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On correctness classes of locally bounded generalized entropy solutions to Cauchy problem for first order quasilinear equations [1]

Abstract

We study scalar conservation laws with power-like growth restriction on the flux vector. For such equations we found correctness classes for the Cauchy problem among locally bounded generalized entropy solutions. These classes are determined by some exponents of admissible growth with respect to space variables. Example are given, which show that enlargement of the growth exponent leads to failure of the correctness, neither uniqueness nor existence of a solution remains valid.

We consider the Cauchy problem for a first order quasilinear equation

$$u_t + \mathrm{div}_x \varphi(u) = 0, \tag{1}$$

$\varphi = (\varphi_1, \dots, \varphi_n) \in C^1(\mathbb{R}, \mathbb{R}^n), \qquad u = u(t,x), \qquad (t,x) \in \Pi_T = (0,T) \times \mathbb{R}^n,$ $0 < T \le +\infty$, with initial condition

$$u(0,x) = u_0(x) \in L^\infty_{loc}(\mathbb{R}^n). \tag{2}$$

We suppose that the derivative $\varphi'(u)$ satisfies the power-like growth restiction

$$|\varphi'(u)| \le C_0(1 + |u|^{p-1}),\ p > 1,\ C_0 = \mathrm{const}, \tag{3}$$

here and below $|v|$ denotes the Euclidean norm of a vector $v \in \mathbb{R}^n$. Remark that (3) implies the estimate

$$|\varphi(u)| \le C_1(1 + |u|^p),\ C_1 = \mathrm{const}. \tag{4}$$

Let us remind the definition of a generalized entropy solution to the problem (1), (2) in the sense of S.N. Kruzhkov [8].

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Definition 1. A locally bounded function $u = u(t,x) \in L^\infty_{loc}(\Pi_T)$ is called a generalized entropy solution (briefly - g.e.s.) to the Cauchy problem (1), (2) if

a) $\forall k \in \mathbb{R}$

$$|u-k|_t + \mathrm{div}_x[\mathrm{sign}\,(u-k)(\varphi(u)-\varphi(k))] \le 0 \tag{5}$$

in the sense of distributions on Π_T (in $\mathcal{D}'(\Pi_T)$);

b) $\operatorname*{ess\,lim}_{t\to 0+} u(t,\cdot) = u_0$ in $L^1_{loc}(\mathbb{R}^n)$, i.e. there exists a set $\mathcal{E} \subset (0,T)$ of full Lebesgue measure such that $u(t,\cdot) \in L^1_{loc}(\mathbb{R}^n)$ for $t \in \mathcal{E}$, and $u(t,\cdot) \to u_0$ in $L^1_{loc}(\mathbb{R}^n)$ as $t \to 0+$, $t \in \mathcal{E}$.

Condition (5) means that for any test function $f = f(t,x) \in C_0^\infty(\Pi_T)$, $f \ge 0$

$$\int_{\Pi_T} \{|u-k| f_t + \mathrm{sign}\,(u-k)(\varphi(u)-\varphi(k), \nabla_x f)\} dtdx \ge 0,$$

here and below $(\cdot,\cdot)$ is the scalar product on $\mathbb{R}^n$. It is well-known (see for instance [8]) that condition (5) can be rewritten in the equivalent form: for any convex function $\eta(u) \in C^1(\mathbb{R})$ (the entropy)

$$\eta(u)_t + \mathrm{div}_x \psi(u) \le 0 \ \text{ in } \mathcal{D}'(\Pi_T), \tag{6}$$

where $\psi(u) = (\psi_1(u), \ldots, \psi_n(u))$ is the corresponding entropy flux, which is defined up to an additive constant by the equality $\psi'(u) = \eta'(u)\varphi'(u)$. If we require that condition (6) holds for all convex nondecreasing (nonincresing) entropies then we obtain the notion of generalized entropy subsolution (g.e.sub-s.) and, respectively, generalized entropy supersolution (g.e.super-s.) to the problem (1), (2). Rigorous definitions can be found in papers [1, 2, 3, 9, 11, 12].

As directly follows from the definitions, a function $u(t,x) \in L^\infty(\Pi_T)$ is a g.e.s. of the problem (1), (2) if and only if it is g.e.sub-s. and g.e.super-s. of this problem simultaneously. In [11] was shown that maximum of a finite family of g.e.sub-s. (respectively, minimum of a finite family of g.e.super-s.) to the problem (1), (2) is a g.e.sub-s. (g.e.super-s.) of this problem as well with corresponding initial data. Below in the example of nonexistence we shall need the following particular case of this property:

if $u = u(t,x)$ is a g.e.sub-s. (g.e.super-s.) of the problem (1), (2) *and $c \in \mathbb{R}$ then* $\max(u,c)$ *is a g.e.sub-s. of this problem with initial function*

$\max(u_0, c)$ *(respectively,* $\min(u, c)$ *is a g.e.super-s. of* (1), (2) *with initial function* $\min(u_0, c)$ *).*

It is well-known (see [8]) that for the bounded initial function $u_0(x) \in L^\infty(\mathbb{R}^n)$ there exists the unique bounded g.e.s. $u(t,x) \in L^\infty(\Pi_T)$ of (1), (2). More generally, the following "comparison principle" holds (see papers [2, 3, 9, 11, 12], where only continuity of flux vector is assumed).

Proposition 1. *Let* $u_0, v_0 \in L^\infty(\mathbb{R}^n)$, $u_0(x) \le v_0(x)$ *a.e. on* $\mathbb{R}^n$ *and* $u(t,x)$ *be a g.e.sub-s.,* $v(t,x)$ *be a g.e.super-s. of the problem* (1), (2) *with initial data* u_0, v_0 *respectively. Then* $u \le v$ *a.e. on* Π_T.

Applying the comparison principle to a g.e.s. $u(t,x)$ of (1), (2) and the constants $\pm M$, $M = \|u_0\|_\infty$ we derive the "maximum principle": $|u(t,x)| \le M$ a.e. on Π_T.

As was shown in [12], in the case when the flux vector satisfies (3) with $p = 1$ (i.e. it is Lipschitz continuous) the Cauchy problem is well-posed even in a wider class of locally integrable solutions. Namely, for any initial function $u_0(x) \in L^1_{loc}(\mathbb{R}^n)$ there exists the unique g.e.s. $u(t,x) \in L^1_{loc}(\Pi_T)$; besides, the maximum principle for spatial norms $\|u(t,\cdot)\|_q$, $1 \le q \le \infty$ is satisfied. Existence and maximum principles remain true for more general situation when the flux vector is only continuous and satisfies the linear growth restriction, for uniqueness additional conditions are necessary on character of continuity of the flux vector. As was established in [5, 6], if $p > 1$ then the Cauchy problem (1), (2) is ill-posed in the whole space of locally bounded functions: no one of positive results (such as existence, uniqueness, maximum principle) remains generally true. In this connection it is actual to extract correctness classes of the problem (1), (2) in the case $p > 1$, and that is the aim of the presented paper.

Correctness classes. Theorems of existence and uniqueness.
Set $\alpha = (p-1)^{-1}$ and consider spaces

$$\begin{aligned} B^0_\alpha = \{\ u = u(x) \in L^\infty_{loc}(\mathbb{R}^n) \mid \exists M = M_u \\ |u(x)| \le M(1 + |x|^\alpha) \ \text{ a.e. on } \mathbb{R}^n\ \}, \\ \dot{B}^0_\alpha = \{\ u = u(x) \in B^\alpha \mid \operatorname*{ess\,lim}_{|x|\to\infty} |u(x)||x|^{-\alpha} = 0\ \}, \\ B_\alpha = \{\ u = u(t,x) \in L^\infty_{loc}(\Pi_T) \mid \exists M = M(t) \in L^\infty_{loc}([0,T)) \\ |u(t,x)| \le M(t)(1 + |x|^\alpha) \ \text{ a.e. on } \Pi_T\ \}. \end{aligned}$$

It turns out that the space B_α is actually the correctness class for the problem (1), (2) with initial data from B^0_α. More precisely the following theorems on uniqueness and existence hold.

Theorem 1. *Let $u_0(x) \in B^0_\alpha$. Then the g.e.s. $u(t,x) \in B_\alpha$ of the problem (1), (2) is unique.*

Proof. Let $u_1(t,x), u_2(t,x) \in B_\alpha$ be two g.e.s. of the problem (1), (2), and $\tau < T$. Then there exists a constant $M > 0$ (depending on τ) such that

$$|u_1(t,x)| \le M(1+|x|^\alpha),\ |u_2(t,x)| \le M(1+|x|^\alpha) \tag{7}$$

a.e. in Π_τ. As was shown in [8], for $u_1 = u_1(t,x)$, $u_2 = u_2(t,x)$ the following relation takes place

$$|u_1 - u_2|_t + \operatorname{div}_x[\operatorname{sign}(u_1-u_2)(\varphi(u_1)-\varphi(u_2))] \le 0 \ \text{ in } \mathcal{D}'(\Pi_\tau). \tag{8}$$

This relation is proved by the known method of "doubling variables" and it is the base for the proof of uniqueness for a bounded g.e.s. in [8]. By locality of (8) it is clear that this inequality remains true for locally bounded g.e.s. u_1, u_2 as well.

Then, as follows from (7), for a.e. $t \in [0,\tau)$, $|x| > 1$ we have an estimate $\max(|u_1(t,x)|, |u_2(t,x)|) \le 2M|x|^\alpha$. Taking also into account condition (3) we obtain that for a.e. $t \in [0,\tau)$, $|x| > 1$

$$\begin{aligned} |\varphi(u_2)-\varphi(u_1)| = \left|\int_0^1 \varphi'((1-s)u_1+su_2)(u_2-u_1)ds\right| \le \\ \int_0^1 |\varphi'((1-s)u_1+su_2)|ds \cdot |u_2-u_1| \le \\ C_0(1+(\max(|u_1|,|u_2|))^{p-1})|u_1-u_2| \le \\ C_0(1+(2M)^{p-1}|x|)|u_1-u_2| \le C|x||u_1-u_2|, \end{aligned} \tag{9}$$

where $C = C_0(1+(2M)^{p-1})$. Choose a function $\beta(s) \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\beta(s) \subset [0,1]$, $\beta(s) \ge 0$, $\displaystyle\int \beta(s)ds = 1$ and set for $\nu \in \mathbb{N}$ $\delta_\nu(s) = \nu\beta(\nu s)$, $\theta_\nu(t) = \displaystyle\int_{-\infty}^t \delta_\nu(s)ds$. Obviously the sequence $\delta_\nu(s)$ converges to Dirac δ-measure in $\mathcal{D}'(\mathbb{R})$ as $\nu \to \infty$; $0 \le \theta_\nu(t) \le 1$, $\theta_\nu(t) = 0$ for $t \le 0$, $\theta_\nu(t) = 1$ for $t \ge 1/\nu$, and therefore the sequence $\theta_\nu(t)$ converges pointwise as $\nu \to \infty$ to the Heaviside function $\theta(t) = \begin{cases} 0 & , \quad t \le 0 \\ 1 & , \quad t > 0 \end{cases}$.

Let $R > 0$ be such that $r = Re^{-C\tau} > 1$. Choose a function $\rho(u) \in C_0^\infty(\mathbb{R}_+)$ such that $\rho(u) \geq 0$, $\rho'(u) \leq 0$ and $\rho(u) = 1$ for $u \in [0, R]$. Set for $0 < t_0 < t_1 < \tau$, $\nu \in \mathbb{N}$ $\chi_\nu(t) = \theta_\nu(t - t_0) - \theta_\nu(t - t_1)$ and choose a test function $f = f(t, x) = \chi_\nu(t) p(t, x)$, where $p(t, x) = \rho(|x|e^{Ct})$. It is easy to see that for large enough ν the function $f \in C_0^\infty(\Pi_\tau)$, $f \geq 0$. Applying the inequality (8) to the test function f we obtain, after simple transforms, that

$$\int_{\Pi_\tau} |u_1 - u_2|(\delta_\nu(t - t_0) - \delta_\nu(t - t_1))p(t, x)dtdx +$$
$$\int_{\Pi_\tau} [|u_1 - u_2|C|x| + \operatorname{sign}(u_1 - u_2) \times$$
$$(\varphi(u_1) - \varphi(u_2), x/|x|)]e^{Ct}\rho'(|x|e^{Ct})\chi_\nu(t)dtdx \geq 0. \tag{10}$$

Remark that $|x|e^{Ct} \leq R$ for $|x| \leq 1$ and therefore $\rho'(|x|e^{Ct}) = 0$, and for $|x| > 1$ by (9)

$$|(\varphi(u_1) - \varphi(u_2), x/|x|)| \leq |\varphi(u_1) - \varphi(u_2)| \leq C|x||u_1 - u_2|.$$

Thus, with account of the condition $\rho' \leq 0$, the second integral in (10) is nonpositive and we conclude that

$$\int_{\Pi_\tau} |u_1 - u_2|(\delta_\nu(t - t_0) - \delta_\nu(t - t_1))p(t, x)dtdx \geq 0$$

or in the notation $I(t) = \int_{\mathbb{R}^n} |u_1(t, x) - u_2(t, x)|p(t, x)dx$

$$\int I(t)\delta_\nu(t - t_1)dt \leq \int I(t)\delta_\nu(t - t_0)dt. \tag{11}$$

Let $E \subset (0, \tau)$ be a set of full measure of Lebesgue points of the function $I(t)$. Then for $t_0, t_1 \in E$ (11) implies in the limit as $\nu \to \infty$ the inequality $I(t_1) \leq I(t_0)$. Further, by the initial condition $\operatorname*{ess\,lim}_{t_0 \to 0} I(t_0) = \int_{\mathbb{R}^n} |u_0(x) - u_0(x)|p(0, x)dx = 0$ we can conclude that $\forall t = t_1 \in E$

$$I(t) = \int_{\mathbb{R}^n} |u_1(t, x) - u_2(t, x)|p(t, x)dx = 0,$$

that is $|u_1(t, x) - u_2(t, x)|p(t, x) = 0$ a.e. in Π_τ. In particular, $u_1(t, x) = u_2(t, x)$ for a.e. $(t, x) \in \Pi_\tau$ such that $|x| < r = Re^{-C\tau}$. At last, in view of arbitrariness of $r > 1$, $\tau \in (0,T)$ $u_1(t,x) = u_2(t,x)$ a.e. in Π_T, which completes the proof.

Remark that for a g.e.s. $u \in B_\alpha$ the initial function u_0 must lay in B_α^0. This immediately follows from the Definition of B_α and the initial condition b).

As will be shown below, the parameter α in Theorem 1 is exact, for the greater value of this parameter uniqueness of g.e.s. (even the zero solution) is no longer valid.

Now we turn to the problem of existence of g.e.s. We have the following

Theorem 2. *Let $u_0(x) \in B_\alpha^0$. Then in some layer Π_T there exists a g.e.s. $u(t,x) \in B_\alpha$ of the problem (1), (2). Moreover, if $u_0(x) \in \dot{B}_\alpha^0$ then we can set $T = +\infty$, i.e. the solution exists in the whole subspace $t > 0$.*

Proof. Let $M_k = \operatorname*{ess\,sup}_{|x|<2^k} |u_0(x)| 2^{-k\alpha}$, $k \in \mathbb{N}$. By definition of the class B_α^0 $\exists M > 0$ such that $|u_0(x)| \le M(1+|x|^\alpha)$ a.e. on $\mathbb{R}^n$. Therefore for a.e. x, $|x| \le 2^k$ we have an estimate $|u_0(x)| \le M(1+2^{k\alpha}) \le 2M2^{k\alpha}$. Thus, the sequence M_k is bounded: $M_k \le 2M$. Besides if $u_0 \in \dot{B}_\alpha^0$ then $M_k \to 0$ as $k \to \infty$. Indeed, let $\varepsilon > 0$. Choose a natural number r such that $2M2^{-\frac{r\alpha}{2}} < \varepsilon$ and $|u_0(x)| < \varepsilon |x|^\alpha$ for a.e. x, $|x| > 2^{\frac{r\alpha}{2}}$. Then for $k \ge r$ a.e. in a ball $|x| \le 2^{\frac{k}{2}}$

$$|u_0(x)| \le M(1+|x|^\alpha) \le M(1+2^{\frac{k\alpha}{2}}) \le 2M2^{\frac{k\alpha}{2}} = 2M2^{-\frac{k\alpha}{2}}2^{k\alpha} < \varepsilon 2^{k\alpha},$$

while for a.e. x, $2^{\frac{k}{2}} < |x| < 2^k$

$$|u_0(x)| < \varepsilon |x|^\alpha < \varepsilon 2^{k\alpha}.$$

Therefore, $M_k < \varepsilon \; \forall k \ge r$. Since $\varepsilon > 0$ is arbitrary we conclude that $M_k \to 0$.

Now set $\gamma_k = \sup\limits_{r \ge k} M_r$. Then $0 \le M_k \le \gamma_k \le 2M$, the sequence γ_k is nonincreasing, $\gamma_{k+1} \le \gamma_k \; \forall k \in \mathbb{N}$, and $\lim\limits_{k\to\infty} \gamma_k = 0$ in the case $u_0 \in \dot{B}_\alpha^0$.

It is natural to construct the desired g.e.s. of the problem (1), (2) as a limit of the sequence of bounded g.e.s., which can be chosen as follows. We consider for $k \in \mathbb{N}$ cut-off functions

$$u_{0k}(x) = \begin{cases} u_0(x) & , \quad |x| < 2^k \\ 0 & , \quad |x| \ge 2^k \end{cases}$$

and let $u_k = u_k(t,x) \in L^\infty(\Pi)$ be the corresponding unique bounded g.e.s. of the Cauchy problem for equation (1) in the half-space $\Pi = \Pi_\infty$, with initial data u_{0k}. Since $\|u_{0k}\|_\infty \le \gamma_k 2^{k\alpha}$ then by the maximum principle $\|u_k\|_\infty \le \gamma_k 2^{k\alpha}$. Taking into account condition (3) and the fact that for a.e. $(t,x) \in \Pi$

$$\max(|u_k(t,x)|, |u_{k+1}(t,x)|) \le \gamma_k 2^{(k+1)\alpha}$$

we conclude that a.e. on Π

$$\begin{aligned} |\varphi(u_{k+1}) - \varphi(u_k)| \le \max_{|u| \le \gamma_k 2^{(k+1)\alpha}} |\varphi'(u)||u_{k+1} - u_k| \le \\ C_0(1 + (\gamma_k 2^{(k+1)\alpha})^{p-1})|u_{k+1} - u_k| = \\ C_0(1 + (\gamma_k)^{p-1} 2^{(k+1)})|u_{k+1} - u_k| = \\ C_0(2^{-k} + 2(\gamma_k)^{p-1})2^k |u_{k+1} - u_k| = \eta_k 2^k |u_{k+1} - u_k|, \end{aligned} \tag{12}$$

where we denote $\eta_k = C_0(2^{-k} + 2(\gamma_k)^{p-1})$. Remark that the sequence η_k decreases and, in the case $u_0 \in \dot{B}^0_\alpha$, tends to zero as $k \to \infty$.

Set $h_k = (2\eta_k)^{-1}$ and show that $u_{k+1} = u_k$ a.e. in the frustum of a cone

$$D_k = \{ \ (t,x) \mid |x| < 2^k(1 - \eta_k t), 0 < t < h_k \ \}.$$

In fact, since u_k, u_{k+1} are g.e.s. then the Kruzhkov's relation (similar to (8)) holds

$$|u_{k+1} - u_k|_t + \operatorname{div}_x[\operatorname{sign}(u_{k+1} - u_k)(\varphi(u_{k+1}) - \varphi(u_k))] \le 0 \text{ in } \mathcal{D}'(\Pi). \tag{13}$$

We apply this relation to a test function $f = f(t,x) = \chi_\nu(t)p(t,x)$, where the function $\chi_\nu(t) = \theta_\nu(t-t_0) - \theta_\nu(t-t_1)$ approximating the indicator function of the interval $(t_0, t_1]$, $0 < t_0 < t_1 < h_k$ was defined in the proof of Theorem 1, while the function $p(t,x)$ has a form: $p(t,x) = \rho(|x| - 2^k(1 - \eta_k t))$. It is assumed here that $\rho(r) \in C^\infty(\mathbb{R})$, $\rho'(r) \le 0$, $\rho(r) = 0$ for $r \ge 0$, $\rho(r) > 0$ for $r < 0$, and $\rho(r) = 1$ for $r \le -1$.

It is easy to see that $f \ge 0$ and $f \in C_0^\infty(\Pi)$ for large enough $\nu \in \mathbb{N}$. We take into account also that $t < h_k$ on the support of $\chi_\nu(t)$ for large ν and consequently $2^k(1 - \eta_k t) > 2^{k-1} \ge 1$. Therefore, $\rho(|x| - 2^k(1 - \eta_k t)) = 1$ in a vicinity of the singular line $x = 0$, so $f \in C^\infty$.

By (13)

$$\int_\Pi |u_{k+1} - u_k|(\delta_\nu(t - t_0) - \delta_\nu(t - t_1))p(t,x)dtdx +$$

$$\int_\Pi [|u_{k+1}-u_k|2^k\eta_k + \operatorname{sign}(u_{k+1}-u_k)\times$$
$$(\varphi(u_{k+1})-\varphi(u_k), x/|x|)]\rho'(|x|-2^k(1-\eta_k t))\chi_\nu(t)dtdx \ge 0. \qquad (14)$$

Then, in view of (12), a.e. on Π

$$|(\varphi(u_{k+1})-\varphi(u_k), x/|x|)| \le |\varphi(u_{k+1})-\varphi(u_k)| \le 2^k\eta_k|u_{k+1}-u_k|.$$

This, together with the condition $\rho'(\cdot)\le 0$, implies that the second integral in (14) is nonpositive and therefore

$$\int_\Pi |u_{k+1}-u_k|(\delta_\nu(t-t_0)-\delta_\nu(t-t_1))p(t,x)dtdx \ge 0$$

or using the notation $I(t)=\int_{\mathbb{R}^n}|u_{k+1}(t,x)-u_k(t,x)|p(t,x)dx$

$$\int I(t)\delta_\nu(t-t_1)dt \le \int I(t)\delta_\nu(t-t_0)dt.$$

In the limit as $\nu\to\infty$ we obtain that $I(t_1)\le I(t_0)$ for all $t_0,t_1\in E$, $t_0<t_1$, where $E\subset(0,h_k)$ is a set of full measure consisting of Lebesgue points of the function $I(t)$. By initial conditions $u_{k+1}(0,x)=u_{0k+1}$, $u_k(0,x)=u_{0k}$ we have also the relation

$$\operatorname*{ess\,lim}_{t_0\to 0} I(t_0)=\int_{\mathbb{R}^n}|u_{0k+1}(x)-u_{0k}(x)|\rho(|x|-2^k)dx=0$$

because $\rho(|x|-2^k)=0$ for $|x|\ge 2^k$. Thus, $\forall t=t_1\in E\ I(t)\le \operatorname*{ess\,lim}_{t_0\to 0} I(t_0)=0$. This means that $|u_{k+1}(t,x)-u_k(t,x)|p(t,x)=0$ a.e. on D_k. Taking into account the condition $p(t,x)>0$ on D_k, one can conclude that $u_{k+1}(t,x)=u_k(t,x)$ a.e. on D_k.

Since the sequence η_k decreases while the sequence h_k increases $D_k\subset D_{k+1}$ $\forall k\in\mathbb{N}$ and $\bigcup_{k\in\mathbb{N}} D_k=\Pi_T$ with $T=\lim_{k\to\infty} h_k\le+\infty$. Hence there exists an unique (up to equality on a set of full measure) $u=u(t,x)\in L^\infty_{loc}(\Pi_T)$ such that $u(t,x)=u_k(t,x)$ on D_k. Obviously, $u(t,x)$ is a g.e.s. of the origional problem (1), (2). We show that $u\in B_\alpha$. Let $\tau<T$. We can find such $r\in\mathbb{N}$ that $\tau<h_k$ $\forall k\ge r$. Let $(t,x)\in\Pi_\tau$ be a Lebesgue point of $u(t,x)$, $M_1=\gamma_{r+1}2^{(r+1)\alpha}$. In the case $|x|<2^r$ we see that $(t,x)\in D_{r+1}$ and

$$|u(t,x)|=|u_{r+1}(t,x)|\le\gamma_{r+1}2^{(r+1)\alpha}=M_1\le M_1(1+|x|^\alpha).$$

While for $|x| \geq 2^r$ we can find $k \geq r$ such that $2^k \leq |x| < 2^{k+1}$. In particular, $(t,x) \in D_{k+2}$ and

$$|u(t,x)| = |u_{k+2}(t,x)| \leq \gamma_{k+2} 2^{(k+2)\alpha} \leq \gamma_{r+1} 2^{2\alpha} 2^{k\alpha} \leq M_1 |x|^\alpha \leq M_1(1+|x|^\alpha).$$

Thus, for a.e. $(t,x) \in \Pi_\tau$ $|u(t,x)| \leq M_1(1+|x|^\alpha)$ and therefore $u \in B_\alpha$ in the layer Π_T. Remark also that $T = +\infty$ if $u_0(x) \in \dot{B}^0_\alpha$ because in this case $h_k \to \infty$. The proof is complete.

Remark 1. If $u_0 \in B^0_\alpha \setminus \dot{B}^0_\alpha$ then a g.e.s. can generally exist only in a layer Π_T of finite width T. Indeed, for $n = 1$ consider the problem

$$u_t - (|u|^p/p)_x = 0, \quad u(0,x) = |x|^{\alpha-1}x. \tag{15}$$

One can easily verify that the function $u(t,x) = |x/(1-t)|^{\alpha-1}x/(1-t)$ belongs to B_α in Π_1 and it is a g.e.s. of (15). We demonstrate that there is no g.e.s. $\tilde{u} = \tilde{u}(t,x) \in B_\alpha$ of the problem (15) defined in a larger layer Π_T, $T > 1$. If in the contrary such g.e.s. exists then by Theorem 1 $\tilde{u} = u(t,x)$ a.e. in Π_1 and, since the function $u(t,x)$ is not bounded near points of the line $t = 1$, $\tilde{u} \notin L^\infty_{loc}(\Pi_T)$. The latter contradicts to the requirement $\tilde{u} \in B_\alpha$.

Example of nonuniqueness of g.e.s.

In a half-plane $\Pi = \{\ (t,x) \mid x \in \mathbb{R},\ t > 0\ \}$ we consider the equation

$$u_t + \varphi(u)_x = 0,\ u = u(t,x) \tag{16}$$

with the power-like flux function $\varphi(u) = |u|^{p-1}u$, $p > 1$.

In [5, 6] for equation (16) with cubic nonlinearity ($p = 3$) the family of nontrivial g.e.s. of (16), (2) with zero initial data was found, on the basis of one construction of A.Yu. Goritsky from [4]. It is clear that the unique bounded g.e.s. of this problem is the zero function.

Here we present generalization of these results to the case of an arbitrary parameter $p > 1$ and construct nontrivial g.e.s. of (16), (2) with zero initial data. These g.e.s. lie in the classes B_μ, where $\mu > \alpha = (p-1)^{-1}$ is arbitrary. Thus we show that the parameter α in the definition of the correctness class B_α is exact and can not be chosen greater.

Solutions, which we intend to construct, will be piecewise smooth. As is known (see [10, 7, 8]), a piecewise smooth function $u(t,x)$ is a g.e.s. of the equation (16) if and only if the following conditions hold:

1) $u(t,x)$ is a classical solution of equation (16) in domains of smoothness;

2) on curves of discontinuity $x = x(t)$ the Rankine-Hugoniót condition holds

$$\varphi(u^+) - \varphi(u^-) = x'(t)(u^+ - u^-),$$

where $u^+ = u^+(t) = \lim\limits_{x \to x(t)+} u(t,x)$, $u^- = u^-(t) = \lim\limits_{x \to x(t)-} u(t,x)$ are the on-sided limits of the function $u(t,x)$ on a discontinuity line;

3) the following stability conditions is satisfied: for $u^+ > u^-$ ($u^+ < u^-$) the graph of the function $\varphi(u)$ has no points below (respectively - above) the hord connecting the points of this graph with first coordinates u^-, u^+.

One of characteristic property of equation (16) is that it admits the semi-group of transforms $t \to t/s$, $x \to xs^\gamma$, $u \to us^\beta$, where $s > 0$ is arbitrary, and the fixed parameters β, γ are connected by a relation $(p-1)\beta + \gamma = -1$. Therefore, equation (16) has invariant solution of the kind

$$u(t,x) = t^\beta v(xt^\gamma). \tag{17}$$

In the particular case $\beta = 0$, $\gamma = -1$ solutions of the form (17) are well-known *self-similar solutions* $u = v(x/t)$. Excepting this case assume that $\beta \neq 0$.

The conditions 1)-3) for functions (17) can be reformulated in terms of the piecewise smooth function $v(\xi)$, $\xi \in \mathbb{R}$. Obviously, $u(t,x)$ is a g.e.s. of equation (16) if and only if on intervals of smoothness $v(\xi)$ satisfies the ordinary differential equation

$$\beta v d\xi + (\gamma\xi + p|v|^{p-1})dv = 0, \tag{18}$$

and at any point of discontinuity $\xi = \bar\xi$ the Rankine-Hugoniót condition

$$\varphi(v^+) - \varphi(v^-) + \gamma\bar\xi(v^+ - v^-) = 0 \tag{19}$$

is satisfied together with the stability condition formulated in the same way as condition 3), in which we only replace u^+ and u^- to v^+ and v^- respectively, where $v^+ = v(\bar\xi+) = \lim\limits_{\xi \to \bar\xi+} v(\xi)$, $v^- = v(\bar\xi-) = \lim\limits_{\xi \to \bar\xi-} v(\xi)$ are one-sided limits of the function $v(\xi)$ at the point $\bar\xi$.

Equation (18) is linear with respect to ξ, its general solution

$$\xi = \xi_C(v) = p|v|^{p-1} - C|v|^{-\gamma/\beta} = (p - C|v|^{1/\beta})|v|^{p-1}, \tag{20}$$

where C is an arbitrary constant. Equality (20) defines the function $v(\xi)$ on intervals of strict monotonicity of the function $\xi(v)$. As in [5, 6] (where

$p=3$, $\gamma=-\beta=1$), suppose that $\beta<0$, $\gamma>0$, $C=C_1>0$. Then the function $\xi(v)$ strictly increases on the interval $(v_1^+,+\infty)$, $v_1^+=\left(\frac{p(1-p)\beta}{\gamma C_1}\right)^\beta$, the corresponding inverse function $v=v_{C_1}(\xi)\in C^1(\xi_1,+\infty)$, where $\xi_1=\xi(v_1^+)=-\frac{p}{\gamma}(v_1^+)^{p-1}=\min_{v\geq 0}\xi(v)$. Obviously, functions $\pm v$ satisfy to equation (18). Extend the function $v(\xi)$ outside of the interval $(\xi_1,+\infty)$ assuming that ξ_1 is a point of admissible discontinuity. We have $v^+=v_1^+$, and by condition (19) $v^-=-av_1^+$, where $a>1$ is a unique solution of the equation $a^p-pa+(1-p)=0$. We find a constant C_2 such that $v^-=-v_2(\xi_1)$, where we denote $v_2(\xi)=v_{C_2}(\xi)$. The constant $C=C_2$ is uniquely determined by the equality $\xi_C(av_1^+)=\xi_1$. It is directly verified that $C_2=lC_1$, where $l=\frac{\gamma+a^{1-p}}{a^{1/\beta}(1-p)\beta}$. Since $\min\xi(v)<\xi_1$ for $C=C_2$ then $l>1$ (this also follows from increasing of the function $h(a)=\gamma+a^{1-p}-a^{1/\beta}(1-p)\beta$ with respect to $a\geq 1$ and the equality $h(1)=0$). We need also show that $-v^-=av^+>v_2^+$, where $v_2^+=\left(\frac{p(1-p)\beta}{\gamma C_2}\right)^\beta$ is the minimum point of $\xi(v)$, to garantee that $-v^-$ belongs to the interval of increasing for $\xi(v)$ and therefore $v^-=-v_2(\xi_1)$, as required.

To verify this inequality $av^+>v_2^+$ remark that it is equivalent to the inequality $a^{-1/\beta}>l$ and reduces to the condition $a^{1-p}<(1-p)\beta-\gamma=1$, which is satisfied because $a>1$.

It is easy to see that the function $v_2(\xi)$ is defined for $\xi>\xi_2=r\xi_1$, where $r=l^{\beta(1-p)}>1$.

In the same way we can take ξ_2 as the next point of discontinuity and prolong $v(\xi)$ on the interval $(\xi_3,+\infty)$, $\xi_3=r\xi_2$. Iterating this process of prolongation, we get finally the function

$$v(\xi)=\begin{cases} v_1(\xi) & \text{for} \quad \xi>\xi_1, \\ (-1)^{k-1}v_k(\xi) & \text{for} \quad \xi_{k-1}\geq\xi>\xi_k, k>1, \end{cases}$$

where $v_k(\xi)=v_{C_k}(\xi)$, $C_k=C_1l^{k-1}$, $\xi_k=\xi_1r^{k-1}$, which is defined for all $\xi\in\mathbb{R}$. By the construction condition (19) is satisfied at the points of discontinuity ξ_k. The stability condition is also satisfied. Indeed, the chord to the graph of $\varphi(v)$ connecting the points with first coordinates $v_k^+=v(\xi_k+)=(-l^{-\beta})^{k-1}v_1^+$, $v_k^-=v(\xi_k-)=-av_k^+$ is tangent to this graph

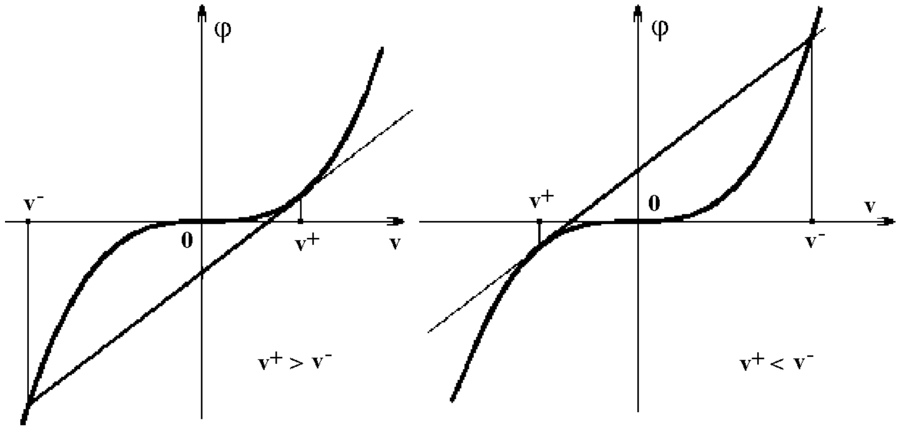

Figure 1:

at the point $(v_k^+, \varphi(v_k^+))$, by the condition $(a^p + 1)/(a + 1) = p$, and the mutual arrangement of the graph and the chord is as required, see Figure 1.

To complete the construction connect the function above with the zero function on the right by an admissible point of discontinuity ξ_0. By Rankine-Hugoniót condition (19) we conclude that $(v_1(\xi_0))^{p-1} = -\gamma\xi_0$, and $\xi_0 = -\frac{1}{\gamma}\left(\frac{p\gamma+1}{\gamma C_1}\right)^{(p-1)\beta}$, where $0 > \xi_0 > \xi_1$. The corresponding value $v_1(\xi_0) = \left(\frac{p\gamma+1}{\gamma C_1}\right)^{\beta} > v_1^+$.

Since $u = 0$ is an inflection point of the function $\varphi(u)$ then the stability condition is satisfied at the point of discontinuity ξ_0. Thus, the function $\bar{v}(\xi) = \begin{cases} 0 & , \ \xi > \xi_0 \\ v(\xi) & , \ \xi \le \xi_0 \end{cases}$ generates the g.e.s. $u = t^{\beta}v(xt^{\gamma})$ of equation (16), see Figure 2.

Since $u(t,x) = 0$ in a neighborhood of the line $t = 0$ then $u(t,x)$ satisfies initial condition (2) with zero initial fiunction in the sense of relation b) in Definition 1. Thus, for any fixed values β, γ we construct a one-parameter (with parameter C_1) family of nontrivial g.e.s. of the problem (16), (2) with zero initial data. Now, we show that the following estimate holds for

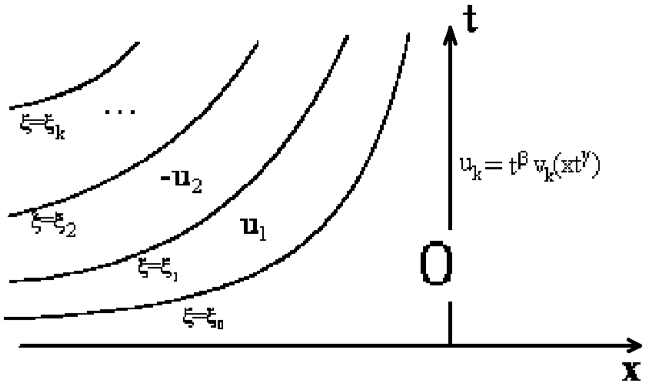

Figure 2:

the constructed solutions

$$|u(t,x)| \leq C|x|^{-\frac{\beta}{\gamma}} = C|x|^{\frac{\gamma+1}{(p-1)\gamma}},\ C = \text{const.} \tag{21}$$

Firstly, we prove the estimate

$$|v(\xi)|^{p-1} \leq M|\xi|,\ M = \gamma\left(1 + \frac{(p-1)}{pa}\right). \tag{22}$$

In fact, if $\xi > \xi_0$ then $v(\xi) = 0$ and (22) is valid. For $\xi_1 < \xi \leq \xi_0$ we have that $v(\xi) = v_1(\xi) \leq v_1(\xi_0) = \left(\dfrac{p\gamma+1}{\gamma C_1}\right)^{\beta}$. Therefore, $C_1|v(\xi)|^{1/\beta} - p \geq 1/\gamma > 0$. Then by equality (20)

$$|v(\xi)|^{p-1} \leq \gamma|\xi|.$$

At last, if $\xi_k < \xi \leq \xi_{k-1}$, $k > 1$ then $|v(\xi)| = v_k(\xi) \leq v_k(\xi_{k-1}) = av_{k-1}^{+}$ and taking into account the relation $\xi_{k-1} = -\dfrac{p}{\gamma}(v_{k-1}^{+})^{p-1}$ we obtain estimate (22):

$$|v(\xi)|^{p-1} \leq a^{p-1}(v_{k-1}^{+})^{p-1} = \frac{\gamma a^{p-1}}{p}|\xi_{k-1}| \leq$$
$$\frac{\gamma a^{p-1}}{p}|\xi| = \gamma\left(1 + \frac{(p-1)}{pa}\right)|\xi|.$$

To prove (21) remark firstly that for $\xi = xt^\gamma > \xi_0$ $u(t,x) = 0$ and (21) is satisfied. For $\xi \le \xi_0$ (22) implies that for $M_1 = M^{1/(p-1)}$

$$|u(t,x)| = t^\beta |v(\xi)| \le M_1 t^\beta |\xi|^{1/(p-1)} =$$
$$M_1 |\xi|^{-\frac{1}{\gamma(p-1)}} (t^\beta |\xi|^{-\frac{\beta}{\gamma}}) \le M_1 |\xi_0|^{-\frac{1}{\gamma(p-1)}} |x|^{-\frac{\beta}{\gamma}} = \mathrm{const}|x|^{-\frac{\beta}{\gamma}},$$

as required.

Remark that in (21) the parameter $\gamma > 0$ is arbitrary and $\dfrac{\gamma+1}{(p-1)\gamma} \to \alpha = (p-1)^{-1}$ as $\gamma \to +\infty$. Therefore, for any $\mu > \alpha$ one can choose the parameters in such way that the constructed solutions $u(t,x) \in B_\mu$ on a half-plane Π. We conclude that for larger value of α uniqueness of a g.e.s (even trivial one) is no longer valid.

Example of nonexistence of a g.e.s. Previously, nonexistence of g.e.s. was demonstrated in [6]. In this section we extend the example from [6], and show that for $u_0 \in B^0_\mu$, $\mu > \alpha$ g.e.s. of the problem (1), (2) can be absent in any strip Π_T. Hence, we establish that in the existence theorem the parameter α is exact.

For $n = 1$ we consider the problem

$$u_t + \varphi(u)_x = 0, \quad \text{where } \varphi(u) = (\max(0,u))^p, \ p > 1, \tag{23}$$
$$u(0,x) = |x|^\mu, \ \mu > \alpha = (p-1)^{-1}, \tag{24}$$

and show that this problem has no locally bounded g.e.s. in any strip Π_T. Assuming the contrary, choose $T > 0$ and a g.e.s. $u = u(t,x) \in L^\infty_{loc}(\Pi_T)$ of the problem (23), (24). Firstly, show that $u \ge 0$ a.e. on Π_T. Consider the function $v = \min(u,0)$. Then $v = v(t,x)$ is a g.e.super-s. of equation (23) with zero initial function and, as follows from condition (6) with $\eta(u) = -u$, $v_t + \varphi(v)_x \ge 0$ in $\mathcal{D}'(\Pi_T)$. Since $v \le 0$ then $\varphi(v) = 0$ and we conclude that $v_t \ge 0$ in $\mathcal{D}'(\Pi_T)$. But $v(0,x) = 0$ (in the sense of condition b) in Definition 1) and $v \le 0$. Thus, in fact $v = 0$, i.e. $u \ge 0$ a.e. on Π_T.

Now, we set for $k \in \mathbb{N}$

$$u_{0k} = u_{0k}(x) = \begin{cases} 0 & , \quad x \ge -k, \\ k^\mu & , \quad x < -k \end{cases}$$

and let $u_k = u_k(t,x)$ be a unique bounded g.e.s. of the Cauchy problem for equation (23) with initial data u_{0k}. It is easy to verify that this solution has the form of "shock wave"

$$u_k(t,x) = \begin{cases} 0 & , \quad x \ge -k + k^{\mu(p-1)}t, \\ k^{\mu} & , \quad x < -k + k^{\mu(p-1)}t. \end{cases}$$

By the condition $\mu > \alpha$ we see that $-k + k^{\mu(p-1)}t \to +\infty$ as $k \to \infty$ $\forall t > 0$ and therefore $u_k(t,x) \to +\infty$ on Π_T.

Let $v_k = v_k(t,x) = \min(u, k^{\mu})$. Then v_k is a bounded (by nonnegativity of u) g.e.super-s. of equation (23) with initial data $v_{0k} = \min(u_0, k^{\mu}) \ge u_{0k}$. Since u_k is a g.e.sub-s. of this equation with initial data u_{0k} then by comparison principle (see Proposition 1) $u_k(t,x) \le v_k(t,x) \le u(t,x)$ a.e. on Π_T $\forall k \in \mathbb{N}$. Passing to the limit as $k \to \infty$ we claim that $u(t,x) = +\infty$ a.e. on Π_T, which is impossible. Thus, the problem under consideration has no locally bounded g.e.s.

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