\beta}(\Phi(F)) = \frac{\partial^2 W(F)}{\partial F_{i\alpha}\partial F_{j\beta}}\,,$$

we utilize the condition on E as in Theorem 4.2 to show that $\mathcal{E}_{pm}$ generates a positive definite quantity, equivalent to the norm $\psi(t)$. $\square$

We conclude the section with the relative energy estimate and the resulting control of the relaxation process. We introduce the modulated relative energy for system (6.4),

$$\begin{aligned}\mathcal{E}_{pmd} :=& \frac{1}{2}|v-\widehat{v}|^2 + +\epsilon(v_i - \widehat{v}_i)\partial_t(v_i-\widehat{v}_i) + \frac{1}{2}\epsilon^2\lambda|\partial_t(v-\widehat{v})|^2 \\ &+ g(\Xi) - g(\widehat{\Xi}) - \frac{\partial g}{\partial \Xi^A}(\widehat{\Xi})\,(\Xi^A - \widehat{\Xi}^A) \\ &+ \frac{1}{2}\epsilon^2\lambda E\sum_{\alpha=1}^{3}|\partial_\alpha(v-\widehat{v})|^2 + \epsilon\lambda\partial_\alpha(v_i-\widehat{v}_i)(\widetilde{T}_{i\alpha}(\Xi) - \widetilde{T}_{i\alpha}(\widehat{\Xi}))\,,\end{aligned} \tag{6.13}$$

its flux

$$
\begin{aligned}
\mathcal{F}_{\alpha,\,pmd} := & \left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \right) (v_i - \widehat{v}_i) \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \\
& + \epsilon E (v_i - \widehat{v}_i) \partial_\alpha (v_i - \widehat{v}_i) + \epsilon^2 \lambda E \partial_t (v_i - \widehat{v}_i) \partial_\alpha (v_i - \widehat{v}_i) \\
& + \epsilon \lambda \partial_t (v_i - \widehat{v}_i)(\widetilde{T}_{i\alpha}(\Xi) - \widetilde{T}_{i\alpha}(\widehat{\Xi}))
\end{aligned}
$$

and compute first a structural identity for the modulated relative energy.

Lemma 6.3. *Let (v, Ξ) and $(\widehat{v}, \widehat{\Xi})$ be smooth solutions of (6.2) and (6.1) respectively. Then we have*

$$
\begin{aligned}
& \partial_t \mathcal{E}_{pmd} - \partial_\alpha \mathcal{F}_{\alpha,\,pmd} \\
& \quad + \epsilon \left(E \sum_{\alpha=1}^{3} |\partial_\alpha (v - \widehat{v})|^2 - \lambda \partial_\alpha (v_i - \widehat{v}_i) \widetilde{W}_{i\alpha}^{j\beta}(\Xi) \partial_\beta (v_j - \widehat{v}_j) \right) \\
& \quad + \epsilon (\lambda - 1) |\partial_t (v - \widehat{v})|^2 \\
& = Q + \epsilon \lambda \partial_\alpha (v_i - \widehat{v}_i) \left(\widetilde{W}_{i\alpha}^{j\beta}(\Xi) - \widetilde{W}_{i\alpha}^{j\beta}(\widehat{\Xi}) \right) \partial_\alpha \widehat{v}_j \\
& \quad + \epsilon (v_i - \widehat{v}_i)(E \partial_\alpha \partial_\alpha \widehat{v}_i - \partial_t^2 \widehat{v}_i) + \epsilon^2 \lambda \partial_t (v_i - \widehat{v}_i)(E \partial_\alpha \partial_\alpha \widehat{v}_i - \partial_t^2 \widehat{v}_i)\,, \qquad (6.14)
\end{aligned}
$$

where $\widetilde{W}_{i\alpha}^{j\beta}(\Xi)$ is defined in Lemma 6.1, Q is given by (5.12) and λ is an arbitrary constant.

Proof. To control the quadratic part of the energy $\frac{1}{2}|v|^2 + g(\Xi)$ with respect to $(v - \widehat{v}, \Xi - \widehat{\Xi})$, we use the same arguments of Lemma 5.2 to obtain

$$
\begin{aligned}
& \partial_t \left[\frac{1}{2} |v - \widehat{v}|^2 + g(\Xi) - g(\widehat{\Xi}) - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} (\Xi^A - \widehat{\Xi}^A) \right] \\
& \quad - \partial_\alpha \left[\left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \right) (v_i - \widehat{v}_i) \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \right] \\
& \quad = Q + \epsilon (v_i - \widehat{v}_i)(E \partial_\alpha \partial_\alpha v_i - \partial_t^2 v_i)\,. \qquad (6.15)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\epsilon (v_i - \widehat{v}_i)(E \partial_\alpha \partial_\alpha v_i - \partial_t^2 v_i) = & \, \epsilon |\partial_t (v - \widehat{v})|^2 - \partial_t [\epsilon (v_i - \widehat{v}_i) \partial_t (v_i - \widehat{v}_i)] \\
& + \partial_\alpha [\epsilon E (v_i - \widehat{v}_i) \partial_\alpha (v_i - \widehat{v}_i)] - \epsilon E \sum_{\alpha=1}^{3} |\partial_\alpha (v - \widehat{v})|^2 \\
& + \epsilon (v_i - \widehat{v}_i)(E \partial_\alpha \partial_\alpha \widehat{v}_i - \partial_t^2 \widehat{v}_i)\,. \qquad (6.16)
\end{aligned}
$$

As we already did in the previous cases, we must correct the above estimate by adding the acoustic energy of the wave equation $(6.2)_2$, namely

$$\begin{aligned}
&\partial_t \left[\frac{1}{2}\lambda\epsilon^2 |\partial_t (v-\widehat{v})|^2 + \frac{1}{2}\lambda\epsilon^2 E \sum_{\alpha=1}^{3} |\partial_\alpha (v-\widehat{v})|^2 \right] \\
&\quad - \partial_\alpha \left[\lambda\epsilon^2 E \partial_t (v_i - \widehat{v}_i) \partial_\alpha (v_i - \widehat{v}_i) \right] + \lambda\epsilon |\partial_t (v-\widehat{v})|^2 \\
&\quad - \lambda\epsilon \partial_t (v_i - \widehat{v}_i) \partial_\alpha (\widetilde{T}_{i\alpha}(\Xi) - \widetilde{T}_{i\alpha}(\widehat{\Xi})) \\
&\quad = \lambda\epsilon^2 \partial_t (v_i - \widehat{v}_i)(E\partial_\alpha\partial_\alpha \widehat{v}_i - \partial_t^2 \widehat{v}_i)\,,
\end{aligned} \tag{6.17}$$

where λ is an arbitrary constant. Now, we interchange the x and t derivatives in the last term of the left hand side of (6.17), by using the identities

$$\begin{aligned}
&\partial_t \Big(\partial_\alpha (v_i - \widehat{v}_i)(\widetilde{T}_{i\alpha}(\Xi) - \widetilde{T}_{i\alpha}(\widehat{\Xi})) \Big) - \partial_\alpha \Big(\partial_t (v_i - \widehat{v}_i)(\widetilde{T}_{i\alpha}(\Xi) - \widetilde{T}_{i\alpha}(\widehat{\Xi})) \Big) \\
&\quad = \partial_\alpha (v_i - \widehat{v}_i) \partial_t (\widetilde{T}_{i\alpha}(\Xi) - \widetilde{T}_{i\alpha}(\widehat{\Xi})) - \partial_t (v_i - \widehat{v}_i) \partial_\alpha (\widetilde{T}_{i\alpha}(\Xi) - \widetilde{T}_{i\alpha}(\widehat{\Xi}))
\end{aligned} \tag{6.18}$$

and

$$\begin{aligned}
\partial_t (\widetilde{T}_{i\alpha}(\Xi) - \widetilde{T}_{i\alpha}(\widehat{\Xi})) &= \widetilde{W}^{j\beta}_{i\alpha}(\Xi)\partial_\beta v_j - \widetilde{W}^{j\beta}_{i\alpha}(\widehat{\Xi})\partial_\beta \widehat{v}_j \\
&= \widetilde{W}^{j\beta}_{i\alpha}(\Xi)\partial_\beta (v_j - \widehat{v}_j) + (\widetilde{W}^{j\beta}_{i\alpha}(\Xi) - \widetilde{W}^{j\beta}_{i\alpha}(\widehat{\Xi}))\partial_\beta \widehat{v}_j\,.
\end{aligned} \tag{6.19}$$

The identity resulting from (6.17), (6.18) and (6.19) is combined with the identities (6.15) and (6.16) to obtain (6.14). □

The relative energy established in the above lemma allows us to control the following L^2 distance between the relaxation and the limit solutions

$$\psi_d(t) := \int_{\mathbb{R}^3} \Big(|v-\widehat{v}|^2 + |F-\widehat{F}|^2 + \epsilon^2 \Big(|\partial_t (v-\widehat{v})|^2 + \sum_{\alpha=1}^{3} |\partial_\alpha (v-\widehat{v})|^2 \Big)\Big) dx$$

in the framework of solutions verifying (6.10), provided the uniform polyconvexity and subcharacteristic conditions (6.9) and (6.11) hold.

Theorem 6.4. *Let (v^ϵ, F^ϵ), $F^\epsilon = \nabla y^\epsilon$ and $(\widehat{v}, \widehat{F})$, $\widehat{F} = \nabla \widehat{y}$ be smooth solutions of (6.4) and (6.3) defined on $\mathbb{R}^3 \times [0,T]$ that decay sufficiently fast as $|x| \to +\infty$ and emanate from initial data $(v_0^\epsilon, F_0^\epsilon)$, $F_0^\epsilon = \nabla y^\epsilon(\cdot, 0)$ and $(\widehat{v}_0, \widehat{F}_0)$, $\widehat{F}_0 = \nabla \widehat{y}(\cdot, 0)$. Assume the a-priori bounds*

$$|F^\epsilon(x,t)|,\ |\widehat{F}(x,t)| \le M\,,$$

and that conditions (6.9) and (6.11) hold. Then, there exists a constant $C = C(T, \gamma, \Gamma, M, \widehat{v}, \widehat{F}) > 0$ *such that*

$$\psi_d(t) \le C \left(\psi_d(0) + \epsilon^2 \right) .$$

If also $\psi_d^\epsilon(0) \to 0$ *as* $\epsilon \downarrow 0$*, then*

$$\sup_{t\in[0,T]} \left(\|v^\epsilon(\cdot,t) - \widehat{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} + \|F^\epsilon(\cdot,t) - \widehat{F}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \right) \longrightarrow 0 \,,$$

as $\epsilon \downarrow 0$*.*

Proof. As in the proof of Theorem 6.2, we take a fixed $\lambda > 1$ such that

$$E > \lambda\Gamma \,, \qquad E > \lambda\frac{\Gamma^2}{\gamma} \,,$$

and we consider relation (6.14) for this λ. Let us denote by $\varphi(t)$ the quantity

$$\varphi(t) = \int_{\mathbb{R}^3} \mathcal{E}_{pmd}(v, \Xi; \widehat{v}, \widehat{\Xi}) dx \,,$$

where $\mathcal{E}_{pmd}$ is defined in (6.13). Then, integrating (6.14) in x and t we have

$$\begin{aligned}\varphi(t) + \epsilon \int_0^t \int_{\mathbb{R}^3} \Big(\big(E\delta_{i\alpha}\delta_{j\beta} - \lambda \widetilde{W}^{j\beta}_{i\alpha}(\Xi) \big) \partial_\alpha (v_i - \widehat{v}_i) \partial_\beta (v_j - \widehat{v}_j) \Big) \, dxds \\ \le \varphi(0) + \int_0^t \int_{\mathbb{R}^3} |\mathcal{R}^\epsilon| dxds \,, \end{aligned} \tag{6.20}$$

where

$$\begin{aligned}\mathcal{R}^\epsilon = Q + \epsilon\lambda\partial_\alpha(v_i - \widehat{v}_i) \left(\widetilde{W}^{j\beta}_{i\alpha}(\Xi) - \widetilde{W}^{j\beta}_{i\alpha}(\widehat{\Xi}) \right) \partial_\alpha \widehat{v}_j \\ + \epsilon(v_i - \widehat{v}_i)(E\partial_\alpha\partial_\alpha\widehat{v}_i - \partial_t^2\widehat{v}_i) + \epsilon^2\lambda\partial_t(v_i - \widehat{v}_i)(E\partial_\alpha\partial_\alpha\widehat{v}_i - \partial_t^2\widehat{v}_i)\end{aligned}$$

stands for the right hand side of (6.14). Moreover, the uniform convexity of g implies

$$\begin{aligned}\mathcal{E}_{pmd} \ge C(|v - \widehat{v}|^2 + \epsilon^2 |\partial_t (v - \widehat{v})|^2) + \frac{1}{2}\gamma|\Xi - \widehat{\Xi}|^2 \\ + \frac{1}{2}\epsilon^2\lambda E \sum_{\alpha=1}^{3} |\partial_\alpha (v - \widehat{v})|^2 + \epsilon\lambda\partial_\alpha(v_i - \widehat{v}_i)(\widetilde{T}_{i\alpha}(\Xi) - \widetilde{T}_{i\alpha}(\widehat{\Xi})) \,,\end{aligned}$$

for a positive constant $C = C(E, \gamma, \Gamma)$. Therefore relation (6.20) becomes

$$\begin{aligned}\int_{\mathbb{R}^3} \Bigg(C(|v - \widehat{v}|^2 + \epsilon^2 |\partial_t (v - \widehat{v})|^2) + \frac{1}{2}\gamma|\Xi - \widehat{\Xi}|^2 \\ + \frac{1}{2}\epsilon^2\lambda E \sum_{\alpha=1}^{3} |\partial_\alpha (v - \widehat{v})|^2 + \epsilon\lambda\partial_\alpha(v_i - \widehat{v}_i)(\widetilde{T}_{i\alpha}(\Xi) - \widetilde{T}_{i\alpha}(\widehat{\Xi})) \Bigg) dx\end{aligned}$$

$$+\,\epsilon\int_0^t\int_{\mathbb{R}^3}\left(E\sum_{\alpha=1}^{3}|\partial_\alpha(v-\widehat{v})|^2-\lambda\partial_\alpha(v_i-\widehat{v}_i)\widetilde{W}_{i\alpha}^{j\beta}(\Xi)\partial_\beta(v_j-\widehat{v}_j)\right)dxds$$
$$\leq\varphi(0)+\int_0^t\int_{\mathbb{R}^3}|\mathcal{R}^\epsilon|dxds\,. \tag{6.21}$$

Thus, we exploit relation (6.21) for the special solutions with $\Xi=\Phi(F)$ and $\widehat{\Xi}=\Phi(\widehat{F})$ and, since $\frac{1}{2}\gamma|\Xi-\widehat{\Xi}|^2\geq\frac{1}{2}\gamma|F-\widehat{F}|^2$ and

$$\widetilde{T}_{i\alpha}(\Phi(F))=T_{i\alpha}(F)=\frac{\partial W(F)}{\partial F_{i\alpha}}\,,\qquad \widetilde{W}_{i\alpha}^{j\beta}(\Phi(F))=\frac{\partial^2 W(F)}{\partial F_{i\alpha}\partial F_{j\beta}}\,,$$

we utilize the conditions on E as in Theorem 4.4 to have

$$\psi_d(t)\leq C\left(\psi_d(0)+\int_0^t\int_{\mathbb{R}^3}|\mathcal{R}^\epsilon|\Big|_{(\Xi,\widehat{\Xi})=(\Phi(F),\Phi(\widehat{F}))}dxds\right)\,,$$

where $C=C(E,\gamma,\Gamma,M)>0$. Moreover, the quadratic term $|Q|$ of $|\mathcal{R}^\epsilon|$ is controlled as in the proof of Theorem 5.3 and the last three terms are controlled as in the proof of Theorem 4.4. Therefore, there exists a positive constant $C=C(\gamma,\Gamma,M,\widehat{v},\widehat{F})$ such that

$$\psi_d(t)\leq C\left(\psi_d(0)+\epsilon^2 t+\int_0^t\psi_d(s)ds\right)$$

and the conclusion follows from the Gronwall lemma. □

Acknowledgements

This work was carried out during a visit of the first author at the "Institute of Applied and Computational Mathematics", FORTH and the "Department of Applied Mathematics", University of Crete, Heraklion (Greece). He would like to express his thanks for their hospitality.

This is a joint work within the EU–RTN "HYKE", Contract Number: HPRN-CT-2002-00282. CL is also supported by MIUR-COFIN-2002, Progetto Nazionale "Equazioni Iperboliche e Paraboliche Nonlineari". AET is partially supported by the National Science Foundation.

Appendix A. Comparison of modulated energy and the H-estimate near equilibrium

It is interesting to compare the dissipation structure that emerges from the H-theorem with the structure associated to the modulated energy. Both structures are globally defined in the state space, the first is motivated from thermodynamic ideas and essentially of kinetic theory flavor, while the second is motivated from energy considerations of continuum physics origin. Our objective is to compare the two structures near the "Maxwellian" equilibria, using the Chapman-Enskog expansion.

For the sake of clarity, we discuss the one dimensional model of stress–relaxation

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x \sigma = 0 \\ \partial_t (\sigma - Eu) = -\frac{1}{\epsilon}(\sigma - g(u)) \,, \end{cases} \tag{A.1}$$

where $W'(u) = g(u)$ and W convex. For (A.1), the equation (4.5) for the modulated energy $\mathcal{E}_m$ becomes (for $\lambda = 1$)

$$\begin{aligned} &\partial_t \left(\frac{1}{2} v^2 + W(u) + \epsilon v \partial_t v + \frac{1}{2}\epsilon^2 E |\partial_x v|^2 + \epsilon g(u) \partial_x v \right) \\ &\quad - \partial_x \left[v g(u) + \epsilon E v \partial_x v + \epsilon^2 E \partial_t v \partial_x v + \epsilon \partial_t v g(u) \right] \\ &\quad + \epsilon \left(E - g'(u) \right) |\partial_x v|^2 = 0 \,. \end{aligned} \tag{A.2}$$

The subcharacteristic condition (4.8) implies that the last term is nonnegative and $\mathcal{E}_m$ is coercive. On the other hand, (4.8) implies the free energy function $\Psi(u,a)$, which presently reads

$$\Psi(u,a) = \frac{1}{2} E u^2 + a u - \int_0^a h^{-1}(\xi) d\xi \,,$$

is uniformly convex in (u,a). The associated (weak) dissipation estimate coming from the H–theorem (3.2) is

$$\begin{aligned} &\partial_t \Big(\frac{1}{2} |v|^2 + \Psi(u, \sigma - Eu) \Big) - \partial_x [v\sigma] \\ &\qquad + \frac{1}{\epsilon} (u - h^{-1}(\sigma - Eu))(\sigma - g(u)) = 0 \,, \end{aligned} \tag{A.3}$$

where the last term capturing the dissipation of viscoelastic stresses is nonnegative.

We next investigate the relations of these two dissipative estimates in terms of the Chapman–Enskog expansion of (A.1). Carrying out the expansion up to order ϵ^2 gives

$$\sigma = g(u) + \epsilon (E - g'(u)) \partial_x v + \epsilon^2 \left(g''(u) |\partial_x v|^2 + (g'(u) - E) \partial_x^2 g(u) \right) \,.$$

The corresponding Burnett approximation of (A.1) reads

$$\begin{aligned}
&\partial_t u - \partial_x v = 0 \\
&\partial_t v - \partial_x g(u) = \epsilon \partial_x \big((E - g'(u)) \partial_x v \big) \\
&\qquad\qquad + \epsilon^2 \partial_x \left(g''(u) |\partial_x v|^2 + (g'(u) - E) \partial_x^2 g(u) \right) .
\end{aligned}$$

Consider now the H-estimate (A.3) and perform the Chapman–Enskog expansion. A tedious but straightforward calculation, keeping terms up to order $O(\epsilon^2)$, gives

$$\begin{aligned}
&\partial_t \left(\frac{1}{2} |v|^2 + \Psi(u, h(u)) + \frac{1}{2} \epsilon^2 (E - g'(u)) |\partial_x v|^2 \right) \\
&\quad - \partial_x \Big(v g(u) + \epsilon v (E - g'(u)) \partial_x v + \epsilon^2 v (g''(u) |\partial_x v|^2 + (g'(u) - E) \partial_x^2 g(u)) \Big) \\
&\quad + \epsilon (E - g'(u)) |\partial_x v|^2 \\
&\quad + \epsilon^2 \left(\frac{3}{2} g''(u) (\partial_x v)^3 + 2 (g'(u) - E)(\partial_x v) \partial_x^2 g(u) \right) = 0 \,. \qquad \text{(A.4)}
\end{aligned}$$

Note that (A.4) is an expansion that holds near the equilibrium manifold $\sigma \sim g(u)$, while (A.2) holds on the entire state space. To compare the two (for ϵ near zero), note that

$$\Psi(u, h(u))' = g(u) = W'(u)$$

and that

$$\begin{aligned}
\partial_x [\epsilon v g'(u) \partial_x v] &= \partial_x [\epsilon v \partial_t g(u)] \\
&= \partial_t [\epsilon \partial_x v g(u)] - \partial_x [\epsilon \partial_t v g(u)] + \partial_t [\epsilon v \partial_t v] \,,
\end{aligned}$$

because in the Chapman–Enskog expansion we assume the variables are near equilibrium, that is

$$\partial_t u = \partial_x v \,, \qquad\qquad \partial_t v = \partial_x g(u) \,. \qquad \text{(A.5)}$$

Using the above identities we see that the expansions of the two identities near $\epsilon \sim 0$ coincide up to terms of order $O(\epsilon)$, and deviate from each other already from the terms of order $O(\epsilon^2)$.

References

[1] Ch. Arvanitis, Ch. Makridakis, and A.E. Tzavaras *Stability and convergence of a class of finite element schemes for hyperbolic systems of conservation laws*, SIAM J. Numer. Analysis, to appear.

[2] J.M. Ball, *Convexity conditions and existence theorems in the nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1977), 337–403.

[3] J.M. Ball, *Some open problems in elasticity* In: Geometry, mechanics and dynamics, Springer, New York, 2002, pp. 3-59.

[4] Y. Brenier, *Hydrodynamic structure of the augmented Born-Infeld equations*, Arch. Rational Mech. Anal. **172** (2004), 65-91.

[5] Y. Brenier, R. Natalini, and M. Puel, *On a relaxation approximation of the incompressible Navier–Stokes equations*, Proc. Amer. Math. Soc., **132** (2004), 1021-1028.

[6] R. Caflisch, and G.C. Papanicolaou, *The fluid dynamical limit of a nonlinear model Boltzmann equation*, Comm. Pure Appl. Math. **32** (1979), 589–616.

[7] G.-Q. Chen, C.D. Levermore, and T.-P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure and Appl. Math. **47** (1994), 787–830.

[8] P.G. Ciarlet, *Mathematical Elasticity. Vol. I. Three–dimensional elasticity*, Studies in Mathematics and its Applications, vol. 20, North–Holland, 1988.

[9] B.D. Coleman and M.E. Gurtin, *Thermodynamics with internal state variables*, J. Chem. Physics **47** (1967), 597-613.

[10] F. Coquel and B. Perthame, *Relaxation of energy and approximate Riemann solvers for general pressure laws in fluid dynamics*, SIAM J. Num. Anal. **35** (1998), 2223-2249.

[11] C.M. Dafermos, *The second law of thermodynamics and stability*, Arch. Rational Mech. Analysis textbf70 (1979), 167-179.

[12] C.M. Dafermos, *Quasilinear hyperbolic systems with involutions*, Arch. Rational Mech. Anal. **94** (1986), 373-389.

[13] C.M. Dafermos, *Hyperbolic conservation laws in continuum physics*, Grundlehren der Mathematischen Wissenschaften, vol. 325, Springer-Verlag, Berlin, 2000.

[14] C.M. Dafermos, Ch. V of 2nd edition of [13], (private communication).

[15] S. Demoulini, D.M.A. Stuart, and A.E. Tzavaras, *A variational approximation scheme for three–dimensional elastodynamics with polyconvex energy*, Arch. Ration. Mech. Anal. **157** (2001), 325–344.

[16] C. Faciu and M. Mihailescu-Suliciu, *The energy in one-dimensional rate-type semilinear viscoelasticity*, Int. J. Solids Structures **23** (1987), 1505-1520.

[17] T. Qin, *Symmetrizing nonlinear elastodynamic system*, J. Elasticity **50** (1998), 245-252.

[18] S. Jin and M. Slemrod, *Regularization of the Burnett equations via relaxation*, J. Statist. Phys. **103** (2001), no. 5-6, 1009–1033.

[19] R.T. Rockafellar and R.J-B. Wets, *Varational Analysis*, Grundlehren der Mathematischen Wissenschaften, vol. 317, Springer-Verlag, Berlin, 1998.

[20] D. Serre, *Hyperbolicity of the Nonlinear Models of Maxwell's Equations*, Arch. Rational Mech. Anal. **172** (2004), 309–331.

[21] I. Suliciu, *On the thermodynamics of rate-type fluids and phase transitions. I. Rate-type fluids*, Internat. J. Engrg. Sci. **36** (1998), no. 9, 921–947.

[22] A. Tzavaras, *Materials with Internal Variables and Relaxation to Conservation Laws*, Arch. Rational Mech. Anal. **146** (1999), 129–155.

[23] W.A. Yong, *Singular perturbations of first-order hyperbolic systems with stiff source terms*, J. Differential Equations **155** (1999), 89–132.

[24] Y. Zeng, *Gas dynamics in thermal nonequilibrium and general hyperbolic systems with relaxation*, Arch. Rational Mech. Anal. **150** (1999), 225-279.

Corrado Lattanzio – Sezione di Matematica per l'Ingegneria, Dipartimento di Matematica Pura ed Applicata, Piazzale E. Pontieri, 2, Monteluco di Roio, 67040 L'Aquila, Italy
E-mail address: corrado@univaq.it

Athanasios E. Tzavaras – Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706, USA
E-mail address: tzavaras@math.wisc.edu