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Existence of strong traces for generalized solutions of multidimensional scalar conservation laws [1]

Abstract

In the half-space $t > 0$ a multidimensional scalar conservation law with only continuous flux vector is considered. For the wide class of functions including generalized entropy sub- and super-solutions to this equation we prove existence of the strong trace on the initial hyperspace $t = 0$.

§ 1. Introduction.

In the domain $\Omega \subset \Pi = \mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ = (0, +\infty)$ we consider a first-order conservation law

$$u_t + \mathrm{div}_x \varphi(u) = 0, \tag{1}$$

$\varphi = (\varphi_1, \dots, \varphi_n) \in C(\mathbb{R}, \mathbb{R}^n)$, $u = u(t, x)$, $(t, x) \in \Pi$.

Let us remind the definition of a generalized entropy solution of (1) in the sense of S.N. Kruzhkov [1].

Definition 1. A bounded measurable function $u = u(t, x) \in L^\infty(\Pi)$ is called a generalized entropy solution (briefly - g.e.s.) of the equation (1) in the domain Ω if

a) $\forall k \in \mathbb{R}$

$$|u - k|_t + \mathrm{div}_x[\mathrm{sign}\,(u - k)(\varphi(u) - \varphi(k))] \le 0 \tag{2}$$

in the sense of distributions on Ω (in $\mathcal{D}'(\Omega)$).

Condition (2) means that for any test function $f = f(t, x) \in C_0^\infty(\Omega)$, $f \ge 0$

$$\int_\Omega \{|u - k| f_t + \mathrm{sign}\,(u - k)(\varphi(u) - \varphi(k), \nabla_x f)\} dt dx \ge 0.$$

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Here and below we use the notation $(\cdot,\cdot)$ for the scalar product in a finite-dimensional Euclidean space. We also denote by $|\cdot|$ the corresponding Euclidean norm.

Setting in (2) $k=\pm\|u\|_\infty$, one can easily derive that $u_t+\mathrm{div}_x\varphi(u)=0$ in $\mathcal{D}'(\Omega)$ and $u(t,x)$ satisfies (1) in the sense of distributions, i.e. $u(t,x)$ is really a generalized solution (g.s.).

It is well-known (see for example [1, 4]) that condition (2) can be rewritten in the equivalent form: for any convex function $\eta(u)\in C^1(\mathbb{R})$ (entropy)

$$\eta(u)_t+\mathrm{div}_x\psi(u)\le 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{3}$$

where $\psi(u)=(\psi_1(u),\dots,\psi_n(u))\in C(\mathbb{R},\mathbb{R}^n)$ is the corresponding flux vector of the entropy, defined up to an additive constant by the equality

$$\psi(u)=\eta'(u)\varphi(u)-\int_0^u\varphi(s)d\eta'(s)$$

(so that in the case $\varphi\in C^1(\mathbb{R},\mathbb{R}^n)$ the usual relation $\psi'(u)=\eta'(u)\varphi'(u)$ holds).

If we replace the condition (2) by one of the following conditions

$$((u-k)^+)_t+\mathrm{div}_x[\mathrm{sign}^+(u-k)(\varphi(u)-\varphi(k))]\le 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{4}$$

$$((u-k)^-)_t+\mathrm{div}_x[\mathrm{sign}^-(u-k)(\varphi(u)-\varphi(k))]\le 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{5}$$

where $(u-k)^+=\max(u-k,0)$, $\mathrm{sign}^+(u-k)=\mathrm{sign}\,((u-k)^+)=\dfrac{\partial}{\partial u}(u-k)^+$; $(u-k)^-=\max(k-u,0)$, $\mathrm{sign}^-(u-k)=-\mathrm{sign}\,((u-k)^-)=\dfrac{\partial}{\partial u}(u-k)^-$, we obtain notions of a generalized entropy sub-solution (g.e.sub-s.) and a generalized entropy super-solution (g.e.super-s.) respectively (see [2, 3, 4]). Obviously, a function $u(t,x)$ is a g.e.s. of (1) if and only if $u(t,x)$ is a g.e.sub-s. and g.e.super-s of this equation simultaneously.

In [5] the so-called *isentropic solutions* (briefly - i.s.) of the equation (1) were introduced, for which the condition (2) is satisfied with equality sign:

$$|u-k|_t+\mathrm{div}_x[\mathrm{sign}\,(u-k)(\varphi(u)-\varphi(k))]=0 \quad \text{in } \mathcal{D}'(\Omega). \tag{6}$$

Now, we formulate our main assumptions. Let $u=u(t,x)\in L^\infty(\Pi)$ be a function that is defined on the whole subspace Π and satisfies the following general condition: for some dense set $F\subset\mathbb{R}$ of values k

$$\eta_k(u)_t+\mathrm{div}_x\psi_k(u)=-\gamma_k \quad \text{in } \mathcal{D}'(\Pi), \tag{7}$$

$\gamma_k = \gamma_k(t,x) \in M_{loc}(\bar\Pi)$, where $M_{loc}(\bar\Pi)$ is the space of locally finite Borel measures γ on $\bar\Pi = [0,+\infty) \times \mathbb{R}^n$ such that

$$\mathrm{Var}(\gamma)(\{0\} \times \mathbb{R}^n) = 0, \tag{8}$$

and $\eta_k(u) = |u-k|$, $\psi_k(u) = \mathrm{sign}\,(u-k)(\varphi(u) - \varphi(k))$ are the Kruzhkov's entropy pairs. From (7) with $k > \|u\|_\infty$ it follows that

$$u_t + \mathrm{div}_x \varphi(u) = -\gamma \ \text{ in } \mathcal{D}'(\Pi), \tag{9}$$

$\gamma \in M_{loc}(\bar\Pi)$.

Remark that a g.e.s. $u(t,x)$ satisfies (7) for all $k \in \mathbb{R}$, moreover in this case $\gamma_k \ge 0$. Indeed, as it follows from the known representation of nonnegative distributions, $\eta_k(u)_t + \mathrm{div}_x \psi_k(u) = -\gamma_k$ in $\mathcal{D}'(\Pi)$, where γ_k is a nonnegative locally finite measure on Π. Any of the measures γ_k can be considered as a Borel measure on $\bar\Pi$, after the unique prolongation satisfying (8). Show that this measure is locally finite on $\bar\Pi$ as well. For this, choose a function $\beta(t) \in C_0^\infty(\mathbb{R})$ such that $\mathrm{supp}\,\beta \subset [0,1]$, $\beta(t) \ge 0$, $\displaystyle\int \beta(t)dt = 1$ and set for $\nu \in \mathbb{N}$ $\delta_\nu(t) = \nu\beta(\nu t)$, $\theta_\nu(t) = \displaystyle\int_0^t \delta_\nu(s)ds$, so that $\theta_\nu'(t) = \delta_\nu(t)$. Clearly, the sequence $\delta_\nu(t)$ converges as $\nu \to \infty$ to the Dirac δ-measure in $\mathcal{D}'(\mathbb{R})$ while the sequence $\theta_\nu(t)$ converges pointwise to the Heaviside function

$$\theta(t) = \begin{cases} 0 & , \quad t \le 0, \\ 1 & , \quad t > 0. \end{cases}$$

Let $f(t,x) \in C_0^\infty(\bar\Pi)$, $f(t,x) \ge 0$. Applying the equality $\eta_k(u)_t + \mathrm{div}_x \psi_k(u) = -\gamma_k$ to the test function $f(t,x)\theta_\nu(t-t_0) \in C_0^\infty(\Pi)$, where $t_0 > 0$, we derive that

$$\int_\Pi f(t,x)\theta_\nu(t-t_0)d\gamma_k(t,x) = \int_\Pi \eta_k(u)f(t,x)\delta_\nu(t-t_0)dtdx + \int_\Pi [\eta_k(u)f_t(t,x) + (\psi_k(u), \nabla_x f)]\theta_\nu(t-t_0)dtdx.$$

Supposing that t_0 is a Lebesgue point of the function

$$t \to \int_{\mathbb{R}^n} \eta_k(u(t,x))f(t,x)dx,$$

we obtain, in the limit as $\nu \to \infty$, that

$$\int_{(t_0,+\infty)\times\mathbb{R}^n} f(t,x)d\gamma_k(t,x) = \int_{\mathbb{R}^n} \eta_k(u(t_0,x))f(t_0,x)dx +$$
$$\int_{(t_0,+\infty)\times\mathbb{R}^n} [\eta_k(u)f_t(t,x) + (\psi_k(u), \nabla_x f)]dtdx \leq$$
$$\int_{\mathbb{R}^n} \eta_k(u(t_0,x))f(t_0,x)dx +$$
$$\int_{\Pi}[\eta_k(u)|f_t| + |\psi_k(u)| \cdot |\nabla_x f|]dtdx \leq C,$$

where the constant C does not depend on t_0. Passing to the limit as $t_0 \to 0$ we derive the estimate

$$\int_{\Pi} f(t,x)d\gamma_k(t,x) \leq C,$$

which implies, in view of arbitrariness of the nonnegative test function $f(t,x) \in C_0^\infty(\bar{\Pi})$, that the measure γ_k is locally finite on $\bar{\Pi}$.

The condition (7) is also satisfied for g.e.sub-s. (and g.e.super-s.) $u(t,x)$. Really, assume for definiteness that $u(t,x)$ is a g.e.sub-s. of (1). From (4) it follows, in the same way as for g.e.s., that $\forall k \in \mathbb{R}$

$$((u-k)^+)_t + \mathrm{div}_x[\mathrm{sign}^+(u-k)(\varphi(u)-\varphi(k))] = -\beta_k,$$

where $\beta_k \in M_{loc}(\bar{\Pi})$, $\beta_k \geq 0$.

In particular, taking in this relation $k < -\|u\|_\infty$, we obtain that $u_t + \mathrm{div}_x\varphi(u) = -\beta$ in $\mathcal{D}'(\Pi)$, where $\beta \in M_{loc}(\bar{\Pi})$, $\beta \geq 0$. Taking into account that

$$\eta_k(u) = |u-k| = 2(u-k)^+ - (u-k),$$
$$\psi_k(u) = \mathrm{sign}\,(u-k)(\varphi(u)-\varphi(k)) = (2\mathrm{sign}\,(u-k)-1)(\varphi(u)-\varphi(k)),$$

we derive the equality $\eta_k(u)_t + \mathrm{div}_x\psi_k(u) = -(2\beta_k - \beta)$ that is (7) is satisfied for all $k \in \mathbb{R}$.

One could easily show that a g.e.s. (g.e.sub-s., g.e.super-s.) $u(t,x)$ of equation (1) in the arbitrary domain $\Omega \subset \Pi$ with the regular enough boundary will satisfy (7) if to extend $u(t,x)$ on Π setting $u(t,x) = 0$ for $(t,x) \notin \Omega$.

The main result of the present paper is the following theorem on existence of the strong trace of $u(t,x)$ on the initial hyperplane $t = 0$:

Theorem 1. *Suppose a function $u(t,x)$ satisfies the entropy condition* (7). *Then there exists a function $u_0 = u_0(x) \in L^\infty(\mathbb{R}^n)$ such that $\operatorname*{ess\,lim}_{t\to 0} u(t,\cdot) = u_0$ in $L^1_{loc}(\mathbb{R}^n)$ (strongly).*

Remark, that the result of Theorem 1 for g.e.s. was recently established in [6], but under very restrictive assumptions. Namely, it is required in [6] that the flux $\varphi(u) \in C^3(\mathbb{R},\mathbb{R}^n)$ and satisfies the nondegeneracy condition: $\forall \xi \in \mathbb{R}^n \setminus \{0\}$ the function $u \to (\xi, \varphi'(u))$ is not constant on sets of positive Lebesgue measure. In the present paper we keep the main scheme of the paper [6] but, instead of the key regularization result from [7], we utilize more refined approach based on the techniques of H-measures and induction in the spatial dimension. The notion of the H-measure was introduced in the papers [8, 9] and was further developed in [10, 11, 12]. Localization principles for H-measures associated with sequences of measure valued functions allow to apply the initially "linear" device of H-measures to nonlinear problems. In particular, on the base of this device in the indicated papers [10, 11, 12] the property of strong precompactness was established for families of entropy (and measure valued) solutions to general multidimensional scalar conservation laws with only continuous flux vectors satisfying the weak nondegeneracy condition.

§ **2. Preliminaries.**

We prove firstly existence of the weak trace (in the $*$-weak topology of $L^\infty(\mathbb{R}^n)$).

Let $u = u(t,x) \in L^\infty(\Pi)$, $\eta(u) \in C(\mathbb{R})$, $\psi(u) \in C(\mathbb{R},\mathbb{R}^n)$, and

$$\eta(u)_t + \operatorname{div}_x \psi(u) = -\gamma \quad \text{in } \mathcal{D}'(\Pi), \tag{10}$$

where $\gamma = \gamma(t,x)$ is a locally finite Borel measure on $\bar{\Pi}$. We denote

$$E = \{\ t \in \mathbb{R}_+ \mid (t,x) \text{ is a Lebesgue point of } u(t,x) \text{ for a.e. } x \in \mathbb{R}^n\ \}. \tag{11}$$

Obviously, E is the set of full measure in $\mathbb{R}_+$. We have the following proposition:

Proposition 1. *There exists a function $v_0(x) \in L^\infty(\mathbb{R}^n)$ such that $\eta(u)(t,\cdot) \to v_0$ $*$-weakly in $L^\infty(\mathbb{R}^n)$ as $t \to 0$, $t \in E$.*

Proof. Let $M = \operatorname*{ess\,sup}_{\Pi} |\eta(u(t,x))|$. Then $\forall t \in E$ $\ \|\eta(u(t,\cdot))\|_\infty \le M$ and by weak precompactness of bounded sets in $L^\infty(\mathbb{R}^n)$ we can find a sequence $t_m \in E$, $m \in \mathbb{N}$, $t_m \underset{m\to\infty}{\to} 0$ and a function $v_0(x) \in L^\infty(\mathbb{R}^n)$ such that $\eta(u(t_m,\cdot)) \to v_0$ as $m \to \infty$ $*$-weakly in $L^\infty(\mathbb{R}^n)$.

If $\tau \in E$ then $t_m < \tau$ for large enough m. We fix such $t_m < \tau$ and set $\chi_\nu(t) = \theta_\nu(t - t_m) - \theta_\nu(t - \tau)$, where the sequence $\theta_\nu(t)$, $\nu \in \mathbb{N}$ was defined above, in the Introduction.

Let $f(x) \in C_0^\infty(\mathbb{R}^n)$. Then for large enough $\nu \in \mathbb{N}$ the function $f(x)\chi_\nu(t) \in C_0^\infty(\Pi)$. Applying relation (10) to this test function, we obtain, after simple transforms, that

$$\int_0^{+\infty}\left(\int_{\mathbb{R}^n}\eta(u(t,x))f(x)dx\right)\delta_\nu(t-\tau)dt - \int_0^{+\infty}\left(\int_{\mathbb{R}^n}\eta(u(t,x))f(x)dx\right)\delta_\nu(t-t_m)dt = \int_\Pi(\psi(u),\nabla_x f)\chi_\nu(t)dtdx - \int_\Pi f(x)\chi_\nu(t)d\gamma. \tag{12}$$

Passing in (12) to the limit as $\nu \to \infty$ and taking into account that $t_m, \tau \in E$ are Lebesgue points of the function $I(t) = \int_{\mathbb{R}^n}\eta(u(t,x))f(x)dx$, and also that the sequence $\chi_\nu(t)$ converges pointwise to the indicator function of the interval $(t_m, \tau]$ and is bounded, we derive the equality

$$I(\tau) - I(t_m) = \int_{(t_m,\tau]\times\mathbb{R}^n}(\psi(u),\nabla_x f)dtdx - \int_{(t_m,\tau]\times\mathbb{R}^n} f(x)d\gamma(t,x),$$

which implies in the limit as $m \to \infty$ that

$$I(\tau) - I(0+) = \int_{\mathbb{R}^n}\eta(u(\tau,x))f(x)dx - \int_{\mathbb{R}^n}v_0(x)f(x)dx = \int_{(0,\tau]\times\mathbb{R}^n}(\psi(u),\nabla_x f)dtdx - \int_{(0,\tau]\times\mathbb{R}^n} f(x)d\gamma(t,x) \underset{\tau\to 0}{\to} 0, \tag{13}$$

i.e. $\forall f(x) \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n}\eta(u(\tau,x))f(x)dx \underset{\tau\to 0}{\to} \int_{\mathbb{R}^n}v_0(x)f(x)dx.$$

This, together with boundedness of $u(\tau,\cdot)$, $\tau \in E$ and density of $C_0^\infty(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$, implies that $\eta(u(t,\cdot)) \to v_0$ $*$-weakly in $L^\infty(\mathbb{R}^n)$ as $t \to 0$, $t \in E$. The proof is complete.

Corollary 1. *If $u(t,x)$ satisfies (9) then there exists a function $u_0(x) \in L^\infty(\mathbb{R}^n)$ such that $u(t,\cdot) \to u_0$ $*$-weakly in $L^\infty(\mathbb{R}^n)$ as $t \to 0$, $t \in E$.*

Remark that existence of the weak trace of $u(t,x)$ also follows from the general result of the paper [13].

Next, we prove existence of the strong trace for isentropic solutions of the equation (1).

Proposition 2. *Let $u(t,x)\in L^\infty(\Pi)$ be an i.s. of (1). Then $u(t,\cdot)\to u_0$ in $L^1_{loc}(\mathbb{R}^n)$ as $t\to 0$, $t\in E$.*

Proof. If $u=u(t,x)$, $v=v(t,x)$ are two i.s. of the equation (1) then by the Kruzhkov method of "doubling variables" (see [1]) one can derive the relation

$$|u-v|_t+\operatorname{div}_x[\operatorname{sign}(u-v)(\varphi(u)-\varphi(v))]=0 \ \text{ in } \mathcal{D}'(\Pi). \tag{14}$$

Consider this relation with i.s. $u=u(t,x)$ and $v=u(t,x+\Delta x)$, where $\Delta x\in\mathbb{R}^n$. Let $f(x)\in C_0^\infty(\mathbb{R}^n)$, $f(x)\ge 0$, $\tau,T\in E$, $\tau<T$. Here $E\subset\mathbb{R}_+$ is a set of full measure defined by (11).

Now, we take $g(t,x)=f(x)\chi_\nu(t)$, where $\chi_\nu(t)=\theta_\nu(t-\tau)-\theta_\nu(t-T)$, $\nu\in\mathbb{N}$, and apply (14) to the test function $g(t,x)\in C_0^\infty(\Pi)$. After simple transforms we obtain that

$$\int_0^{+\infty}\int_{\mathbb{R}^n}|u(t,x)-v(t,x)|f(x)dx\delta_\nu(t-T)dt-$$
$$\int_0^{+\infty}\int_{\mathbb{R}^n}|u(t,x)-v(t,x)|f(x)dx\delta_\nu(t-\tau)dt=$$
$$\int_\Pi \operatorname{sign}(u-v)(\varphi(u)-\varphi(v),\nabla_x f)\chi_\nu(t)dtdx.$$

It is easy to see that the values $\tau,T\in E$ are Lebesgue points of the function $t\to\int_{\mathbb{R}^n}|u(t,x)-v(t,x)|f(x)dx$ and from the relation above we derive, in the limit as $\nu\to\infty$, the equality

$$\int_{\mathbb{R}^n}|u(T,x)-v(T,x)|f(x)dx-\int_{\mathbb{R}^n}|u(\tau,x)-v(\tau,x)|f(x)dx=$$
$$\int_{(\tau,T)\times\mathbb{R}^n}\operatorname{sign}(u-v)(\varphi(u)-\varphi(v),\nabla_x f)dtdx,$$

which implies the estimate:

$$\int_{\mathbb{R}^n}|u(\tau,x)-v(\tau,x)|f(x)dx\le\int_{\mathbb{R}^n}|u(T,x)-v(T,x)|f(x)dx+$$
$$\int_{(0,T)\times\mathbb{R}^n}|\varphi(u)-\varphi(v)|\cdot|\nabla_x f|dtdx. \tag{15}$$

Denote

$$\omega_f^0(h) = \sup_{|\Delta x|<h} \int_{\mathbb{R}^n} |u(T, x + \Delta x) - u(T, x)| f(x) dx,$$

$$\omega_f(h) = \sup_{|\Delta x|<h} \int_{(0,T)\times\mathbb{R}^n} |\varphi(u(t, x + \Delta x)) - \varphi(u(t, x))| \cdot |\nabla_x f| dt dx.$$

By continuity on the average $\omega_f^0(h), \omega_f(h) \to 0$ as $h \to 0$ for all $f \in C_0^\infty(\mathbb{R}^n)$. Then from (15) it follows that $\forall \Delta x \in \mathbb{R}^n$, $|\Delta x| < h$

$$\int_{\mathbb{R}^n} |u(\tau, x + \Delta x) - u(\tau, x)| f(x) dx \le \omega_f^0(h) + \omega_f(h) \underset{h\to 0}{\to} 0$$

for all $\tau \in (0, T) \cap E$, $f \in C_0^\infty(\mathbb{R}^n)$, $f \ge 0$. Thus, the family $u(\tau, \cdot)$, $\tau \in (0, T) \cap E$ is equicontinuous in $L^1(\mathbb{R}^n, f(x)dx)$ and uniformly bounded (by the condition $u \in L^\infty(\Pi)$). By the Hausdorff criterion this family is precompact in $L^1(\mathbb{R}^n, f(x)dx)$ and in $L^1_{loc}(\mathbb{R}^n)$ (because f is arbitrary). Therefore there is a function $u_0 \in L^\infty(\mathbb{R}^n)$, which is a strong limit point of $u(\tau, \cdot)$ as $\tau \to 0$. Obviously, u_0 is a weak limit point for $u(\tau, \cdot)$ as well and must coincide with the limit function from Proposition 1. In particular, the strong limit function u_0 is uniquely defined, and we derive existence of the limit $u(t, \cdot) \to u_0$ in $L^1_{loc}(\mathbb{R}^n)$ as $t \to 0$, $t \in E$. The proof is complete.

In the general case we have the following simple sufficient condition for existence of the strong trace:

Proposition 3. *Let $u(t, x) \in L^\infty(\Pi)$ to satisfy (7) and there is a sequence $t_m \in E$ such that $t_m \underset{m\to\infty}{\to} 0$, and $u(t_m, \cdot) \underset{m\to\infty}{\to} u_0$ in $L^1_{loc}(\mathbb{R}^n)$. Then $u(t, \cdot) \to u_0$ in $L^1_{loc}(\mathbb{R}^n)$ as $t \to 0$, $t \in E$.*

Proof. By Proposition 1 and Corollary 1 as $t \to 0$, $t \in E$

$$u(t, \cdot) \to u_0,\ \eta_k(u) \to v_k\ \ast\text{-weakly in } L^\infty(\mathbb{R}^n)\ \forall k \in F,$$

where $u_0, v_k \in L^\infty(\mathbb{R}^n)$. Since $|\eta_k(u) - \eta_r(u)| \le |k - r|$ we can conclude that the limit relation $\eta_k(u) \underset{t\to 0, t\in E}{\to} v_k$ $\ast$-weakly in $L^\infty(\mathbb{R}^n)$ remains valid for all $k \in \mathbb{R}$.

By our assumptions, as $m \to \infty$ $u(t_m, \cdot) \to u_0$ in $L^1_{loc}(\mathbb{R}^n)$. Therefore, also $\eta_k(u(t_m, \cdot)) \to \eta_k(u_0)$ in $L^1_{loc}(\mathbb{R}^n)$. Thus, it is necessary that $v_k = \eta_k(u_0)$. Let $M = \|u\|_\infty$. By weak convergence $\eta_k(u(t, \cdot)) \to \eta_k(u_0)$ $\forall k \in \mathbb{R}$ we have the relation

$$\int_{-M}^{M} \eta_k(u(t, \cdot)) dk \underset{t\to 0, t\in E}{\to} \int_{-M}^{M} \eta_k(u_0(\cdot)) dk\ \ast\text{-weakly in } L^\infty(\mathbb{R}^n).$$

Taking into account that $u_0(\cdot), u(t,\cdot) \in [-M, M]$ for $t \in E$ a.e. in $\mathbb{R}^n$, and $\forall u \in [-M, M]$

$$\int_{-M}^{M} \eta_k(u)dk = \int_{-M}^{M} |u-k|dk = u^2 + M^2,$$

we obtain that $u^2(t,\cdot) \to u_0^2(\cdot)$ $*$-weakly in $L^\infty(\mathbb{R}^n)$ as $t \to 0$, $t \in E$. This, together with $*$-weak convergence $u(t,\cdot) \to u_0(\cdot)$, implies strong convergence $u(t,\cdot) \to u_0$ in $L^1_{loc}(\mathbb{R}^n)$. Indeed, if $\rho(x) \in L^1(\mathbb{R}^n)$, $\rho(x) \ge 0$ then

$$\int_{\mathbb{R}^n} (u(t,x) - u_0(x))^2 \rho(x)dx = \int_{\mathbb{R}^n} (u^2(t,x) - u_0^2(x))\rho(x)dx - \\ 2\int_{\mathbb{R}^n} (u(t,x) - u_0(x))u_0(x)\rho(x)dx \to 0$$

as $t \to 0$, $t \in E$. The proof is complete.

Let $u(t,x) \in L^\infty(\Pi)$ to satisfy the condition (7). We consider a sequence $\varepsilon_m > 0$, $m \in \mathbb{N}$ such that $\varepsilon_m \to 0$ as $m \to \infty$ and set $u_m = u_m(t,x) = u(\varepsilon_m t, \varepsilon_m x + y)$, where $y \in \mathbb{R}^n$ is treated as a fixed parameter. By Corollary 1 there exists the function $u_0(x) \in L^\infty(\Pi)$ such that $u(t,\cdot) \to u_0$ $*$-weakly in $L^\infty(\mathbb{R}^n)$ as $t \to 0$, $t \in E$. We set also $u_{0m}(x) = u_0(\varepsilon_m x + y)$.

The following lemma was essentially proved in [6] (Lemma 3).

Lemma 1. *There exists a subsequence of ε_m (we keep the notation ε_m) such that for a.e. $y \in \mathbb{R}^n$ $u_{0m}(x) \to u_0(y) = \mathrm{const}$ in $L^1_{loc}(\mathbb{R}^n)$ as $m \to \infty$.*

Evidently, $u_m(t,x)$ satisfies equalities like (7), namely

$$\eta_k(u_m)_t + \mathrm{div}_x \psi_k(u_m) = -S_m^y(\gamma_k) = -\varepsilon_m \gamma_k(\varepsilon_m t, \varepsilon_m x + y) \text{ in } \mathcal{D}'(\Pi). \quad (16)$$

The operator S_m^y is defined on the space $M_{loc}(\bar{\Pi})$ by the equality $S_m^y(\gamma) = \varepsilon_m \gamma(\varepsilon_m t, \varepsilon_m x + y)$ understood in the sense of distributions. This means that $\forall f(t,x) \in C_0(\Pi)$

$$< S_m^y(\gamma), f >= (\varepsilon_m)^{-n} \int_\Pi f(t/\varepsilon_m, (x-y)/\varepsilon_m) d\gamma(t,x).$$

The following lemma can be proved in just the same way as Lemma 2 in [6].

Lemma 2. *If $\gamma \in M_{loc}(\bar{\Pi})$ then, after possible extraction of a subsequence, for a.e. $y \in \mathbb{R}^n$ $S_m^y(\gamma) \underset{m\to\infty}{\to} 0$ in $M_{loc}(\bar{\Pi})$. As usual, the topology* *in* $M_{loc}(\bar\Pi)$ *is generated by seminorms* $\|\gamma\|_K = \mathrm{Var}(\gamma)(K)$, *where* K *is a compact set in* $\bar\Pi$.

Applying Lemma 2 to the measures γ_k, $k \in F$, we see that there exists a subsequence of the sequence ε_m (we keep the notation ε_m for this subsequence) and a set $Y \subset \mathbb{R}^n$ of full measure such that $S_m^y(\gamma_k) \to 0$ in $M_{loc}(\bar\Pi)$ as $m \to \infty$ $\forall y \in Y$. Below we shall assume that the dense set $F \subset \mathbb{R}$ is countable. Then, using the standard diagonal extraction, we can choose the indicated subsequence being common for all $k \in F$.

Thus, we can assume that the sequence ε_m and the set $Y \subset \mathbb{R}^n$ of full measure are chosen in such way that the assertions of Lemma 1 and Lemma 2 (for each $\gamma = \gamma_k$, $k \in F$) hold.

Our interest to the sequence $u_m(t,x)$ is stipulated by the following theorem:

Theorem 2. *Existence of the strong trace* $\operatorname{ess\,lim}_{t\to 0} u(t,\cdot) = u_0$ *in* $L^1_{loc}(\mathbb{R}^n)$ *is equivalent to the condition that, after possible extraction of a subsequence, for a.e.* $y \in \mathbb{R}^n$ *the sequence* $u_m(t,x)$ *converges in* $L^1_{loc}(\Pi)$.

Proof. Suppose $\operatorname*{ess\,lim}_{t\to 0} u(t,\cdot) = u_0$ in $L^1_{loc}(\mathbb{R}^n)$ and $y \in Y$, We denote $\rho(y) = e^{-|y|}$. Then for a.e. $(t,x) \in \Pi$

$$\int |u(\varepsilon_m t, \varepsilon_m x + y) - u_0(\varepsilon_m x + y)|\rho(y)dy =$$
$$\int |u(\varepsilon_m t, z) - u_0(z)|\rho(z - \varepsilon_m x)dz \underset{m\to\infty}{\to} 0.$$

Integrating this relation over (t,x) weighted $p(t,x) = e^{-t-|x|}$, and using the Lebesgue dominated convergence theorem we derive the relation

$$\int \left(\int_\Pi |u(\varepsilon_m t, \varepsilon_m x + y) - u_0(\varepsilon_m x + y)|p(t,x)dtdx \right) \rho(y)dy \underset{m\to\infty}{\to} 0.$$

From this it follows that, after possible extraction of a subsequence, $u_m(t,x) - u_{0m}(x) \to 0$ as $m \to \infty$ in $L^1_{loc}(\Pi)$ for a.e. $y \in Y$. By Lemma 1 we see also that $u_{0m}(x) \to u_0(y)$ in $L^1_{loc}(\Pi)$ and we can conclude that for a.e. y $u_m(t,x) \to u_0(y)$ in $L^1_{loc}(\Pi)$, as required.

Conversely, suppose that $\forall y \in Y_1$, where $Y_1 \subset Y$ is a set of full measure, the sequence $u_m(t,x)$ converges in $L^1_{loc}(\Pi)$ to some function $v(\cdot, y) \in L^\infty(\Pi)$.

Passing to the limit as $m \to \infty$ in (16) and taking into account that for $k \in F$ $S^y_m(\gamma_k) \to 0$ in $\mathcal{D}'(\Pi)$, we obtain that $\forall k \in F$

$$\eta_k(v)_t + \mathrm{div}_x \psi_k(v) = 0 \ \text{ in } \mathcal{D}'(\Pi). \tag{17}$$

Since the left-hand side of this equality is continuous by k in $\mathcal{D}'(\Pi)$, while $F \subset \mathbb{R}$ is dense we conclude that (17) holds for all $k \in \mathbb{R}$, i.e. $v(t,x,y)$ is an i.s. of the equation (1) for all $y \in Y_1$.

Now, we fix $y \in Y_1$. By the relation like (13) with $u = u_m(t,x)$, $\eta(u) = u$ we see that for a.e. $\tau > 0$ $\forall f(x) \in C_0^\infty(\mathbb{R}^n)$

$$I_m(\tau) - I_m(0+) = \int_{\mathbb{R}^n} u_m(\tau,x) f(x)dx - \int_{\mathbb{R}^n} u_{0m}(x)f(x)dx = \\ \int_{(0,\tau]\times\mathbb{R}^n} (\varphi(u_m), \nabla_x f)dtdx - \int_{(0,\tau]\times\mathbb{R}^n} f(x)d\gamma^{(m)}(t,x),$$

where $\gamma^{(m)} = S^y_m(\gamma)$, and γ is the measure from (9).

From this equality we derive the estimate

$$|I_m(\tau) - I_m(0+)| \le C_f \cdot \tau + |\gamma^{(m)}|((0,T] \times K), \\ C_f = \mathrm{const}, \quad K = \mathrm{supp}\, f, \tag{18}$$

which holds for a.e. $\tau \in (0,T)$. Here the constant C_f depends only on $\|u\|_\infty$.

Observe that $\gamma = \gamma_k$ for large enough $k \in F$ and by our assumptions the conclusion of Lemma 2 holds, that is $|\gamma^{(m)}|((0,T] \times K) \to 0$ as $m \to \infty$. Further, for a.e. $\tau > 0$ $u_m(\tau,\cdot) \to v(\tau,\cdot)$ as $m \to \infty$ in $L^1_{loc}(\mathbb{R}^n)$ (after possible extraction of a subsequence), and also $u_{0m}(x) \to u_0(y)$ in $L^1_{loc}(\mathbb{R}^n)$ (by Lemma 1). Therefore, from (18) it follows, in the limit as $m \to \infty$, that for a.e. $\tau > 0$, $\forall f(x) \in C_0^\infty(\mathbb{R}^n)$

$$\left| \int_{\mathbb{R}^n} v(\tau,x,y)f(x)dx - \int_{\mathbb{R}^n} u_0(y)f(x)dx \right| \le C_f \tau \to 0,$$

i.e. $\operatorname*{ess\,lim}_{t\to 0} v(t,\cdot,y) = u_0(y)$ $*$-weakly in $L^\infty(\mathbb{R}^n)$. By Proposition 2 the i.s. $v(\cdot,y)$ has the strong trace, therefore, $\operatorname*{ess\,lim}_{t\to 0} v(t,\cdot,y) = u_0(y)$ in $L^1_{loc}(\mathbb{R}^n)$. In particular, $v(\cdot,y)$ is a g.e.s. to the Cauchy problem for (1) with the constant initial data $u_0(y)$. By the maximum principle (see for instance [2, 3, 4]) this solution is constant as well: $v(\cdot,y) = u_0(y)$ a.e. in Π. Remark that in the case under consideration the flux vector is only continuous and a g.e.s.

to the Cauchy problem can be nonunique if $n > 1$ (see [2, 4]). But, in the case of constant initial data uniqueness of g.e.s. follows from the maximum principle (or the comparison principle). Thus, $u_m(t,x,y) - u_0(y) \to 0$ as $m \to \infty$ in $L^1_{loc}(\Pi)$ $\forall y \in Y_1$. By Lemma 1 also $u_m(t,x,y) - u_{0m}(x,y) \to 0$ in $L^1_{loc}(\Pi)$ for a.e. $y \in \mathbb{R}^n$. Then, for $f(x) \in C_0^\infty(\mathbb{R}^n)$, $\rho(y) = e^{-|y|}$

$$\int_{\mathbb{R}^n} \int_{\Pi_T} |u_m(t,x,y) - u_{0m}(x,y)| f(x) dt dx \rho(y) dy \underset{m\to\infty}{\to} 0, \tag{19}$$

where $\Pi_T = (0,T) \times \mathbb{R}^n$, $T > 0$.

Changing the order of integration in (19) and making the change of variables $z = \varepsilon_m x + y$ in integral over y, we obtain the equality

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{\Pi_T} |u_m(t,x,y) - u_{0m}(x,y)| f(x) dt dx \rho(y) dy = \\ &\int_{\Pi_T} \int_{\mathbb{R}^n} |u(\varepsilon_m t, \varepsilon_m x + y) - u_0(\varepsilon_m x + y)| \rho(y) dy f(x) dt dx = \\ &\int_{\Pi_T} \int_{\mathbb{R}^n} |u(\varepsilon_m t, z) - u_0(z)| \rho(z - \varepsilon_m x) dy f(x) dt dx = \\ &C \int_{\Pi_T} \int_{\mathbb{R}^n} |u(\varepsilon_m t, z) - u_0(z)| \rho(z) dt dz + \alpha_m, \\ &\alpha_m \underset{m\to\infty}{\to} 0, \quad C = \int_{\mathbb{R}^n} f(x) dx. \end{aligned}$$

Then from (19) it follows that

$$\int_{\Pi_T} \int_{\mathbb{R}^n} |u(\varepsilon_m t, z) - u_0(z)| \rho(z) dt dz \underset{m\to\infty}{\to} 0$$

and, after possible extraction of a subsequence, for a.e. $t \in (0,T)$ $u(\varepsilon_m t, \cdot) \to u_0$ in $L^1_{loc}(\mathbb{R}^n)$. Obviously, the value t can be chosen in such way that $t_m = \varepsilon_m t \in E$ $\forall m \in \mathbb{N}$. By Proposition 3 we conclude that $u(t,\cdot) \to u_0$ in $L^1_{loc}(\mathbb{R}^n)$ as $t \to 0$, $t \in E$. The theorem is proved.

§ 3. Reduction of the space dimension.

Suppose that one of the components of the flux vector, say φ_n, is absent, i.e. equation (1) has a form

$$u_t = \sum_{i=1}^{n-1} \varphi_i(u)_{x_i} = u_t + \operatorname{div}_{x'} \varphi(u) = 0,$$
$$x' = (x_1, \ldots, x_{n-1}),\ \varphi(u) = (\varphi_1(u), \ldots, \varphi_{n-1}(u)) \in \mathbb{R}^{n-1}.$$

It is natural to consider the equation

$$u_t + \mathrm{div}_{x'}\varphi(u) = 0, \ u = u(t, x') \tag{20}$$

in the half-space $\Pi' = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ with reduced space dimension.

Below, to justify the inductive jump in the proof of Theorem 1, we shall need the following result

Theorem 3. *For a.e. $x_n \in \mathbb{R}$ the function $\tilde{u}(t, x') = u(t, x', x_n)$ satisfies condition (7) for equation (20).*

To prove this theorem we shall need the following lemma.

Lemma 3. *Let $N \in \mathbb{N}$ and γ be a locally finite measure on $\mathbb{R}^N \times \mathbb{R}$. Then for a.e. $\lambda \in \mathbb{R}$ there exist a locally finite measure γ_λ on $\mathbb{R}^N$ such that $\forall f(x) \in C_0(\mathbb{R}^N)$*

$$\lim_{\nu\to\infty} \int_{\mathbb{R}^{N+1}} f(x)\delta_\nu(s-\lambda)d\gamma(x,s) = \int_{\mathbb{R}^N} f(x)d\gamma_\lambda(x).$$

Proof. We denote $|\gamma| = \mathrm{Var}\,\gamma$ and assume firstly that

$$|\gamma|(\mathbb{R}^N \times [a,b]) < +\infty \textit{ for any segment } [a,b] \subset \mathbb{R}. \tag{$*$}$$

Then we can define a projection $m = \mathrm{pr}^* |\gamma|$, which is an image of $|\gamma|$ under the map $\mathrm{pr}(x,s) = s$. It is clear that m is a locally finite nonnegative measure on $\mathbb{R}$. Further, for any function $f(x) \in C_0(\mathbb{R}^n)$ the correspondence

$$g \to \int_{\mathbb{R}^{N+1}} f(x)g(s)d\gamma(x,s), \quad g \in C_0(\mathbb{R})$$

defines a bounded linear functional l_f on $C_0(\mathbb{R})$, moreover

$$| < l_f, g > | \le \|f\|_\infty \int_{\mathbb{R}^{N+1}} |g(s)|d|\gamma|(x,s) = \|f\|_\infty \int_{\mathbb{R}} |g(s)|dm(s). \tag{21}$$

In particular,

$$| < l_f, g > | \le \|f\|_\infty m(K)\|g(s)\|_\infty, \ K = \mathrm{supp}\, g,$$

which implies that the functional l_f is in fact continuous. By the known representation theorem l_f is given as the integral

$$< l_f, g >= \int_{\mathbb{R}} g(s)d\mu_f(s)$$

over some locally finite measure μ_f, and, as it follows from (21), its variation $|\mu_f| \le \|f\|_\infty \cdot m$. The latter implies that the measures μ_f are absolutely continuous with respect to the measure m and by the Radon-Nikodym theorem $\mu_f = \alpha_f(s)m$, where $\alpha_f(s)$ are measurable over the measure m, and also $|\alpha_f| \le \|f\|_\infty$ m-a.e. on $\mathbb{R}$.

Now, we consider the Lebesgue decomposition $m = \beta(s)ds + \sigma$, where $\beta(s) \in L^1_{loc}(\mathbb{R})$, $\beta(s) \ge 0$, and σ is a singular measure. By known properties of measures (see for instance [14]) there exists a set of full Lebesgue measure $A \subset \mathbb{R}$, on which the measure σ has the null derivative over the Lebesgue measure ds that is $\forall \lambda \in A$

$$\lim_{\nu\to\infty} \int \delta_\nu(s-\lambda)d\sigma(s) = 0. \tag{22}$$

We choose a countable dense set $G \subset C_0(\mathbb{R}^N)$ and consider the set $A' \subset A$ consisting of common Lebesgue points for the function $\beta(s)$ and the countable family of functions $\rho_f(s) = \alpha_f(s)\beta(s)$ with $f \in G$. Since

$$|\rho_f(s) - \rho_h(s)| = |\alpha_{f-h}(s)|\beta(s) \le \|f-h\|_\infty \beta(s) \ \forall g, h \in C_0(\mathbb{R}^N)$$

and G is dense in $C_0(\mathbb{R}^N)$ we conclude that $\lambda \in A'$ are Lebesgue points of all functions $\rho_f(s)$, $f \in C_0(\mathbb{R}^N)$. For $\lambda \in A'$ we have

$$\int_{\mathbb{R}^{N+1}} f(x)\delta_\nu(s-\lambda)d\gamma(x,s) = \int_{\mathbb{R}} \delta_\nu(s-\lambda)\alpha_f(s)dm(s) =$$
$$\int_{\mathbb{R}} \delta_\nu(s-\lambda)\rho_f(s)ds + \int_{\mathbb{R}} \delta_\nu(s-\lambda)\alpha_f(s)d\sigma(s). \tag{23}$$

Since λ is a Lebesgue point of the functions $\rho_f(s)$ then

$$\int_{\mathbb{R}} \delta_\nu(s-\lambda)\rho_f(s)ds \to \rho_f(\lambda) \ \text{ as } \nu \to \infty.$$

Further, in view of (22)

$$\left|\int_{\mathbb{R}} \delta_\nu(s-\lambda)\alpha_f(s)d\sigma(s)\right| \le \|\alpha_f\|_\infty \int_{\mathbb{R}} \delta_\nu(s-\lambda)d\sigma(s) \le$$
$$\|f\|_\infty \int_{\mathbb{R}} \delta_\nu(s-\lambda)d\sigma(s) \underset{\nu\to\infty}{\to} 0$$

and from (23) it follows that

$$\int_{\mathbb{R}^{N+1}} f(x)\delta_\nu(s-\lambda)d\gamma(x,s) \to \rho_f(\lambda) \ \text{ as } \nu \to \infty. \tag{24}$$

For a fixed $\lambda \in A'$ the correspondence $f \to \rho_f(\lambda)$ determines a bounded linear functional on $C_0(\mathbb{R}^N)$ and, taking account of the estimate $|\rho_f(\lambda)| = |\alpha_f(\lambda)\beta(\lambda)| \le \beta(\lambda)\|f\|_\infty$, we have the representation

$$\rho_f(\lambda) = \int_{\mathbb{R}^N} f(x)d\gamma_\lambda(x), \tag{25}$$

where γ_λ is a finite Borel measure such that $|\gamma_\lambda|(\mathbb{R}^N) \le \beta(\lambda)$, $\lambda \in A'$. From (24), (25) it directly follows the conclusion of the Lemma.

In the general case of arbitrary locally finite measure γ we introduce measures $\gamma^l = \gamma|_{B_l}$, where B_l is a ball $|x| \le l$, $l \in \mathbb{N}$. We see that the measures γ^l can be considered on the whole space $\mathbb{R}^N$ and satisfy the condition (*). As we have already proved, there are sets $A_l \subset \mathbb{R}$ of full Lebesgue measure and finite Borel measures γ^l_λ such that $\forall \lambda \in A_l$

$$\lim_{\nu\to\infty} \int_{\mathbb{R}^{N+1}} f(x)\delta_\nu(s-\lambda)d\gamma^l(x,s) = \int_{\mathbb{R}^N} f(x)d\gamma^l_\lambda(x). \tag{26}$$

It is clear that the set $A = \cap_{l\in\mathbb{N}} A_l$ has full measure and for $\lambda \in A$ relation (26) holds for all $l \in \mathbb{N}$. This, in particular, implies that measures γ^l_λ are compatible: $\gamma^{l'}_\lambda|_{B_l} = \gamma^l_\lambda$ for $l' > l$. Therefore, there exists a locally finite measure γ_λ on $\mathbb{R}^N$ such that $\gamma_\lambda|_{B_l} = \gamma^l_\lambda$ $\forall l \in \mathbb{N}$. From (26) it follows that for $\lambda \in A$ $\forall f(x) \in C_0(\mathbb{R}^N)$

$$\lim_{\nu\to\infty} \int_{\mathbb{R}^{N+1}} f(x)\delta_\nu(s-\lambda)d\gamma(x,s) = \int_{\mathbb{R}^N} f(x)d\gamma_\lambda(x),$$

as was to be proved.

Proof of Theorem 3. Remark firstly that the measures γ_k from (7) can be extended on the whole space $\mathbb{R}^{n+1}$, so that the extended measure of a Borel set $A \subset \mathbb{R}^{n+1}$ equals to $\gamma_k(A \cap \Pi)$. As it is easy to see, the extended measures are locally finite and by Lemma 3 there exists a set $A \subset \mathbb{R}$ of full measure such that for $\lambda \subset A$ the following limit relation holds: $\forall k \in F$, $f(t,x') \in C_0(\Pi')$

$$\lim_{\nu\to\infty} \int_{\Pi} f(t,x')\delta_\nu(s-\lambda)d\gamma_k(t,x,s) = \int_{\Pi'} f(t,x')d\gamma_{k,\lambda}(t,x'), \tag{27}$$

where $\gamma_{k,\lambda}(t,x')$ are locally finite measures on Π'. We also utilize the fact that the set F is countable, which allows us to choose the set A being common for all $k \in F$.

Let $A_1 \subset A$ be a set of $\lambda \in A$ such that for a.e. $(t, x') \in \Pi'$ (t, x', λ) is a Lebesgue point of the function $u(t, x) = u(t, x', s)$. Obviously, A_1 is a set of full measure on $\mathbb{R}$. Let $\lambda \in A_1$, $f(t, x') \in C_0^\infty(\Pi')$. We choose a test function $g(t, x) = f(t, x')\delta_\nu(x_n - \lambda)$. By condition (7), applying to the test function g, $\forall k \in F$

$$\int_\Pi [\eta_k(u) f_t + (\psi_k(u), \nabla_{x'} f)]\delta_\nu(x_n - \lambda) dt dx' dx_n = \int_\Pi f(t, x')\delta_\nu(x_n - \lambda) d\gamma_k(t, x', x_n).$$

Passing in this equality to the limit as $\nu \to \infty$ and taking account of (27), we find that

$$\int_{\Pi'} [\eta_k(u(\cdot, \lambda)) f_t + (\psi_k(u(\cdot, \lambda)), \nabla_{x'} f)] dt dx' = \int_{\Pi'} f(t, x')\gamma_{k,\lambda}(t, x').$$

Since $f(t, x') \in C_0^\infty(\Pi')$ is arbitrary, we obtain that $\forall k \in F$

$$\frac{\partial}{\partial t}\eta_k(u(\cdot, \lambda)) + \operatorname{div}_{x'}\psi_k(u(\cdot, \lambda)) = -\gamma_{k,\lambda} \quad \text{in } \mathcal{D}'(\Pi'),$$

i.e. the function $\tilde{u} = u(\cdot, \lambda)$ satisfies to (7) for equation (20). The proof is complete.

§ 4. H-measures associated with sequences of measure valued functions.

We shall need the notion of measure valued function (see [15, 16]). Let Ω be a domain in some Euclidean space $\mathbb{R}^N$. We recall that a measure valued function on Ω is a weakly measurable map $x \to \nu_x$ of the set Ω into the space of Borel probability measures having compact supports on $\mathbb{R}$. Weak measurability of ν_x means that for any continuous function $f(\lambda)$ the function $x \to \int f(\lambda) d\nu_x(\lambda)$ is Lebesgue measurable on Ω.

A measure valued function ν_x is said to be bounded if there is a constant $M > 0$ such that $\operatorname{supp} \nu_x \subset [-M, M]$ for a.e. $x \in \Omega$. The minimal of such M will be denoted by $\|\nu_x\|_\infty$.

Finally, measure valued functions of the kind $\nu_x(\lambda) = \delta(\lambda - u(x))$, where $\delta(\lambda - u)$ is the Dirac measure at the point u, are called regular and they are identified with the corresponding functions $u(x)$. Thus, the set $\mathrm{MV}(\Omega)$ of bounded measure valued functions on Ω includes the space $L^\infty(\Omega)$.

We shall use also the following notions:

Definition 2. Suppose $\nu_x^m \in \mathrm{MV}(\Omega)$, $m \in \mathbb{N}$; $\nu_x \in \mathrm{MV}(\Omega)$. We shall say that

1) the sequence ν_x^m is bounded if for some constant $M > 0$ for all natural m $\|\nu_x^m\|_\infty \le M$;

2) the sequence ν_x^m converges to ν_x *weakly* if

$$\forall f(\lambda) \in C(\mathbb{R}) \quad \int f(\lambda) d\nu_x^m(\lambda) \underset{m\to\infty}{\to} \int f(\lambda) d\nu_x(\lambda) \quad *\text{-weakly in } L^\infty(\Omega);$$

3) the sequence ν_x^m converges to ν_x *strongly* if

$$\forall f(\lambda) \in C(\mathbb{R}) \quad \int f(\lambda) d\nu_x^m(\lambda) \underset{m\to\infty}{\to} \int f(\lambda) d\nu_x(\lambda) \quad \text{in } L^1_{loc}(\Omega).$$

Measure valued functions are naturally arised as weak limits of bounded sequences in $L^\infty(\Omega)$ in the sense of the following theorem of Tartar (see [16]).

Theorem 4. *Let $u_m(x) \in L^\infty(\Omega)$, $m \in \mathbb{N}$ be a bounded sequence. Then there exist a subsequence $u_r(x) = u_{m_r}(x)$ and a measure valued function $\nu_x \in \mathrm{MV}(\Omega)$ such that $u_r(x) \to \nu_x$ weakly (that is the sequence of regular measure valued functions $\nu_x^r = \delta(\lambda - u_r(x))$, identified with u_r, converges weakly to ν_x).*

Besides, ν_x is regular, i.e. $\nu_x = \delta(\lambda - u(x))$ if and only if $u_r(x) \underset{r\to\infty}{\to} u(x)$ in $L^1_{loc}(\Omega)$ that is $\nu_x^r \to \nu_x$ strongly.

The notion of H-measure was introduced in [8, 9] and was further extended in [10] to the case of sequences of bounded measure valued functions.

Let $\nu_x^m \in \mathrm{MV}(\Omega)$, $m \in \mathbb{N}$ be a bounded sequence of measure valued functions weakly convergent to a measure valued function $\nu_x \in \mathrm{MV}(\Omega)$. For $x \in \Omega$ and $p \in \mathbb{R}$ we set

$$V_m(x,p) = \nu_x^m((p,+\infty)), \quad V_0(x,p) = \nu_x((p,+\infty)).$$

It was shown in [10] that for $m \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{R}$ the functions $V_m(x,p) \in L^\infty(\Omega)$ and $0 \le V_m(x,p) \le 1$. Let

$$P = P(\nu_x) = \left\{ p_0 \in \mathbb{R} \,|\, V_0(x,p) \underset{p\to p_0}{\to} V_0(x,p_0) \text{ in } L^1_{loc}(\Omega) \right\}.$$

The following lemma was proved in [10]:

Lemma 4. *The complement* $CP = \mathbb{R} \setminus P$ *is at most countable and for* $p \in P$ $V_m(x,p) \underset{m\to\infty}{\to} V_0(x,p)$ $*$*-weakly in* $L^\infty(\Omega)$.

Let $U_m^p(x) = V_m(x,p) - V_0(x,p)$. By Lemma 4 for $p \in P$ $U_m^p(x) \to 0$ as $m \to \infty$ $*$-weakly in $L^\infty(\Omega)$. We introduce the following notation:
$F(u)(\xi)$, $\xi \in \mathbb{R}^N$ is the Fourier transform of $u(x) \in L^2(\mathbb{R}^N)$;
$S = S^{N-1} = \{\, \xi \in \mathbb{R}^N \,|\, |\xi| = 1 \,\}$ denotes the unit sphere in $\mathbb{R}^N$;
$u \to \bar{u}$, $u \in \mathbb{C}$, is complex conjugation.

Proposition 4 (see [10]). 1) *There exists a family of complex-valued locally finite Borel measures* $\{\mu^{pq}\}_{p,q\in P}$ *on* $\Omega \times S$ *and a subsequence* $U_r(x) = \{U_r^p(x)\}_{p\in P}$, $U_r^p(x) = U_m^p(x)$ *for* $m = m_r$ *such that* $\forall \Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S)$

$$\langle \mu^{pq}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle =$$
$$\lim_{r\to\infty} \int_{\mathbb{R}^N} F(\Phi_1 U_r^p)(\xi)\overline{F(\Phi_2 U_r^q)(\xi)}\psi(\xi/|\xi|)d\xi.$$

2) *The map* $(p,q) \to \mu^{pq}$ *is continuous as a map from* $P \times P$ *into the space* $M_{loc}(\Omega \times S)$ *of locally finite Borel measures on* $\Omega \times S$.

Definition 3. The family $\{\mu^{pq}\}_{p,q\in P}$ is called the H-measure corresponding to the subsequence $\nu_x^r = \nu_x^m$, $m = m_r$.

In the following lemma some important properties of the H-measure are collected (see [10]):

Lemma 5.

1) $\forall p \in P$ $\mu^{pp} \geq 0$;

2) $\forall p,q \in P$ $\mu^{pq} = \overline{\mu^{qp}}$;

3) *for* $p_1, \dots, p_l \in P$ *and for any bounded Borel set* $C \subset \Omega \times S$ *the matrix* $a_{ij} = \mu^{p_i p_j}(C)$, $i,j = 1, \dots, l$ *is positively definite;*

4) $\mu^{pq} = 0$ $\forall p,q \in P$ *if and only if the sequence* ν_x^r *is strongly convergent as* $r \to \infty$.

Now, let $\varphi(u) \in C(\mathbb{R}, \mathbb{R}^N)$ be a continuous vector function. Suppose that the sequence ν_x^r satisfies the condition:

C) $\forall p \in \mathbb{R}$ the sequence of distributions

$$\mathcal{L}_r^p = \mathrm{div}_x \int_{(p,+\infty)} (\varphi(\lambda) - \varphi(p))d\nu_x^r(\lambda)$$

is precompact in $H_{loc}^{-1}(\Omega)$.

Here, as usual, the space $H^{-1}_{loc}(\Omega)$ consists of distributions $u(x)$ such that for all $f(x) \in C_0^\infty(\Omega)$ the product $fu \in H_2^{-1}$. The topology in $H^{-1}_{loc}(\Omega)$ is generated by seminorms $\|fu\|_{H_2^{-1}}$, $f \in C_0^\infty(\Omega)$.

From the results of [11] (see Lemma 2 and the proof of Theorem 4) it directly follows the theorem, which plays key role in the proof of our main Theorem 1.

Theorem 5. *Suppose that the H-measure $\{\mu^{pq}\}_{p,q\in P}$ corresponding to the sequence ν_x^r is not trivial and the condition C) is satisfied.*

Then there exists a nondegenerate segment $I = [p_0, p_0 + \delta]$ and values $q \in P$ and $\xi \in S$ such that $(\xi, \varphi(u)) = \mathrm{const}$ on I and $\mu^{pq} \neq 0$ $\forall p \in P \cap I$.

In particular from Theorem 5 it follows that under the nondegeneracy condition

$\forall \xi \in S$ *the function* $u \to (\xi, \varphi(u))$ *is not constant on nondegenerate intervals*

$\mu^{pq} \equiv 0$ and, therefore, the sequence ν_x^r is strongly convergent.

In [12] this result was generalized to the case when the flux vector φ depends also on the spatial variables x.

§ 5. Proof of Theorem 1.

We shall use the method of mathematical induction in the spatial dimension n.

If $n = 0$ (the base) then we use that $u = u(t)$ satisfies to condition (9): $u' = -\gamma$ in $\mathcal{D}'(\mathbb{R})$, where γ is a locally finite measure on $[0, +\infty)$. This implies that $u(t)$ has bounded variation on any interval $(0, T)$, $T > 0$. Therefore, we can find $u_0 \in \mathbb{R}$ such that $\lim\limits_{t\to 0-} u(t) = u_0$, as required.

Assuming that the conclusion of Theorem 1 is true for $n - 1$ space variables, we prove it for dimension n. So, suppose that the function $u = u(t, x)$ satisfies (7) for the origional equation (1).

We introduce the set $\mathcal{I}$ of segments $I = [a, b]$, $a, b \in F$, such that for some nonzero vector $\bar{\xi} = (\xi_0, \xi) \in \mathbb{R}^{n+1}$ the function $u \to \xi_0 u + (\varphi(u), \xi)$ is constant on I. Since the set F is assumed to be countable, the set $\mathcal{I}$ is also countable (or empty).

Let $I = [a, b] \in \mathcal{I}$. We set $v = v(t, x) = \max(a, \min(u, b))$ so that $v(t, x) \in I$. Let us show that the function $v(t, x)$ also satisfies to condition (7). Take $k \in F$. Then $k' = \max(a, \min(k, b)) \in F$ and one can easily verify that

$$\eta_k(v) = |v-k| = \eta_{k'}(u) - (\eta_a(u)+\eta_b(u))/2 + (b-a)/2,$$
$$\psi_k(v) = \operatorname{sign}(v-k)(\varphi(v)-\varphi(k)) =$$
$$\psi_{k'}(u) - (\psi_a(u)+\psi_b(u))/2 + (\varphi(b)-\varphi(a))/2,$$

which implies that in $\mathcal{D}'(\Pi)$

$$\eta_k(v)_t + \operatorname{div}_x\psi_k(v) = (\gamma_a+\gamma_b)/2 - \gamma_{k'} \in M_{loc}(\bar{\Pi})$$

for all $k \in F$, i.e. $v(t,x)$ satisfies (7).

Now, recall that $\xi_0 v + (\varphi(v),\xi) = \text{const}$ for $v \in I$. In particular, from this follows that $\xi \neq 0$. Hence, we can find a nondegenerate linear change of variables $z = z(t,x) = ct + Ax \in \mathbb{R}^n$, $t = t$ (here $c \in \mathbb{R}^n$, A is a nondegenerate $n \times n$-matrix) such that $z_n = \xi_0 t + (\xi, x)$. After this change equation (1) reduces to the form

$$u_t + \operatorname{div}_z\tilde{\varphi}(u) = 0, \quad \text{where } \tilde{\varphi}_n(u) = \xi_0 u + (\varphi(u),\xi) = \text{const } \forall u \in I.$$

Therefore, for $u = u(t,z) \in I$ the last component $\tilde{\varphi}_n(u)$ vanishes, and the equation can be written as follows

$$u_t + \sum_{i=1}^{n-1} \tilde{\varphi}_i(u) = 0. \tag{28}$$

We consider the function v as a function of new variables (t,z), so that $v(t,z(t,x)) = v(t,x)$. Clearly, the function $v(t,z) = v(t,z',z_n)$ satisfies condition (7) for equation (28), here $z' = (z_1,\ldots,z_{n-1}) \in \mathbb{R}^{n-1}$.

By Theorem 3, for a.e. $z_n \in \mathbb{R}$ the function $v(\cdot,z_n) \in L^\infty(\Pi')$ satisfies condition (7) for equation (28), defined on the domain $\Pi' = \mathbb{R}_+ \times \mathbb{R}^{n-1}$. As it is well-known (see for instance [14]), the set

$$\mathcal{M} = \{\ (t,z) = (t,z',z_n) \in \Pi \mid (t,z) \text{ is a Lebesgue point } u,$$
$$(t,z') \text{ is a Lebesgue point } u(\cdot,z_n)\ \}$$

has full measure. Therefore, for a set E_1 of full measure of values $t > 0$ the sections $\mathcal{M}_t = \{z \in \mathbb{R}^n \mid (t,z) \in \mathcal{M}\ \}$ have full measure on $\mathbb{R}^n$. Observe that $E_1 \subset E$, where the set E is defined by (11). Now, we choose a sequence $t_r \in E_1$, $t_r \underset{r\to\infty}{\to} 0$, and introduce the sets $Z_r \subset \mathbb{R}$ consisting of values $s \in \mathbb{R}$

such that $(z', s) \in \mathcal{M}_{t_r}$ for a.e. $z' \in \mathbb{R}^{n-1}$. Clearly, Z_r have full measure, consequently $Z = \cap_{r\in\mathbb{N}} Z_r$ is also a set of full measure. We see also that for $s \in Z$ all $t_r \in E(s)$, where $E(s)$ is defined by (11) for the function $u(\cdot, s)$ on $\Pi' = \mathbb{R}_+ \times \mathbb{R}^{n-1}$.

As we have already established, for a.e. $z_n \in Z$ $v(\cdot, z_n)$ satisfies condition (7) for equation (28) on Π'. By the inductive assumption $v(t_r, \cdot, z_n) \to v_0(\cdot, z_n)$ in $L^1_{loc}(\mathbb{R}^{n-1})$ as $r \to \infty$, where $v_0 = v_0(z', z_n) \in L^\infty(\mathbb{R}^n)$. Using the Lebesgue dominated convergence theorem, we conclude that as $r \to \infty$ also $v(t_r, z) \to v_0(z)$ in $L^1_{loc}(\mathbb{R}^n)$. This easily implies the same limit relation under origional variables x: $v(t_r, x) \underset{r\to\infty}{\to} v_0(x)$ in $L^1_{loc}(\mathbb{R}^n)$. By Proposition 3 $v(t,x)$ has the strong trace on the hyperspace $t = 0$, i.e. $v(t, \cdot) \to v_0$ in $L^1_{loc}(\mathbb{R}^n)$ as $t \to 0$, $t \in E$.

Then, by Theorem 2 we can find a subsequence of the sequence ε_m from § 2 such that the corresponding sequence $v_m = v_m(t, x) = v(\varepsilon_m t, \varepsilon_m x + y)$ converges strongly (that is in $L^1_{loc}(\Pi)$) for a.e. $y \in \mathbb{R}^m$.

The established property remains valid for any choice of the segment $I \in \mathcal{I}$. Since $\mathcal{I}$ is countable then, using the diagonal extraction, we can choose the subsequence ε_m and the set $Y \subset \mathbb{R}^n$ of full measure to be common for all $I \in \mathcal{I}$. More exactly, we can assume that the conclusions of Lemma 1 and Lemma 2 (for all $\gamma = \gamma_k$, $k \in F$) hold, and $\forall I \in \mathcal{I}$, $y \in Y$ $v_m = v_m(t, x)$ is strongly convergent as $m \to \infty$, where $v_m = Q_I(u_m)$, and for $I = [a, b]$ $Q_I(u) = \max(a, \min(u, b))$, $u_m = u_m(t, x) = u(\varepsilon_m t, \varepsilon_m x + y)$.

Now, we fix $y \in Y$. According to Theorem 4 we can extract a subsequence $\varepsilon_r = \varepsilon_{m_r}$ such that the corresponding sequence of the regular measure valued functions $\nu^r_{t,x}(\lambda) = \delta(\lambda - u_r(t, x))$ (with $u_r(t, x) = u_{m_r}(t, x)$) converges as $r \to \infty$ weakly to a measure valued function $\nu_{t,x}$, and for this sequence the H-measure

$$\mu^{pq} = \mu^{pq}(t, x, \bar{\xi}) \in M_{loc}(\Pi \times S),\ p, q \in P$$

is determined, where $S = S^n = \{\, \bar{\xi} = (\xi_0, \xi) \in \mathbb{R}^{n+1} \,|\, |\bar{\xi}| = 1 \,\}$ is the unit sphere, and the set $P \subset \mathbb{R}$ is defined as in the previous section.

We introduce, as in § 4,

$$V_r(t, x, p) = \nu^r_{t,x}((p, +\infty)) = \operatorname{sign}^+(u_r(t, x) - p),$$
$$V_0(t, x, p) = \nu_{t,x}((p, +\infty)),\ U^p_r(t, x) = V_r(t, x, p) - V_0(t, x, p),$$

here $(t, x) \in \Pi$, $p \in P$. Let us show that the sequence $\nu^r_{t,x}$ satisfies to condition C). Indeed,

$$\mathcal{L}_r^p = \frac{\partial}{\partial t}\int_{(p,+\infty)}(\lambda - p)d\nu_x^r(\lambda) + \mathrm{div}_x\int_{(p,+\infty)}(\varphi(\lambda)-\varphi(p))d\nu_x^r(\lambda) = \frac{\partial}{\partial t}(u_r-p)^+ + \mathrm{div}_x[\mathrm{sign}^+(u_r-p)(\varphi(u_r)-\varphi(p))]$$

and from the equalities

$$(u-p)^+ = (\eta_p(u)+(u-p))/2, \quad \mathrm{sign}^+(u_r-p)(\varphi(u_r)-\varphi(p)) = (\psi_p(u)+\varphi(u)-\varphi(p))/2,$$

relations (7), (9), and (16) it follows that for $p \in F$

$$\mathcal{L}_r^p = -S_r^y(\beta_p), \quad \text{where } \beta_p = (\gamma_p + \gamma)/2.$$

By Lemma 2 the sequence $S_r^y(\beta_p) \to 0$ as $r \to \infty$ in $M_{loc}(\bar{\Pi})$ and, according to the Murat's lemma ([17]), $\mathcal{L}_r^p \to 0$ in H_{loc}^{-1}.

Further, if $f(t,x)$ is a function in Sobolev space $H_2^1(\Pi)$ having compact support $K \subset \Pi$ then, as it easily verified,

$$| < \mathcal{L}_r^p - \mathcal{L}_r^q, f > | \le C_K(|p-q|\cdot\|f_t\|_2 + |\varphi(p)-\varphi(q)|\cdot\|\nabla_x f\|_2) \le C_K(|p-q| + |\varphi(p)-\varphi(q)|)\|f\|_{H_2^1},$$

which implies that the sequence $\mathcal{L}_r^p$ is equicontinuous over the parameter p in $H_{loc}^{-1}(\Pi)$ and, since $\mathcal{L}_r^p \to 0$ in H_{loc}^{-1} for the dense set $p \in F$, this limit relation remains valid for all p. Thus, condition C) is satisfied and we can use the conclusion of Theorem 5. We shall show that the H-measure $\mu^{pq} \equiv 0$. Otherwise, by Theorem 5 one can find a nondegenerate segment $I = [a,b] \in \mathcal{I}$ and $q \in P$ such that $\mu^{pq} \ne 0$ $\forall p \in P \cap I$. We set $v_r = Q_I(u_r)$. As we have seen above, the sequence v_r is strongly convergent. Hence, the H-measure corresponding to this sequence $\tilde{\mu}^{pq} = 0$. Now, since for $p \in P \cap I$ $V_r(t,x,p) = \mathrm{sign}^+(u_r(t,x)-p) = \mathrm{sign}^+(v_r(t,x)-p)$, we conclude that $\mu^{pq} = \tilde{\mu}^{pq} = 0$ for $p,q \in P \cap I$. In particular, $\mu^{pp} = 0$ $\forall p \in P \cap I$. But, as it follows from property 3 in Lemma 5, $\mu^{pq} = 0$ $\forall p \in P \cap I, q \in P$. This contradicts to our choice of the segment I. Thus, $\mu^{pq} \equiv 0$ and by property 4 in Lemma 5 the sequence u_r is strongly convergent as $r \to \infty$. Besides, as it follows from the proof of Theorem 2, the limit function is uniquely defined and equals to $u_0(y)$. So, this limit function does not depend on the choice of the subsequence u_r, and we conclude that the origional sequence $u_m(t,x)$ converges strongly to $u_0(y)$. Thus, for all $y \in Y$

$$u_m = u_m(t,x) = u(\varepsilon_m t, \varepsilon_m x + y) \underset{m\to\infty}{\to} u_0(y).$$

Then, by Theorem 2 $\operatorname*{ess\,lim}_{t\to 0} u(t,\cdot) = u_0$ in $L^1_{loc}(\mathbb{R}^n)$, so that $u = u(t,x)$ has the strong trace at $t = 0$.

According to the method of mathematical induction, the proof is complete.

Remark. The obtained result can be easily extended to more general case when the flux vector in (1) depends also on variables (t,x). Note also that existence of the strong trace could be proved on an arbitrary regular enough noncharacteristic hypersurface. This case reduces to the model case of the hyperspace $t = 0$ after a suitable change of variables.

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