A non standard free-boundary problem arising from stratigraphy

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In this paper, we are interested in the mathematical analysis of a geological stratigraphic model, taking into account a limited weathering condition. Firstly, we present the physical model and the mathematical formulation, which lead to an original conservation law. Then, the definition of a solution and some mathematical tools in order to resolve the problem are given. At last, we treat the $1-D$ case and we present some open problems.

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1. Introduction

In this paper, we are interested in the mathematical study of a stratigraphic model recently developed by the Institut Français du Pétrole (IFP). The model concerns geologic basin formation by the way of erosion and sedimentation. A novel approach that leads to mathematical questions which are difficult to answer in the framework of ill-posed free-boundary problems.

By taking into account large scale in time and space and by knowing *a priori*, the tectonics, the eustatism and the sediments flux at the basin boundary, the model has to state about the transport of sediments. One may find, on the one hand, in D. Greanjeon *and al.*[11] and R. Eymard *and al.*[9] the physical and the numerical modelling of the multi-lithological case, and, on the other hand, in S.N. Antontsev *and al.*[1] and G. Gagneux *and al.*[10] a mathematical analysis of the mono-lithological case.

Let us consider in the sequel a sedimentary basin, denoted by Ξ with base Ω a smooth, bounded domain in $\mathbb{R}^N$ $(N = 1, 2)$, determined by a known vertical position given by $H(t, x)$ at each time t and position x, and, for any positive T, one notes $Q =]0, T \times \Omega$.

One denotes by u the sediments height, thus the topography is then given by $u + H$ and one is led to consider a gravitational model where:

i) the sediments flux $\overrightarrow{q}$ is assumed to be proportional to $K\nabla h(u+H)$, where K is a viscosity rate

and

ii) the erosion speed, $\partial_t u$ in its nonpositive part, is underestimated by $-E$, where E is a given nonnegative bounded measurable function in Q (a weathering limited process): *i.e.* $\partial_t u + E \geq 0$ a.e. in Q.

The original aspect of this model is its weather limited condition on the erosion rate, leading to an ill-posed diffusion equation.

In order to join together the constraint and a conservative formulation, D. Greanjeon[11] proposes to correct the diffusive flux $-K\,\nabla h(H+u)$ by introducing a dimensionless multiplier λ: the new flux is then $-\lambda K\,\nabla h(H+u)$, where λ is an unknown function with values *a priori* in $[0, 1]$.

In the realistic geological problem, $\Gamma = \partial\Omega = \Gamma_e \cup \Gamma_s$ and the boundary conditions are:
$-\lambda\partial_n h(H+u) = f$ on the inflow boundary Γ_e;
and some unilateral constraints on the outflow boundary:
$\partial_n h(H+u) + f \geq 0$, $\partial_t u + E \geq 0$ and $(\partial_n h(H+u) + f)(\partial_t u + E) = 0$;
where f is a given bounded measurable function on $\Sigma =]0, T[\times\Omega$.

Therefore, the mathematical modelling has to express respectively:

• the mass balance of the sediment:

$$\partial_t u - div(\lambda\nabla h(H+u)) = 0 \quad in \quad Q. \tag{1.1}$$

• the boundary conditions on $\partial\Omega = \overline{\Gamma_e} \cup \overline{\Gamma_s}$:

$$-\lambda\partial_n h(H+u) = f \quad on \quad]0, T[\times\Gamma_e, \tag{1.2}$$

$$\left.\begin{array}{l} \partial_t u + E \geq 0, \quad \lambda\partial_n h(H+u) + f \geq 0 \\ and\ (\lambda\partial_n h(H+u) + f)(\partial_t u + E) = 0 \end{array}\right\} \quad on \quad]0, T[\times\Gamma_s. \tag{1.3}$$

• the weather limited condition (moving obstacle):

$$\partial_t u \geq -E \quad in \quad Q. \tag{1.4}$$

• the initial condition:

$$u(0,.) = u_0 \quad in \quad \Omega, \tag{1.5}$$

and the closure of the model by defining the role of the flux limitor λ.

In order to simplify, one considers in the sequel:

i) homogeneous Dirichlet conditions on the boundary (G. Gagneux, D. Etienne and G. Vallet[8] have considered boundary conditions of unilateral type. The mathematical analysis is inspired by the chapter 2 of G. Duvaut and J.L. Lions[7] and by the "new problems" of J.-L Lions[12] p.420, dealing of thermic.)

and

ii) $H = 0$, $K = Id$ and $h = Id$.

Therefore, the problem becomes :

Look for u, *a priori* in $H^1(Q) \cap L^\infty(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$, such that,

$$\partial_t u + E \geq 0 \quad in \quad Q, \quad u(0,.) = u_0 \quad in \quad \Omega,$$

with u_0 in $H_0^1(\Omega) \cap L^\infty(\Omega)$ and

$$\partial_t u(t,x) - Div\,\{\lambda(t,x)\nabla u(t,x)\} = 0 \quad in \quad Q. \tag{1.6}$$

In order to give a mathematical modelling of λ, Th. Gallouët and R. Masson[9] propose to consider the following global constraint:

$$\partial_t u + E \geq 0, \quad 1 - \lambda \geq 0 \quad and \quad (\partial_t u + E)(1 - \lambda) = 0 \quad in\, Q. \tag{1.7}$$

It means that, the flux has to be corrected because of the constraint (*i.e.* if $\lambda < 1$, then $\partial_t u + E = 0$). If the erosion rate constraint is inactive, the flux is equal to the diffusive one.

Then S.N. Antontsev, G. Gagneux and G. Vallet[2] propose the following conservative formulation that contains implicitly (1.7). See G. Vallet[15] too.

If H denotes the maximal monotone graph of the Heaviside function (*i.e.* $H(x) = 0$ if $x < 0$, $H(x) = 1$ if $x > 0$ and $H(0) = [0, 1]$) then (λ, h) is formally a solution of :

$$0 = \partial_t u - div(\lambda \nabla u) \quad where \quad \lambda \in H(\partial_t u + E) \quad in\, Q. \tag{1.8}$$

Remark 1.1. Let us consider the following conjecture :

If $F \in H(div, \Omega) = \{F \in (L^2(\Omega))^N,\ div(F) \in L^2(\Omega)\}$, then $div(F) = 0$ a.e. in $\{F = 0\}$.

This result is well known if $N = 1$, that is the Saks lemma. It seems reasonable if $N > 1$ (W. P. Ziemer, personal communication).

Therefore, if one denotes by $F = \lambda \nabla u$, $\{\partial_t u + E < 0\} \subset \{F = 0\}$ and one gets:

$$\begin{aligned} 0 = -div(\lambda \nabla u) 1_{\{\partial_t u + E < 0\}} &= -\partial_t u.1_{\{\partial_t u + E < 0\}} \\ &= -(\partial_t u + E)^- + E.1_{\{\partial_t u + E < 0\}} \\ &\geq (\partial_t u + E)^-. \end{aligned}$$

So, $\partial_t u + E \geq 0$ and the equivalence of the two formulations is easily proved.

From where our interest for the study of equations (resp. differential inclusions) of the type

$$0 = \partial_t u - div(a(\partial_t u + E)\nabla u), \quad resp. \quad 0 \in \partial_t u - div(H(\partial_t u + E)\nabla u).$$

In our knowledge, there are no mathematical studies of such equations, while S.N. Antontsev[2] points out the presence of conservation laws of the shape

$$0 = \partial_t u - div(a(u, \partial_t u)\nabla u),$$

in fluid mechanics for describing a one-dimensional unsteady vertical filtration flow in inhomogeneous multi-stratum soil: see Y. Mualem and G. Dagan[13] and A. Poulovassilis and E. C. Childs[14].

Let us make some formal qualitative remarks on the equation:

$$0 = \partial_t u - div(a(\partial_t u)\nabla u).$$

i) Note that $0 = \partial_t u - a'(\partial_t u)\nabla\partial_t u \nabla u - a(\partial_t u)\Delta u$.
Thus, $-\frac{|a'(\partial_t u)\nabla u|^2}{4} \leq 0$ and the equation is of degenerated hyperbolic type.
ii) The equation is of degerenated type, but u cam be searched in $H^1(Q) \cap L^\infty(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$, since the test function

$$v = \int_0^{\partial_t u} \frac{1}{a(s)}\, ds,$$

leads to such a regularity if $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

2. Definition of a solution and existence

So, we are looking for (u, λ) in $[H^1(Q) \cap L^\infty(0, T; H_0^1(\Omega)) \cap L^\infty(Q)] \times L^\infty(Q)$ a solution to the initial value problem:

$$\begin{cases} \partial_t u - div(\lambda\nabla u) = 0, & \lambda \in H(\partial_t u + E) \quad in\ Q, \\ u(0,.) = u_0, & u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases}$$

where u_0 is the initial topography.

The equation has to be understood in the variational sense:

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \{\partial_t u v + \lambda\nabla u \nabla v\}\, dx = 0, \quad a.e.\ t \in]0, T[.$$

Let us give some remarks on the sedimentation process *i.e.* if $E = 0$.

i) First, note that if (u, λ) is a solution with $\lambda = 1$ a.e., then, at least in the distribution sense, $\Delta u_0 \geq 0$ is necessary. So, in the general case, λ needs to take values less than 1.

ii) Since $\partial_t u \geq 0$, one has, for a.e. t, $\lambda\nabla u^+ = 0$ a.e. in Ω.
Then, if for example $u_0 \geq 0$ in Ω, $\lambda\nabla u = 0$ a.e. in Ω, $\partial_t u = 0$ and $u(t,.) = u_0$ a.e. in Ω. Thus, if $\nabla u_0 \neq 0$ in Ω, the problem degenerates.

iii) At last, assume that $u_0 \leq 0$ with $\Delta u_0 \geq 0$. Then:
The couple $(u_0, 0)$ is a solution. Note that if $E = 0$, this couple, and more generally $(u_0, 1_{\{x, \nabla u_0 = 0\}})$, is always a solution.

A second solution is $(u, 1)$, where u is the solution to the heat equation, since in that case it is compatible with the constraint according to a minimal principle.

Since λ is needed to correct the theoretical flux, it needs to be as close as possible to 1. Then, one is looking for a maximal λ in the sense: if (w, μ) is another solution then $\mu \leq \lambda$. So, the definition of an admissible solution is:

Definition 2.1. For any u_0 in $H_0^1(\Omega) \cap L^\infty(\Omega)$ a solution to (1.8) is a couple (u, λ) of $[H^1(Q) \cap L^\infty(0, T; H_0^1(\Omega))] \times L^\infty(Q)$ such that:

$$\lambda \in H(\partial_t u + E), \quad u(t = 0) = u_0,$$

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \{\partial_t u v + \lambda \nabla u \nabla v\} \, dx = 0 \quad a.e.\ t \in]0, T[, \tag{2.1}$$

and λ is maximal in the sense: if (w, μ) is another solution then $\mu \leq \lambda$ a.e. in Q.

2.1. *The sedimentation case*

Let us assume that $E = 0$ and that a solution (u, λ) exists. The aim of this section is to give some qualitative properties of such a solution, in particular, for understanding the hyperbolic effects of an equation that seems to be parabolic.

Proposition 2.1. *Assume that (λ, u) solves (2.1).*
Then, $\lambda \nabla u^+ = 0$ and $u^+ = u_0^+$ a.e. in Q.
Thus, for any t, $u(t, .) = u_0$ a.e. in $\{u_0 \geq 0\}$ (so $u(t, .) \leq 0$ a.e. in $\{u_0 \leq 0\}$).

Proof. Consider $v = u^+$ as a test function in equation (2.1). Then, one has

$$\int_\Omega \partial_t u \, u^+ dx + \int_\Omega \lambda \nabla^2 u^+ \, dx = 0,$$

and,

$$\int_\Omega \partial_t u^+ \, (u^+ - u_0^+) \, dx + \int_\Omega \lambda \nabla^2 u^+ \, dx = - \int_\Omega \partial_t u^+ \, u_0^+ \, dx.$$

Since $\partial_t u \geq 0$, $\partial_t u^+ \geq 0$ too and one gets,

$$\int_\Omega \partial_t (u^+ - u_0^+) \, (u^+ - u_0^+) \, dx + \int_\Omega \lambda \nabla^2 u^+ \, dx \leq 0.$$

Therefore, for any t in $]0, T[$, one obtains

$$\frac{1}{2} \int_\Omega (u^+ - u_0^+)^2(t) \, dx + \int_0^t \int_\Omega \lambda \nabla^2 u^+ \, dx ds \leq 0,$$

and the result holds. □

Corollary 2.1. *(Barrier effect) Assume that there exists a compact set K and an open set ω with $K \subset \omega \subset \Omega$ and $\omega \backslash K \subset \{u_0 \geq 0\}$; then, for any t, $u(t,.) = u_0$ in ω.*

Proof. Thanks to the theorem of Urysohn, there exists v in $H_0^1(\Omega)$ such that $1_K \leq v \leq 1_\omega$. Then, one has, a.e. in $]0,T[$,

$$\int_{\omega\backslash K} \partial_t u\, v dx + \int_K \partial_t u\, v dx + \int_{\omega\backslash K} \lambda \nabla u \nabla v\, dx + \int_K \lambda \nabla u \nabla v\, dx = 0.$$

Note that $\nabla v = 0$ a.e. in K, that $\lambda \nabla u = 0$ in $\omega \backslash K$ since it is a subset of $\{u_0 \geq 0\}$ and that $\partial_t u \geq 0$.

Then, one gets $\int_K \partial_t u\, dx \leq 0$.

Therefore $\partial_t u = 0$ a.e. in ω and the result holds. □

In order to prove the existence of a solution, a method of time - discretization is used, with a technique of artificial viscosity. Finally, one supposes that $E \in L^2(0,T,H^1(\Omega))$.

Thus, considering two positive real parameters ε and h, one notes $E_k = \frac{1}{2h}\int_{](k-1)h,(k+1)h} E(s)\, ds$ and H_ε is any lipschitzian-continuous function satisfying

$$\forall x \in \mathbb{R}, \quad \max[\varepsilon, \min(\frac{(1-\varepsilon)x}{\varepsilon} + 1, 1)] \leq H_\varepsilon(x) \leq \max[\varepsilon, \min(\frac{(1-\varepsilon)x}{\varepsilon} + \varepsilon, 1)],$$

and $A_\varepsilon(x) = \int_0^x H_\varepsilon(\sigma)\, d\sigma$.

Proposition 2.2. *For any u_0 in $H_0^1(\Omega) \cap L^\infty(\Omega)$ and any nonnegative E in $H^1(\Omega)$, there exists a unique u_ε in $H_0^1(\Omega)$ such that $\forall v \in H_0^1(\Omega)$,*

$$\int_\Omega \{\frac{u_\varepsilon - u_0}{h} v + H_\varepsilon(\frac{u_\varepsilon - u_0}{h} + E) \nabla u_\varepsilon . \nabla v\}\, dx = 0.$$

Moreover, $\inf_\Omega \text{ess}\, u_0 \leq u_\varepsilon \leq \sup_\Omega \text{ess}\, u_0$.

Proof. *Claim 1: Existence by using Schauder-Tykonov fixed point theorem.*
Let us denote by S the application from $H_0^1(\Omega)$ to $H_0^1(\Omega)$, defined for any g in $H_0^1(\Omega)$ by: $S(g) = u_g$ where u_g is the unique solution to Problem:

$$P_g : \begin{cases} u_g \in H_0^1(\Omega), \quad \forall v \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \{\frac{u_g - u_0}{h} v + H_\varepsilon(\frac{g - u_0}{h} + E) \nabla u_g . \nabla v\}\, dx = 0. \end{cases}$$

Since $H_\varepsilon \geq \varepsilon > 0$, such an application exists thanks to Lax-Milgram theorem.
Note that, using u_g as a test-function, one gets

$$||u_g||^2_{L^2(\Omega)} + 2h\varepsilon ||\nabla u_g||^2_{[L^2(\Omega)]^N} \leq ||u_0||^2_{L^2(\Omega)},$$

and $S(g) \in K = \overline{B}_{H_0^1(\Omega)}(0,R)$ where $R = \frac{||u_0||_{L^2(\Omega)}}{\sqrt{\min(1,2h\varepsilon)}}$.
Consider now a sequence (g_n) in K that converges weakly towards an element g of

K. Then, g_n converges in $L^2(\Omega)$ and, *a.e.* in Ω, up to a sub-sequence still denoted by g_n.
As H_ε is continuous and bounded, thanks to the dominated convergence theorem, for any v in $H_0^1(\Omega)$, $H_\varepsilon(\frac{g_n-u_0}{h}+E)\nabla v$ converges towards $H_\varepsilon(\frac{g-u_0}{h}+E)\nabla v$ in $[L^2(\Omega)]^N$.
Since $(u_{g_n})\subset K$, an other sub-sequence can be extracted that converges weakly towards an element w in $H_0^1(\Omega)$. Then, w is a solution of problem P_g and as this solution is unique, $w=u_g$. Thus, all the sequence u_{g_n} converges weakly towards u_g in $H_0^1(\Omega)$.

Therefore, the theorem of Schauder-Tykonov leads to the existence of a solution u_ε.

Claim 2: Uniqueness.

Let us set $w=\frac{u_\varepsilon-u_0}{h}+E$. Then, $u_\varepsilon=h(w-E)+u_0$ and $\forall v\in H_0^1(\Omega)$,

$$\int_\Omega \{(w-E)\,v+h\nabla A_\varepsilon(w).\nabla v+H_\varepsilon(w)\nabla(u_0-hE).\nabla v\}\,dx=0. \tag{2.2}$$

If $u_{\varepsilon,i}$ $i=1,2$ represent two possible solutions with $w_i=\frac{u_{\varepsilon,i}-u_0}{h}+E$ then,

$$\forall v\in H_0^1(\Omega),\qquad \int_\Omega\{(w_2-w_1)\,v+h\nabla[A_\varepsilon(w_2)-A_\varepsilon(w_1)].\nabla v\}\,dx + \int_\Omega [H_\varepsilon(w_2)-H_\varepsilon(w_1)]\nabla(u_0-hE).\nabla v\,dx=0.$$

By considering $v=p_\delta(A_\varepsilon(w_2)-A_\varepsilon(w_1))$ where $p_\delta(x)=\max(0,\min(1,x/\delta))$, one gets

$$\begin{aligned}&\int_\Omega\{(w_2-w_1)\,v+hp'_\delta(A_\varepsilon(w_2)-A_\varepsilon(w_1))\nabla^2[A_\varepsilon(w_2)-A_\varepsilon(w_1)]\}\,dx\\ \le&\int_\Omega\frac{[H_\varepsilon(w_2)-H_\varepsilon(w_1)]^2}{2h}p'_\delta(A_\varepsilon(w_2)-A_\varepsilon(w_1))\nabla^2(u_0-hE)\,dx\\ &+\int_\Omega\frac{hp'_\delta(A_\varepsilon(w_2)-A_\varepsilon(w_1))}{2}\nabla^2[A_\varepsilon(w_2)-A_\varepsilon(w_1)]\;dx,\end{aligned}$$

i.e.

$$\begin{aligned}&\int_\Omega\{(w_2-w_1)\,v+\frac{h}{2}p'_\delta(A_\varepsilon(w_2)-A_\varepsilon(w_1))\nabla^2[A_\varepsilon(w_2)-A_\varepsilon(w_1)]\}\,dx\\ \le&\,||(H_\varepsilon\circ A_\varepsilon^{-1})'||_\infty^2\int_\Omega\frac{\delta^2}{2h}p'_\delta(A_\varepsilon(w_2)-A_\varepsilon(w_1))\nabla^2(u_0-hE)\,dx.\end{aligned}$$

Since, the second hand part of this inequality converges towards zero when δ goes to 0^+, one concludes that $w_2\le w_1$ a.e. in Ω. In the same way $w_2\le w_1$ a.e. in Ω can be proved and the solution is unique.

Claim 3: Maximum principle.

In order to prove this result, let us set $m = \inf_\Omega \text{ess } u_0$, $M = \sup_\Omega \text{ess } u_0$ and $f(x) = (x-M)^+ - (x-m)^-$. Since $u_0 \in H_0^1(\Omega)$, $m \le 0 \le M$ and $f(u_\varepsilon) \in H_0^1(\Omega)$ too. Thus,

$$\int_\Omega \{\frac{u_\varepsilon - u_0}{h} f(u_\varepsilon) + H_\varepsilon(\frac{u_\varepsilon - u_0}{h} + E) f'(u_\varepsilon)\nabla^2 u_\varepsilon\}\, dx = 0.$$

Note that $(u_\varepsilon - u_0)f(u_\varepsilon) \ge 0$ and that $f'^2 = f'$.

So, one gets $\varepsilon \int_\Omega \nabla^2 f(u_\varepsilon)\, dx = 0$ and the result holds. □

Proposition 2.3. *For any u_0 in $H_0^1(\Omega) \cap L^\infty(\Omega)$ and any nonnegative E in $H^1(\Omega)$, there exists (λ, u) in $L^\infty(\Omega) \times H_0^1(\Omega)$ such that $\lambda \in H(\frac{u-u_0}{h} + E)$ and $\forall v \in H_0^1(\Omega)$,*

$$\int_\Omega \{\frac{u-u_0}{h} v + \lambda \nabla u . \nabla v\}\, dx = 0. \tag{2.3}$$

Moreover, $\inf_\Omega \text{ess } u_0 \le u_\varepsilon \le \sup_\Omega \text{ess } u_0$ and $u \ge u_0 - hE$ a.e. in Ω.

First, let us give some *a priori* estimates.

Lemma 2.1. *Denote by $w_\varepsilon = \frac{u_\varepsilon - u_0}{h} + E$.*
i) (w_ε) and $(A_\varepsilon(w_\varepsilon))$ are bounded generalised sequences respectively in $L^2(\Omega)$ and $H^1(\Omega)$.
ii) $(w_\varepsilon)^-$ converges towards 0 in $L^2(\Omega)$.

Proof. i) Let us set $v = w_\varepsilon - E$. Then, (2.2) gives

$$\int_\Omega \{(w_\varepsilon - E)^2 + h\nabla A_\varepsilon(w_\varepsilon).\nabla w_\varepsilon + \nabla(u_0 - 2hE).\nabla A_\varepsilon(w_\varepsilon)\}\, dx$$
$$= \int_\Omega H_\varepsilon(w_\varepsilon)\nabla(u_0 - hE).\nabla E\, dx.$$

So, since $0 \le H_\varepsilon \le 1$, one gets

$$\begin{aligned} &\int_\Omega \{(w_\varepsilon - E)^2 + h\nabla^2 A_\varepsilon(w_\varepsilon)\}\, dx \\ \le\ & \frac{1}{2h}||\nabla(2hE - u_0)||^2_{L^2(\Omega)^N} + \frac{h}{2}||\nabla A_\varepsilon(w_\varepsilon)||^2_{L^2(\Omega)^N} \\ &+ ||\nabla(u_0 - hE)||_{L^2(\Omega)^N}||\nabla E||_{L^2(\Omega)^N}, \end{aligned}$$

and the result holds since $|A_\varepsilon| \le |id|$.
ii) Let us set $v = -(w_\varepsilon + \varepsilon)^- \in H_0^1(\Omega)$ (since $E \ge 0$). Then, one has

$$\int_\Omega \{v^2 - H_\varepsilon(w_\varepsilon)\nabla u_\varepsilon.\nabla(w_\varepsilon + \varepsilon)^-\}\, dx = -\int_\Omega (E+\varepsilon)(w_\varepsilon + \varepsilon)^-\, dx \le 0,$$

i.e.

$$\int_\Omega \{v^2 + H_\varepsilon(w_\varepsilon)\nabla^2 v\}\, dx \le \int_\Omega H_\varepsilon(w_\varepsilon)\nabla(u_0 - hE).\nabla(w_\varepsilon + \varepsilon)^-\, dx.$$

Since $H_\varepsilon(w_\varepsilon) = \varepsilon$ if $w_\varepsilon \le -\varepsilon$, one gets

$$2\int_\Omega \{v^2 + \varepsilon\nabla^2 v\}\, dx \le \varepsilon \int_{\Omega\cap\{w_\varepsilon < -\varepsilon\}} \nabla^2(u_0 - hE)\, dx \le C\varepsilon,$$

and the result holds since $(w_\varepsilon)^- \le (w_\varepsilon + \varepsilon)^- + \varepsilon$. □

Proof. (of the proposition.)

A sub-sequence can be extracted, still denoted with ε, such that:
i) w_ε converges weakly towards w in $L^2(\Omega)$,
ii) $A_\varepsilon(w_\varepsilon)$ converges weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$ and a.e. in Ω towards χ
iii) $(w_\varepsilon)^-$ converges towards 0 in $L^2(\Omega)$.
Thanks to the hypothesis on function H_ε:
On the one hand, one gets

$$|A_\varepsilon(w_\varepsilon) - w_\varepsilon^+| \le \varepsilon|w_\varepsilon^-| + \varepsilon.$$

Thus, w_ε^+ converges in $L^2(\Omega)$ towards χ, as well as w_ε since w_ε^- converges to 0 in $L^2(\Omega)$.
Then, one has: $w = \chi \ge 0$.
On the other hand, up to a sub-sequence, still indexed by ε, one gets:

$$\exists\lambda \in L^\infty(\Omega), \quad 0 \le \lambda \le 1 \; a.e. \; in \; \Omega, \quad H_\varepsilon(w_\varepsilon) \rightharpoonup \lambda \; in \; L^\infty(\Omega) \; weak - *.$$

Let us denote by $A^{+(-)}$ the set $A^{+(-)} = \{x \in \Omega,\ w(x) > 0\ (< 0)\}$.
One may assumes that w_ε converges towards w a.e. in Ω, so:
i) for a.e. x in A^+, $w_\varepsilon(x)$ converges towards $w(x) > 0$. So, for $0 < \varepsilon < w(x)/2$, $H_\varepsilon(w_\varepsilon(x)) = 1$ and it converges to 1 a.e. in A^+;
ii) for a.e. x in A^-, $w_\varepsilon(x)$ converges towards $w(x) < 0$. So, for $0 > -\varepsilon > w(x)/2$, $H_\varepsilon(w_\varepsilon(x)) = \varepsilon$ and it converges to 0 a.e. in A^-.

Then, one gets that

$$\int_\Omega H_\varepsilon(w_\varepsilon)1_{A^+}\, dx \;\text{ converges to }\; \int_\Omega 1_{A^+}\, dx,$$
$$\int_\Omega H_\varepsilon(w_\varepsilon)1_{A^-}\, dx \;\text{ converges to }\; 0.$$

Since

$$\int_\Omega H_\varepsilon(w_\varepsilon)1_{A^{+(-)}}\, dx \text{ has to converge toward } \int_\Omega \lambda 1_{A^{+(-)}}\, dx,$$

with $0 \le \lambda \le 1$, one gets that $\lambda \in H(w)$.

Passing to limits is then possible and, for any v in $H_0^1(\Omega)$,

$$\int_\Omega \{(w - E)v + h\nabla w.\nabla v + \lambda\nabla(u_0 - hE).\nabla v\}\, dx = 0.$$

Let us denote by $u = h(w - E) + u_0$.
Since $A_\varepsilon(w_\varepsilon)$ converges weakly in $H^1(\Omega)$ towards w with $A_\varepsilon(w_\varepsilon)_{|\partial\Omega} = A_\varepsilon(E_{|\partial\Omega})$, we deduce that $w \in H^1(\Omega)$ and that $w_{|\partial\Omega} = E_{|\partial\Omega}$, *i.e.* $u_{|\partial\Omega} = 0$.

So, u belongs to $H_0^1(\Omega)$ and is solution, for any function v of $H_0^1(\Omega)$, to

$$\int_\Omega \{\frac{u-u_0}{h}v + h(1-\lambda)\nabla w.\nabla v + \lambda \nabla u.\nabla v\}\,dx = 0.$$

In order to achieve the demonstration, note that $(1-\lambda)\nabla w = -\nabla w^- = 0$ since $w \geq 0$ a.e. in Ω. □

Proposition 2.4. *There exists a sequence* $(\lambda_k, u^k)_k$ *in* $L^\infty(\Omega)\times H_0^1(\Omega)$ *such that* $\lambda_k \in H(\dfrac{u^k-u^{k-1}}{h}+E_k)$, $u^0 = u_0$ *and* $\forall v \in H_0^1(\Omega)$,

$$\int_\Omega \{\frac{u^k-u^{k-1}}{h}v + \lambda_k \nabla u^k.\nabla v\}\,dx = 0. \tag{2.4}$$

Moreover, $\inf_\Omega \text{ess}\, u_0 \leq \inf_\Omega \text{ess}\, u^{k-1} \leq u^k \leq \sup_\Omega \text{ess}\, u^{k-1} \leq \inf_\Omega \text{ess}\, u_0$,
and $u^k \geq u^{k-1} - hE_k$ *a.e. in* Ω.

Proof. This proposition is just an induction based on the previous one. □

Let us note:
i) $\hat{u}_h(t,x) = \sum_{k=0}^N [\dfrac{u^k-u^{k-1}}{h}(t-kh) + u^{k-1}]1_{[kh,(k+1)h]}$ where $u^{-1} = u^0$ and $h = \frac{T}{N}$.
i) $\lambda_h(t,x) = \sum_{k=0}^N \lambda^k 1_{[kh,(k+1)h[}$.

Lemma 2.2. *Discrete Gronwall lemma (Cf. D. Bainov[3] p.165)*
Assume that g_0, k_n *and* p_n *are nonnegative real numbers and that*

$$\begin{array}{ll} & x_0 \leq g_0, \\ \forall n \geq 1, & x_n \leq g_0 + \sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} k_s x_s, \end{array}$$

then,

$$\forall n \geq 1, \quad x_n \leq (g_0 + \sum_{s=0}^{n} p_s)e^{\sum_{s=0}^n k_s}.$$

Lemma 2.3. *Independently of* h*, for any integer* n*, one computes that*

$$\frac{2}{h}\sum_{k=1}^n ||u^k-u^{k-1}||^2_{L^2(\Omega)} + ||u^n||^2_{H_0^1(\Omega)} + \sum_{k=1}^n ||u^k-u^{k-1}||^2_{H_0^1(\Omega)} \leq C.$$

Proof. Let us consider the test-function $v = \dfrac{u_\varepsilon^k - u_\varepsilon^{k-1}}{h}$. Then, one gets

$$\int_\Omega \{|\frac{u^k-u^{k-1}}{h}|^2 + \lambda_k \nabla u^k.\nabla(\frac{u^k-u^{k-1}}{h}+E_k)\}\,dx = \int_\Omega \lambda_k \nabla u^k.\nabla E_k\,dx.$$

Since $\lambda_k \in H(\dfrac{u^k-u^{k-1}}{h}+E_k)$ with $\frac{u^k-u^{k-1}}{h}+E_k \geq 0$, one has :

$$\lambda_k \nabla u^k.\nabla(\frac{u^k - u^{k-1}}{h} + E_k) = \nabla u^k.\nabla(\frac{u^k - u^{k-1}}{h} + E_k)^+$$
$$= \nabla u^k.\nabla(\frac{u^k - u^{k-1}}{h} + E_k).$$

Then,

$$\int_\Omega \{|\frac{u^k - u^{k-1}}{h}|^2 + \frac{1}{2h}[\nabla^2 u^k + \nabla^2(u^k - u^{k-1}) - \nabla^2 u^{k-1}] + \nabla u^k \nabla E_k\}\, dx$$
$$= \int_\Omega \lambda_k \nabla u^k.\nabla E_k\, dx,$$

i.e.

$$\frac{2}{h}\int_\Omega \{|u^k - u^{k-1}|^2 + [\nabla^2 u^k + \nabla^2(u^k - u^{k-1}) - \nabla^2 u^{k-1}]\}\, dx$$
$$\leq 2h \int_\Omega |\nabla u^k.\nabla E_k|\, dx.$$

So, one gets that

$$\frac{2}{h}\sum_{k=1}^n ||u^k - u^{k-1}||^2_{L^2(\Omega)} + ||\nabla u^n||^2_{L^2(\Omega)^N} + \sum_{k=1}^n ||\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)^N}$$
$$\leq ||\nabla u_0||^2_{L^2(\Omega)^N} + h\sum_{k=1}^n ||\nabla u^k||^2_{L^2(\Omega)^N} + h\sum_{k=1}^n ||\nabla E_k||^2_{L^2(\Omega)^N},$$

and thereby,

$$||\nabla u^n||^2_{L^2(\Omega)^N} \leq \frac{1}{1-h}||\nabla u_0||^2_{L^2(\Omega)^N} + \frac{h}{1-h}\sum_{k=1}^{n-1} ||\nabla u^k||^2_{L^2(\Omega)^N}$$
$$+ \frac{h}{1-h}\sum_{k=1}^n ||\nabla E_k||^2_{L^2(\Omega)^N}.$$

Thanks to the discrete Gronwall lemma, one concludes

$$||\nabla u^n||^2_{L^2(\Omega)^N} \leq (\frac{1}{1-h}||\nabla u_0||^2_{L^2(\Omega)^N} + \frac{h}{1-h}\sum_{k=1}^N ||\nabla E_k||^2_{L^2(\Omega)^N})e^{\frac{nh}{1-h}} \leq C,$$

and the lemma holds. □

In brief, all these lemmas lead to the following result:

Proposition 2.5. *The sequence* $(\hat{u}_h)$ *is bounded in* $H^1(Q) \cap L^\infty(0,T;H^1_0(\Omega))$. *Thus, it is relatively compact in* $C([0,T],L^2(\Omega))$.

Moreover,

$$\lambda_h = \sum_{k=0}^{N} \lambda_k I_{[kh,(k+1)h[} \in H(\partial_t \hat{u}_h + \sum_{k=0}^{N} E_k I_{[kh,(k+1)h[}),$$

$$\partial_t \hat{u}_h + \sum_{k=0}^{N} E_k I_{[kh,(k+1)h[} \geq 0 \quad a.e.\ in\ Q,$$

and for any v in $L^2(0,T,H_0^1(\Omega))$, one has the approximating equation of continuity:

$$\int_Q \{\partial_t \hat{u}_h v + \lambda^h \nabla \hat{u}_h . \nabla v\}\, dxdt = o(h). \tag{2.5}$$

Remark 2.1. On the one hand, each accumulation point provides a "mild solution" in the sense of Ph. Bénilan *and al.*[5] ; on the other hand, the double weak convergence does not allow us to pass to limits in the diffusion term $\int_Q \lambda^h \nabla \hat{u}_h . \nabla v\, dxdt$.

Let us note that in the geological practice (see R. Eymard and al.[9]) for a delta propagation test, $T = 1.6$ Myears, the uniform time step $h = 25$ centuries and $E = 5\ m/Myears$.

Proposition 2.6. *If one conjectures that λ_h may converge a.e. in Q for a sub-sequence to λ, then*

i) for any v in $H_0^1(\Omega)$, and it follows that:

$$\int_\Omega \{\partial_t u\, v + \lambda \nabla u . \nabla v\}\, dx = 0, \quad a.e.\ t \in]0,T[,$$

ii) $\lambda \in H(\partial_t u + E)$.

Proof. i) Thanks to the theorem of Lebesgue, for any v in $L^2(0,T,H_0^1(\Omega))$, $\lambda_h \nabla v$ converges in $[L^2(\Omega)]^N$ to $\lambda \nabla v$ and claim i) is proved.
ii) $\lambda_h \in H(\partial_t \hat{u}_h + \sum_{k=0}^N E_k I_{[kh,(k+1)h[})$ with $\partial_t \hat{u}_h + \sum_{k=0}^N E_k I_{[kh,(k+1)h[} \geq 0$, so one has

$$\int_Q \{(1-\lambda_h)(\partial_t \hat{u}_h + \sum_{k=0}^{N} E_k I_{[kh,(k+1)h[})\}\, dxdt = 0.$$

Since passing to limits is possible, one gets $\int_Q (1-\lambda)(\partial_t u + E)\, dxdt = 0$.
Since $(1-\lambda)(\partial_t u + E) \geq 0$ a.e. in Q, it comes that $(1-\lambda)(\partial_t u + E) = 0$ a.e. in Q and result ii) holds consequently since $\partial_t u + E \geq 0$. □

Remark 2.2. Looking for *a priori* estimate in $BV(Q)$ for (λ_h) is a classical way to obtain the a.e. convergence conjectured in the previous proposition.

Unfortunately, such a result is out of reach as soon as $N > 1$[a]

In the forthcoming paragraph, some relevant cases, where such a conjecture can be proven, are proposed.

[a]Cf. J. Rauch's works and Strichartz's inequality in W. Littman's one as mentioned by C. Bardos[4].

3. The sedimentation 1-D case

In this section, $\Omega =]-1,1[$ and $E = 0$ is assumed.

Let us start with this essential remark for the sequel:

Remark 3.1. Since $E = 0$, the sequence $(u^k)_k$ is nondecreasing, so the function $\varphi_k : x \mapsto (\lambda^1 u^{k\prime})(x)$ is continuous and nondecreasing in $[-1, 1]$.

This allows us to treat the following examples for illustrating some heuristics.

3.1. *Between two hills*

If $u_0 \geq 0$ in $]a, b[\cup]c, d[$ for given $-1 \leq a < b \leq c < d \leq 1$, then $\lambda^1 u^{1\prime} = 0$ and $u^1 = u_0$ in $]a, d[$.

By induction, the solution to problem (1.8) is $(u_0, 1_{\{\nabla u_0 = 0\}})$ in $]a, d[$.

In particular, if $u_0 \geq 0$ in $]-1, -1+\varepsilon[\cup]1-\varepsilon, 1[$ for a given positive ε, the solution to problem (1.8) is $(u_0, 1_{\{\nabla u_0=0\}})$ in Ω.

3.2. *A convex basin against a hill*

Assume now that $u_0 \geq 0$ in $]-1, 0]$, $u_0 \leq 0$ and convex in $[0, 1[$.

Then u_0 is decreasing in $[0, \alpha]$, constant in $[\alpha, \beta]$ (if needed, otherwise $\alpha = \beta$) and increasing in $[\beta, 1]$.

According to the foregoing, $u^1 = u_0$ in $[-1, 0]$, and since $u_0 \leq 0$ in $]0, 1[$, thanks to the maximum principle, one has $u^1 \leq 0$ in $]0, 1[$.

Let us note $x(h) = \sup\{x \in [-1, 1],\ \lambda^1 u^{1\prime}(x) = 0\}$.

In order to have a non trivial solution, one is looking for $x(h) < 1$.

Thanks to the above proposition, for any $x > x(h)$, one has $\lambda^1 u^{1\prime}(x) > 0$. In particular, u^1 is an increasing function in $]x(h), 1[$. Since $u^1(0) \geq 0$, one notices that inevitably $x(h) > 0$, otherwise, one would have a contradiction with $u^1(1) = 0$.

So, one has to look for $x(h) > 0$, $u^1 \in H^1(x(h), 1)$ and $\lambda^1 \in H(\dfrac{u^1 - u_0}{h})$ with $\lambda^1 > 0$ in $]x(h), 1[$, such that

$$\begin{cases} u^1 - h(\lambda^1 u^{1\prime})' = u_0 \ in\ ]x(h), 1[, \\ \qquad\qquad with \\ u^1(x(h)) = u_0(x(h)),\ u^{1\prime}(x(h)) = 0 \ and\ u^1(1) = 0. \end{cases}$$

As a maximal value of λ^1 is required, one sets $\lambda^1 = 1$ in $]x(h), 1[$ and then, u^1 is given by:

$$u^1(x) = u_0(x) - \int_{x(h)}^{x} u_0'(y) ch(\frac{y - x}{\sqrt{h}})\, dy,$$

where the unique point $x(h)$ is defined by:

$$\int_{x(h)}^{1} u_0'(y) ch(\frac{y - 1}{\sqrt{h}})\, dy = 0. \tag{3.1}$$

Moreover, one obviously notes that $x(h) \in]0, \alpha[$ and that $u^1 \geq u_0$.

Assume now that there exists a solution (v, μ) such that $\mu \neq 0$ in $]0, x(h)[$.

As u_0 is decreasing in $]0, x(h)[$, inevitably $v \neq u_0$ in $]0, x(h)[$.

Thus, there exists a in $]0, x(h)[$ and $\varepsilon > 0$ such that $v > u_0$ in $]a, a+\varepsilon[$. Therefore, $\mu = 1$ in $]a, a+\varepsilon[$, $\mu v' > 0$ in $]a, 1[$ and v is an increasing function in $]a, 1[$. As u_0 is nonincreasing in $]0, \alpha[$, $\mu = 1$ in $]a, \alpha[$.

Remark that $v > u$ in $]a, x(h)[$ and denote by $b = \inf\{x \in]a, 1], \; v(x) = u(x)\}$ (Remind that $v(1) = u(1) = 0$).

Since $v \geq u > u_0$ in $]x(h), b[$, one has $\mu = 1$ and $u - v$ is a solution to:

$$\begin{cases} u - v - h(u-v)" = 0 \; in \;]x(h), b[, \\ \qquad with \\ (u-v)(b) = 0, \quad (u-v)(x(h)) = u_0(x(h)) - v(x(h)) < 0, \\ \qquad and \\ (u-v)'(x(h)) = -v'(x(h)) < 0. \end{cases}$$

Thus, $u - v$ is concave on $[x(h), b]$ with $(u-v)'(x(h)) < 0$ and $(u-v)(x(h)) < 0$. Therefore, $(u - v)(b) < 0$ and one has a contradiction.

So, $\lambda^1 = 1_{]x(h),1[}$ is the only maximal solution.

That allows us to build explicitly the iteration u^1 and one is able to remark that u^1 is nonpositive and convex over $]0, 1[$; decreasing on $]0, x(h)[$ and increasing on $]x(h), 1[$.

Moreover, as u_0 is a convex function, one has $u^1 \geq u_0$ and this constructed solution is the maximal solution with respect to any possible value of λ^1 in $H(\dfrac{u^1 - u_0}{h})$.

So, it is possible to pursue the construction of u^k and λ^k by induction, in the following way : there exists a nonincreasing sequence $x^k(h)$ in $[0, \alpha]$ such that

$$\lambda^k = 1_{]x^k(h),1[} \quad and \quad u^k = u_0 1_{]-1,x^k(h)]} + w^k 1_{]x^k(h),1[},$$

where w^k is the solution to :

$$\begin{cases} w^k - hw^k" = u_0 \quad in \quad]x^k(h), 1[, \\ \qquad with \\ w^k(x^k(h)) = u_0(x^k(h)), \; w^{k\prime}(x^k(h)) = 0 \quad and \quad w^k(1) = 0. \end{cases}$$

So, according to the notation of the property (2.5), $(\lambda_h)_h$ is a bounded sequence in $BV(Q) \cap L^\infty(Q)$ and in particular $var(\lambda_h) \leq T + 1$.

Therefore , it is possible to extract from (λ_h) a sub-sequence that converges a.e. in Q and in any $L^p(Q)$ (for any finite p) towards λ with $0 \leq \lambda \leq 1$ a.e. in Q.

Furthermore, by a monotone argument, $(\lambda - 1)\partial_t u = 0$ a.e. in Q.

Then, one has : $\lambda \in H(\partial_t u)$ and as the control in (2.5) is then possible, one constructs a solution to problem (1.8), with the supplementary information, appropriate for the 1 - D case: thanks to Ascoli's theorem, $u \in C^0(\overline{Q})$.

Let us note that the a.e. convergence with values of $(\lambda_h)_h$ in $\{0,1\}$ implies that $\lambda(t,x) \in \{0,1\}$ and that *a.e.* $\lambda = 1_\omega$ where $\omega \subset Q$ is a finite perimeter set.

Moreover, one proves that the free boundary $\partial\omega \cap Q$ is the graph of a continuous, nonincreasing function $t \mapsto \xi(t)$.

3.3. *A convexo-concave basin against a hill*

Let us examine now to the case: $u_0 \geq 0$ in $]-1,0]$, $u_0 \leq 0$ in $[0,1]$, convex in $[0,\beta[$ with u_0 decreasing in $[0,\alpha]$, nondecreasing in $[\alpha,\beta]$, increasing and concave in $[\beta,1]$.

By using the same ideas, the following algorithm is proposed to build (λ^1, u^1).

Consider $x_1(h)$ in $]\alpha,1[$ and denote by $x_0(h)$ the unique point in $]0,\alpha[$ such that

$$\int_{x_0(h)}^{1} u_0'(y) ch(\frac{y - x_1(h)}{\sqrt{h}})\, dy = 0. \tag{3.2}$$

Therefore, $u^1(x) = u_0(x) - \int_{x(h)}^{x} u_0'(y) ch(\frac{y-x}{\sqrt{h}})\, dy$ is the unique solution to

$$\begin{cases} u^1 - hu^{1\prime\prime} = u_0 \ in \ ]x_0(h), x_1(h)[, \\ \qquad\qquad with \\ u^1(x_0(h)) = u_0(x_0(h)), \quad u^{1\prime}(x_0(h)) = 0, \quad u^1(x_1(h)) = u_0(x_1(h)), \\ \qquad\qquad and \\ \qquad u^1 = u_0 \quad in \quad]-1,1[\setminus]x_0(h), x_1(h)[. \end{cases}$$

At first, assume that $x_1(h)$ is in $]\alpha,\beta[$ such that $u^1 \geq u_0$. Note that since u_0 is not convex on $]\beta,1[$, $x_1(h) = 1$ is not obvious.

Remark that $\lambda^1 = 0$ in $]-1, x_0(h)[$, $\lambda^1 = 1$ in $]x_0(h), x_1(h)[$ and $h(\lambda^1 u^{1\prime})' = u^1 - u_0 = 0$ in $]x_1(h),1[$. As, $\lambda^1 u^{1\prime}$ is a nondecreasing continuous function, for any $x \geq x_1(h)$, one has :

$$u^{1\prime}(x_1(h)^-) = (\lambda^1 u^{1\prime})(x_1(h)^-) = (\lambda^1 u^{1\prime})(x_1(h)^+) = \lambda^1(x) u_0'(x) \leq u_0'(x).$$

As u_0 is concave in $]\beta,1[$, $\lambda^1(x) u_0'(x) \leq u_0'(1)$ and it remains only to consider $\lambda^1(x) = \frac{u_0'(1)}{u_0'(x)}$ in order to construct a solution (λ^1, u^1).

Note that, if $u_0'(1) = 0$, then $\lambda^1 = 0$ in $]x_1(h),1[$. Thus, $x_0(h) = x_1(h)$ and $u^1 \equiv u_0$.

At last, one only has to choose $x_1(h)$ as close as possible to 1 within the above constraints.

Since u^1 has got the same properties than u_0, one is able to construct a nondecreasing sequence of intervals $]x_1^k(h), x_2^k(h)[$ such that $\lambda^k = 0$ in $]-1, x_1^k(h)[$ with $u^k = u_0$, $\lambda^k = 1$ in $]x_1^k(h), x_2^k(h)[$, and $u^k = u_0$ in $]x_2^k(h), 1[$.

So, according to the notation of the property (2.5), $\lambda_h = 1_{\omega_h} + \lambda_h 1_{\Omega - \omega_h}$ where 1_{ω_h} is a bounded sequence in $BV(Q)$ and $u_h = u_0$ in $\Omega - \omega_h$.

Once again, the passing to limits in (2.5) is allowed.

3.4. Numerical simulations

In this section, one presents some numerical simulations obtained by using a fixed point technique in the second order operator. Since H is a graph and not a function, the properties of the function φ_k are used to make a correction of the parameter λ at each iteration step.

The first goal of these numerical simulations is to show that the explicit solutions of the erosion equation in the 1-D case can be calculted numericaly.

In order to solve the equation (2.1), we discretise the time derivative by using an implicit Euler scheme and the space derivative by using a P1-conform finite element method. The function λ is approached by constants by piece.

let us define $\Omega = \cup_{i=0}^{n-1}[x_i, x_{i+1}]$, $V = vect(v_i)$ the space of the hat-functions and $u^k(x) = \sum_{i=0}^{n} u_i^k v_i(x)$ an approximation of $u(kh, x)$.

$$\int_\Omega \frac{u^{k+1} - u^k}{h} v_j dx + \int_\Omega \lambda(u^{k+1}, u^k) u^{k+1\prime} v_j' dx = 0, \qquad \forall v_j \in V \subset H_0^1(\Omega).$$

At each time step, a nonlinear equation must be solved and a fixed point algorithm is used :

$$\left| \begin{array}{l} \text{For } l = 1, 2, \ldots \\ \displaystyle\int_\Omega \frac{u^{k+1,l+1} - u^k}{h} v_j dx + \int_\Omega \lambda(u^{k+1,l}, u^k) u^{k+1,l+1\prime} v_j' dx = 0, \qquad \forall v_j \in V \subset H_0^1(\Omega), \\ \text{with } u^{k+1,0} = u^k. \end{array} \right.$$

Supposed given $\lambda(u^{k+1,l}, u^k)$, then $u^{k+1,l+1}$ is the solution of a linear system and the process is stopped when the series converge.

Let us denote $u_j^k = u^k(x_j)$ and define the discrete operators by: ,

$$D_t u_j = \frac{u_j^{k+1,l} - u_j^k}{h}, D_{x,0} u_j = \frac{u_j^k - u_{j-1}^k}{x_j - x_{j-1}}, D_{x,1} u_j = \frac{u_j^{k+1,l} - u_{j+1}^{k+1,l}}{x_j - x_{j-1}}.$$

In order to construct $\lambda(u^{k+1,l}, u^k)$, relations (1.7) are taking into account and when λ is different from 0 and 1, the continuity of φ_k is used (remark 3.1). So we have the following algoritm:

- $\lambda_j = 0$ $(1 \le j \le n)$.
- if $D_t u_{j-1} > 0$ and $D_t u_j > 0$ then $\lambda_j = 1$. And we denote by $I = \cup_{i=1}^{m} \left[x_{p_i^1}, x_{p_i^2}\right]$ the set where $\lambda = 1$.
- for $j \in \left[p_i^1, p_i^2\right]$, $\lambda_j = \dfrac{D_{x,0} u_{p_i^1}}{D_{x,1} u_j}$ if $D_{x,1} u_{p_i^1} > 0$ and $\lambda_j = \dfrac{D_{x,0} u_{p_i^2}}{D_{x,1} u_j}$ if $D_{x,1} u_{p_i^2} < 0$ (the λ_j obtained can be greater than 1).
- if $\lambda_j > 1$ then $\lambda_j = 1$.

Remark 3.2. At each step of the fixed point process, the finite element matrix must be assembled again. If an explicit time scheme is used, the fixed point method does not converge.

i) Case 1: a convex basin against a hill.

In Figure 1, one presents the numerical simulation when u_0 is a convex function in its negative part. In full line, one has the iterations u^k at time $t = kh = 0; 0.2; 0.6; 0.12; 0.24; 0.48$ and 1.62. In dotted line, the iterations λ^k at the same times are presented, and all the qualitative properties announced in section 3.2 are illustrated.

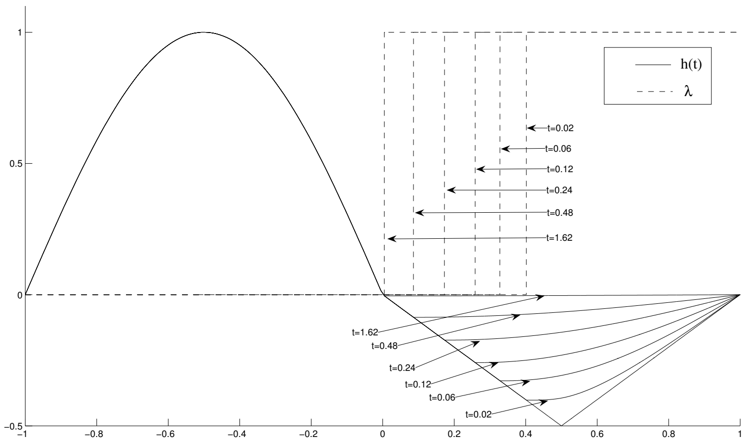

Fig. 1. Numerical simulation of case 1

In particular, one is able to see explicitly in Figure 2 the changing type of the equation.

In Figure 3, one gives an estimation of the error between the theoretical value of x_1 given by the formulae (3.1) and the numerical one, given by the scheme.

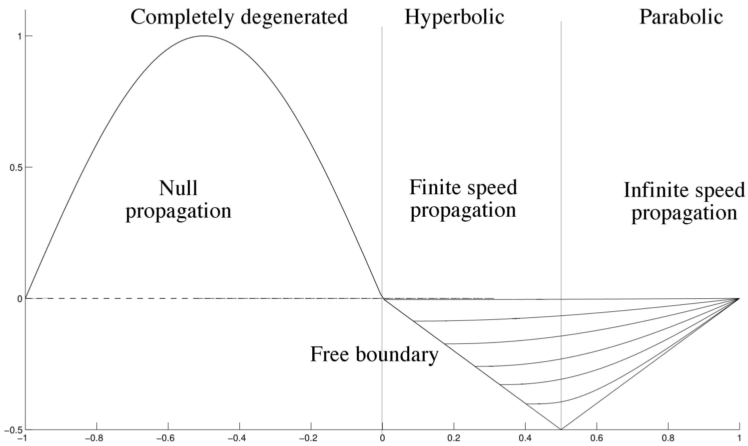

Fig. 2. A changing type Equation

dt	**0,005**	**0,01**	**0,015**	**0,02**	**0,03**	**0,04**
x1 theoretical	0,45099	0,43069	0,41513	0,40206	0,38035	0,36238
x1 numerical	0,45095	0,43074	0,41514	0,40213	0,38033	0,36233
error	4E-05	5E-05	1E-05	7E-05	2E-05	5E-05
dt	0,02	0,03	0,04	0,05	0,1	0,2
x1 theoretical	0,40206	0,38035	0,36238	0,34693	0,2909	0,22629
x1 numerical	0,40213	0,38033	0,36233	0,34692	0,2909	0,22629
error	7E-05	2E-05	5E-05	1E-05	<1E-05	<1E-05

Fig. 3. Error between the theoretical x_1 and the numerical one

Figure 4 is devoted to the numerical simulation of the function φ_k introduced in the remark 3.1. It is an increasing function when λ^k is equal to 1, else it is a constant function.

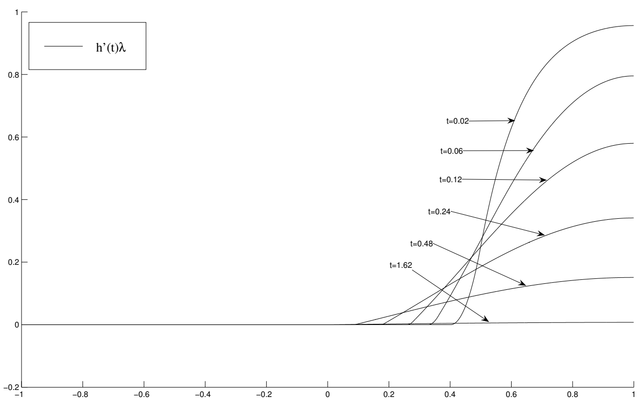

Fig. 4. Flux function φ_k of the case 1

ii) Case 2: a convexo-concave basin against a hill

In this case, see Figure 5, u_0 is a convex, then a concave function in its negative part. One obtains the two points x_0 and x_1 as presented in the equation (3.2).

Then, when u^k becomes a convex function in its negative part, one finds again the behaviour of the first case.

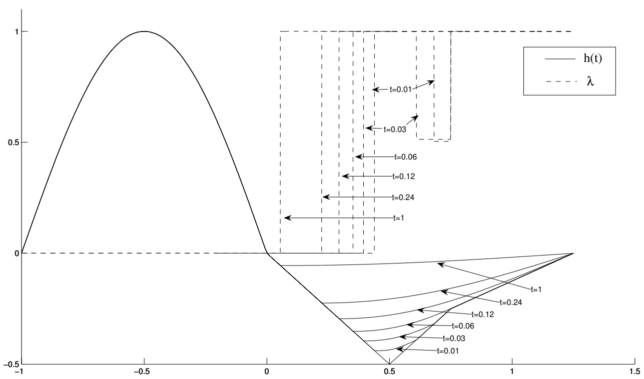

Fig. 5. Numerical simulation of case 2

In Figure 6, the curves of the functions φ_k introduced in the remark 3.1 are presented.

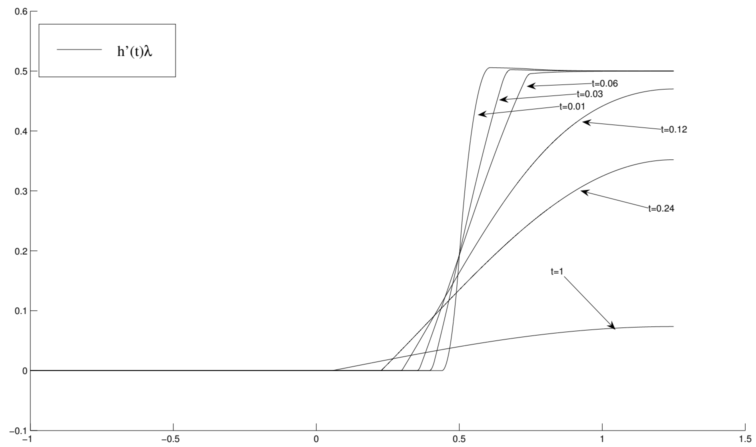

Fig. 6. Flux function φ_k of the case 2

iii) Cases 3,4 and 5 (Figures 7 to 12) represent the numerical simulations of some others initial conditions.

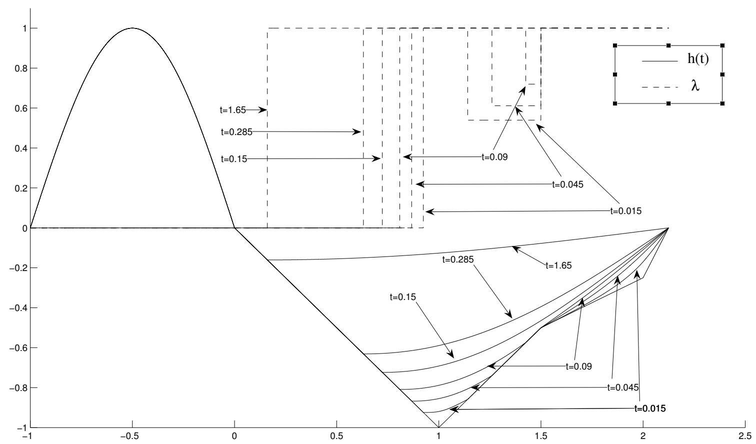

Fig. 7. Numerical simulation of the case 3

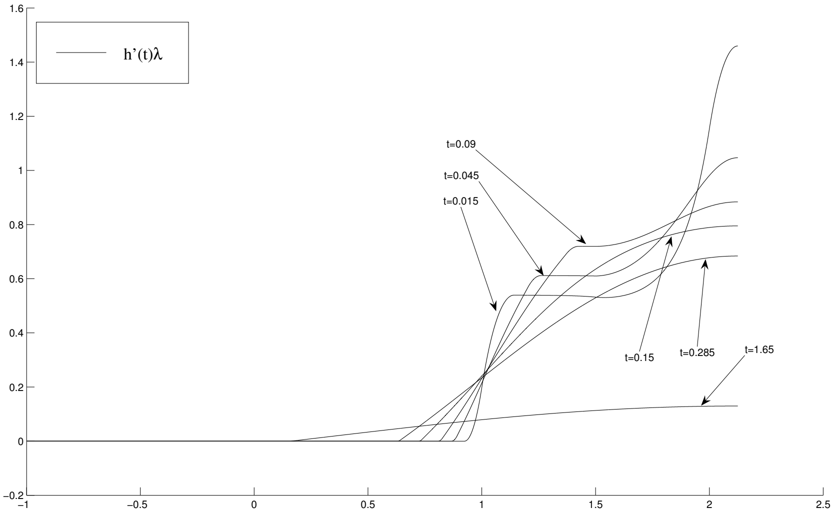

Fig. 8. Flux function φ_k of the case 3

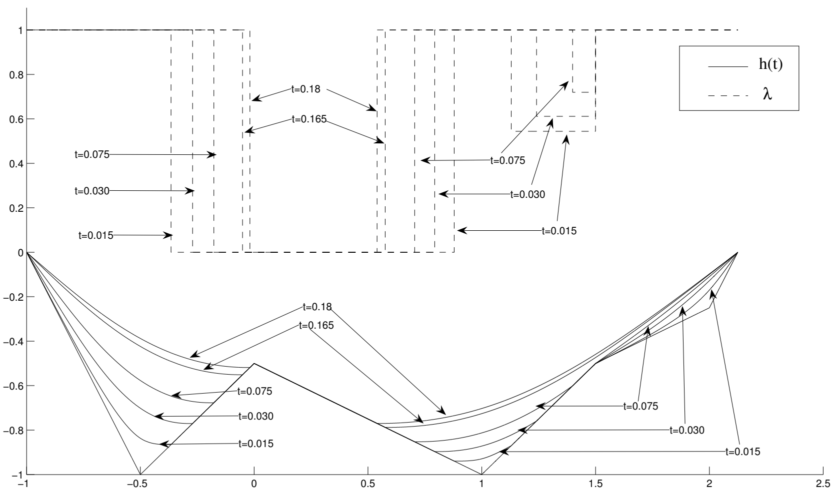

Fig. 9. Numerical simulation of the case 4

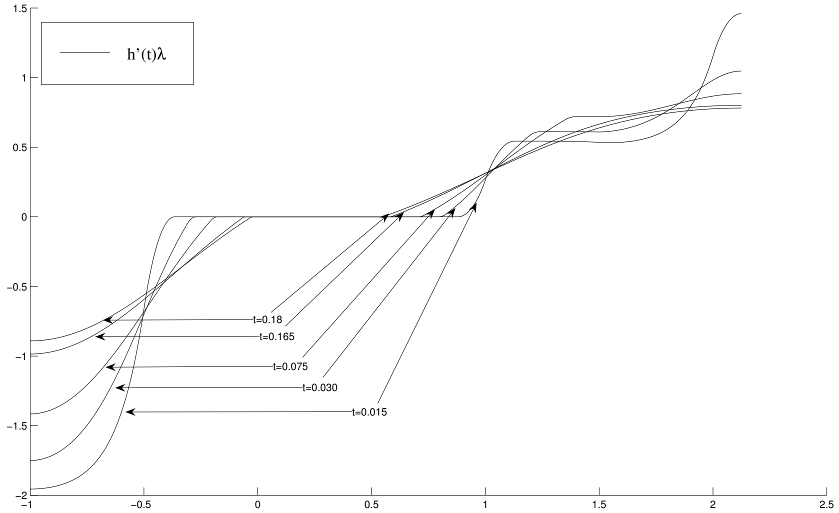

Fig. 10. Flux function φ_k of the case 4

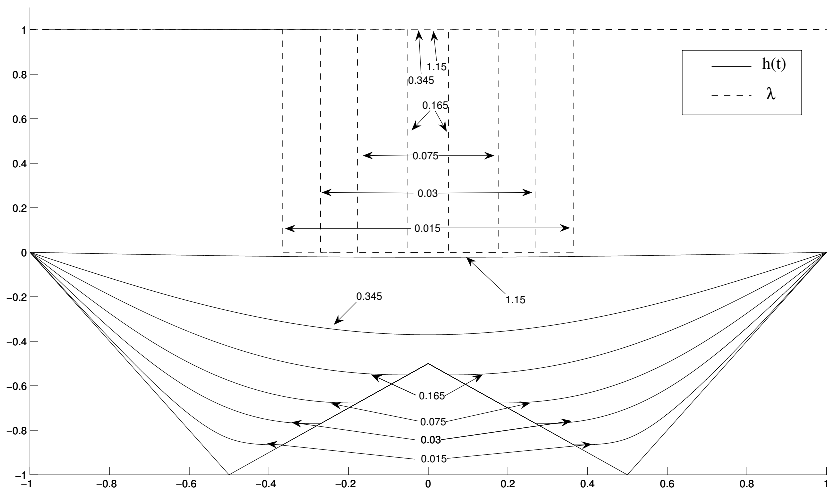

Fig. 11. Numerical simulation of the case 5

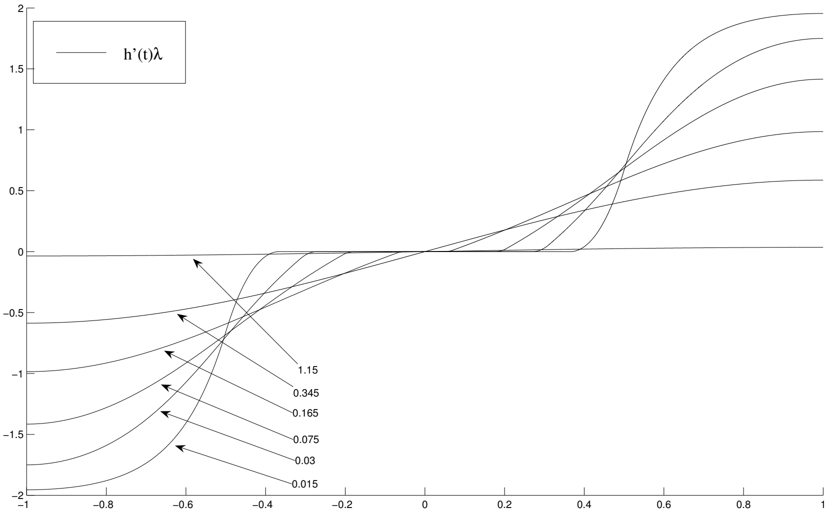

Fig. 12. Flux function φ_k of the case 5

4. Conclusion and open problems

In this paper, a new conservation law coming from geological problematic has been presented. Its general study remains still open. In particular, the way to understand the apparition of the hyperbolic zone and of the parabolic one and the resulting free-boundaries.

Besides the research of a maximal solution, an important point lies in the obtaining of a variational solution (*i.e.* a solution to (2.1)). We have presented a way to prove the existence of such a solution in the $1-D$ case, but passing to limits in (2.5) is still a problem in the general case.

A question may be: under suitable data u_0, are there $T > 0$ and a set $\omega \subset Q$ such that $u = u_0$ in $Q \backslash \omega$ and u is the solution of the heat equation in ω?

Is the free boundary $\partial\omega \cap Q$ characterised, if one notes $\widetilde{u} = u_{|\omega}$, by a double condition of type Dirichlet -Neuman for $\widetilde{u}$ on $\partial\omega \cap Q$?

One finds, under a generalized shape, Bernoulli's problem as presented for example by A. Beurling[6].

G. Gagneux, D. Etienne and G. Vallet[8] propose an adaptation of this study to the geological boundary conditions (1.3) and (1.4). One has now to consider the case of a nonlinear function h in the mass balance equation (1.1)

At last, an other problem concerns the numerical simulation of this geological phenomena in situations of practical importance. On the one hand, in a general case, what kind of method one has to use for type changing equations. On the other hand, it is a nonlinear equation involving a maximal monotone graph.

A general procedure must be devoted to the construction of accurate schemes for approximating such non standard free-boundary problems.

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